

MISO Broadcast Channel with Hybrid CSIT: Beyond Two Users

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Abstract

We study the impact of heterogeneity of channel-state-information available at the transmitters (CSIT) on the capacity of broadcast channels with a multiple-antenna transmitter and k single-antenna receivers (MISO BC). In particular, we consider the k -user MISO BC, where the CSIT with respect to each receiver can be either instantaneous/perfect, delayed, or not available; and we study the impact of this heterogeneity of CSIT on the degrees-of-freedom (DoF) of such network. We first focus on the 3-user MISO BC; and we completely characterize the DoF region for all possible heterogeneous CSIT configurations, assuming linear encoding strategies at the transmitters. The result shows that the state-of-the-art achievable schemes in the literature are indeed sum-DoF optimal, when restricted to linear encoding schemes. To prove the result, we develop a novel bound, called *Interference Decomposition Bound*, which provides a lower bound on the interference dimension at a receiver which supplies delayed CSIT based on the average dimension of constituents of that interference, thereby decomposing the interference into its individual components. Furthermore, we extend our outer bound on the DoF region to the general k -user MISO BC, and demonstrate that it leads to an approximate characterization of linear sum-DoF to within an additive gap of 0.5 for a broad range of CSIT configurations. Moreover, for the special case where only one receiver supplies delayed CSIT, we completely characterize the linear sum-DoF.

I. INTRODUCTION

Channel state information at the transmitters (CSIT) plays a crucial role in the design and operation of multi-user wireless networks. Timely and accurate knowledge of the channels can potentially help the transmitters mitigate the interference that they cause at the unintended receivers, therefore enabling them to increase the communication rate to their intended receivers. The common procedure for obtaining channel state information (CSI) is to send training symbols (or pilots) at the transmitters, and then estimate the channels at the receivers and feed the estimates back to the transmitters. As a result of this feedback mechanism, CSI may not always be perfect and instantaneous. For instance, CSIT may be outdated due to the fast fading nature of the channels or slow feedback mechanism, it can be noisy (imperfect), or not available at all. Therefore, one can expect that in a large network there would be various types of CSI available at the transmitters with respect to different receivers. This results in communication scenarios with *heterogeneous* or *hybrid* CSIT.

As a result, there has been a growing interest in studying the impact of CSIT on the capacity of wireless networks, especially the broadcast channel. In particular, it was shown in [3] that even when the transmitter(s) only have access to delayed CSIT, there is significant potential for degrees-of-freedom (DoF) gain. They studied the problem of k -user multiple-input single-output broadcast channel (MISO BC) with delayed CSIT, and showed that for such network $\text{DoF} = \frac{k}{1 + \frac{1}{2} + \dots + \frac{1}{k}}$. This work was followed by several other works which studied other network configurations under the assumption of delayed CSIT, including interference channel [4]–[7], X-channel [8], [9], multi-hop networks [10], and other variations of delayed CSIT [11].

Most of these prior works assume that the entire network state information is obtained with delay. However, in a large network, one can expect various types of CSIT available at the transmitters with respect to different receivers. As a result, there have also been several works on studying the impact of heterogeneous (or hybrid)

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CSIT on the capacity of wireless networks, where the CSIT with respect to each receiver can now be either instantaneous/perfect (P), delayed (D), or not available (N) [12]–[18]. However, studying networks under the assumption of heterogeneous CSIT becomes quite challenging, to the extent that only the DoF for 2-user MISO BC is characterized [15], [19]; and beyond the 2-user network configuration even the DoF is unknown and the problem remains widely open.

To make progress on the MISO BC beyond 2 users, in this paper we focus on characterizing the degrees of freedom when restricted to linear schemes (also called LDoF). Our motivation to focus on the linear degrees of freedom is based on recent progress made in [9], [20], where the concept of LDoF was introduced; and it was shown that for 2-user X-channel with delayed CSIT, LDoF can be characterized, while the information theoretic DoF remains open. Linear schemes are also of significant practical interests due to their low complexity; and in fact, the majority of DoF-optimal schemes developed so far for networks with delayed CSIT are linear.

We consider the problem of MISO BC with hybrid CSIT, with a multiple-antenna transmitter and k single-antenna receivers ($k > 2$), and study its linear degrees of freedom. The channels are time-varying, and the CSIT provided by each receiver is either instantaneous (P), delayed (D), or none (N). We first study the case of $k = 3$, and fully characterize the LDoF for all 3^3 possible hybrid CSIT configurations. The result is obtained by developing a general outer bound on the LDoF region, and a matching achievable scheme for each of the CSIT configurations.

The outer bound, which is the main contribution of this paper, is based on three main ingredients. The first ingredient is a novel lemma, called *Interference Decomposition Bound*. It essentially lower bounds the interference dimension at a receiver with delayed CSIT by the average dimension of its constituents, thereby decomposing the interference into its individual components. As a result of Interference Decomposition Bound, we can then focus on analyzing the dimension of constituents of interference at receivers which supply delayed CSIT, in order to derive an upper bound on LDoF. Proof of Interference Decomposition Bound is based on temporal analysis of dimensions of transmit signals at different receivers, leading to necessary conditions on the increments of such dimensions using the delayed CSIT constraint.

The second main ingredient of the converse proof is *MIMO Rank Ratio Inequality for Broadcast Channel*, which provides a lower bound on the dimension of interference components at receivers supplying delayed CSIT. In particular, the bound states that if the transmitter employs linear precoding schemes, the dimension of each interference component at a single-antenna receiver supplying delayed CSIT is at least half of the dimension of the corresponding signal at any other single-antenna receiver. This inequality can be viewed as a variation of the Rank Ratio Inequality proved in [9], which shows that if two *distributed* single-antenna transmitters employ linear strategies, the dimensions of received linear subspace at a single-antenna receiver supplying delayed CSIT is at least $\frac{2}{3}$ of the dimension of the same signal at any other single-antenna receiver. Note that the key difference between the two lemmas lies in the assumption of *distributed* antennas in Rank Ratio Inequality, which changes the proof techniques required to establish the inequality.

Finally, the third ingredient of the converse, called *Least Alignment Lemma*, provides a lower bound on the dimension of interference components at receivers supplying no CSIT. In particular, the bound states that once the transmitter(s) in a network has no CSIT of a certain receiver, the least amount of alignment will occur at that receiver, meaning that transmit signals will occupy the maximal signal dimensions at that receiver. As a result, Least Alignment Lemma implies that the dimension of interference caused at a receiver which supplies no CSIT by the message intended for another receiver is at least equal to the dimension of the message itself. Using the three main ingredients we develop a converse proof which characterizes the linear DoF region for all 3^3 possible hybrid CSIT configurations of the 3-user MISO BC.

We next extend the key proof ingredients of the converse for 3-user MISO BC to the general k -user setting. In particular, we extend the Interference Decomposition Bound to the k -user setting to provide lower bound on the dimension of any interfering signal in an arbitrary receiver supplying delayed CSIT. In addition, we present a generalized version of MIMO Rank Ratio Inequality for BC, which provides a lower bound on the dimension of *joint* received signals at any arbitrary subset of receivers supplying delayed CSIT. Additionally, we extend the Least Alignment Lemma and show that under linear schemes, for arbitrary transmit signals the dimension of received signal at a receiver supplying no CSIT cannot be less than any other receiver.

By extending the converse tools to the general k -user setting, we provide a new outer bound on the linear DoF region of the general k -user MISO BC with arbitrary hybrid CSIT configuration. We demonstrate that our new outer bound leads to an approximate linear sum-DoF characterization to within an additive gap of 0.5 for networks

with more number of receivers supplying instantaneous CSIT than delayed CSIT; and the approximation gap decays exponentially with the increase in number of receivers supplying instantaneous CSIT. Furthermore, by using the outer bound and providing a new multi-phase achievable scheme, we present the exact characterization of linear sum-DoF for networks in which only one receiver supplies delayed CSIT.

Notation. We use small letters (e.g. x) for scalars, arrowed letters (e.g. \vec{x}) for vectors, capital letters (e.g. X) for matrices, and calligraphic font (e.g. \mathcal{X}) for sets. We also use bold letters (e.g. \mathbf{x}) for random entities, and non-bold letters for deterministic values (e.g., realizations of random variables).

II. SYSTEM MODEL

We consider the Gaussian k -user multiple-input single-output broadcast channel (MISO BC) as depicted in Figure 1. It consists of a transmitter with m antennas, and k single-antenna receivers, $\text{Rx}_1, \text{Rx}_2, \dots, \text{Rx}_k$, where $m \geq k$. The transmitter has a separate message for each of the receivers.

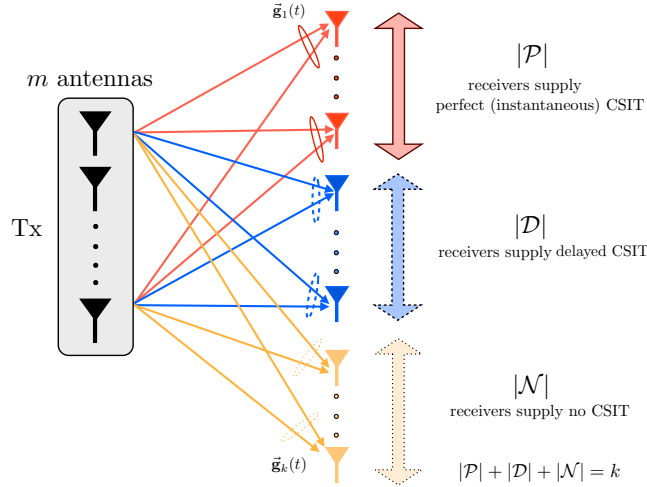


Fig. 1. Network configuration for k -user MISO BC.

Consider communication over n time slots. The received signal at Rx_j ($j \in \{1, 2, \dots, k\}$) at time t is given by

$$\mathbf{y}_j(t) = \vec{\mathbf{g}}_j(t) \vec{\mathbf{x}}(t) + \mathbf{z}_j(t), \quad (1)$$

where $\vec{\mathbf{x}}(t) \in \mathbb{C}^m$ is the transmit signal vector at time t ; $\vec{\mathbf{g}}_j(t) \in \mathbb{C}^{1 \times m}$ denotes the channel coefficients of the channel from Tx to Rx_j ; and $\mathbf{z}_j(t)$ denotes the additive white noise which is distributed as $\mathcal{CN}(0, 1)$. The elements of the channel coefficients vector $\vec{\mathbf{g}}_j(t)$ are i.i.d, drawn from a continuous distribution and also i.i.d across time and users. $\mathcal{G}(t)$ denotes the set of all k channel vectors at time t . In addition, we denote by \mathcal{G}^n the set of all channel coefficients from time 1 to n , i.e.,

$$\mathcal{G}^n = \{\vec{\mathbf{g}}_j(t) : j = 1, 2, \dots, k, \quad t = 1, \dots, n\}. \quad (2)$$

We denote the vector of transmit signals in a block of length n by $\vec{\mathbf{x}}^n$, where $\vec{\mathbf{x}}^n$ is the result of concatenation of transmit signal vectors $\vec{\mathbf{x}}(1), \dots, \vec{\mathbf{x}}(n)$. We assume Tx obeys an average power constraint, $\frac{1}{n} E\{\|\vec{\mathbf{x}}^n\|^2\} \leq P_0$.

We focus on scenarios in which channel state information available at the transmitter (CSIT) with respect to different receivers can be instantaneous (P), delayed (D), or none (N). We refer to these scenarios as *fixed hybrid scenarios*, or *hybrid* in short. In particular, CSIT with respect to Rx_j , $j = 1, 2, \dots, k$, is denoted by $I_j \in \{P, N, D\}$, as defined in [15]. In this notation, $I_j = P$ indicates that Tx has access to instantaneous CSIT with respect to Rx_j ; i.e., at time t , Tx has access to $\{\vec{\mathbf{g}}_j(1), \dots, \vec{\mathbf{g}}_j(t)\}$. Similarly, $I_j = D$ indicates delayed CSIT with respect to Rx_j ; i.e., at time t , Tx has access to $\{\vec{\mathbf{g}}_j(1), \dots, \vec{\mathbf{g}}_j(t-1)\}$. Finally, $I_j = N$ indicates no CSIT, which means the channel to Rx_j is not known to the Tx at all. We assume that the type of CSIT for each receiver is fixed and does not alternate over time (nevertheless, channels are time-varying). Therefore, there are 3^k different fixed hybrid scenarios. As an example, we use PDD to denote the 3-user MISO BC where the first receiver provides instantaneous CSIT, while the other two provide delayed CSIT.

Definition 1. We denote the set of indices of users in states $\mathcal{P}, \mathcal{D}, \mathcal{N}$ by $\mathcal{P}, \mathcal{D}, \mathcal{N}$, respectively. In addition, for an ordered set \mathcal{S} we denote by $\pi_{\mathcal{S}}$ the ordered set obtained by a permutation of the elements of \mathcal{S} , where we denote the elements of the new ordered set by $\pi_{\mathcal{S}}(1), \pi_{\mathcal{S}}(2), \dots, \pi_{\mathcal{S}}(|\mathcal{S}|)$.

Note that according to Definition 1, $\mathcal{P} \cup \mathcal{D} \cup \mathcal{N} = \{1, 2, \dots, k\}$ and $\mathcal{P} \cap \mathcal{D} = \mathcal{D} \cap \mathcal{N} = \mathcal{P} \cap \mathcal{N} = \emptyset$. Based on the above description of channel state information, the channel outcomes available to Tx at time t are denoted by the following set:

$$\tilde{\mathcal{G}}^t = \{\mathbf{G}_i^t; i \in \mathcal{P}\} \cup \{\mathbf{G}_j^{t-1}; j \in \mathcal{D}\}. \quad (3)$$

We restrict ourselves to linear coding strategies as defined in [9], in which degrees-of-freedom (DoF) represents the dimension of the linear subspace of transmitted signals. More specifically, consider a communication scheme with block length n , in which the Tx wishes to deliver a vector $\bar{\mathbf{x}}_j \in \mathbb{C}^{m_j(n)}$ of $m_j(n) \in \mathbb{N}$ information symbols to Rx _{j} ($j \in \{1, 2, \dots, k\}$). Each information symbol is a random variable with variance P_0 . These information symbols are then modulated with precoding matrices $\mathbf{V}_j(t) \in \mathbb{C}^{m \times m_j(n)}$ at times $t = 1, 2, \dots, n$. Note that the precoding matrix $\mathbf{V}_j(t)$ depends only upon the outcome of $\tilde{\mathcal{G}}^t$ due to the hybrid CSIT constraint:

$$\mathbf{V}_j(t) = f_{j,t}^{(n)}(\tilde{\mathcal{G}}^t). \quad (4)$$

Based on this linear precoding, Tx will then send $\bar{\mathbf{x}}(t) = \sum_{j=1}^k \mathbf{V}_j(t) \bar{\mathbf{x}}_j$ at time t . We can rewrite $\bar{\mathbf{x}}(t)$ as following.

$$\bar{\mathbf{x}}(t) = [\mathbf{V}_1(t) \dots \mathbf{V}_k(t)] [\bar{\mathbf{x}}_1; \dots; \bar{\mathbf{x}}_k], \quad (5)$$

where $[A; B]$ denotes the vertical concatenation of matrices A and B (i.e., $\begin{bmatrix} A \\ B \end{bmatrix}$).

We denote by $\mathbf{V}_j^n \in \mathbb{C}^{nm \times m_j(n)}$ the overall precoding matrix of Tx for Rx _{j} , such that the rows $1+(t-1)m, \dots, tm$ of \mathbf{V}_j^n constitute $\mathbf{V}_j(t)$. In addition, we denote the precoding function used by Tx by $f^{(n)} = \{f_{j,t}^{(n)}\}_{t=1, \dots, n, j=1, \dots, k}$.

