Study of an Abating Aggregation Operator in Many-Valued Logic

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***Abstract*—This paper considers a parametrised aggregation operator, originally introduced in the formal framework of many-valued logic and in the applicative context of information scoring. It studies this operator, outside this applicative context, looking at specific configurations of interest: highlighting the wide range of its instantiations, from the lower to the upper extreme cases; showing some t-norms it can encode, as specific cases; and also how it allows rich and flexible intermediate behaviours.**

I. Introduction

Aggregation is a vast research domain, especially in the context of fuzzy logic [2], [5], that led to wide ranges of operators and typologies, as well as categorisations of their properties. In these, one can distinguish between logical operators, for instance expressing conjunction (e.g. t-norms), disjunction (e.g. t-conorms) or implication, and compromise operators (e.g. averages or OWA) or hybrids in particular offering full reinforcement properties (e.g. MICA or symmetric sum). In the theoretical framework of many-valued logic [4], [6], aggregation is mainly performed using logical operators [4], corresponding to t-norms, t-conorms or implications, and arithmetical ones [8], beyond a purely logical interpretation.

In this many-valued framework, this paper proposes to study the operator introduced in [7] in the applicative context of information scoring: this task (see e.g. [3]) aims at measuring the quality of a piece of information and often relies on the decomposition of the definition of quality into several dimensions, for which a huge variety of possibilities can be considered (see e.g. [1], [3]). Each dimension is then individually assessed and the results are aggregated into the final score. The approach introduced in [7] proposes to measure the dimensions in an extended many-valued logic framework, which offers clarity and legibility, and develops original aggregation operators in this context.

More precisely, the global quality of a piece of information is dynamically measured as a score whose value shifts with the consecutive integration of different dimensions. These factors are projected on the current score, resulting in an updated value, aggregation of the previous value and the dimension evaluation. These consecutive corrections entail two types of behaviours, we consider here the projections that have an abating influence: they are defined as resulting in either a decrease in value or, at most, leaving it unchanged.

This paper proposes to study this operator outside its original applicative framework, through properties and examples: Section II offers a reminder of the definition of this abating operator, which this paper proposes to name AbOp. Section III theoretically examines parameter constraints required to guarantee the desired abating behaviour, proposing a matrix representation of the operator. Section IV examines specific cases of AbOp, discussing various configurations of interest: it thus illustrates the wide range of covered operators, including extreme lower and upper cases, as well as classical t-norms and, beyond, rich intermediate operators. Section V concludes the paper.

II. Definition of the Many-Valued Increasing Abating Operator AbOp

This section presents the AbOp operator studied in this paper, introduced in [7] in an extended many-valued logic framework. This paper restricts its study to the classical case [4], [6] which uses M truth degrees of a totally ordered set $\mathcal{L}_M = \{\tau_0, \dots, \tau_{M-1}\}$, where $\tau_\alpha \leq \tau_\beta \Leftrightarrow \alpha \leq \beta$. These span, at a granularity varying with M, the different levels of veracity they represent from τ_0, meaning 'false', to τ_{M-1}, for 'true'. At a very general level, an aggregation operator is then defined as a mapping $F : \mathcal{L}_M \times \mathcal{L}_M \longrightarrow \mathcal{L}_M$.

A. Formalisation of the Desired Behaviour

Due to the information scoring context considered in [7] and not detailed further here, the considered aggregation operator is required to have an abating influence on its first argument and to be increasing in both its arguments. Formally, these constraints respectively impose that, for all $\tau_\alpha, \tau_\beta, \tau_\gamma \in \mathcal{L}_M$:

$$C_1 : \quad F(\tau_\alpha, \tau_\beta) \leq \tau_\alpha \tag{1}$$

$$C_2 : \quad \text{if } \tau_\alpha \leq \tau_\beta, \qquad F(\tau_\alpha, \tau_\gamma) \leq F(\tau_\beta, \tau_\gamma) \tag{2}$$

$$C_3 : \quad \text{if } \tau_\alpha \leq \tau_\beta, \qquad F(\tau_\gamma, \tau_\alpha) \leq F(\tau_\gamma, \tau_\beta) \tag{3}$$

As underlined in [7], it can be observed that conjunctive aggregation operators such as t-norms satisfy these conditions. However, they also impose other properties, as they are also commutative, associative and have τ_{M-1} for a neutral element. These properties are superfluous in the original context and would lead to unwanted constraints, limiting the expressiveness of the model. A general definition of the operator is thus proposed in [7], as described below.

