Response to Comments on "PCA Based Hurst Exponent Estimator for fBm Signals Under Disturbances"

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Abstract

In this response, we try to give a repair to our previous proof given in Appendix of [9] by using orthogonal projection. Moreover, we answer the question raised in [16]: If a centered Gaussian process G_t admits two series expansions on different Riesz bases, we may possibly study the asymptotic behavior of one eigenvalue sequence from the knowledge on the asymptotic behaviors of another.

1   The Backgrounds

Many thanks to the note of Prof. Zanten [16], a flaw was found to lie in Appendix of Li, et al. [9], which tries to give another proof for the asympotitics of the eigenvalues for Karhunen-Loève expansion of fBm process. Fortunately, all the theorems in the mainbody of [9] still holds, due to the nice proof of [7]-[8]. In the rest of this response, we will try to fix our uncompleted proof in [9] and answer a related question raised in [16].

Let us briefly recall some backgrounds of our discussions. Suppose $B = \{B_t, t \geq 0\}$ is a standard fBm process within a finite time interval $[0, 1]$ (can be scaled to $[0, T]$, but it does not matter our proof) and with Hurst exponent H $(0 < H < 1)$.

The autocorrelation function of B_t can be written as [1]

$$R_b(s, t) = E\left[B_s B_t\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |s - t|^{2H}\right) \tag{1}$$

According to Mercer's theorem [2]-[3], we have

$$R_b(s, t) = \sum_{n=1}^{\infty} \lambda_n \phi_n(s)\phi_n(t) \tag{2}$$

$$\int_0^1 R_b(s, t)\phi_n(t)dt = \lambda_n \phi_n(s) \tag{3}$$

1

where $\{\phi_n(t)\}_{n=1}^\infty$ is a set of orthonormal functions in the interval $[0,1]$, where λ_n are the corresponding eigenvalues of the nth orthonormal functions.

As shown in [4]-[9], Eq.(3) is the continuous Karhunen-Loève (K-L) expansion for fBm process and λ_n is the associated eigenvalues. Our main problem here is to discuss the asympotics of λ_n.

2 Prof. Bronski's Proof

In [7]-[8], Prof. Bronski had showed that $\lambda_n \sim n^{-2H-1}$ as follows.

Clearly, from Eq.(2)-(3), we get a integral kernel $[T_\kappa \phi](x) = \int_0^1 R_b(x,y)\phi(y)dy$ on $L_2([0,1] \times [0,1])$. Moreover, this operator T_κ is a non-negative symmetric, Hilbert-Schmidt and compact.

We can then prove the rigorous estimates of the eigenvalues by considering the Nyström approximation of this kernel [10]-[12] on a special orthonormal basis $\phi_n(x) = \{\sqrt{2}\sin((n+\frac{1}{2})\pi x)_{n=0}^\infty\}$. Particularly in [7]-[8], the operator T_κ is approximated by an operator A from the sequence space l_2 to l_2, which has matrix elements

$$A_{n,m} = \langle \phi_n(x)A\phi_m(y)\rangle = 2\int_0^1\int_0^1 R_b(x,y)\sin((n+\frac{1}{2})\pi x)\sin((m+\frac{1}{2})\pi x)dxdy \tag{4}$$

We can also consider $x^T A y$ with the kernel matrix $A_{n,m}$ as a n-degenerate approximation of the Mercer kernel function $R_b(x,y)$.

By examining the leading order diagonal piece D and the higher order off-diagonal piece OD of A ($A = D + OD$), Bronski proved that $OD_{n,m}$ has higher order and can be neglected with respect to $D_{n,m}$. Thus, the eigenvalues of A is mainly determined by D.

In [7]-[8], Bronski further proved that

$$\frac{\sin(\pi H)\Gamma(2H+1)}{n^{2H+1}} + \epsilon_{left} \leq \lambda_n(A) \leq \frac{\sin(\pi H)\Gamma(2H+1)}{n^{2H+1}} + \epsilon_{right} \tag{5}$$

where ϵ_{left} and ϵ_{right} are neglectable items.