Based on the above setting, the received signal at Rx _{j} ($j \in \{1, 2, \dots, k\}$) after the n time steps of the communication will be

$$\bar{\mathbf{y}}_j^n = \mathbf{G}_j^n [\mathbf{V}_1^n \dots \mathbf{V}_k^n] [\bar{\mathbf{x}}_1; \dots; \bar{\mathbf{x}}_k] + \bar{\mathbf{z}}_j^n, \quad (6)$$

where $\mathbf{G}_j^n \in \mathbb{C}^{n \times nm}$ is the block diagonal channel coefficients matrix where the channel coefficients of timeslot t (i.e. $\bar{\mathbf{g}}_j^n(t)$) are in the row t , and in the columns $1+(t-1)m, \dots, tm$ of \mathbf{G}_j^n , and the rest of the elements of \mathbf{G}_j^n are zero.¹

Now, consider the decoding of $\bar{\mathbf{x}}_j$ at Rx _{j} (i.e., decoding the $m_j(n)$ information symbols for Rx _{j}). The corresponding interference subspace at Rx _{j} will be

$$\mathcal{I}_j = \text{colspan}(\mathbf{G}_j^n [\cup_{i \neq j} \mathbf{V}_i^n]),$$

where $[\cup_{i \neq j} \mathbf{V}_i^n]$ is the matrix formed by row concatenation of matrices \mathbf{V}_i^n for $i \neq j$, and $\text{colspan}(\cdot)$ of a matrix corresponds to the sub-space that is spanned by its columns. Let $\mathcal{I}_j^\perp \subseteq \mathbb{C}^n$ denote the orthogonal subspace of \mathcal{I}_j . Then, in the regime of asymptotically high transmit powers (i.e., ignoring the noise), the decodability of information symbols at Rx _{j} corresponds to the constraint that the image of $\text{colspan}(\mathbf{G}_j^n \mathbf{V}_j^n)$ on \mathcal{I}_j^\perp has dimension $m_j(n)$:

$$\dim(\text{Proj}_{\mathcal{I}_j^\perp} \text{colspan}(\mathbf{G}_j^n \mathbf{V}_j^n)) = \dim(\text{colspan}(\mathbf{G}_j^n \mathbf{V}_j^n)) = m_j(n), \quad (7)$$

which can be shown by simple linear algebra to be equivalent to the following:

$$\text{rank}[\mathbf{G}_j^n [\cup_{i=1}^k \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_j^n [\cup_{i \neq j} \mathbf{V}_i^n]] = \text{rank}[\mathbf{G}_j^n \mathbf{V}_j^n] = m_j(n). \quad (8)$$

Based on this setting, we now define the linear degrees-of-freedom of the k -user MISO broadcast channel with hybrid CSIT.

¹For $j \in \{1, \dots, k\}$, we define $\mathbf{G}_j^0[\mathbf{V}_1^0 \dots \mathbf{V}_k^0] \triangleq \vec{0}$; therefore, for instance, $\text{rank}[\mathbf{G}_j^0[\mathbf{V}_1^0 \dots \mathbf{V}_k^0]] = 0$.

Definition 2. k -tuple (d_1, d_2, \dots, d_k) degrees-of-freedom are linearly achievable if there exists a sequence $\{f^{(n)}\}_{n=1}^{\infty}$ such that for each n and the corresponding choice of $(m_1(n), m_2(n), \dots, m_k(n))$, $(\mathbf{V}_1^n, \mathbf{V}_2^n, \dots, \mathbf{V}_k^n)$ satisfy the decodability condition of (8) with probability 1; i.e., for all $j \in \{1, \dots, k\}$,

$$\text{rank}[\mathbf{G}_j^n [\cup_{i=1}^k \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_j^n [\cup_{i \neq j} \mathbf{V}_i^n]] \stackrel{a.s.}{=} \text{rank}[\mathbf{G}_j^n \mathbf{V}_j^n] \stackrel{a.s.}{=} m_j(n), \quad (9)$$

and

$$d_j = \lim_{n \rightarrow \infty} \frac{m_j(n)}{n}. \quad (10)$$

We also define the linear degrees-of-freedom region $\text{LDoF}_{\text{region}}$ as the closure of the set of all linearly achievable k -tuples (d_1, d_2, \dots, d_k) . Furthermore, the linear sum-degrees-of-freedom (LDoF_{sum}) is defined as follows:

$$\text{LDoF}_{\text{sum}} \triangleq \max \sum_{j=1}^k d_j, \quad \text{s.t.} \quad (d_1, d_2, \dots, d_k) \in \text{LDoF}_{\text{region}}. \quad (11)$$

In what follows we first focus on the case of $k = 3$, and completely characterize the $\text{LDoF}_{\text{region}}$ for 3-user MISO BC with hybrid CSIT. We then extend our bounds and present new outer bounds on the $\text{LDoF}_{\text{region}}$ of the general k -user MISO BC with hybrid CSIT.

III. 3-USER MISO BROADCAST CHANNEL WITH HYBRID CSIT

In this section we focus on 3-user MISO broadcast channel with hybrid CSIT. In particular, we first state the complete characterization of $\text{LDoF}_{\text{region}}$ for all hybrid CSIT configurations; and then, we present the proof based on 3 key lemmas.

Theorem 1. Given a hybrid CSIT configuration, i.e., a partition of 3 users into disjoint sets \mathcal{P}, \mathcal{D} , and \mathcal{N} as defined in Definition 1, the $\text{LDoF}_{\text{region}}$ is characterized as follows:

$$\text{LDoF}_{\text{region}} = \left\{ \begin{array}{l} (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \\ \forall i \in \mathcal{D}, \forall \pi_{\mathcal{P} \cup \mathcal{D} \setminus i}, \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_{\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}(j)}}{2^j} + d_i + \sum_{j \in \mathcal{N}} d_j \leq 1, \\ \forall \pi_{\mathcal{D}}, \sum_{j \in \mathcal{P}} \frac{d_j}{3} + \sum_{j=1}^{|\mathcal{D}|} \frac{d_{\pi_{\mathcal{D}}(j)}}{j} + \sum_{j \in \mathcal{N}} d_j \leq 1, \\ \forall i \in \mathcal{P} \cup \mathcal{D}, d_i + \sum_{j \in \mathcal{N}} d_j \leq 1 \end{array} \right\}. \quad (12)$$

The $\text{LDoF}_{\text{region}}$ and the corresponding LDoF_{sum} for different CSIT configurations are summarized in Table I.

Note that although there are 3^3 different CSIT configurations for 3-user MISO BC, many of them are permutations of one another, e.g. PPD, PDP, DPP . As a result, there are only 10 distinct CSIT configurations which are presented in Table I.

Remark 1. The bound in Theorem 1 strictly improves the state-of-the-art bounds, and also leads to complete characterization of $\text{LDoF}_{\text{region}}$ for $k = 3$. For instance, for PPD (i.e. Rx_1 supplying instantaneous CSIT, while Rx_2, Rx_3 supply delayed CSIT) the prior results suggest that $\text{LDoF}_{\text{sum}} \leq \frac{17}{9}$ [17], [21], while by Theorem 1, LDoF_{sum} is indeed equal to $\frac{9}{5}$. Similarly, for the case of PPD , the prior results [17], [21] imply that $\text{LDoF}_{\text{sum}} \leq \frac{7}{3}$, while by Theorem 1, $\text{LDoF}_{\text{sum}} = \frac{9}{4}$.

Remark 2. Theorem 1 implies that the state-of-the-art achievable schemes presented in [18] for PPD and PDD are both optimal from the perspective of LDoF_{sum} .

Remark 3. It is worth noting that in any CSIT configuration which involves receivers with state N , the inequalities that constitute the LDoF region have coefficient 1 for the degrees-of-freedom of receivers with state N . In other

CSIT States	Linear Degrees of Freedom Region (LDoF _{region})	LDoF _{sum}
PPP	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1 \right\}$	3
PPD	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1, \frac{d_1}{4} + \frac{d_2}{2} + d_3 \leq 1 \right\}$	$\frac{9}{4}$
PPN	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, d_1 + d_3 \leq 1, d_2 + d_3 \leq 1 \right\}$	2
PDD	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \right.$ $\left. \frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1, \frac{d_1}{2} + d_2 + \frac{d_3}{4} \leq 1, \right.$ $\left. \frac{d_1}{3} + \frac{d_2}{2} + d_3 \leq 1, \frac{d_1}{3} + d_2 + \frac{d_3}{2} \leq 1 \right\}$	$\frac{9}{5}$
PDN	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \frac{d_1}{2} + d_2 + d_3 \leq 1, d_1 + d_3 \leq 1 \right\}$	$\frac{3}{2}$
DDD	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \right.$ $\left. \frac{d_1}{3} + \frac{d_2}{2} + d_3 \leq 1, \frac{d_1}{3} + d_2 + \frac{d_3}{2} \leq 1, \frac{d_1}{2} + \frac{d_2}{3} + d_3 \leq 1, \right.$ $\left. \frac{d_1}{2} + d_2 + \frac{d_3}{3} \leq 1, d_1 + \frac{d_2}{2} + \frac{d_3}{3} \leq 1, d_1 + \frac{d_2}{3} + \frac{d_3}{2} \leq 1 \right\}$	$\frac{18}{11}$
DDN	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, \frac{d_1}{2} + d_2 + d_3 \leq 1, d_1 + \frac{d_2}{2} + d_3 \leq 1 \right\}$	$\frac{4}{3}$
PNN, DNN, NNN	$\text{LDoF}_{\text{region}} = \left\{ (d_1, d_2, d_3) \mid 0 \leq d_1, d_2, d_3 \leq 1, d_1 + d_2 + d_3 \leq 1 \right\}$	1

TABLE I
LDoF_{region} AND LDoF_{sum} FOR ALL POSSIBLE CONFIGURATIONS OF HYBRID CSIT FOR 3-USER MISO BC.

words, receivers that supply no CSIT do not contribute to the LDoF_{sum}, and unless all receivers have state N , removing the no CSIT receivers from the network will not decrease the LDoF_{sum}.

In the remainder of this section we prove Theorem 1. To this aim, we first present the converse proof in Section III-A, and then discuss the achievability in Section III-B.

A. Proof of Converse for 3-User MISO Broadcast Channel with Hybrid CSIT

We first provide the three main ingredients that are key in proving the converse for 3-user MISO broadcast channel with hybrid CSIT. We then show how those main ingredients are used to prove the converse for two representative CSIT configurations (i.e. PDD and PDN). The proof of converse for other CSIT configurations can be found in Appendix A. The first two ingredients of the converse proof deal with lower bounding received signal dimension at a receiver which supplies delayed CSIT, while the third ingredient captures the impact of no CSIT.

The first key ingredient is Interference Decomposition Bound, which essentially provides a lower bound on the interference dimension at a receiver supplying delayed CSIT, based on the constituents of that interference, as well as the received signal dimension at other receivers. It is stated below; and its proof is provided in Appendix B.

Lemma 1. (Interference Decomposition Bound) Consider $k = 3$, and a fixed linear coding strategy $f^{(n)}$, with corresponding precoding matrices $\mathbf{V}_1^n, \mathbf{V}_2^n, \mathbf{V}_3^n$ as defined in (4). If $I_3 = D$ (i.e., if Rx_3 supplies delayed CSIT),

$$\frac{\text{rank}[\mathbf{G}_1^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_1^n\mathbf{V}_2^n] + \text{rank}[\mathbf{G}_3^n\mathbf{V}_2^n]}{2} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]]. \quad (13)$$

Remark 4. The R.H.S. of Interference Decomposition Bound represents the dimension of interference caused at Rx_3 , which supplies delayed CSIT, by the messages intended for Rx_1, Rx_2 . On the other hand, the third term on the L.H.S. (i.e. $\text{rank}[\mathbf{G}_3^n\mathbf{V}_2^n]$) is the dimension of the remaining interference at Rx_3 after removing the contribution of the message of Rx_1 ; and the first two terms (i.e. $\text{rank}[\mathbf{G}_1^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_1^n\mathbf{V}_2^n]$) can be shown by (9) and

sub-modularity of rank (stated in Lemma 4) to equal $\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]$, which is the dimension of message of R_{x_1} . Hence, Interference Decomposition Bound provides an inequality which connects the dimension of interference at a receiver to the average dimension of its constituents. Note that statement of Lemma 1 does not assume any specific CSIT with respect to any receiver except R_{x_3} .

The second main ingredient, called MIMO Rank Ratio Inequality for BC, provides a lower bound on the dimension of received signal at a receiver supplying delayed CSIT. It is stated below; and its proof is provided in Appendix D.

Lemma 2. (MIMO Rank Ratio Inequality for BC) Consider $k = 3$, and a linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \mathbf{V}_2^n, \mathbf{V}_3^n$ as defined in (4). If $I_3 = D$ (i.e., if R_{x_3} supplies delayed CSIT), then, for each beamforming matrix \mathbf{V}_i^n , where $i \in \{1, 2, 3\}$, and each $\ell \in \{1, 2, 3\}$, we have

$$\frac{\text{rank}[[\mathbf{G}_\ell^n; \mathbf{G}_3^n] \mathbf{V}_i^n]}{2} \stackrel{\text{a.s.}}{\leq} \text{rank}[\mathbf{G}_3^n \mathbf{V}_i^n], \quad (14)$$

where $[\mathbf{G}_\ell^n; \mathbf{G}_3^n]$ denotes the column concatenation of matrices \mathbf{G}_ℓ^n and \mathbf{G}_3^n .

Remark 5. Lemma 2 implies that for any transmit signal \mathbf{V}_i^n , the corresponding received signal dimension at a receiver with delayed CSIT is at least half of the corresponding received signal dimension at any other receiver. Note that statement of Lemma 2 does not assume any specific CSIT with respect to any receiver except R_{x_3} .

The third main ingredient of converse, Least Alignment Lemma, demonstrates that when using linear schemes, once the transmitter has no CSIT with respect to a certain receiver, the least amount of alignment will occur at that receiver, meaning that transmit signals will occupy the maximal signal dimensions at that receiver. The lemma is stated below; and its proof is provided in Appendix E.

Lemma 3. (Least Alignment Lemma) Consider $k = 3$, and a linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \mathbf{V}_2^n, \mathbf{V}_3^n$ as defined in (4). For $\mathcal{S} \subseteq \{1, 2, 3\}$ let $\mathbf{V}^n \triangleq [\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]$ denote the row concatenation of the precoding matrices \mathbf{V}_i^n , where $i \in \mathcal{S}$. If $I_3 = N$ (i.e., if R_{x_3} supplies no CSIT),

$$\forall \ell \in \{1, 2, 3\}, \quad \text{rank}[\mathbf{G}_\ell^n \mathbf{V}^n] \stackrel{\text{a.s.}}{\leq} \text{rank}[\mathbf{G}_3^n \mathbf{V}^n].$$

Remark 6. Note that the statement of Lemma 3 does not assume any specific CSIT with respect to any receiver except R_{x_3} .

Remark 7. Lemma 3 can be seen as a variation of the corresponding result in the context of secrecy problems in [22], [23]. Moreover, as shown in [19], Least Alignment Lemma also holds for non-linear schemes; and for this extension the reader is referred to [19].

We now prove the converse for two representative CSIT configurations PDD and PDN , highlighting the applications of the above three lemmas. Converse proofs for other CSIT configurations can be found in Appendix A.

1) *Proof of Converse for PDD:* According to Table I, it is sufficient to show that $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ and $\frac{d_1}{3} + \frac{d_2}{2} + d_3 \leq 1$; since the other two inequalities (i.e. $\frac{d_1}{2} + d_2 + \frac{d_3}{4} \leq 1$, and $\frac{d_1}{3} + d_2 + \frac{d_3}{2} \leq 1$) can be proven similarly using symmetry. Moreover, the bound $\frac{d_1}{3} + \frac{d_2}{2} + d_3 \leq 1$ follows directly from the existing state-of-the-art arguments used in [3], [17]. Henceforth, we focus on proving $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$.

Suppose (d_1, d_2, d_3) degrees-of-freedom are linearly achievable. Hence, by Definition 2 there exists a sequence $\{f^{(n)}\}_{n=1}^\infty$ such that for each n and the corresponding choice of $(m_1(n), m_2(n), m_3(n))$, $(\mathbf{V}_1^n, \mathbf{V}_2^n, \mathbf{V}_3^n)$ satisfy the conditions in (9) and (10). Therefore, in order to prove $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$, it is sufficient to show that

$$\frac{m_1(n)}{2} + \frac{m_2(n)}{4} + m_3(n) \stackrel{\text{a.s.}}{\leq} n. \quad (15)$$

Note that since in the PDD configuration receiver 3 supplies delayed CSIT, we can invoke Lemma 1, which states that:

$$\begin{aligned} 2 \times \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \ \mathbf{V}_2^n]] &\stackrel{\text{a.s.}}{\geq} \text{rank}[\mathbf{G}_1^n [\mathbf{V}_1^n \ \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_1^n \mathbf{V}_2^n] + \text{rank}[\mathbf{G}_3^n \mathbf{V}_2^n] \\ &\stackrel{(9)}{\stackrel{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n \mathbf{V}_2^n]. \end{aligned} \quad (16)$$

We now further bound each side of the above inequality. We first upper bound the left-hand-side of the above inequality:

$$\text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \stackrel{(9)}{\underset{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - m_3(n) \leq n - m_3(n). \quad (17)$$

On the other hand, for the right-hand-side of (16) we have

$$\begin{aligned} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n \mathbf{V}_2^n] &\stackrel{(9)}{\underset{\text{a.s.}}{=}} m_1(n) + \text{rank}[\mathbf{G}_3^n \mathbf{V}_2^n] \stackrel{(\text{Lemma 2})}{\underset{\text{a.s.}}{\geq}} m_1(n) + \frac{1}{2} \text{rank}[[\mathbf{G}_2^n; \mathbf{G}_3^n] \mathbf{V}_2^n] \\ &\geq m_1(n) + \frac{1}{2} \text{rank}[\mathbf{G}_2^n \mathbf{V}_2^n] \stackrel{(9)}{\underset{\text{a.s.}}{=}} m_1(n) + \frac{1}{2} m_2(n). \end{aligned} \quad (18)$$

Hence, by considering (16)-(18), we obtain

$$m_1(n) + \frac{1}{2} m_2(n) + 2m_3(n) \stackrel{\text{a.s.}}{\leq} 2n, \quad (19)$$

which proves (15), and therefore, completes the converse proof for PDD.