Constraints C_1 to C_3, on the other hand, do exclude other types of aggregation operators, in particular implications (which are decreasing in their first argument), as well as disjunctive, compromise and reinforcement operators, which do not offer the desired abating effect.

B. Definition of the AbOp Operator

1) Formal Expression: To be as general as possible, the AbOp operator, defined in [7], depends on a set of parameters:

$$\kappa \subseteq \{\kappa_\alpha^\gamma \in \mathcal{L}_M, \alpha, \gamma \in [\![0, M-1]\!]\} \tag{4}$$

$$\text{such that } \forall \alpha, \exists \gamma, \kappa_\alpha^\gamma = \tau_{M-1} \tag{5}$$

and is formally expressed as:

$$F_\kappa(\tau_\alpha, \tau_\beta) = \min\{\tau_\gamma \in \mathcal{L}_M | \tau_\beta \leq \kappa_\alpha^\gamma\} \tag{6}$$

This definition is discussed and illustrated below, both graphically and using examples.

The condition in Eq. (5) guarantees the function given in Eq. (6) is well-defined, i.e. that $F(\tau_\alpha, \tau_\beta)$ can be computed. Indeed, it is necessary that, for all α, β, the set from which the minimum is taken is not empty. For any τ_β and any α, since $\tau_\beta \leq \tau_{M-1} = \kappa_\alpha^\gamma$ for some γ, the function can be computed.

2) Graphical Representation: Figure 1, reproduced from [7], illustrates such a function in the specific case of a 5-valued logic, i.e. $\mathcal{L}_5$: the horizontally aligned discs represent the degrees of $\mathcal{L}_5$, from τ_0 to τ_4, corresponding either to possible values of τ_α or possible outputs. The arrow labels represent the parameters κ_α^γ.

The computation of $F(\tau_\alpha, \tau_\beta)$ then consists in starting from the disc labelled τ_α and following the arrows whose labels κ_α^γ are greater than τ_β. The output is the lowest accessible τ_γ, i.e. the leftmost pointed to disc, with $\kappa_\alpha^\gamma \geq \tau_\beta$.

Figures 2 to 6 illustrate various instantiations of this general schema and are commented in Section IV. As an example, when considering Figure 4, the computation of $F(\tau_3, \tau_2)$ starts from the disc labelled τ_3 and considers the arrows with labels greater than $\tau_\beta = \tau_2$, i.e. the ones pointing to τ_3 and τ_2. The output is the minimal value, i.e. $F_{\kappa_Z}(\tau_3, \tau_2) = \tau_2$. In the operator instantiation graphically represented in Figure 2, however, a single arrow starting from τ_3, pointing to τ_0, can be considered. It satisfies the constraint as its label $\kappa_3^0 = \tau_{M-1} = \tau_4 > \tau_\beta = \tau_2$: the output is, thus, $F_{\kappa_D}(\tau_3, \tau_2) = \tau_0$.

3) Parameter Interpretation: The κ parameters can thus be interpreted as the conditions on the transition from the initial value τ_α to the output value τ_γ: the transition is allowed if the value τ_β to be aggregated with τ_α is less than the transition cost, i.e. if $\tau_\beta \leq \kappa_\alpha^\gamma$.

When more than one transition is possible, the most 'pessimistic', i.e. the one leading to the lowest value, is chosen.

III. Parameter Constraints

The general definition given in Eq. (4) to (6) suggests that M^2 parameters must be specified to define an instantiation of AbOp. These can be stored in a square matrix, that must contain at least one τ_{M-1} on each row, due to the condition expressed in Eq. (5).

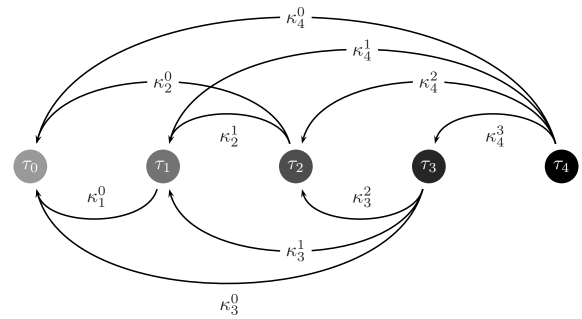

Fig. 1. An example of $F(\tau_\alpha, \tau_\beta)$, in $\mathcal{L}_5$, reproduced from [7].