Thus, we reach the conclusion we desired

$$\lambda_n(T_\kappa) \approx \lambda_n(A) \sim n^{-2H-1} \tag{6}$$

3 Another Proof that Actually Detours

3.1 The Proof in Appendix of [9]

In [9], we consider the series expansion of the fBm process on a set of orthonormal basis functions $\phi_n(t)$

$$B_t = \sum_{n=1}^\infty c_n \phi_n(t) \tag{7}$$

2

where c_n is the corresponding coefficient satisfying

$$E\{c_n c_m\} = \lambda_n \delta[n - m] \tag{8}$$

If we can obtain the representation of c_n, we can directly get λ_n via Eq.(5).

However, in [9] (or equivalently http://arxiv.org/abs/0805.3002v1), we instead study another series expansion of the fBm process on a set of special basis functions $\psi_n(t)$ proposed in [13]-[15]

$$B_t = \sum_{n=1}^{\infty} b_n \psi_n(t) \tag{9}$$

where $\{\psi_n(t)\}_{n=1}^{\infty}$ is a set of linearly independent but not orthogonal basis functions. Thus, the expansion coefficients b_n is not equivalent to the eigenvalues of the Karhunen-Loève expansion.

The Appendix of [9] proves the $E\{b_n\} \sim n^{-2H-1}$. But as shown in [16], it is just a intermediate result for our final goal.

3.2   A Remedy

Because the problem lies in the orthogonality, we will give a remedy for our proof by orthogonal projection.

More precisely, we will project these functions $\{\psi_n(t)\}_{n=1}^{\infty}$ to a set of orthonormal basis functions $\{\phi_n(t)\}_{n=1}^{\infty}$ as

$$\psi_n(t) = \sum_{k=1}^{\infty} \mu_{n,k} \phi_k(t) \tag{10}$$

where $\mu_{n,k} = \mathrm{proj}_{\phi_n(t)}(\psi_k(t)) = \left(\int_0^1 \phi_n(t) \psi_k(t) dt \right)$.

Based on Eq.(7), (9)-(10), we have

$$B_t = \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=1}^{\infty} b_n \psi_n(x) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \mu_{n,k} b_k \right) \phi_n(x) \tag{11}$$

Thus, we can study the eigenvalue asymptotics of c_n from

$$c_n = \sum_{k=1}^{\infty} \mu_{n,k} b_k \tag{12}$$

This method is similar to what had been applied in [17]-[18]. We will discuss when such projection is valid at the end of this response.

The success of Prof. Bronski [7]-[8] inspired us to choose the orthonormal basis functions $\phi_n(t) = \{\sqrt{2} \sin((n + \frac{1}{2})\pi t)_{n=0}^{\infty}\}$. Because the Karhunen-Loève expansion for brownian motion $H = \frac{1}{2}$ is well known [19], [13], we will focus on the cases that $H \neq \frac{1}{2}$.

As pointed out in [9]-[15], we can expand a standard fBm process B_t as

$$B_t = \sum_{n=1}^{\infty} \left(z_n \frac{\sin(x_n t)}{x_n} + w_n \frac{1 - \cos(y_n t)}{y_n} \right) \tag{13}$$

where x_n are the positive zeros of the Bessel function J_{-H} of the first kind, y_n are the positive zeros of the Bessel function J_{1-H} of the first kind. As shown in [9], we have

$$x_n = n\pi + h_1 + O(n^{-1}), \qquad y_n = n\pi + h_2 + O(n^{-1}) \tag{14}$$

where h_1 and h_2 are constants.

z_n and w_n are independent sequences of independent, centered Gaussian random variables on a common probability space, with

$$E[z_n] = E[w_n] = 0 \tag{15}$$

$$E[z_n^2] = \frac{2c_H^2}{x_n^{2H} J_{1-H}^2(x_n)}, \qquad E[w_n^2] = \frac{2c_H^2}{y_n^{2H} J_{-H}^2(y_n)} \tag{16}$$

where $c_H^2 = \frac{\Gamma(1+2H)\sin(\pi H)}{\pi}$.