Remark 8. Note that in order to prove $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ for PDD, we did not rely on any specific CSIT assumption with respect to Rx₂. Therefore, the bound $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ also holds for the case of PPD. Moreover, note that by symmetry one can conclude that $\frac{d_1}{4} + \frac{d_2}{2} + d_3 \leq 1$ also holds for PPD. Hence, since according to Table I, $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ and $\frac{d_1}{4} + \frac{d_2}{2} + d_3 \leq 1$ constitute the LDoF region for PPD, the above derivations suffice in proving the converse for the CSIT configuration PPD as well.

2) *Proof of Converse for PDN:* According to Table I, it is sufficient to show that $\frac{d_1}{2} + d_2 + d_3 \leq 1$ and $d_1 + d_3 \leq 1$. Suppose (d_1, d_2, d_3) degrees-of-freedom are linearly achievable. Hence, by Definition 2 there exists a sequence $\{f^{(n)}\}_{n=1}^{\infty}$ such that for each n and the corresponding choice of $(m_1(n), m_2(n), m_3(n))$, $(\mathbf{V}_1^n, \mathbf{V}_2^n, \mathbf{V}_3^n)$ satisfy the conditions in (9) and (10). Therefore, in order to prove $\frac{d_1}{2} + d_2 + d_3 \leq 1$ and $d_1 + d_3 \leq 1$, it is sufficient to show that

$$\frac{m_1(n)}{2} + m_2(n) + m_3(n) \stackrel{\text{a.s.}}{\leq} n, \quad (20)$$

and

$$m_1(n) + m_3(n) \stackrel{\text{a.s.}}{\leq} n. \quad (21)$$

We have,

$$\begin{aligned} \frac{m_1(n)}{2} + m_2(n) + m_3(n) &\stackrel{(9)}{\underset{\text{a.s.}}{=}} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + m_2(n) + m_3(n) \\ &\stackrel{(9)}{\underset{\text{a.s.}}{=}} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_3^n]] + m_3(n) \\ &\stackrel{(a)}{\leq} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_2^n \mathbf{V}_1^n] + m_3(n) \\ &\leq \frac{\text{rank}[[\mathbf{G}_1^n; \mathbf{G}_2^n] \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_2^n \mathbf{V}_1^n] + m_3(n) \\ &\stackrel{(b)}{\underset{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + m_3(n) \\ &\stackrel{(9)}{\underset{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_2^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \\ &\stackrel{(\text{Lemma 3})}{\underset{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] \leq n, \end{aligned}$$

where (a) follows from the sub-modularity of rank of matrices (see Lemma 4 stated below); and (b) follows from Lemma 2 applied to Rx₂ as the receiver which supplies delayed CSIT. Therefore, the proof of (20) is complete. We now prove (21).

$$\begin{aligned} m_1(n) + m_3(n) &\stackrel{(9)}{\underset{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + m_3(n) \\ &\stackrel{(9)}{\underset{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \\ &\stackrel{(\text{Lemma 4})}{\leq} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n \mathbf{V}_1^n] \end{aligned}$$

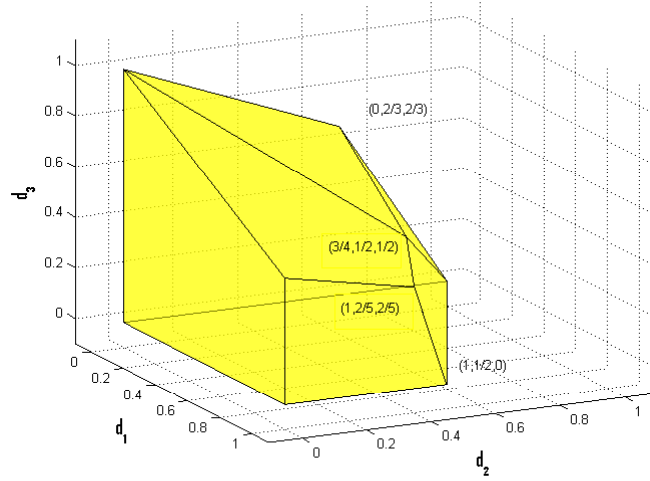


Fig. 2. LDoF Region for PDD .

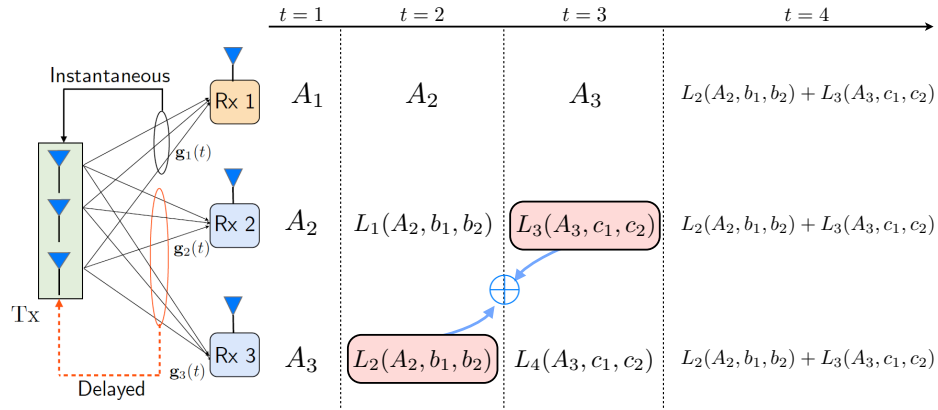


Fig. 3. Achieving $(d_1, d_2, d_3) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$ for PDD .

$$\stackrel{\text{(Lemma 3)}}{\underset{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_3^n]] \leq n,$$

which completes the proof of (21). We now state the sub-modularity of rank of matrices (see [24] for more details).

Lemma 4. (Sub-modularity of rank) Consider a matrix $A^{m \times n} \in \mathbb{C}^{m \times n}$. Let A_S , $S \subseteq \{1, 2, \dots, n\}$ denote the sub-matrix of A created by those columns in A which have their indices in S . Then, for any $S_1, S_2 \subseteq \{1, 2, \dots, n\}$, $\text{rank}[A_{S_1}] + \text{rank}[A_{S_2}] \geq \text{rank}[A_{S_1 \cap S_2}] + \text{rank}[A_{S_1 \cup S_2}]$.

Note that a similar statement is true for sub-modularity of rank with respect to the rows of a matrix, instead of the columns as stated in Lemma 4.

B. Proof of Achievability for Theorem 1

The regions described in Theorem 1 result in polytopes in \mathbb{R}^3 ; and therefore, the LDoF regions can be completely described via their extreme points. Many of such extreme points can be trivially achieved (e.g. the point $(1, 1, 0)$ for PDD); therefore, we only focus on the non-trivial extreme points and provide reference for each of them in Table II.

The only non-trivial extreme point that has not yet been achieved in the literature according to Table II belongs to PDD , and is $(\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$. The LDoF region suggested by Theorem 1 for PDD is shown in Fig. 2. Therefore, we only prove the achievability of $(\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$ for PDD . The scheme is illustrated in Fig. 3. We will show how to deliver 3 symbols (a_1, a_2, a_3) to Rx1, 2 symbols (b_1, b_2) to Rx2, and 2 symbols (c_1, c_2) to Rx3 over 4 time slots in order to achieve $(d_1, d_2, d_3) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$.

CSIT States	Non-trivial extreme points of the LDoF region and reference to the achievable scheme
PPD	$\left(1, 0, \frac{1}{2}\right), \left(0, 1, \frac{1}{2}\right)$ achieved in Section III-A of [13] $\left(1, 1, \frac{1}{4}\right)$ achieved in Section IV-D of [18]
PDD	$\left(1, 0, \frac{1}{2}\right), \left(0, 1, \frac{1}{2}\right)$ achieved in Section III-A of [13] $\left(1, \frac{2}{5}, \frac{2}{5}\right)$ achieved in Section IV-C of [18] $\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{2}\right)$ achieved in Section III-B of this paper $\left(0, \frac{2}{3}, \frac{2}{3}\right)$ achieved in Section III-A of [3]
PDN	$\left(1, \frac{1}{2}, 0\right)$ achieved in Section III-A of [13]
DDD	$\left(\frac{2}{3}, \frac{2}{3}, 0\right), \left(\frac{2}{3}, 0, \frac{2}{3}\right), \left(0, \frac{2}{3}, \frac{2}{3}\right)$ achieved in Section III-A of [3] $\left(\frac{6}{11}, \frac{6}{11}, \frac{6}{11}\right)$ achieved in Section III-B of [3]
DDN	$\left(\frac{2}{3}, \frac{2}{3}, 0\right)$ achieved in Section III-A of [3]

TABLE II

ACHIEVABILITY RESULTS FOR EXTREME POINTS OF DIFFERENT CONFIGURATIONS OF HYBRID CSIT FOR 3-USER MISO BC

At $t = 1$, we simply send the uncoded 3 symbols (a_1, a_2, a_3) , which are desired by Rx_1 . Therefore, the transmit and received signals at the receivers are as follows (for the sake of DoF analysis, we ignore the additive noise):

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{y}_j(1) = \vec{\mathbf{g}}_j(1) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad j = 1, 2, 3. \quad (22)$$

Denote the linear combinations received by $\text{Rx}_1, \text{Rx}_2, \text{Rx}_3$ at $t = 1$ by A_1, A_2, A_3 . Notice that Rx_1 requires A_2, A_3 to be able to (almost surely) decode (a_1, a_2, a_3) . Using delayed CSIT from Rx_2, Rx_3 , transmitter can reconstruct A_2, A_3 .

At $t = 2$, the transmitter sends the symbols A_2, b_1, b_2 as

$$\vec{\mathbf{x}}(2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} A_2 + \begin{bmatrix} \vec{\mathbf{g}}_1(2)^\perp \end{bmatrix}^\top \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad (23)$$

where $[\vec{\mathbf{g}}_1(2)^\perp]$ is a 2×3 matrix, where $\vec{\mathbf{g}}_1(2)[\vec{\mathbf{g}}_1(2)^\perp]^\top = [0 \ 0]$. Therefore, the received signals at the Rx_j is ($j = 1, 2, 3$):

$$\mathbf{y}_j(2) = \vec{\mathbf{g}}_j(2) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} A_2 + \begin{bmatrix} \vec{\mathbf{g}}_1(2)^\perp \end{bmatrix}^\top \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right). \quad (24)$$

Note that by the end of time slot 2 Rx_1 is able to decode A_2 . We denote the linear combinations received by Rx_2, Rx_3 at $t = 2$ by $L_1(A_2, b_1, b_2), L_2(A_2, b_1, b_2)$, respectively.

At $t = 3$, the transmitted and received signals are:

$$\vec{x}(3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} A_3 + \begin{bmatrix} \vec{g}_1(3)^\perp \end{bmatrix}^\top \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \mathbf{y}_j(3) = \vec{g}_j(3) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} A_3 + \begin{bmatrix} \vec{g}_1(3)^\perp \end{bmatrix}^\top \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right),$$

which suggests that Rx₁ would be able to decode A_3 . We denote the linear combinations received by Rx₂, Rx₃ at $t = 3$ by $L_3(A_3, c_1, c_2)$, $L_4(A_3, c_1, c_2)$, respectively.

Note that if Rx₂ is given $L_2(A_2, b_1, b_2)$, it can use its past received signals (i.e., A_2 and $L_1(A_2, b_1, b_2)$) together with $L_2(A_2, b_1, b_2)$ to decode both b_1, b_2 . Therefore, Rx₂ needs $L_2(A_2, b_1, b_2)$. On the other hand, Rx₂ has access to $L_3(A_3, c_1, c_2)$. Similarly, Rx₃ needs $L_3(A_3, c_1, c_2)$ to be able to decode both c_1, c_2 , and it has access to $L_2(A_2, b_1, b_2)$. Therefore, at $t = 4$, the transmitter sends $L_2(A_2, b_1, b_2) + L_3(A_3, c_1, c_2)$, which is of interest to both Rx₂, Rx₃; this is because Rx₃ can then cancel $L_2(A_2, b_1, b_2)$ from its received signal at $t = 4$ to obtain $L_3(A_3, c_1, c_2)$ which it needs. Similarly, Rx₂ can cancel $L_3(A_3, c_1, c_2)$ from its received signal at $t = 4$ to obtain $L_2(A_2, b_1, b_2)$ which it needs. Consequently, all receivers will be able to decode their desired symbols by the end of the fourth time slot; hence, the DoF tuple $(\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$ is achievable. See Fig. 3 for an illustration of the achievable scheme.

IV. k -USER MISO BC WITH HYBRID CSIT

In this section we focus on the general k -user MISO BC with hybrid CSIT. In particular, we first present an outer bound on the LDoF region of the general k -user MISO BC for any arbitrary hybrid CSIT configuration. Then, we show that the bound provides an approximate characterization of LDoF_{sum} for the case of $|\mathcal{P}| \geq |\mathcal{D}|$, and exact characterization of LDoF_{sum} for $|\mathcal{D}| = 1$. We then present the key tools needed for proving the general outer bound; and finally, we prove the outer bound on the LDoF region.

Theorem 2. *Given a hybrid CSIT configuration, i.e., a partition of k users into disjoint sets \mathcal{P}, \mathcal{D} , and \mathcal{N} as defined in Definition 1, the $\text{LDoF}_{\text{region}}$ is contained in the following region:*

$$\text{LDoF}_{\text{region}} \subseteq \left\{ (d_1, \dots, d_k) \mid 0 \leq d_1, \dots, d_k \leq 1, \right. \\ \left. \forall i \in \mathcal{D}, \forall \pi_{\mathcal{P} \cup \mathcal{D} \setminus i}, \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_{\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}(j)}}{2^j} + d_i + \sum_{j \in \mathcal{N}} d_j \leq 1, \right. \quad (25)$$

$$\left. \forall \pi_{\mathcal{D}}, \sum_{j \in \mathcal{P}} \frac{d_j}{k} + \sum_{j=1}^{|\mathcal{D}|} \frac{d_{\pi_{\mathcal{D}}(j)}}{j} + \sum_{j \in \mathcal{N}} d_j \leq 1, \right. \quad (26)$$

$$\left. \forall i \in \mathcal{P} \cup \mathcal{D}, d_i + \sum_{j \in \mathcal{N}} d_j \leq 1 \right\}. \quad (27)$$

Theorem 2 enables us to approximately characterize LDoF_{sum} to within $\frac{|\mathcal{P}|}{2^{|\mathcal{P}|}}$ for a broad range of CSIT configurations ($|\mathcal{P}| \geq |\mathcal{D}|$). This gap (i.e. $\frac{|\mathcal{P}|}{2^{|\mathcal{P}|}}$) is less than or equal to 0.5, and decays exponentially to zero as $|\mathcal{P}|$ increases. Moreover, Theorem 2 allows us to exactly characterize LDoF_{sum} for the case of $|\mathcal{D}| = 1$. These results are stated more precisely in the following two Propositions.

Proposition 1. *For general k -user MISO BC with $|\mathcal{P}| \geq |\mathcal{D}|$,*

$$|\mathcal{P}| \leq \text{LDoF}_{\text{sum}} \leq |\mathcal{P}| + \frac{|\mathcal{P}|}{2^{|\mathcal{P}|}} \leq |\mathcal{P}| + \frac{1}{2}.$$

Proposition 2. *For general k -user MISO BC with $|\mathcal{D}| = 1$,*

$$\text{LDoF}_{\text{sum}} = |\mathcal{P}| + \frac{1}{2^{|\mathcal{P}|}}. \quad (28)$$

Proofs of Propositions 1, 2 are provided in Appendix F and Appendix G, respectively. We will now prove Theorem 2. In particular, we first present the key ingredients of the proof, which are the generalizations of Lemmas 1-3. We then prove (25)-(27).

A. Key Ingredients for Proof of Theorem 2

Similar to the proof for the case of 3-user MISO BC with hybrid CSIT, we need to extend the Lemmas 1-3. We present the generalizations here, and then prove Theorem 2. We first present the generalized version of Interference Decomposition Bound in Lemma 1. The proof is provided in Appendix B.