However, the form of the function, given in Eq. (6), and the constraints specifying the desired behaviour, Eq. (1), (2) and (3) restrict the choices, reducing the number of required parameters. This restriction is crucial since, without it, it would be more relevant and no more expensive to define, explicitly and directly, the desired value of $F(\tau_\alpha, \tau_\beta)$ for all pairs $(\tau_\alpha, \tau_\beta)$.

This section discusses and formalises the various restrictions to be considered, together with their consequences on the matrix representation.

A. Parameter Non-Redundancy

1) Constraint Expression: First an ordering constraint can be imposed on the parameters, such that

$$\forall \tau_\alpha, \tau_\gamma, \tau_\delta \in \mathcal{L}_M \quad \text{if } \tau_\gamma \leq \tau_\delta, \text{ then } \kappa_\alpha^\gamma < \kappa_\alpha^\delta \tag{7}$$

As shown in Prop. 1 below, this monotonicity constraint does not restrict the set of operators which can be defined and it makes the proofs of the properties of the next sections easier.

Property 1: Let κ be a set of parameters in which there exist α, γ and δ such that $\tau_\gamma \leq \tau_\delta$ and $\kappa_\alpha^\gamma \geq \kappa_\alpha^\delta$.

Let $\kappa' = \kappa \setminus \{\kappa_\alpha^\delta\}$.

Then $F_\kappa = F_{\kappa'}$

Proof: Informally, the principle is that for any τ_α, τ_β, if $\tau_\beta \leq \kappa_\alpha^\delta$, then $\tau_\beta \leq \kappa_\alpha^\gamma$. Since the minimal value is kept, by definition, this implies that $F_\kappa(\tau_\alpha, \tau_\beta) \neq \tau_\delta$: the δ value cannot be the selected candidate, since γ is a better, more 'pessimistic' fit.

Formally, for any τ_α, τ_β, if $\tau_\beta > \kappa_\alpha^\delta$, then κ_α^δ is not involved in the computation of $F_\kappa(\tau_\alpha, \tau_\beta)$ and therefore $F_\kappa(\tau_\alpha, \tau_\beta) = F_{\kappa'}(\tau_\alpha, \tau_\beta)$. If, on the other hand, $\tau_\beta \leq \kappa_\alpha^\delta$, then:

$$\begin{aligned} F_\kappa(\tau_\alpha, \tau_\beta) &= \min\{\tau_u \in \mathcal{L}_M | \tau_\beta \leq \kappa_\alpha^u\} \\ &= \min(\min\{\tau_u \in \mathcal{L}_M \setminus \{\tau_\delta\} | \tau_\beta \leq \kappa_\alpha^u\}, \tau_\delta) \\ &= \min(F_{\kappa'}(\tau_\alpha, \tau_\beta), \tau_\delta) = F_{\kappa'}(\tau_\alpha, \tau_\beta) \end{aligned}$$

The second line isolates, in the minimum, the special value τ_δ which must be taken into account as a candidate, since, by hypothesis, $\tau_\beta \leq \kappa_\alpha^\delta$. The last equality comes from the fact that $\tau_\beta \leq \kappa_\alpha^\delta$ implies $\tau_\beta \leq \kappa_\alpha^\gamma$ and, therefore, $F_{\kappa'}(\tau_\alpha, \tau_\beta) \leq \tau_\gamma$ which in turn is lower than τ_δ, by hypothesis. ■

Therefore, in the rest of the paper, we suppose that the κ_α^γ values are ordered with respect to their exponents, for each α.

2) Corollaries: The previous property has two consequences of interest, respectively for high and for low values: first, it leads to the fact that if $\kappa_\alpha^\gamma = \tau_{M-1}$, then the values of any κ_α^δ for $\delta > \gamma$ can be left out. Indeed, due to the monotonicity constraint, they can only be equal to τ_{M-1} as well and would, thus, be redundant.