Let us first examine the projection of $\{\frac{\sin(x_n t)}{x_n}\}_{n=1}^{\infty}$. Given $n, k \in \mathbb{N}$, we can obtain the projection coefficients $\hat{\mu}_{n,k}$ as

$$
\begin{aligned}
&\hat{\mu}_{n,k} \\
&= \int_0^1 \frac{\sin(x_k t)}{x_k} \sqrt{2} \sin([n - \tfrac{1}{2}]\pi t) dt \\
&= \frac{\sqrt{2}}{2x_k} \left[\frac{\sin(x_k - [n - \tfrac{1}{2}]\pi)}{x_k - [n - \tfrac{1}{2}]\pi} - \frac{\sin(x_k + [n - \tfrac{1}{2}]\pi)}{x_k + [n - \tfrac{1}{2}]\pi} \right] \\
&= \frac{\sqrt{2} d_1}{x_k} \left(\frac{\left(x_k + [n - \tfrac{1}{2}]\pi\right) + \left(x_k - [n - \tfrac{1}{2}]\pi\right)}{x_k^2 - [n - \tfrac{1}{2}]^2 \pi^2} + O(n^{-2}) \right) \\
&= \frac{2\sqrt{2} d_1}{x_k^2 - [n - \tfrac{1}{2}]^2 \pi^2} + \frac{\sqrt{2} d_1}{x_k} O(n^{-2})
\end{aligned} \tag{17}
$$

where d_1 is a constant.

As shown in [9], we have

$$J_{1-H}(x_k) = d_2 x_k^{-1/2} + O(x_k^{-3/2}) \tag{18}$$

where d_2 is a postiche constant.

Thus, based on Eq.(14) and (18), we have

$$\sum_{k=1}^{\infty} E[z_k^2] \hat{\mu}_{n,k}^2$$

4

$$= \sum_{k=1}^{\infty} \frac{16 c_H^2 d_1^2}{x_k^{2H} J_{1-H}^2(x_k) \left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2} + O(n^{-3})$$

$$= d_3 \sum_{k=1}^{\infty} \frac{x_k^{-2H+1}}{\left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2} + O(n^{-3})$$

(19)

where d_3 is a positive constant.

It is easy to show that

$$\sum_{k=1}^{\infty} \frac{x_k^{-2H+1}}{\left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2} > \frac{x_k^{-2H+1}}{\left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2} \Bigg|_{k=n} = d_3 n^{-2H-1} + O(n^{-3})$$

(20)

and

$$\sum_{k=1}^{\infty} \frac{x_k^{-2H+1}}{\left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2} < d_3 \sum_{k=1}^{\infty} \frac{x_k^{-2H+1}}{\left[(2k-1)n \right]^2} + O(n^{-3}) = d_4 n^{-2H-1} + O(n^{-3})$$

(21)

where d_3 and d_4 are positive constants.

Noticing that $\sum_{k=1}^{\infty} \frac{x_k^{-2H+1}}{\left[x_k^2 - [n-\frac{1}{2}]^2 \pi^2 \right]^2}$ converges, based on (13)-(15), we have

$$\sum_{k=1}^{\infty} E\left[z_k^2 \right] \hat{\mu}_{n,k}^2 = d_5 n^{-2H-1} + O(n^{-3}) \sim n^{-2H-1}$$

(22)

where d_5 is a positive constant.

Similarly, we can prove that the projection coefficients of $\{ \frac{1-\cos(y_n t)}{y_n} \}_{n=1}^{\infty}$ satisfies

$$\sum_{k=1}^{\infty} E\left[w_k^2 \right] \tilde{\mu}_{n,k}^2 \sim n^{-2H-1}$$

(23)

where $\tilde{\mu}_{n,k} = \int_0^1 \frac{1-\cos(y_k t)}{y_k} \sqrt{2} \sin([n-\frac{1}{2}]\pi t) dt$.

Due to the independence of z_k and w_k, we have

$$\lambda_n = E\left[c_n^2 \right] = \sum_{k=1}^{\infty} \left(E\left[z_k^2 \right] \hat{\mu}_{n,k}^2 + E\left[w_k^2 \right] \tilde{\mu}_{n,k}^2 \right) \sim n^{-2H-1}$$

(24)

Therefore, our proof in [9] is repaired.