Lemma 5. (Interference Decomposition Bound) Consider a fixed linear coding strategy $f^{(n)}$, with corresponding precoding matrices $\mathbf{V}_1^n, \mathbf{V}_2^n, \dots, \mathbf{V}_k^n$ as defined in (4). For any $\mathcal{S} \subseteq \{1, 2, \dots, k\}$, any $\ell \in \mathcal{S}$, and any $j \notin \mathcal{S}$ for which $I_j = D$,

$$\frac{\text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}, i \neq \ell} \mathbf{V}_i^n]] + \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}, i \neq \ell} \mathbf{V}_i^n]]}{2} \stackrel{\text{a.s.}}{\leq} \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]], \quad (29)$$

where $[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]$ denotes the row concatenation of the corresponding precoding matrices \mathbf{V}_i^n , where $i \in \mathcal{S}$.

Remark 9. Lemma 1 is a special case of Lemma 5 where $\mathcal{S} = \{1, 2\}$, $j = 3$, $\ell = 1$.

We now present the generalized version of Lemma 2, which is the second main ingredient of the proof, and is proved in Appendix D.

Lemma 6. (MIMO Rank Ratio Inequality for BC) Consider a linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4). Let $\mathbf{Y}_j^n \triangleq \mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]$, where $\mathcal{S} \subseteq \{1, 2, \dots, k\}$. Also, consider distinct receivers $Rx_{i_1}, \dots, Rx_{i_{j+1}}$, where $j = 1, 2, \dots, k-1$ and $i_1, \dots, i_{j+1} \in \{1, \dots, k\}$. If $Rx_{i_1}, \dots, Rx_{i_j}$ supply delayed CSIT, then,

$$\frac{\text{rank}[\mathbf{Y}_{i_1}^n; \dots; \mathbf{Y}_{i_{j+1}}^n]}{j+1} \stackrel{\text{a.s.}}{\leq} \frac{\text{rank}[\mathbf{Y}_{i_1}^n; \dots; \mathbf{Y}_{i_j}^n]}{j}. \quad (30)$$

Remark 10. Lemma 2 is a special case of Lemma 6 where $j = 1$, $i_1 = 3$, $i_2 = \ell$, and $\mathcal{S} = \{i\}$.

Finally, we present the general version of Lemma 3, which is the third main ingredient for the proof of Theorem 2.

Lemma 7. (Least Alignment Lemma) For any linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4), and any $\mathcal{S} \subseteq \{1, 2, \dots, k\}$, if $I_j = N$ for some $j \in \{1, 2, \dots, k\}$,

$$\forall \ell \in \{1, 2, \dots, k\}, \quad \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] \stackrel{\text{a.s.}}{\leq} \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]],$$

where $[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]$ denotes the row concatenation of the precoding matrices \mathbf{V}_i^n , where $i \in \mathcal{S}$.

Using these three ingredients we now proceed to the proof of Theorem 2, and in particular proving the bounds (25)-(27).

B. Proof of Bound (25) in Theorem 2

Without loss of generality, suppose $\mathcal{P} = \{1, \dots, |\mathcal{P}|\}$, and $\mathcal{D} = \{|\mathcal{P}| + 1, \dots, |\mathcal{P}| + |\mathcal{D}|\}$, and $\mathcal{N} = \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$. In addition, let $i = |\mathcal{P}| + |\mathcal{D}|$, and $\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}$ be the identity permutation. Consequently, we can rewrite (25), and our goal is to show

$$\sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_j}{2^j} + d_{|\mathcal{P}|+|\mathcal{D}|} + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k d_j \leq 1. \quad (31)$$

If the k -tuple (d_1, d_2, \dots, d_k) degrees-of-freedom are linearly achievable, then by Definition 2 there exists a sequence $\{f^{(n)}\}_{n=1}^\infty$ such that for each n and the corresponding choice of $(m_1(n), m_2(n), \dots, m_k(n))$, $(\mathbf{V}_1^n, \mathbf{V}_2^n, \dots, \mathbf{V}_k^n)$ satisfy the conditions in (9) and (10). Therefore, it is sufficient to show

$$\sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{m_j(n)}{2^j} + m_{|\mathcal{P}|+|\mathcal{D}|}(n) + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{\text{a.s.}}{\leq} n. \quad (32)$$

We upper bound each of the three terms on the L.H.S. of (32) separately. By induction and application of Lemma 5 and (9), one can prove the following claim, which provides an upper bound for the first term on the L.H.S. of (32), and is proved in Appendix H.

Claim 1.

$$\sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{m_j(n)}{2^j} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]. \quad (33)$$

We now upper bound $m_{|\mathcal{P}|+|\mathcal{D}|}(n)$, which is the second term on the L.H.S. of (32). By (9) we obtain

$$\begin{aligned} m_{|\mathcal{P}|+|\mathcal{D}|}(n) &\stackrel{a.s.}{=} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\cup_{j=1}^k \mathbf{V}_j^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\cup_{j \neq |\mathcal{P}|+|\mathcal{D}|} \mathbf{V}_j^n]] \\ &\stackrel{(\text{Lemma 4})}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] \\ &\stackrel{(a)}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]], \end{aligned} \quad (34)$$

where (a) follows by Least Alignment Lemma (Lemma 7) since receiver $|\mathcal{P}| + |\mathcal{D}| + 1$ supplies no CSIT.

We now upper bound $\sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n)$, which is the third term on the L.H.S. of (32). By (9), for all $i \in \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$,

$$m_i(n) \stackrel{a.s.}{=} \text{rank}[\mathbf{G}_i^n [\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_i^n [\cup_{j \neq i} \mathbf{V}_j^n]] \stackrel{(\text{Lemma 4})}{\leq} \text{rank}[\mathbf{G}_i^n [\mathbf{V}_1^n \dots \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_i^n [\mathbf{V}_1^n \dots \mathbf{V}_{i-1}^n]].$$

Hence, by summing over all the inequalities for $i \in \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$, we obtain

$$\begin{aligned} \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) &\stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n [\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] \\ &\quad + \sum_{i=|\mathcal{P}|+|\mathcal{D}|+1}^{k-1} (\text{rank}[\mathbf{G}_i^n [\mathbf{V}_1^n \dots \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_{i+1}^n [\mathbf{V}_1^n \dots \mathbf{V}_i^n]]). \end{aligned} \quad (35)$$

Note that since receivers with index in $\{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$ supply no CSIT, and due to their channel symmetry, for each $i \in \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k-1\}$ we have

$$\text{rank}[\mathbf{G}_i^n [\mathbf{V}_1^n \dots \mathbf{V}_i^n]] \stackrel{a.s.}{=} \text{rank}[\mathbf{G}_{i+1}^n [\mathbf{V}_1^n \dots \mathbf{V}_i^n]]. \quad (36)$$

Therefore, by (35), (36) we obtain

$$\sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n [\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n [\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]]. \quad (37)$$

Hence, by summing the inequalities in (33), (34), and (37) we obtain

$$\sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{m_j(n)}{2^j} + m_{|\mathcal{P}|+|\mathcal{D}|}(n) + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n [\mathbf{V}_1^n \dots \mathbf{V}_k^n]] \leq n,$$

which proves (32), thus, completing the proof of bound (25) in Theorem 2.

C. Proof of Bound (26) in Theorem 2

Without loss of generality, suppose $\mathcal{P} = \{1, \dots, |\mathcal{P}|\}$, and $\mathcal{D} = \{|\mathcal{P}| + 1, \dots, |\mathcal{P}| + |\mathcal{D}|\}$, and $\mathcal{N} = \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$. In addition, let $\pi_{\mathcal{D}}$ be the reverse of the identity permutation. Consequently, our goal becomes to show

$$\sum_{j=1}^{|\mathcal{P}|} \frac{d_j}{k} + \sum_{j=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \frac{d_j}{|\mathcal{P}| + |\mathcal{D}| + 1 - j} + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k d_j \leq 1. \quad (38)$$

Suppose (d_1, \dots, d_k) are linearly achievable as defined in Definition 2. Then, by (10), it is sufficient to show

$$\sum_{j=1}^{|\mathcal{P}|} \frac{m_j(n)}{k} + \sum_{j=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \frac{m_j(n)}{|\mathcal{P}|+|\mathcal{D}|+1-j} + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} n. \quad (39)$$

We upper bound each of the three terms on the L.H.S. of (39) separately. We first upper bound the first term. By (9), for all $j = 1, \dots, |\mathcal{P}|$,

$$\begin{aligned} m_j(n) &\stackrel{a.s.}{=} \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_j^n[\cup_{i \neq j} \mathbf{V}_i^n]] \\ &\stackrel{(\text{Lemma 4})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_j^n]] - \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_{j-1}^n]] \\ &\stackrel{(a)}{\leq} \text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_j^n]] - \text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_{j-1}^n]], \end{aligned}$$

where (a) follows from the fact that for four matrices A, B, C, D , $\text{rank}[A \ B] - \text{rank}[B] \leq \text{rank}[A \ B; C \ D] - \text{rank}[B; D]$, and it can be proven using straightforward linear algebra.

By summing the above inequalities for $j = 1, \dots, |\mathcal{P}|$, and dividing both sides of the resulting inequality by k we obtain

$$\sum_{j=1}^{|\mathcal{P}|} \frac{m_j(n)}{k} \stackrel{a.s.}{\leq} \frac{\text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{k}. \quad (40)$$

We now upper bound the second term on the L.H.S of (39). For the receivers supplying delayed CSIT, i.e. Rx_j , where $j = |\mathcal{P}| + 1, \dots, |\mathcal{P}| + |\mathcal{D}|$, by (9) we have:

$$\begin{aligned} m_j(n) &\stackrel{a.s.}{=} \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_j^n[\cup_{i \neq j} \mathbf{V}_i^n]] \\ &\stackrel{(\text{Lemma 4})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_j^n]] - \text{rank}[\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_{j-1}^n]] \\ &\stackrel{(b)}{\leq} \text{rank}[[\mathbf{G}_j^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_j^n]] - \text{rank}[[\mathbf{G}_j^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{j-1}^n]], \end{aligned}$$

where (b) follows from the fact that for four matrices A, B, C, D , $\text{rank}[A \ B] - \text{rank}[B] \leq \text{rank}[A \ B; C \ D] - \text{rank}[B; D]$. Hence, if we divide both sides of the above inequality by $|\mathcal{P}| + |\mathcal{D}| + 1 - j$, and sum over all inequalities for $j = |\mathcal{P}| + 1, \dots, |\mathcal{P}| + |\mathcal{D}|$, we obtain

$$\begin{aligned} &\sum_{j=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \frac{m_j(n)}{|\mathcal{P}|+|\mathcal{D}|+1-j} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|} \\ &\quad + \sum_{j=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|-1} \left(\frac{\text{rank}[[\mathbf{G}_j^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_j^n]]}{|\mathcal{P}|+|\mathcal{D}|+1-j} - \frac{\text{rank}[[\mathbf{G}_{j+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_j^n]]}{|\mathcal{P}|+|\mathcal{D}|-j} \right) \\ &\stackrel{(\text{Lemma 6})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|} \\ &\stackrel{(\text{Lemma 7})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|}. \end{aligned} \quad (41)$$

We now upper bound the third term on the L.H.S of (39) exactly the same way as we upper bounded the third term on the L.H.S. of (32). To avoid redundancy, we only restate the resulting bound which was stated in (37).

$$\sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]]. \quad (42)$$

We now merge the upper bounds on individual terms on the L.H.S. of (39). By summing (40), (41), and (42), we obtain

$$\sum_{j=1}^{|\mathcal{P}|} \frac{m_j(n)}{k} + \sum_{j=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \frac{m_j(n)}{|\mathcal{P}|+|\mathcal{D}|+1-j} + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n)$$

$$\begin{aligned}
& \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] + \frac{\text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{k} - \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|} \\
& \stackrel{(c)}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_k^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] \leq n,
\end{aligned} \tag{43}$$

where (c) follows from Claim 2, which is stated below and proved in Appendix I.

Claim 2.

$$\frac{\text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{k} \stackrel{a.s.}{\leq} \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|}. \tag{44}$$

Hence, from (43), the proof of (39) is complete, which concludes the proof of (26) in Theorem 2.

D. Proof of Bound (27) in Theorem 2

The proof of (27) is similar to proof of (25); however, the proof is presented here for completeness. Without loss of generality, suppose $i = |\mathcal{P}| + |\mathcal{D}|$, and $\mathcal{N} = \{|\mathcal{P}| + |\mathcal{D}| + 1, \dots, k\}$. Consequently, our goal is to show

$$d_{|\mathcal{P}|+|\mathcal{D}|} + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k d_j \leq 1. \tag{45}$$

Suppose (d_1, \dots, d_k) are linearly achievable as defined in Definition 2. Then, by (10), it is sufficient to show

$$m_{|\mathcal{P}|+|\mathcal{D}|}(n) + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} n. \tag{46}$$

We upper bound each of the two terms on the L.H.S. of (46) separately. For the first term on the L.H.S of (46), by (9), we obtain

$$\begin{aligned}
m_{|\mathcal{P}|+|\mathcal{D}|}(n) & \stackrel{a.s.}{=} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\cup_{i \neq |\mathcal{P}|+|\mathcal{D}|} \mathbf{V}_i^n]] \\
& \stackrel{(\text{Lemma 4})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] \\
& \stackrel{(\text{Lemma 7})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]].
\end{aligned} \tag{47}$$

We now upper bound the second term on the L.H.S of (46), exactly the same way as we upper bounded the third term on the L.H.S. of (32). To avoid redundancy, we only restate the resulting bound which was stated in (37).

$$\sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|+1}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|}^n]]. \tag{48}$$

We now sum the upper bounds on individual terms on the L.H.S. of (46):

$$m_{|\mathcal{P}|+|\mathcal{D}|}(n) + \sum_{j=|\mathcal{P}|+|\mathcal{D}|+1}^k m_j(n) \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_k^n[\mathbf{V}_1^n \dots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] \leq n, \tag{49}$$

which completes the proof of (46), thus concluding the proof of (27) in Theorem 2.

V. CONCLUSION

In this paper we studied the impact of hybrid CSIT on the linear DoF (LDoF) of broadcast channels with a multiple-antenna transmitter and k single-antenna receivers (MISO BC), where the CSIT supplied by each receiver can be instantaneous (P), delayed (D), or none (N). We first focused on the 3-user MISO BC; and we completely characterized the DoF region for all possible hybrid CSIT configurations, assuming linear encoding strategies at the transmitters. In order to prove the result, we presented 3 key tools, and in particular, developed a novel bound, called *Interference Decomposition Bound*, which provides a lower bound on the interference dimension at a receiver which supplies delayed CSIT based on the average dimension of constituents of that interference, thereby decomposing the interference into its individual components.

We then extended our main proof ingredients to the general k -user setting; and we presented a general outer bound on linear DoF region of the k -user MISO BC with arbitrary CSIT configuration. We demonstrated how the bound provides an approximate characterization of linear sum-DoF to within an additive gap of 0.5 for the broad range of scenarios in which the number of receivers supplying instantaneous CSIT is greater than the number of receivers supplying delayed CSIT. In addition, for the case where only one receiver supplies delayed CSIT, we completely characterized the linear sum-DoF.

There are several future directions the one can pursue in regards to this work. An interesting direction is to improve both the inner and outer bounds for linear DoF of k -user MISO BC, where the number of receivers supplying instantaneous CSIT is less than the number of receivers supplying delayed CSIT. Another interesting future direction is to extend the results to the non-linear setting (DoF). To this aim, one needs to extend the three main ingredients of the proof of outer bounds to the non-linear setting. Least Alignment Lemma has recently been extended to the non-linear setting in [19]. Hence, an interesting direction would be to extend the Interference Decomposition Bound and MIMO Rank Ratio Inequality for BC to the non-linear setting.

APPENDIX A PROOF OF CONVERSE FOR THEOREM 1

For each CSIT configuration considered in Table I we provide the converse proof. Note that the converse proof for the cases PDD and PDN are already provided in Section III-A. Furthermore, since for the case of PPP the only bound is $0 \leq d_1, d_2, d_3 \leq 1$ according to Table I, the proof is trivial. We now prove the converse for Theorem 1 for the rest of the CSIT configurations.

A. PPD

Note that as mentioned in Remark 8, in order to prove $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ for PDD , we did not rely on any specific CSIT assumption with respect to Rx_2 . Therefore, the bound $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ also holds for the case of PPD . Moreover, note that by symmetry one can conclude that $\frac{d_1}{4} + \frac{d_2}{2} + d_3 \leq 1$ also holds for PPD . Hence, since $\frac{d_1}{2} + \frac{d_2}{4} + d_3 \leq 1$ and $\frac{d_1}{4} + \frac{d_2}{2} + d_3 \leq 1$ constitute the LDoF region for PPD according to Table I, the derivations in the converse proof of PDD also prove the converse for PPD .