More generally, for any α, a unique occurrence of each observed value among the κ_α^γ should be kept, others, associated to greater γ values, are redundant.

Second, for low values, this property similarly implies that, for each α, a single occurrence of τ_0 should appear among all κ_α^γ, specifically the one with minimal γ. However, the values of κ_α^δ for lower δ cannot be considered as either implicitly defined or redundant, they must be considered as undefined, since they have no allowed value.

A semantic difference should be made between values which need not be defined, because they are redundant, and values which cannot be defined, because they point to excluded outputs, values which can never be reached. Note that excluded values may be encountered in other situations than just in the τ_0 case discussed here, as illustrated in Section IV-B.

3) Matrix Representation: Due to the previous corollaries, the matrix representation of the κ values can be constrained and restricted further: a single occurrence of each observed value on each row can be imposed.

This choice leads to leave some of the matrix cells empty, as discussed below. A distinction could be made between redundant and excluded values, e.g. introducing a special symbol, outside $\mathcal{L}_M$, to represent the latter. In the following, we do not make this distinction, representing both cases with empty cells. The considered matrix is, thus, quite sparse.

B. Abating Effect

1) Constraint Expression: The desired abating effect expressed by constraint C_1 imposes the following constraint on the AbOp parameters, which refines the condition expressed in Eq. (5):

Property 2: F_κ satisfies the abating constraint given in Eq. (1) iff κ is a parameter set such that:

$$\forall \tau_\alpha \in \mathcal{L}_M,\ \exists \gamma^* \text{ such that } \tau_{\gamma^*} \leq \tau_\alpha \text{ and } \kappa_\alpha^{\gamma^*} = \tau_{M-1} \quad (8)$$

In this expression, the dependency between γ^* and α is not formally denoted, to keep the notations simple.

Proof: If F_κ presents the abating effect, for any τ_α, considering constraint C_1 for τ_{M-1} imposes that $F_\kappa(\tau_\alpha, \tau_{M-1}) = \min\{\tau_\gamma \in \mathcal{L}_M | \tau_{M-1} \leqslant \kappa_\alpha^\gamma\} \leq \tau_\alpha$. Therefore, there must exist γ^* such that $\tau_{M-1} \leq \kappa_\alpha^{\gamma^*}$ and $\tau_{\gamma^*} \leq \tau_\alpha$. The inequality in the min imposes that $\kappa_\alpha^{\gamma^*} = \tau_{M-1}$.

Conversely, if κ satisfies Eq. (8), then for any τ_α and τ_β, the γ^* associated to τ_α ensures that $F(\tau_\alpha, \tau_\beta) \leq \tau_{\gamma^*}$. Indeed $\tau_\beta \leq \tau_{M-1} = \kappa_\alpha^{\gamma^*}$ implies that γ^* belongs to the set of γ candidates and thus $F(\tau_\alpha, \tau_\beta) \leq \tau_{\gamma^*}$. Since, by definition, $\tau_{\gamma^*} \leq \tau_\alpha$, this leads to $F(\tau_\alpha, \tau_\beta) \leq \tau_\alpha$. ■

This condition expresses that the unique occurrence of the τ_{M-1} value in the set of κ_α^γ for any α (as shown in the previous section) is associated to a value lower than τ_α. When considering the graphical representation of Fig. 1, it means that there is at least one arrow *to the left* labelled τ_{M-1}, from each disc.

Additionally, it should be noted that, from the non-redundancy constraint studied in the previous section and given in Eq. (7), this property implies that, from a graphical point of view, there is no arrow to the right, pointing up, from any of the α.

2) Consequence on the Matrix Representation: When considering the matrix representation of κ, the constraint expressed in Eq. (8) means that, in each row, the unique occurrence of τ_{M-1} must be on the diagonal or in a column located to the left of the diagonal. As discussed in Section III-A, it is not necessary to give the values in columns located to the right of τ_{M-1}. As a consequence, the matrix takes the form of a lower triangular matrix, with at most $M(M+1)/2$ values, and a single occurrence of τ_{M-1} on each row.

C. Monotonicity in the First Argument

At a matrix level, the previous constraints apply to each row independently. The monotonicity condition considered in this section establishes relations between the rows.