In summary, the nice proof given by Bronski in [7]-[8] is a direct attack on the problem, and our proof detours. However, the appendix in [9] plus this response gives another view on the asympotics of K-L expansion of fBm process and meanwhile shows how the important results obtained in [7]-[8] and [13]-[15] can be linked together.

4 Some Discussions

Finally, we would like to discuss the interesting question raised in [16]: *If a centered Gaussian process G_t admits two different series expansions, under what conditions do the two eigenvalue sequences have the same asymptotic behavior?*

We think this question can be partly solved by evaluating the mapping operator T between the two sets of basis functions. If the mapping T consists of **appropriate** projection coefficients, the asymptotics of the eigenvalues can still be held.

The general cases are obviously too difficult to solve in this short response. In the follows, we will briefly discuss a special case: when one basis is a Riesz basis and the other is a orthonormal basis.

Suppose we have a expansion of the integral kernel K is in $L_2([0,1] \times [0,1])$ on a Riesz basis $\{\psi_n(t)\}_{n=1}^\infty$ (no need to be orthogonal) as $K(s,t) = \sum_{n=1}^\infty \tau\psi_n(s)\psi_n(t)$; and meanwhile we have the K-L expansion of K on a orthonormal basis $\{\phi_n(t)\}_{n=1}^\infty$ in the same space as $K(s,t) = \sum_{n=1}^\infty \lambda\phi_n(s)\phi_n(t)$.

Based on the property of Riesz basis [20]-[23], we can always find a linear bounded bijective operator T satisfying $\{\psi_n(t)\}_{n=1}^\infty = \{T\phi_n(t)\}_{n=1}^\infty$.

The basis function used in [13]-[15] can be viewed as a special Riesz basis, which satisfying the above requirement. Thus, we can study the asymptotics of K-L eigenvalues by using orthogonal projection.

Since $\{\psi_n(t)\}_{n=1}^\infty$ is a Riesz basis, it will associate with a set of Riesz sequence. Hence, if A is the infinite matrix representing this bounded linear operator T, the sequence $\{A_{k,n}\}_{k=1}^\infty$ formed by the nth column of A is a Bessel sequence in l^2.

Assume that λ_n and τ_n have the same asymptotics. According to [24]-[29], the upper bound for the decaying rate of the eigenvalues for a smooth Mercer kernel is $O(n^{-1})$. Thus, we have $\lambda_n, \tau_n \sim n^{-p}$, $p > 1$.

Since $\{A_{k,n}\}_{k=1}^\infty$ is a Bessel sequence, $\sum_{k=1}^\infty A_{k,n}^2 \tau_k \leq C_1 \sum_{k=1}^\infty \tau_k = C_2$ when $\tau_n \sim n^{-p}$, C_1 and C_2 are constants. Thus, $\sum_{k=1}^\infty A_{k,n}^2 \tau_k$ converges. Based on the mapping relation, we have

$$\lambda_n = \sum_{k=1}^\infty A_{k,n}^2 \tau_k \tag{25}$$

or equivalently

$$d_6 n^{-p} = \sum_{k=1}^\infty A_{k,n}^2 k^{-p} + O(n^{-p}) \tag{26}$$

which indicates that given a $n \in \mathbb{N}$, the maximum value of $A_{k,n}$ in terms of k locates at a point k^* that is approximately proportional to n (say, $k^* = \lfloor d_7 n \rfloor$). Here d_6 and d_7 are two positive constants.

Similarly, if $\{\psi_n(t)\}_{n=1}^\infty$ and $\{\phi_n(t)\}_{n=1}^\infty$ are two different Riesz bases, we can always find two linear bounded bijective operators U and V satisfying $\{U\psi_n(t)\}_{n=1}^\infty = \{V\phi_n(t)\}_{n=1}^\infty$. Thus, if G_t admits two series expansions on

different Riesz bases, we may possibly study the asymptotic behavior on one basis from the knowledge on the asymptotic behaviors of another.

Besides, when $H \to 0$, the decaying rate of the eigenvalues will approach the bound $\lambda_n \sim O(n^{-1})$ as $\lambda_n \sim n^{-2H-1}$. This can be another example in practice supporting Weyl's conclusion: the rate $O(n^{-1})$ for the eigenvalues of a smooth Mercer-like kernel cannot be improved in general [24]-[29].

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