B. PPN

According to Table I, it is sufficient to show that $d_1 + d_3 \leq 1$ and $d_2 + d_3 \leq 1$. We only show $d_1 + d_3 \leq 1$; since $d_2 + d_3 \leq 1$ can be proven similarly due to symmetry. Suppose (d_1, d_2, d_3) is linearly achievable as defined in Definition 2. Thus, according to (10), it is sufficient to show that $m_1(n) + m_3(n) \stackrel{a.s.}{\leq} n$. By the Decodability condition in (9) we have,

$$\begin{aligned} m_1(n) + m_3(n) &\stackrel{(9)}{=} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + m_3(n) \\ &\stackrel{(9)}{=} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \\ &\stackrel{(\text{Lemma 4})}{\leq} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n \mathbf{V}_1^n] \\ &\stackrel{(\text{Lemma 3})}{\stackrel{a.s.}{\leq}} \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_3^n]] \leq n, \end{aligned}$$

which completes the proof of converse for the case of PPN .

C. PNN, DNN, NNN

According to Table I, it is sufficient to show that $d_1 + d_2 + d_3 \leq 1$. In addition, note that it is sufficient to prove $d_1 + d_2 + d_3 \leq 1$ for the case of PNN ; since any upper bound for PNN is also a valid bound for DNN and NNN . Suppose (d_1, d_2, d_3) is linearly achievable as defined in Definition 2. Then, according to (10), it is sufficient to show that $m_1(n) + m_2(n) + m_3(n) \stackrel{a.s.}{\leq} n$. By the Decodability condition in (9) we have,

$$\begin{aligned} m_1(n) + m_2(n) + m_3(n) &\stackrel{a.s.}{=} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + m_2(n) + m_3(n) \\ &\stackrel{(9)}{=} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_3^n]] + m_3(n) \end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{(Lemma 4)}}{\leq} \text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n] + \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_2^n \mathbf{V}_1^n] + m_3(n) \\
& \stackrel{\text{(a)}}{\stackrel{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + m_3(n) \\
& \stackrel{\text{(9)}}{\stackrel{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \\
& \stackrel{\text{(Lemma 3)}}{\stackrel{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] \leq n,
\end{aligned}$$

where (a) follows by applying Lemma 3 and considering Rx₂ as the receiver which supplies no CSIT. Hence, the proof of converse for the cases PNN, DNN, NNN is complete.

D. DDD

The bounds stated in Table I for DDD have been proven in [3] for general encoding schemes via network enhancement and using the fact that in physically degraded broadcast channel feedback does not increase the capacity. Therefore, the same bounds also hold for the class of linear schemes. See [3] for the bounds on the DoF of k -user MISO broadcast channel with delayed CSIT.

E. DDN

Note that according to Table I and due to symmetry of the first two users it is sufficient to show that $\frac{d_1}{2} + d_2 + d_3 \leq 1$. The other inequality (i.e. $d_1 + \frac{d_2}{2} + d_3 \leq 1$) can be proven similarly. Suppose (d_1, d_2, d_3) is linearly achievable, as defined in Definition 2. Thus, according to (10), it is sufficient to show that $\frac{m_1(n)}{2} + m_2(n) + m_3(n) \stackrel{\text{a.s.}}{\leq} n$. We have

$$\begin{aligned}
\frac{m_1(n)}{2} + m_2(n) + m_3(n) & \stackrel{\text{(9)}}{\stackrel{\text{a.s.}}{=}} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + m_2(n) + m_3(n) \\
& \stackrel{\text{(9)}}{\stackrel{\text{a.s.}}{=}} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_3^n]] + m_3(n) \\
& \stackrel{\text{(Lemma 4)}}{\leq} \frac{\text{rank}[\mathbf{G}_1^n \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_2^n \mathbf{V}_1^n] + m_3(n) \\
& \leq \frac{\text{rank}[[\mathbf{G}_1^n; \mathbf{G}_2^n] \mathbf{V}_1^n]}{2} + \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - \text{rank}[\mathbf{G}_2^n \mathbf{V}_1^n] + m_3(n) \\
& \stackrel{\text{(a)}}{\stackrel{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + m_3(n) \\
& \stackrel{\text{(9)}}{\stackrel{\text{a.s.}}{=}} \text{rank}[\mathbf{G}_2^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] - \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \\
& \stackrel{\text{(Lemma 3)}}{\stackrel{\text{a.s.}}{\leq}} \text{rank}[\mathbf{G}_3^n [\mathbf{V}_1^n \quad \mathbf{V}_2^n \quad \mathbf{V}_3^n]] \leq n,
\end{aligned}$$

where (a) follows by applying Lemma 2 to Rx₂ as the receiver which supplies delayed CSIT. Hence, the proof of converse for the case of DDN is complete.

APPENDIX B

PROOF OF INTERFERENCE DECOMPOSITION BOUND (PROOF OF LEMMAS 1,5)

Note that Lemma 1 is a special case of Lemma 5 where $k = 3$, $\mathcal{S} = \{1, 2\}$, $j = 3$, and $\ell = 1$. Therefore, in order to prove Lemma 1 and Lemma 5 it is sufficient to prove only Lemma 5. We first restate Lemma 5 here for convenience.

Lemma 5. (Interference Decomposition Bound) Consider a fixed linear coding strategy $f^{(n)}$, with corresponding precoding matrices $\mathbf{V}_1^n, \mathbf{V}_2^n, \dots, \mathbf{V}_k^n$ as defined in (4). For any $\mathcal{S} \subseteq \{1, 2, \dots, k\}$, any $\ell \in \mathcal{S}$, and any $j \notin \mathcal{S}$ for which $I_j = D$,

$$\frac{\text{rank}[\mathbf{G}_\ell^n [\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_\ell^n [\cup_{\substack{i \in \mathcal{S} \\ i \neq \ell}} \mathbf{V}_i^n]] + \text{rank}[\mathbf{G}_j^n [\cup_{\substack{i \in \mathcal{S} \\ i \neq \ell}} \mathbf{V}_i^n]]}{2} \stackrel{\text{a.s.}}{\leq} \text{rank}[\mathbf{G}_j^n [\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]. \quad (50)$$

To prove Lemma 5, we first introduce some definitions. Consider a fixed linear encoding function $f^{(n)}$, with corresponding precoding matrices $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4).

Definition 3. For $\mathcal{S} \subseteq \{1, \dots, k\}, \ell \in \mathcal{S}, j \in \{1, \dots, k\}$, we define

$$\begin{aligned}\mathcal{T}_1 &\triangleq \{t \in \{1, \dots, n\} \mid \text{rank}[\mathbf{G}_\ell^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_\ell^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]] + 1\} \\ \mathcal{T}_2 &\triangleq \{t \in \mathcal{T}_1 \mid \text{rank}[\mathbf{G}_\ell^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]] \in \text{rowspan}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]\}.\end{aligned}$$

Remark 11. \mathcal{T}_1 is the subset of time slots in which the dimension of received signal at Rx_ℓ increases, while \mathcal{T}_2 is the subset of \mathcal{T}_1 in which the received signal at Rx_ℓ is already recoverable by using the past received signals at Rx_j . The definitions of $\mathcal{T}_1, \mathcal{T}_2$ focus only on the contribution of \mathbf{V}_i^n , where $i \in \mathcal{S}$, on the dimension of received signals at different receivers; because the statement of Lemma 5 only involves \mathbf{V}_i^n , where $i \in \mathcal{S}$.

We now state two lemmas that are the main building blocks of the proof of Lemma 5.

Lemma 8.

$$\text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] - |\mathcal{T}_2| \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]. \quad (51)$$

Lemma 9.

$$|\mathcal{T}_2| - \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] \leq \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_j^n[\cup_{i \neq \ell} \mathbf{V}_i^n]]. \quad (52)$$

Note that proof of Lemma 5 is immediate from summing the inequalities in Lemma 8 and Lemma 9. Hence, we will prove Lemma 8 and Lemma 9.

A. Proof of Lemma 8

Before proving Lemma 8, we first provide its proof sketch for the special case of $k = 3, j = 3, \ell = 1, \mathcal{S} = \{1, 2\}$, the same special case as considered in Lemma 1, to emphasize the underlying ideas. For such special case, Lemma 8 reduces to the following inequality:

$$\text{rank}[\mathbf{G}_1^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] - |\mathcal{T}_2| \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]], \quad (53)$$

which can be re-written in the following equivalent form:

$$n - \text{rank}[\mathbf{G}_3^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] \stackrel{a.s.}{\leq} n - \text{rank}[\mathbf{G}_1^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]] + |\mathcal{T}_2|. \quad (54)$$

Note that the L.H.S. of (54) is basically the number of time slots $t \in \{1, \dots, n\}$ in which $\text{rank}[\mathbf{G}_3^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ does not increase (compared to $\text{rank}[\mathbf{G}_3^{t-1}[\mathbf{V}_1^{t-1} \quad \mathbf{V}_2^{t-1}]]$). Let us denote the set of such time slots by \mathcal{T} . First, note that in each $t \in \mathcal{T}$, either $\text{rank}[\mathbf{G}_1^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ increases by 1 (compared to $\text{rank}[\mathbf{G}_1^{t-1}[\mathbf{V}_1^{t-1} \quad \mathbf{V}_2^{t-1}]]$), or it remains constant. Accordingly, we partition \mathcal{T} into two sets, and upper bound the cardinality of each set. The number of those time slots $t \in \mathcal{T}$ in which $\text{rank}[\mathbf{G}_1^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ remains constant is at most $n - \text{rank}[\mathbf{G}_1^n[\mathbf{V}_1^n \quad \mathbf{V}_2^n]]$, which constitutes the first two terms on the R.H.S of (54).

We now upper bound the number of time slots $t \in \mathcal{T}$, in which $\text{rank}[\mathbf{G}_1^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ increases by 1. In each such time slot, Rx_3 receives an equation which is already recoverable by using its past received equations (since $\text{rank}[\mathbf{G}_3^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ does not increase). But note that due to the assumption of delayed CSIT for Rx_3 , the transmitter does not know the channels to Rx_3 when transmitting its signals at time slot t ; and the received signal at Rx_3 would be a random linear combination of transmit signals. In order for this random linear combination to be known at Rx_3 , Rx_3 must have already been able to recover each of the individual signals transmitted at time t , based on its past received signals. Note that if Rx_3 knows each individual transmit signal at time t , it also knows any linear combination of them. Hence, it can already recover what Rx_1 receives at time t . Therefore, the number of time slots $t \in \mathcal{T}$ in which $\text{rank}[\mathbf{G}_1^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ increases by 1 is upper bounded by the number of time slots in which the received signal at Rx_1 is already recoverable by Rx_3 , and $\text{rank}[\mathbf{G}_1^t[\mathbf{V}_1^t \quad \mathbf{V}_2^t]]$ increases by 1, which in turn, by the definition of \mathcal{T}_2 is equal to $|\mathcal{T}_2|$, the last term on the R.H.S. of (54). Thus, the proof sketch is complete.

The following is the general mathematical proof for Lemma 8, which relies on the above approach. Let us denote the indicator function by $I(\cdot)$. We then have

$$\begin{aligned}
n - \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] &= \sum_{t=1}^n I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) \\
&= \sum_{t \in \mathcal{T}_1} I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) + \sum_{t \in \mathcal{T}_1^c} I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) \\
&\stackrel{(a)}{=} \sum_{t \in \mathcal{T}_1} I(\text{rowspan}[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)] \subseteq \text{rowspan}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) + \sum_{t \in \mathcal{T}_1^c} I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) \\
&\leq \sum_{t \in \mathcal{T}_1} I(\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)] \in \text{rowspan}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) + \sum_{t \in \mathcal{T}_1^c} I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) \\
&= |\mathcal{T}_2| + \sum_{t \in \mathcal{T}_1^c} I(\text{rank}[\mathbf{G}_j^t[\cup_{i \in \mathcal{S}} \mathbf{V}_i^t]] = \text{rank}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]) \\
&\leq |\mathcal{T}_2| + \sum_{t \in \mathcal{T}_1^c} 1 = |\mathcal{T}_2| + n - |\mathcal{T}_1| \\
&\stackrel{(b)}{=} |\mathcal{T}_2| + n - \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]],
\end{aligned}$$

where (a) is due to Lemma 10, which is stated and proved in Appendix C²; and (b) follows immediately from the definition of \mathcal{T}_1 . By rearranging the above inequality, the proof of Lemma 8 will be complete.

B. Proof of Lemma 9

We first state a claim which is useful in lower bounding the R.H.S. of the inequality in Lemma 9, and it can be proved using simple linear algebra; hence the proof is omitted for brevity.

Claim 3. For two matrices A, B of the same row size, $\text{rank}[A \ B] - \text{rank}[B] = \dim(\text{span}([\vec{s} \ \vec{0}] \text{ s.t. } [\vec{s} \ \vec{0}] \in \text{rowspan}[A \ B]))$.

We are now ready to prove Lemma 9. Let $[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{i \neq \ell, t \in \mathcal{T}_2}$ denote the matrix constructed by rows $\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]$, where $t \in \mathcal{T}_2$. We have

$$\begin{aligned}
\text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] - \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] &\stackrel{(\text{Claim 3})}{=} \dim(\text{span}([\vec{s} \ \vec{0}] \text{ s.t. } [\vec{s} \ \vec{0}] \in \text{rowspan}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]])) \\
&\stackrel{(a)}{\geq} \dim(\text{span}([\vec{s} \ \vec{0}] \text{ s.t. } [\vec{s} \ \vec{0}] \in \text{rowspan}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{t \in \mathcal{T}_2})) \\
&\stackrel{(\text{Claim 3})}{=} \text{rank}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{t \in \mathcal{T}_2}] - \text{rank}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{i \neq \ell, t \in \mathcal{T}_2}] \\
&\stackrel{(b)}{=} |\mathcal{T}_2| - \text{rank}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{i \neq \ell, t \in \mathcal{T}_2}] \\
&\geq |\mathcal{T}_2| - \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]],
\end{aligned}$$

where (a) follows from the fact that for each $t \in \mathcal{T}_2$, $\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)] \in \text{rowspan}[\mathbf{G}_j^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]]$; hence, for each $t \in \mathcal{T}_2$, $\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)] \in \text{rowspan}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]$; and therefore, $\text{rowspan}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{t \in \mathcal{T}_2}] \subseteq \text{rowspan}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]$. Furthermore, (b) holds since $\text{rank}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{t \in \mathcal{T}_2}] = |\mathcal{T}_2|$, which is due to the following: since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, if $t \in \mathcal{T}_2$, then $t \in \mathcal{T}_1$. Therefore, using the definition of \mathcal{T}_1 , we get

$$\forall t \in \mathcal{T}_2, \quad \vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)] \notin \text{rowspan}(\mathbf{G}_\ell^{t-1}[\cup_{i \in \mathcal{S}} \mathbf{V}_i^{t-1}]). \quad (55)$$

Consequently, the vectors $\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]$, where $t \in \mathcal{T}_2$, are linearly independent; and therefore, we have $\text{rank}[[\vec{\mathbf{g}}_\ell(t)[\cup_{i \in \mathcal{S}} \mathbf{V}_i(t)]]_{t \in \mathcal{T}_2}] = |\mathcal{T}_2|$. Hence, the proof of Lemma 9 is complete.

²Lemma 10 is a variation of Lemma 6 in [9] which was stated for the setting with *distributed* transmit antennas. Proof of Lemma 10 follows similar steps as in the proof of Lemma 6 in [9]; it is provided in Appendix C for completeness.