1) Constraint Expression: The desired increasing monotonicity expressed by constraint C_2 in Eq. (2) imposes the following ordering condition:

$$\begin{aligned} &\forall \alpha_1, \forall \alpha_2 \text{ such that } \alpha_1 \leq \alpha_2, \quad \forall \beta \\ &\quad \exists \gamma^* \text{ such that } \tau_\beta \leq \kappa_{\alpha_1}^{\gamma^*} \\ &\qquad \text{and } \gamma^* \leq \gamma \text{ for all } \gamma \text{ such that } \tau_\beta \leq \kappa_{\alpha_2}^\gamma \end{aligned} \quad (9)$$

This constraint is a direct rewriting of condition C_2 in Eq. (2) using the definition of F_κ so does not require a formal proof.

2) Consequence on the Matrix Representation: This constraint means that, given a reference position a_{ij} of the matrix, all values located in the lower left submatrix (i.e. a_{kl} such that $k \geq i, l \leq j$) must be less than the value of a_{ij}.

In particular considering the special case where $\tau_\beta = \tau_{M-1}$, the condition implies that there must exist γ^* such that $\kappa_{\alpha_1}^{\gamma^*} = \tau_{M-1}$ and such that γ^* is lower than all γ for which $\kappa_{\alpha_2}^{\gamma^*} = \tau_{M-1}$: the τ_{M-1} value on row α_1 must be to the left of the τ_{M-1} value on row α_2.

Considering the other extreme case where $\tau_\beta = \tau_0$, the condition implies that there cannot be an undefined value in row α_1 if the value is defined for row α_2.

D. Monotonicity in the Second Argument

The desired increasing monotonicity expressed by constraint C_3 in Eq. (3) does not impose any constraint, it directly follows from the definition. Indeed, for any α, β_1 and β_2 with $\beta_1 \leq \beta_2$, it holds that

$$\begin{array}{llcl} & \tau_{\beta_2} \leq \kappa_\alpha^\gamma & \Rightarrow & \tau_{\beta_1} \leq \kappa_\alpha^\gamma \\ \text{thus} & \{\tau_\gamma | \tau_{\beta_2} \leq \kappa_\alpha^\gamma\} & \subseteq & \{\tau_\gamma | \tau_{\beta_1} \leq \kappa_\alpha^\gamma\} \\ \text{thus} & \min\{\tau_\gamma | \tau_{\beta_2} \leq \kappa_\alpha^\gamma\} & \geq & \min\{\tau_\gamma | \tau_{\beta_1} \leq \kappa_\alpha^\gamma\} \\ \text{i.e.} & F(\tau_\alpha, \tau_{\beta_2}) & \geq & F(\tau_\alpha, \tau_{\beta_1}) \end{array}$$

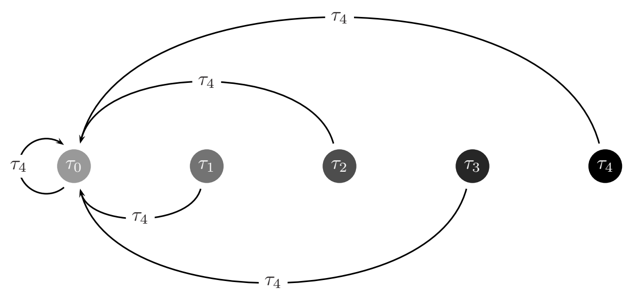

Fig. 2. Graphical representation of the drastic case F_{κ_D} discussed in Section IV-A.

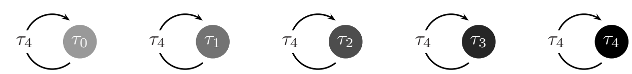

Fig. 3. Graphical representation of the extreme upper case F_{κ_U} discussed in Section IV-B.

E. Summary of the Matrix Constraints

This subsection recaps the conditions imposed on the matrix representation of the κ parameter set:

- it is a sparse lower triangular matrix,
- each row has exactly one occurrence of the value τ_{M-1},
- the values on each row are unique and sorted in increasing order,
- the values on two rows also satisfy an ordering constraint.

From the matrix point of view, the instantiation of AbOp thus consists in choosing, in each row, the position of τ_{M-1}, possibly the position of τ_0 and of their intermediate values, satisfying the inter-row constraints and possibly leaving some empty cells.

IV. Specific Configurations of Interest

This section proposes to study the general AbOp operator defined in the previous section through examples illustrating the variety of behaviours it offers. The two extreme cases, respectively lower and upper bounds of the range of operators, are studied first, then instantiations built on fuller matrices, corresponding to implementations of classic t-norms by the operator, are introduced. Finally, some intermediate behaviours, highlighting the flexibility of the operator are considered.

In each case, the matrix representation is given, as well as a graphical representation, in the special case of $\mathcal{L}_5$, and some properties and characteristics are proved and discussed.

A. Extreme Lower Case: Drastic Case

The extreme lower case is defined when the required τ_{M-1} value in each row is assigned to the minimal possible value, i.e. the first column: formally

$$\forall \alpha \in \mathcal{L}_M, \quad \kappa_\alpha^0 = \tau_{M-1} \tag{10}$$

Then no other value need be specified. This corresponds to the very sparse lower triangular matrix

$$\kappa_D = \begin{bmatrix} \tau_{M-1} & & & \\ \tau_{M-1} & & & \\ \vdots & & & \\ \tau_{M-1} & & & \end{bmatrix}$$

and the graphical representation given in Fig. 2, in $\mathcal{L}_5$.

This configuration is named drastic because of the following property:

Property 3:

$$\forall \tau_\alpha, \tau_\beta \in \mathcal{L}_M, \quad F_{\kappa_D}(\tau_\alpha, \tau_\beta) = \tau_0$$

Proof: For any $\tau_\alpha, \tau_\beta \in \mathcal{L}_M$, $\tau_\beta \leq \kappa_\alpha^0 = \tau_{M-1}$, thus 0 is a candidate and it satisfies the minimality constraint: $F_{\kappa_D}(\tau_\alpha, \tau_\beta) = \min(\min\{\tau_\gamma \in \mathcal{L}_M \setminus \{\tau_0\} | \tau_\beta \leq \kappa_\alpha^\gamma\}, \tau_0) = \tau_0$ ■

Thus this aggregation operator always outputs the minimal value τ_0, setting one extremity of the operator range.

In itself, this instantiation obviously has very limited expressiveness and usefulness, but it explicitly sets the lower limit of the range of aggregation operators AbOp can represent.

B. Extreme Upper Case

The extreme upper case is defined when, on the contrary, τ_{M-1} is assigned to the maximal possible value, i.e. according to Eq. (8)

$$\forall \alpha \in \mathcal{L}_M, \quad \kappa_\alpha^\alpha = \tau_{M-1} \tag{11}$$

and all other values are left undefined. This corresponds to the diagonal matrix:

$$\kappa_U = \begin{bmatrix} \tau_{M-1} & & & \\ & \tau_{M-1} & & \\ & & \ddots & \\ & & & \tau_{M-1} \end{bmatrix}$$

Its graphical representation given in Fig. 3, for $\mathcal{L}_5$.

The operator expression can then be simplified to:

Property 4:

$$\forall \tau_\alpha, \tau_\beta \in \mathcal{L}_M, \quad F_{\kappa_U}(\tau_\alpha, \tau_\beta) = \tau_\alpha$$

Proof: For any $\tau_\alpha, \tau_\beta \in \mathcal{L}_M$, $\tau_\beta \leq \kappa_\alpha^\alpha = \tau_{M-1}$, and this value $\gamma = \alpha$ is the only candidate since the others are undefined. ■

This aggregation operator restricts AbOp to an identity function in its first argument thus, indeed, constitutes the extreme upper case of an operator with the abating property. Therefore, F_{κ_U} sets the other bound of the range of aggregation operators AbOp can represent.

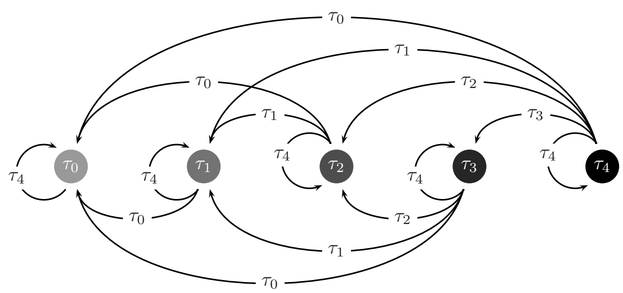

Fig. 4. Graphical representation of the reference case F_{κ_Z} discussed in Section IV-C.