APPENDIX C
STATEMENT AND PROOF OF LEMMA 10

Lemma 10. Consider a fixed linear coding strategy $f^{(n)}$ with corresponding precoding matrices $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4). Consider an arbitrary index j , where $j \in \{1, \dots, k\}$, and assume $I_j \in \{D, N\}$; i.e., the transmitter has either delayed or no CSIT with respect to R_{x_j} . In addition, consider an arbitrary set of receiver indices \mathcal{S} , where $\mathcal{S} \subseteq \{1, \dots, k\}$. For any $t \in \{1, 2, \dots, n\}$, let $\mathcal{A}_t, \mathcal{B}_t$, denote the following sets of channel realizations:

- $\mathcal{A}_t \triangleq \{\mathcal{G}^n \mid \text{rank}[G_j^t[\cup_{i \in \mathcal{S}} V_i^t]] = \text{rank}[G_j^{t-1}[\cup_{i \in \mathcal{S}} V_i^{t-1}]]\}.$
- $\mathcal{B}_t \triangleq \{\mathcal{G}^n \mid \text{rowspan}[\cup_{i \in \mathcal{S}} V_i(t)] \subseteq \text{rowspan}[G_j^{t-1}[\cup_{i \in \mathcal{S}} V_i^{t-1}]]\}.$

Then,

$$\Pr(\mathcal{G}^n \in \cup_{t=1}^n (\mathcal{A}_t \cap \mathcal{B}_t^c)) = 0.$$

Proof: Note that due to Union Bound, it is sufficient to show that for any $t \in \{1, 2, \dots, n\}$, $\Pr(\mathcal{G}^n \in \mathcal{A}_t \cap \mathcal{B}_t^c) = 0$. Consider an arbitrary $t \in \{1, 2, \dots, n\}$. Due to Total Probability Law, it is sufficient to show that for any channel realization of the first $t-1$ timeslots, denoted by \mathcal{G}^{t-1} , we have

$$\Pr(\mathcal{G}^n \in \mathcal{A}_t \cap \mathcal{B}_t^c \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) = 0. \quad (56)$$

Consider an arbitrary channel realization of the first $t-1$ time slots \mathcal{G}^{t-1} and precoding matrices V_1^t, \dots, V_k^t (which are now deterministic because they are only function of the channel realizations for the first $t-1$ time slots). Also, suppose that given \mathcal{G}^{t-1} , \mathcal{B}_t^c occurs; since otherwise, the proof of (56) would be complete. We denote the row h of the matrix $[\cup_{i \in \mathcal{S}} V_i(t)]$ by $[\cup_{i \in \mathcal{S}} V_{i,h}(t)]$. Note that by assuming \mathcal{B}_t^c occurs, and denoting $\mathcal{L} = \text{rowspan}[G_j^{t-1}[\cup_{i \in \mathcal{S}} V_i^{t-1}]]$, the following is true (according to the definition of \mathcal{B}_t):

$$\begin{aligned} & \exists h \in \{1, \dots, m\} \quad \text{s.t.} \quad [\cup_{i \in \mathcal{S}} V_{i,h}(t)] \notin \mathcal{L} \\ \Rightarrow & \exists h \in \{1, \dots, m\} \quad \text{s.t.} \quad \text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,h}(t)] \neq 0. \end{aligned} \quad (57)$$

Therefore, the $m \times (\sum_{i \in \mathcal{S}} m_i(n))$ matrix $[\text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,1}(t)]; \dots; \text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,m}(t)]]$ is non-zero, which means that its null space has dimension strictly lower than m , the number of its rows. Hence, we have,

$$\begin{aligned} \Pr(\mathcal{G}^n \in \mathcal{A}_t \cap \mathcal{B}_t^c \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) & \stackrel{(a)}{=} \Pr(\mathcal{G}^n \in \mathcal{A}_t \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) \\ & \stackrel{(b)}{=} \Pr(\text{Proj}_{\mathcal{L}^\perp}[\vec{g}_j(t)[\cup_{i \in \mathcal{S}} V_i(t)]] = 0 \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) \\ & \stackrel{(c)}{=} \Pr(\vec{g}_j(t)[\text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,1}(t)]; \dots; \text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,m}(t)]] = 0 \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) \\ & = \Pr(\vec{g}_j(t)^\top \in \text{nullspace}([\text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,1}(t)]; \dots; \text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,m}(t)]]^\top) \mid \mathcal{G}^{t-1} = \mathcal{G}^{t-1}) \\ & \stackrel{(d)}{=} 0, \end{aligned}$$

where (a) holds since we assumed that for realization \mathcal{G}^{t-1} , \mathcal{B}_t^c occurs; (b) holds according to the definition of \mathcal{A}_t ; (c) holds due to linearity of orthogonal projection; and (d) holds since, as mentioned before, the matrix $[\text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,1}(t)]; \dots; \text{Proj}_{\mathcal{L}^\perp}[\cup_{i \in \mathcal{S}} V_{i,m}(t)]]^\top$ is non-zero, meaning that its null space, which is a subspace in \mathbb{R}^m , has dimension strictly lower than m . Therefore, the probability that the random vector $\vec{g}_j(t)$ lies in a subspace in \mathbb{R}^m of strictly lower dimension (than m) is zero. ■

APPENDIX D
PROOF OF MIMO RANK RATIO INEQUALITY FOR BC (PROOF OF LEMMAS 2,6)

Note that Lemma 2 is a special case of Lemma 6 where $k = 3$, $j = 1$, $\mathcal{S} = \{i\}$, and $i_1 = 3, i_2 = \ell$. Therefore, in order to prove Lemma 2 and Lemma 6 it is sufficient to prove only Lemma 6. We first re-state Lemma 6 here for convenience.

Lemma 6. (MIMO Rank Ratio Inequality for BC) Consider a linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4). Let $\mathbf{Y}_j^n \triangleq \mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]$, where $\mathcal{S} \subseteq \{1, 2, \dots, k\}$. Also, consider distinct receivers

$Rx_{i_1}, \dots, Rx_{i_{j+1}}$, where $j = 1, 2, \dots, k-1$ and $i_1, \dots, i_{j+1} \in \{1, \dots, k\}$. If $Rx_{i_1}, \dots, Rx_{i_j}$ supply delayed CSIT, then,

$$\frac{\text{rank}[\mathbf{Y}_{i_1}^n; \dots; \mathbf{Y}_{i_{j+1}}^n]}{j+1} \stackrel{\text{a.s.}}{\leq} \frac{\text{rank}[\mathbf{Y}_{i_1}^n; \dots; \mathbf{Y}_{i_j}^n]}{j}. \quad (58)$$

Proof: Without loss of generality, we suppose that $i_1 = 1, i_2 = 2, \dots, i_{j+1} = j+1$. Thus, we need to show that

$$\frac{\text{rank}[\mathbf{Y}_1^n; \dots; \mathbf{Y}_{j+1}^n]}{j+1} \stackrel{\text{a.s.}}{\leq} \frac{\text{rank}[\mathbf{Y}_1^n; \dots; \mathbf{Y}_j^n]}{j}. \quad (59)$$

Let us denote

$$\text{rank}[A|B] \triangleq \text{rank}[A; B] - \text{rank}[B]. \quad (60)$$

Hence, by sub-modularity property of rank (Lemma 4), for matrices A, B, C with the same number of columns,

$$\text{rank}[A|B] \geq \text{rank}[A|B; C]; \quad (61)$$

$$\text{rank}[A|C] + \text{rank}[B|C] \geq \text{rank}[A; B|C]. \quad (62)$$

Moreover, we denote $\mathbf{Y}_j(t) \triangleq \vec{\mathbf{g}}_j(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)]$ and $\mathbf{Y}^t \triangleq [\mathbf{Y}_1^t; \dots; \mathbf{Y}_j^t]$. For each $i = 1, \dots, j$, we have

$$\begin{aligned} & \text{rank}[\mathbf{Y}_i(t)|[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]] \\ & \stackrel{(60)}{=} I(\vec{\mathbf{g}}_i(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \notin \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]) \end{aligned} \quad (63)$$

$$= 1 - I(\vec{\mathbf{g}}_i(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \in \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]) \quad (64)$$

$$\stackrel{(a)}{\stackrel{\text{a.s.}}{=}} 1 - I(\text{rowspan}[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \subseteq \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]) \quad (65)$$

$$\stackrel{(b)}{\geq} 1 - I(\vec{\mathbf{g}}_{j+1}(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \in \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]) \quad (66)$$

$$\stackrel{(60)}{=} \text{rank}[\mathbf{Y}_{j+1}(t)|[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]] \quad (67)$$

$$\stackrel{(61)}{\geq} \text{rank}[\mathbf{Y}_{j+1}(t)|[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]] \quad (68)$$

$$\stackrel{(60)}{=} \text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_{j+1}(t)]|\mathbf{Y}^{t-1}] - \text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]|\mathbf{Y}^{t-1}] \quad (69)$$

$$\stackrel{(61)}{\geq} \text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_{j+1}(t)]|[\mathbf{Y}^{t-1}; \mathbf{Y}_{j+1}^{t-1}]] - \text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]|\mathbf{Y}^{t-1}], \quad (70)$$

where to see why (a) holds, we first present the following variant of Lemma 10: if \mathcal{A}_t denotes the event: $\vec{\mathbf{g}}_i(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \in \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]$, and \mathcal{B}_t denotes the event: $\text{rowspan}[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \subseteq \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]$, then

$$\Pr(\mathcal{A}_t \cap \mathcal{B}_t^c) = 0, \quad (71)$$

which can be proven using the same steps as in proof of Lemma 10; therefore, its proof is omitted for brevity. As a result, $I(\mathcal{A}_t) = I(\mathcal{A}_t \cap \mathcal{B}_t) + I(\mathcal{A}_t \cap \mathcal{B}_t^c) \stackrel{(71)}{\stackrel{\text{a.s.}}{=}} I(\mathcal{A}_t \cap \mathcal{B}_t) = I(\mathcal{B}_t)$, where the last equality holds since occurrence of \mathcal{B}_t implies occurrence of \mathcal{A}_t . Therefore, $1 - I(\mathcal{A}_t) \stackrel{\text{a.s.}}{=} 1 - I(\mathcal{B}_t)$. In addition, note that the left-hand-side of (a) is $1 - I(\mathcal{A}_t)$, and the right-hand-side of (a) is $1 - I(\mathcal{B}_t)$. Hence, (a) holds. Moreover, (b) holds due to the fact that if

$$\text{rowspan}[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \subseteq \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)],$$

then,

$$\vec{\mathbf{g}}_{j+1}(t)[\cup_{h \in \mathcal{S}} \mathbf{V}_h(t)] \in \text{rowspan}[\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)].$$

By summing both sides of (70) over $i = 1, \dots, j$, and using (60) we obtain,

$$\text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]|\mathbf{Y}^{t-1}] \stackrel{\text{a.s.}}{\geq} j \left(\text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_{j+1}(t)]|[\mathbf{Y}^{t-1}; \mathbf{Y}_{j+1}^{t-1}]] - \text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]|\mathbf{Y}^{t-1}] \right); \quad (72)$$

and by rearranging the above inequality and dividing both sides by $j(j+1)$ we obtain

$$\frac{\text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_j(t)]|\mathbf{Y}^{t-1}]}{j} \stackrel{a.s.}{\geq} \frac{\text{rank}[[\mathbf{Y}_1(t); \dots; \mathbf{Y}_{j+1}(t)]|\mathbf{Y}^{t-1}; \mathbf{Y}_{j+1}^{t-1}]}{j+1}. \quad (73)$$

Finally, by summing both sides of the above inequality over all $t = 1, \dots, n$, and using the definition in (60), the proof of (59) would be complete, which concludes the proof of Lemma 6. \blacksquare

APPENDIX E

PROOF OF LEAST ALIGNMENT LEMMA (PROOF OF LEMMAS 3,7)

Note that Lemma 3 is a special case of Lemma 7 where $k = 3$ and $j = 3$. Therefore, in order to prove Lemma 3 and Lemma 7 it is sufficient to prove only Lemma 7. We first re-state Lemma 7 here for convenience.

Lemma 7. (Least Alignment Lemma) *For any linear coding strategy $f^{(n)}$, with corresponding $\mathbf{V}_1^n, \dots, \mathbf{V}_k^n$ as defined in (4), and any $\mathcal{S} \subseteq \{1, 2, \dots, k\}$, if $I_j = N$ for some $j \in \{1, 2, \dots, k\}$,*

$$\forall \ell \in \{1, 2, \dots, k\}, \quad \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]].$$

Proof: Define $m(n) \triangleq \sum_{i \in \mathcal{S}} m_i(n)$. We first state a lemma that will be later useful in proving Lemma 7.

Lemma 11. ([25]) *For $n \in \mathbb{N}$, a multi-variate polynomial function on \mathbb{C}^n to \mathbb{C} , is either identically 0, or non-zero almost everywhere.*

We now prove Lemma 7. Denote by $[1 : n]$ the set $\{1, \dots, n\}$. For any matrix $B_{n \times m(n)}$ and $I_1 \subseteq [1 : n]$, and $I_2 \subseteq [1 : m(n)]$, we denote by B_{I_1, I_2} the sub-matrix of B whose rows and columns are specified by I_1 and I_2 , respectively. Define the set of channel realizations \mathcal{A} as:

$$\mathcal{A} \triangleq \{\mathcal{G}^n | \text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] > \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]\}. \quad (74)$$

In order to prove $\text{rank}[\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]$, we only need to show $\Pr(\mathcal{A}) = 0$. Since a matrix $B_{n \times m(n)}$ has rank r if and only if the maximum size of a square sub-matrix of B with non-zero determinant is r ,

$$\begin{aligned} \mathcal{A} \subseteq \{\mathcal{G}^n | \exists I_1 \subseteq [1 : n], I_2 \subseteq [1 : m(n)], |I_1| = |I_2|, \\ \text{s.t. } \det([\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \neq \det([\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) = 0\}, \end{aligned}$$

which can be rewritten as

$$\mathcal{A} \subseteq \cup_{\substack{I_1 \subseteq [1:n] \\ I_2 \subseteq [1:m(n)] \\ |I_1|=|I_2|}} \{\mathcal{G}^n | \det([\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \neq 0, \det([\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) = 0\}. \quad (75)$$

Let X^n denote a diagonal matrix of size $n \times n$ where the elements on the diagonal are variables in \mathbb{C} . Then, for any $I_1 \subseteq [1 : n]$, $I_2 \subseteq [1 : m(n)]$, where $|I_1| = |I_2|$, $\det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2})$ is a multi-variate polynomial function in the elements of X^n . Note that if for some realization $X^n = \mathbf{G}_\ell^n$, $\det([\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \neq 0$, then the polynomial function defined by $\det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2})$ is not identical to zero (i.e., $\det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \stackrel{\text{identical}}{\neq} 0$). So, by (75), we have

$$\begin{aligned} \mathcal{A} \subseteq \cup_{\substack{I_1 \subseteq [1:n] \\ I_2 \subseteq [1:m(n)] \\ |I_1|=|I_2|}} \{\mathcal{G}^n | \det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \stackrel{\text{identical}}{\neq} 0, \det([\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) = 0\} \\ = \cup_{\substack{I_1 \subseteq [1:n] \\ I_2 \subseteq [1:m(n)] \\ |I_1|=|I_2|}} \{\mathcal{G}^n | \det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \stackrel{\text{identical}}{\neq} 0, \mathbf{G}_j^n \text{ is root of } \det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2})\}. \end{aligned} \quad (76)$$

Note that by Lemma 11, for every $I_1 \in [1 : n]$, $I_2 \in [1 : m(n)]$, $|I_1| = |I_2|$, we have

$$\Pr(\{\mathcal{G}^n | \det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2}) \stackrel{\text{identical}}{\neq} 0, \mathbf{G}_j^n \text{ is root of } \det([X^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]]_{I_1, I_2})\}) = 0. \quad (77)$$

So, since finite union of measure-zero sets has measure zero,

$$\Pr\left(\bigcup_{\substack{I_1 \subseteq [1:n] \\ I_2 \subseteq [1:m(n)] \\ |I_1|=|I_2|}} \{G^n |\det([X^n[\cup_{i \in \mathcal{S}} V_i^n])_{I_1, I_2}]^{\text{identical}}| \neq 0, \quad G_j^n: \text{root of } \det([X^n[\cup_{i \in \mathcal{S}} V_i^n])_{I_1, I_2}])\} = 0, \quad (78)$$

which by (76) implies that $\Pr(\mathcal{A}) = 0$. Therefore, according to the definition of \mathcal{A} in (74),

$$\text{rank} [\mathbf{G}_\ell^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]] \stackrel{a.s.}{\leq} \text{rank} [\mathbf{G}_j^n[\cup_{i \in \mathcal{S}} \mathbf{V}_i^n]], \quad (79)$$

which completes the proof of Least Alignment Lemma. \blacksquare

Remark 12. Using the same line of argument as in the proof of Lemma 7, one can prove Lemma 7 for a more general network setting where there are arbitrary number of transmitters, and the transmitters have arbitrary number of antennas. In addition, the statement of Lemma 7 holds even if Rx_j, Rx_ℓ have multiple but equal number of antennas.

APPENDIX F

PROOF OF PROPOSITION 1 (CONSTANT GAP CHARACTERIZATION FOR $|\mathcal{P}| \geq |\mathcal{D}|$)

In this Appendix we show that for $|\mathcal{P}| \geq |\mathcal{D}|$, Theorem 2 leads to an approximate characterization of LDoF_{sum} to within an additive gap of $\frac{1}{2}$, as presented in Proposition 1. First, note that for the special case of $|\mathcal{P}| = |\mathcal{D}| = 0$, $\text{LDoF}_{\text{region}}$ is completely characterized by $\{(d_1, \dots, d_k) \mid \sum_{i=1}^k d_i \leq 1\}$. Thus, henceforth we assume that $|\mathcal{P}| > 0$.

Moreover, note that a naive lower bound for LDoF_{sum} is $|\mathcal{P}|$; since we can focus only on the $|\mathcal{P}|$ receivers that provide instantaneous CSIT, and for those $|\mathcal{P}|$ receivers we can perform zero-forcing to cancel interference and achieve $|\mathcal{P}|$ as a lower bound on LDoF_{sum} . Using this lower bound we show that for the case where $|\mathcal{P}| \geq |\mathcal{D}|$, the statement of Proposition 1 holds. In particular, we first consider the case where $|\mathcal{D}| = 0$. For this case, by (27) in Theorem 2 we have

$$\forall i \in \mathcal{P} \cup \mathcal{D}, \quad d_i + \sum_{j \in \mathcal{N}} d_j \leq 1, \quad (80)$$

which, together with $\forall i, d_i \leq 1$, yields

$$\text{LDoF}_{\text{sum}} \leq |\mathcal{P}|. \quad (81)$$

Hence, the naive lower bound of $|\mathcal{P}|$ on LDoF_{sum} is tight for the case where $|\mathcal{D}| = 0$. Moreover, LDoF_{sum} for the special case where $|\mathcal{D}| = 1$ is characterized in Proposition 2. Therefore, we only need to prove Proposition 1 for the case of $|\mathcal{P}| \geq |\mathcal{D}| > 1$. Recall that by (25) in Theorem 2,

$$\forall i \in \mathcal{D}, \forall \pi_{\mathcal{P} \cup \mathcal{D} \setminus i}, \quad \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_{\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}(j)}}{2^j} + d_i + \sum_{j \in \mathcal{N}} d_j \leq 1. \quad (82)$$

Without loss of generality, suppose $\mathcal{P} = \{1, \dots, |\mathcal{P}|\}$, and $\mathcal{D} = \{|\mathcal{P}|+1, \dots, |\mathcal{P}|+|\mathcal{D}|\}$, and $\mathcal{N} = \{|\mathcal{P}|+|\mathcal{D}|+1, \dots, k\}$. In addition, let $i = |\mathcal{P}|+|\mathcal{D}|$, and $\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}$ be the identity permutation. Consequently, by (25) in Theorem 2 we obtain:

$$\sum_{i=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_i}{2^i} + d_{|\mathcal{P}|+|\mathcal{D}|} + \sum_{j \in \mathcal{N}} d_j \leq 1, \quad (83)$$

or equivalently,

$$\left(\sum_{i=1}^{|\mathcal{P}|} \frac{d_i}{2^i}\right) + \left(\sum_{i=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{d_i}{2^i} + d_{|\mathcal{P}|+|\mathcal{D}|}\right) + \sum_{j \in \mathcal{N}} d_j \leq 1. \quad (84)$$

Note that in the above inequality there are $|\mathcal{P}|$ different coefficients (i.e. $\frac{1}{2}, \dots, \frac{1}{2^{|\mathcal{P}|}}$) for receivers in \mathcal{P} , and $|\mathcal{D}|$ different coefficients (i.e. $\frac{1}{2^{|\mathcal{P}|+1}}, \dots, \frac{1}{2^{|\mathcal{P}|+|\mathcal{D}|-1}}, 1$) for receivers in \mathcal{D} . Due to symmetry, we can consider all the

possible $|\mathcal{P}|! \times |\mathcal{D}|!$ joint permutations of the receivers in \mathcal{P} and \mathcal{D} , leading to permutations of the corresponding coefficients in (84). By summing over all those resulting inequalities, and diving by $|\mathcal{P}|! \times |\mathcal{D}|!$, we obtain

$$(1 - \frac{1}{2^{|\mathcal{P}|}})(\sum_{i=1}^{|\mathcal{P}|} \frac{d_i}{|\mathcal{P}|}) + (1 + \frac{1}{2^{|\mathcal{P}|}} - \frac{1}{2^{|\mathcal{P}|+|\mathcal{D}|-1}})(\sum_{i=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \frac{d_i}{|\mathcal{D}|}) + \sum_{j \in \mathcal{N}} d_j \leq 1. \quad (85)$$

Note that $\text{LDoF}_{\text{sum}} \leq \max \sum_{i=1}^k d_i$ subject to (85) and $d_i \leq 1$ for all i , which is basically a simple linear program. By solving the linear program, one can easily see that

$$\text{LDoF}_{\text{sum}} \leq \max \left(|\mathcal{P}| + \frac{|\mathcal{D}|}{2^{|\mathcal{P}|} + 1 - \frac{1}{2^{|\mathcal{D}|-1}}}, |\mathcal{P}| + \frac{1}{2^{|\mathcal{P}|}}, 1, \frac{|\mathcal{D}|}{1 + \frac{1}{2^{|\mathcal{P}|}} - \frac{1}{2^{|\mathcal{P}|+|\mathcal{D}|-1}}} \right). \quad (86)$$

Note that since we assumed $|\mathcal{P}| \geq |\mathcal{D}| > 1$, the above inequality simplifies as follows:

$$\text{LDoF}_{\text{sum}} \leq |\mathcal{P}| + \frac{|\mathcal{D}|}{2^{|\mathcal{P}|} + 1 - \frac{1}{2^{|\mathcal{D}|-1}}}, \quad (87)$$

which together with $\text{LDoF}_{\text{sum}} \geq |\mathcal{P}|$ leads to

$$|\mathcal{P}| \leq \text{LDoF}_{\text{sum}} \leq |\mathcal{P}| + \frac{|\mathcal{D}|}{2^{|\mathcal{P}|} + 1 - \frac{1}{2^{|\mathcal{D}|-1}}}. \quad (88)$$

Therefore, the gap between upper and lower bounds on LDoF_{sum} is upper bounded as

$$\text{Gap} = \frac{|\mathcal{D}|}{2^{|\mathcal{P}|} + 1 - \frac{1}{2^{|\mathcal{D}|-1}}} \leq \frac{|\mathcal{D}|}{2^{|\mathcal{P}|}} \leq \frac{|\mathcal{P}|}{2^{|\mathcal{P}|}} \leq \frac{1}{2}.$$

Hence, the proof of Proposition 1 is complete.

APPENDIX G

PROOF OF PROPOSITION 2 ($\text{LDoF}_{\text{sum}} = |\mathcal{P}| + \frac{1}{2^{|\mathcal{P}|}}$ FOR $|\mathcal{D}| = 1$)

We focus on the k -user MISO BC with only one receiver supplying delayed CSIT. We first prove the converse. Assume without loss of generality that $\mathcal{P} = \{1, \dots, |\mathcal{P}|\}$, $\mathcal{D} = \{|\mathcal{P}| + 1\}$ and $\mathcal{N} = \{|\mathcal{P}| + 2, \dots, k\}$. Further, let $i = |\mathcal{P}| + 1$, and $\pi_{\mathcal{P} \cup \mathcal{D} \setminus i}$ denote the identity permutation. Then, by Theorem 2 the solution to the following linear program provides an upper bound on LDoF_{sum} :

$$\begin{aligned} \text{LDoF}_{\text{sum}} \leq \quad & \max \quad \sum_{i=1}^k d_i \\ \text{s.t.} \quad & \sum_{i=1}^{|\mathcal{P}|} \frac{d_i}{2^i} + d_{|\mathcal{P}|+1} + \sum_{j \in \mathcal{N}} d_j \leq 1, \end{aligned} \quad (89)$$

$$0 \leq d_i \leq 1, \quad i = 1, \dots, k, \quad (90)$$

where the first constraint in the linear program is due to (25) in Theorem 2. Thus, by solving the above linear program one can readily see that

$$\text{LDoF}_{\text{sum}} \leq |\mathcal{P}| + \frac{1}{2^{|\mathcal{P}|}}. \quad (91)$$

Hence, the converse proof is complete. We now present the achievable scheme, which is a multi-phase scheme that uses hybrid CSIT available to the transmitter to perform interference alignment. The new achievable scheme generalizes the schemes for PD in [13] and PPD in [18] (see Figure 4 for the special case of $PPPD$).

To achieve LDoF_{sum} of $|\mathcal{P}| + \frac{1}{2^{|\mathcal{P}|}}$, we will ignore the receivers in \mathcal{N} ; and we show that we can linearly achieve $(d_1, \dots, d_{|\mathcal{P}|+1}) = (1, \dots, 1, \frac{1}{2^{|\mathcal{P}|}})$. Therefore, if, with slight abuse of notation, we denote $K \triangleq |\mathcal{P}| + 1$, we need to show that the following DoF tuple is linearly achievable:

$$(d_1, \dots, d_{K-1}, d_K) = \left(1, \dots, 1, \frac{1}{2^{K-1}}\right). \quad (92)$$

To this end, we present a new multi-phase communication scheme which

- operates over 2^{K-1} time slots;
- delivers 2^{K-1} symbols to each of the receivers $1, \dots, K-1$;
- delivers 1 symbol to receiver K .

The overall scheme is split into K phases, indexed as $i = 0, 1, 2, \dots, (K-1)$:

- the duration of i -th phase is $\binom{K-1}{i}$ time slots;
- each of the first $(K-1)$ receivers obtain new (interference-free) linear equations in every time slot;
- receiver K obtains $\binom{K-1}{i}$ equations during phase i (one corresponding to each time slot).

At the end of the i -th phase, receiver K does the following: it uses its received $\binom{K-1}{i-1}$ equations from phase $(i-1)$ and $\binom{K-1}{i}$ equations in phase i to obtain $\binom{K-1}{i}$ new equations with the following specific property: each equation is a linear combination of the desired symbol by Rx_K and $(K-1-i)$ undesired symbols, where each undesired symbol is in fact desired by another receiver.

Throughout the proof of the achievable scheme we only utilize the first K transmit antennas; therefore, without loss of generality we can assume as well that there are only K transmit antennas. We first start with Phase 0, and then explain the transmission strategy for an arbitrary phase i in full detail.

A. Phase 0

Phase 0 is of duration $\binom{K-1}{0} = 1$, i.e., this phase only has 1 time slot. In this phase, the transmitter sends 2 information symbols for each of $\text{Rx}_1, \text{Rx}_2, \dots, \text{Rx}_{K-1}$, denoted by $(\mathbf{s}_1^1, \mathbf{s}_1^2), (\mathbf{s}_2^1, \mathbf{s}_2^2), \dots, (\mathbf{s}_{K-1}^1, \mathbf{s}_{K-1}^2)$, along with one symbol, denoted by \mathbf{s}_K , for the K -th receiver. Let $\vec{\mathbf{g}}_{\mathcal{S}}(1)^\perp$, where $\mathcal{S} \subseteq \{1, \dots, K-1\}$, denote a full row rank matrix of size $(K-|\mathcal{S}|) \times K$, where each row of $\vec{\mathbf{g}}_{\mathcal{S}}(1)^\perp$ is perpendicular to any $\vec{\mathbf{g}}_i(1)$ where $i \in \mathcal{S}$. We need to deliver one equation about $(\mathbf{s}_i^1, \mathbf{s}_i^2)$ interference-free to Rx_i , for $i = 1, \dots, K-1$. To this aim, the transmit signal at time 1 will be:

$$\vec{\mathbf{x}}_1(1) = \sum_{i=1}^{K-1} [\vec{\mathbf{g}}_{\{1, \dots, K-1\} \setminus \{i\}}(1)^\perp]^\top \begin{bmatrix} \mathbf{s}_i^1 \\ \mathbf{s}_i^2 \end{bmatrix} + [\vec{\mathbf{g}}_{\{1, \dots, K-1\}}(1)^\perp]^\top \mathbf{s}_K. \quad (93)$$

As a result, each of the first $K-1$ receivers obtain one equation in 2 desired symbols:

$$\mathbf{y}_i(1) = \vec{\mathbf{g}}_i(1) [\vec{\mathbf{g}}_{\{1, \dots, K-1\} \setminus \{i\}}(1)^\perp]^\top \begin{bmatrix} \mathbf{s}_i^1 \\ \mathbf{s}_i^2 \end{bmatrix}, \quad i = 1, \dots, K-1; \quad (94)$$

and receiver K obtains \mathbf{s}_K along with interference from the other symbols:

$$\mathbf{y}_K(1) = \sum_{i=1}^{K-1} \vec{\mathbf{g}}_K(1) [\vec{\mathbf{g}}_{\{1, \dots, K-1\} \setminus \{i\}}(1)^\perp]^\top \begin{bmatrix} \mathbf{s}_i^1 \\ \mathbf{s}_i^2 \end{bmatrix} + \vec{\mathbf{g}}_K(1) [\vec{\mathbf{g}}_{\{1, \dots, K-1\}}(1)^\perp]^\top \mathbf{s}_K, \quad (95)$$

which can be re-written as:

$$\mathbf{y}_K(1) = L_1(\mathbf{s}_1^1, \mathbf{s}_1^2) + L_2(\mathbf{s}_2^1, \mathbf{s}_2^2) + \dots + L_{K-1}(\mathbf{s}_{K-1}^1, \mathbf{s}_{K-1}^2) + \left(\vec{\mathbf{g}}_K(1) [\vec{\mathbf{g}}_{\{1, \dots, K-1\}}(1)^\perp]^\top \right) \mathbf{s}_K, \quad (96)$$

where $L_i(\mathbf{s}_i^1, \mathbf{s}_i^2) = \vec{\mathbf{g}}_K(1) [\vec{\mathbf{g}}_{\{1, \dots, K-1\} \setminus \{i\}}(1)^\perp]^\top \begin{bmatrix} \mathbf{s}_i^1 \\ \mathbf{s}_i^2 \end{bmatrix}$. We observe that the K -th receiver has obtained 1 equation,

and this equation has $(K-1)$ interfering order-2 symbols, where each order-2 symbol is desirable by one of the other $(K-1)$ receivers. In particular, each order-2 symbol $L_i(\mathbf{s}_i^1, \mathbf{s}_i^2)$ is desired by Rx_i .

The purpose of subsequent phases of the scheme is the following: in each phase i , we deliver the interference symbols of phase $i-1$ to the intended receivers while simultaneously sending new information symbols. This should be done in an iterative manner to create a new set of equations at the K -th receiver with net interference from a smaller set of receivers, where the interference is useful for that set of receivers. With this broad goal in mind, we next describe the transmission strategy for the general phase i .

B. Phase i

Duration of Phase i is $\binom{K-1}{i}$ time slots. Let us index the slots as $j = 1, 2, \dots, \binom{K-1}{i}$.

1) *Transmission in slot j , $j = 1, 2, \dots, \binom{K-1}{i}$* : In each time slot j , the transmitter selects i receivers out of first $(K-1)$ receivers. This splits the set of $(K-1)$ receivers into two disjoint sets, and for simplicity we denote these as:

- \mathcal{R} (Repetition set): this is a set of i receivers. Let us denote the indices of the receivers in this set by (p_1, p_2, \dots, p_i) .
- \mathcal{F} (Fresh set): this is the remaining set of $(K-1-i)$ receivers, and we denote this set of receivers as $(p_{i+1}, \dots, p_{K-1})$

The basic idea behind the scheme can now be explained clearly:

- Note that in phase $(i-1)$, the K -th receiver has obtained $\binom{K-1}{i-1}$ equations, where each equation is a linear combination of $(K-i)$ undesired symbols and the intended symbol (of the K -th receiver).
- Via delayed CSIT, the transmitter can reconstruct all of these equations within noise distortion.
- Out of these $\binom{K-1}{i-1}$ equations, the transmitter focuses on those equations which consist of all symbols from the receivers p_{i+1}, \dots, p_{K-1} (i.e., the receivers belonging to the fresh set \mathcal{F}). In total, there are exactly $\binom{i}{1} = i$ such equations. The reason is that each equation in phase $(i-1)$ has interference from exactly $(K-i)$ receivers. We zoom in on such equations with interference from $(K-1-i)$ receivers p_{i+1}, \dots, p_{K-1} , and thus the remaining flexibility is to choose 1 more interference symbol. The total remaining receivers to select from are $(K-1) - (K-1-i) = i$ and hence the number of ways is $\binom{i}{1} = i$.
- From each of these i equations, the transmitter reconstructs the only symbol in the equation which is desired by one of the receivers in the repetition set (p_1, p_2, \dots, p_i) . Let us denote the reconstructed symbols by $s_{p_1}(j), \dots, s_{p_i}(j)$. Also, we denote those i equations as following:

$$s_{p_1}(j) + LC_1 : \text{where } LC_1 \text{ is a linear combination of symbols for receivers in set } \mathcal{F} \cup \{K\} \quad (97)$$

$$s_{p_2}(j) + LC_2 : \text{where } LC_2 \text{ is a linear combination of symbols for receivers in set } \mathcal{F} \cup \{K\} \quad (98)$$

$$\vdots \quad (99)$$

$$s_{p_i}(j) + LC_i : \text{where } LC_i \text{ is a linear combination of symbols for receivers in set } \mathcal{F} \cup \{K\}. \quad (100)$$

- For each of the $(K-1-i)$ receivers in the fresh set, the transmitter sends 2 precoded fresh (i.e. new) symbols. Let us denote these as $(s_{p_{i+1}}^1(j), s_{p_{i+1}}^2(j)), (s_{p_{i+2}}^1(j), s_{p_{i+2}}^2(j))$ up to $(s_{p_{K-1}}^1(j), s_{p_{K-1}}^2(j))$.

Hence in the j -th slot of phase i , the transmitter sends:

$$\vec{x}_i(j) = \underbrace{\sum_{r=1}^i [\vec{g}_{\{1, \dots, K-1\} \setminus \{p_r(j)\}}(j)^\perp]^\top \begin{bmatrix} s_{p_r}(j) \\ 0 \end{bmatrix}}_{i \text{ repetition symbols}} + \underbrace{\sum_{r=i+1}^{K-1} [\vec{g}_{\{1, \dots, K-1\} \setminus \{p_r(j)\}}(j)^\perp]^\top \begin{bmatrix} s_{p_r}^1(j) \\ s_{p_r}^2(j) \end{bmatrix}}_{2(K-1-i) \text{ fresh symbols}}. \quad (101)$$

Clearly, the set of repetition receivers $\{p_1, p_2, \dots, p_i\}$ receive one symbol without interference. Similarly, the set of receivers $\{p_{i+1}, \dots, p_{K-1}\}$ also receive one clean (interference-free) useful symbol in this slot (which is a linear combination of the two fresh symbols).

2) *Operation at Rx_K in time slot j of phase i* : Let us now focus on Rx_K at the j -th time slot of phase i . Rx_K obtains

$$y_K(i, j) = \sum_{r=1}^i \alpha_r(i, j) s_{p_r}(j) + \sum_{r=i+1}^{K-1} LC(s_{p_r}^1(j), s_{p_r}^2(j)), \quad (102)$$

where $\alpha_r(i, j)$ denotes the coefficient of the symbol $s_{p_r}(j)$ when received at Rx_K ; and $LC(s_{p_r}^1(j), s_{p_r}^2(j))$ denotes the linear combination of $s_{p_r}^1(j), s_{p_r}^2(j)$ received at Rx_K . Note that from phase $(i-1)$, the receiver also has i equations $s_{p_1}(j) + LC_1, \dots, s_{p_i}(j) + LC_i$ as mentioned in (97)-(100). Using these i equations together with (102), receiver K eliminates the i symbols $s_{p_1}(j), s_{p_2}(j), \dots, s_{p_i}(j)$; and it is left with an equation of the following form:

$$LC_{p_{i+1}}(j) + LC_{p_{i+2}}(j) + \dots + LC_{p_{K-1}}(j) + s_K. \quad (103)$$

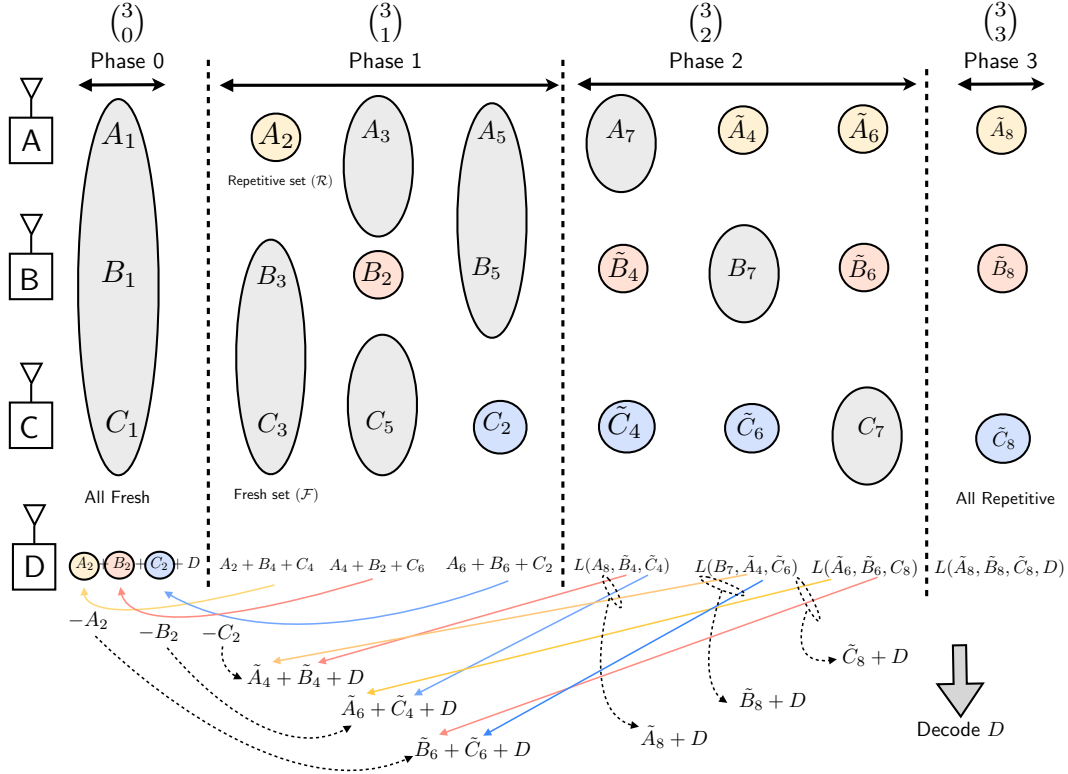


Fig. 4. Scheme for 4-user MISO BC: PPPD Setting.

This equation consists of $(K - 1 - i)$ interfering symbols, where each interfering symbol $LC_{p_r}(j)$ is desired by Rx_{p_r} , for $r = i + 1, i + 2, \dots, (K - 1)$.

Recall that the slot index j varies from 1 to $\binom{K-1}{i}$, each slot corresponding to the partitioning of the set of $K - 1$ receivers into two disjoint sets of size i and $(K - 1 - i)$. Each slot gives the K -th receiver one equation with interference from exactly $(K - 1 - i)$ receivers. Hence, in total, at the end of phase i , the receiver has $\binom{K-1}{i}$ equations, and each equation has interference from symbols desired by exactly $(K - 1 - i)$ receivers. Thus, we can now readily apply this process iteratively.

3) *Phase $K - 1$ (the last phase; corresponding to $i = K - 1$):* Before the last phase $K - 1$, (i.e., just after phase $K - 2$), the K -th receiver has $\binom{K-1}{i-1} = \binom{K-1}{K-2} = K - 1$ equations, and each equation has interference from exactly $(K - 1) - (i - 1) = (K - 1) - (K - 2) = 1$ receiver. Hence, the K -th receiver has $K - 1$ equations of the following form before the last phase:

$$LC'_1 + s_K, LC'_2 + s_K, \dots, LC'_{K-1} + s_K, \quad (104)$$

where LC'_1 is desired by receiver 1, LC'_2 is desired by receiver 2, etc.

In the last phase, whose duration is only 1 slot (since $\binom{K-1}{K-1} = 1$), the transmitter sends LC'_1, \dots, LC'_{K-1} without any interference to receivers $1, \dots, K - 1$ by utilizing instantaneous CSIT. Receiver K obtains a linear combination of LC'_1, \dots, LC'_{K-1} . Hence, the K th receiver has K equations in K variables $LC'_1, LC'_2, \dots, LC'_K$ and s_K . Therefore, it can decode s_K ; and the proof is complete.

C. Illustrative Example – 4 User MISO BC

Here, we present the achievable scheme for $K = 4$ to clearly illustrate the idea behind the iterative scheme. For the case of 4-user MISO BC with PPPD, the goal is to achieve:

$$(d_1, d_2, d_3, d_4) = \left(1, 1, 1, \frac{1}{2^3}\right). \quad (105)$$

Here, the scheme has $K = 4$ phases, with the following phase durations:

- Phase 0: $\binom{3}{0} = 1$ time slots; Tx sends two new information symbols for each of the first three receivers, and one symbol for the fourth receiver. Each of the first three receivers will receive a linear combination of its two desired symbols without any interference.
- Phase 1: $\binom{3}{1} = 2$ time slots; in each time slot, Tx sends the signal received in the past by R_{x_K} with respect to the symbols of one of the first three receivers; and it also sends two new information symbols for each of the other 2 receivers supplying instantaneous CSIT.
- Phase 2: $\binom{3}{2} = 3$ time slots; in each slot, Tx sends fresh information for 1 receiver with instantaneous CSIT and supplies past signals received by R_{x_K} with respect to the remaining 2 receivers supplying instantaneous CSIT.
- Phase 3: $\binom{3}{3} = 1$ time slot; Tx sends past received signals by R_{x_K} which are desired by the three receivers supplying instantaneous CSIT.

See Figure 4 which illustrates the achievable scheme for 4-user MISO BC, where the first 3 receivers supply instantaneous CSIT, while the fourth receiver supplies delayed CSIT.

APPENDIX H PROOF OF CLAIM 1

We first re-state Claim 1 here for convenience.

Claim 1.

$$\sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{m_j(n)}{2^j} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]. \quad (106)$$

To prove Claim 1 we first prove the following inequality by induction, and then show how it leads to proving Claim 1.

$$\sum_{j=1}^{i-1} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_i^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^{i-1}} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]], \quad i = 2, \dots, |\mathcal{P}| + |\mathcal{D}| - 1. \quad (107)$$

We prove (107) by induction on i . For the base case of $i = 2$, the inequality in (107) simplifies to

$$\frac{m_1(n)}{2} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_2^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2} \stackrel{a.s.}{\leq} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]. \quad (108)$$

Hence, the base case of $i = 2$ holds due to Lemma 5 and (9). Suppose that the induction hypothesis is true for $i = s$. We show that it will also hold for $i = s + 1$. By our assumption we have

$$\begin{aligned} \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_1^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] &\stackrel{a.s.}{\geq} \sum_{j=1}^{s-1} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_s^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^{s-1}} \\ &\stackrel{\text{(Lemma 5)}}{\stackrel{a.s.}{\geq}} \sum_{j=1}^{s-1} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_s^n [\mathbf{V}_s^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] - \text{rank}[\mathbf{G}_s^n [\mathbf{V}_{s+1}^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] + \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_{s+1}^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^{s-1}} \\ &\stackrel{\text{(Lemma 4)}}{\stackrel{a.s.}{\geq}} \sum_{j=1}^{s-1} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_s^n [\mathbf{V}_1^n \cdots \mathbf{V}_k^n]] - \text{rank}[\mathbf{G}_s^n [\cup_{i \in \{1, \dots, k\}, i \neq s} \mathbf{V}_i^n]] + \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_{s+1}^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^s} \\ &\stackrel{\text{(9)}}{=} \sum_{j=1}^{s-1} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_s^n \mathbf{V}_s^n] + \text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_{s+1}^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^s} \\ &\stackrel{\text{(9)}}{=} \sum_{j=1}^s \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n [\mathbf{V}_{s+1}^n \cdots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]]}{2^s}. \end{aligned}$$

Hence, the induction hypothesis holds for $i = s + 1$ as well; and as a result, the proof of (107) is complete. We now show how (107) leads to proof of Claim 1. Let $i = |\mathcal{P}| + |\mathcal{D}| - 1$. Then, by (107),

$$\begin{aligned}
\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]] &\stackrel{\text{a.s.}}{\geq} \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-2} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]}{2^{|\mathcal{P}|+|\mathcal{D}|-2}} \\
&\stackrel{\text{(Lemma 6)}}{\stackrel{\text{a.s.}}{\geq}} \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-2} \frac{m_j(n)}{2^j} + \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|-1}^n; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n] \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]}{2^{|\mathcal{P}|+|\mathcal{D}|-1}} \\
&\geq \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-2} \frac{m_j(n)}{2^j} + \frac{\text{rank}[\mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|-1}^n \mathbf{V}_{|\mathcal{P}|+|\mathcal{D}|-1}^n]}{2^{|\mathcal{P}|+|\mathcal{D}|-1}} \\
&\stackrel{\text{(9)}}{\stackrel{\text{a.s.}}{=}} \sum_{j=1}^{|\mathcal{P}|+|\mathcal{D}|-1} \frac{m_j(n)}{2^j},
\end{aligned}$$

which completes the proof of Claim 1.

APPENDIX I PROOF OF CLAIM 2

We first re-state the Claim for convenience.

Claim 2.

$$\frac{\text{rank}[[\mathbf{G}_1^n; \dots; \mathbf{G}_k^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{k} \stackrel{\text{a.s.}}{\leq} \frac{\text{rank}[[\mathbf{G}_{|\mathcal{P}|+1}^n; \dots; \mathbf{G}_{|\mathcal{P}|+|\mathcal{D}|}^n][\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]]}{|\mathcal{D}|}. \quad (109)$$

Proof:

We consider the notations (60), (61); and we use $\mathbf{Y}_j^n \triangleq \mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]$, and $\mathbf{Y}_j(t) \triangleq \vec{\mathbf{g}}_j(t)[\mathbf{V}_1(t) \dots \mathbf{V}_{|\mathcal{P}|}(t)]$. Furthermore, we denote by $\mathbf{Y}_{\mathcal{S}}^n$ the column concatenation of matrices $\mathbf{G}_j^n[\mathbf{V}_1^n \dots \mathbf{V}_{|\mathcal{P}|}^n]$, where $j \in \mathcal{S}$. Therefore, we need to show that

$$\frac{\text{rank}[\mathbf{Y}_1^n; \dots; \mathbf{Y}_k^n]}{k} \stackrel{\text{a.s.}}{\leq} \frac{\text{rank}[\mathbf{Y}_{\mathcal{D}}^n]}{|\mathcal{D}|}. \quad (110)$$

For all $t = 1, \dots, n$, we have

$$\begin{aligned}
(|\mathcal{P}| + |\mathcal{N}|) \times \text{rank}[\mathbf{Y}_{\mathcal{D}}(t) | \mathbf{Y}_{\mathcal{D}}^{t-1}] &\stackrel{(60)}{=} \sum_{i=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} (|\mathcal{P}| + |\mathcal{N}|) \times \text{rank}[\mathbf{Y}_i(t) | [\mathbf{Y}_{\mathcal{D}}^{t-1}; \mathbf{Y}_{|\mathcal{P}|+1}(t); \dots; \mathbf{Y}_{i-1}(t)]] \\
&\stackrel{(a)}{\stackrel{\text{a.s.}}{\geq}} \sum_{i=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \sum_{j \in \mathcal{P} \cup \mathcal{N}} \text{rank}[\mathbf{Y}_j(t) | [\mathbf{Y}_{\mathcal{D}}^{t-1}; \mathbf{Y}_{|\mathcal{P}|+1}(t); \dots; \mathbf{Y}_{i-1}(t)]] \\
&\stackrel{(61)}{\geq} \sum_{i=|\mathcal{P}|+1}^{|\mathcal{P}|+|\mathcal{D}|} \sum_{j \in \mathcal{P} \cup \mathcal{N}} \text{rank}[\mathbf{Y}_j(t) | [\mathbf{Y}_{\mathcal{D}}^n; \mathbf{Y}_{\mathcal{P} \cup \mathcal{N}}^{t-1}]] = |\mathcal{D}| \sum_{j \in \mathcal{P} \cup \mathcal{N}} \text{rank}[\mathbf{Y}_j(t) | [\mathbf{Y}_{\mathcal{D}}^n; \mathbf{Y}_{\mathcal{P} \cup \mathcal{N}}^{t-1}]] \\
&\stackrel{(62)}{\geq} |\mathcal{D}| \times \text{rank}[\mathbf{Y}_{\mathcal{P} \cup \mathcal{N}}(t) | [\mathbf{Y}_{\mathcal{D}}^n; \mathbf{Y}_{\mathcal{P} \cup \mathcal{N}}^{t-1}]], \quad (111)
\end{aligned}$$

where (a) follows from the same arguments as in (63)-(67) which were used to show that

$$\text{rank}[\mathbf{Y}_i(t) | [\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]] \stackrel{\text{a.s.}}{\geq} \text{rank}[\mathbf{Y}_{j+1}(t) | [\mathbf{Y}^{t-1}; \mathbf{Y}_1(t); \dots; \mathbf{Y}_{i-1}(t)]],$$

for the case where $i \in \{1, \dots, j\} \subseteq \mathcal{D}$, and $\mathbf{Y}^{t-1} \triangleq [\mathbf{Y}_1^{t-1}; \dots; \mathbf{Y}_j^{t-1}]$.

By summing both sides of the inequality (111) over all $t = 1, \dots, n$, we obtain

$$\begin{aligned}
(|\mathcal{P}| + |\mathcal{N}|) \times \text{rank}[\mathbf{Y}_{\mathcal{D}}^n] &\stackrel{\text{a.s.}}{\geq} |\mathcal{D}| \times \text{rank}[\mathbf{Y}_{\mathcal{P} \cup \mathcal{N}}^n | \mathbf{Y}_{\mathcal{D}}^n] \\
&\stackrel{(60)}{=} |\mathcal{D}| \times \text{rank}[\mathbf{Y}_1^n; \dots; \mathbf{Y}_k^n] - |\mathcal{D}| \times \text{rank}[\mathbf{Y}_{\mathcal{D}}^n]. \quad (112)
\end{aligned}$$

Finally, by rearranging the above inequality we obtain (110), which proves Claim 2. ■

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