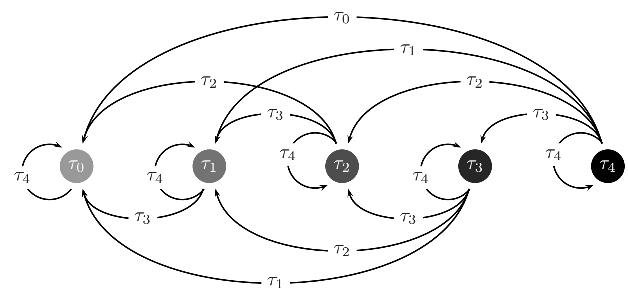

Fig. 5. Graphical representation of the reference case F_{κ_L} discussed in Section IV-D.

C. Reference Full Case: Zadeh t-norm

A less extreme case than the previous two considers a full lower diagonal matrix, which imposes that τ_{M-1} is set on the diagonal. In order to fill the rest of the matrix, consider the values:

$$\kappa_\alpha^\alpha = \tau_{M-1} \quad \text{and} \quad \kappa_\alpha^\gamma = \tau_\gamma \text{ for } \gamma < \alpha \tag{12}$$

This configuration means that, except for the reflexive case κ_α^α, the transition condition between τ_α and τ_γ only depends on the destination, τ_γ, disregarding the starting point, τ_α: this can be considered as a regular abating behaviour, where the abating factor does not depend on the value to be abated.

It corresponds to the matrix whose subdiagonal values are constant columnwise

$$\kappa_Z = \begin{bmatrix} \tau_{M-1} & & & & & \\ \tau_0 & \tau_{M-1} & & & & \\ \tau_0 & \tau_1 & \tau_{M-1} & & & \\ \tau_0 & \tau_1 & \tau_2 & \tau_{M-1} & & \\ \vdots & \vdots & \vdots & & \ddots & \\ \tau_0 & \tau_1 & \tau_2 & \cdots & \cdots & \tau_{M-1} \end{bmatrix}$$

and the graphical representation given in Fig. 4, for $\mathcal{L}_5$. It satisfies all constraints established in Section III and summarised in Section III-E: it is obviously a lower triangular matrix with τ_{M-1} in each row, as well as single ordered occurrences of the other values. In addition, for any row in the matrix, the previous one is obtained by removing the last subdiagonal value, starting from the last row that covers the whole range of $\mathcal{L}_M$ values.

This configuration can be considered as a reference case: as stated in the next property, it constitutes the AbOp instantiation that encodes the classical Zadeh t-norm:

Property 5:

$$\forall \tau_\alpha, \tau_\beta \in \mathcal{L}_M, \quad F_{\kappa_Z}(\tau_\alpha, \tau_\beta) = \min(\tau_\alpha, \tau_\beta)$$

Proof: For any $\tau_\alpha, \tau_\beta \in \mathcal{L}_M$,

$$\begin{aligned} F_{\kappa_Z}(\tau_\alpha, \tau_\beta) &= \min\{\tau_\gamma \in \mathcal{L}_M | \tau_\beta \leq \kappa_\alpha^\gamma\} \\ &= \min(\{\tau_\gamma \in \mathcal{L}_M | \gamma < \alpha, \tau_\beta \leq \kappa_\alpha^\gamma = \tau_\gamma\} \\ &\quad \cup \{\tau_\gamma \in \mathcal{L}_M | \gamma = \alpha, \tau_\beta \leq \kappa_\alpha^\gamma = \tau_{M-1}\} \\ &\quad \cup \{\tau_\gamma \in \mathcal{L}_M | \gamma > \alpha, \tau_\beta \leq \kappa_\alpha^\gamma\}) \\ F_{\kappa_Z}(\tau_\alpha, \tau_\beta) &= \min(\{\tau_\beta\} \cup \{\tau_\alpha\} \cup \emptyset) \\ &= \min(\tau_\beta, \tau_\alpha) \end{aligned}$$

Indeed, as discussed in Section III, when $\tau_\gamma > \tau_\alpha$, κ_α^γ are undefined and, therefore, yield no candidates. ■

This configuration thus shows that AbOp generalises a classic aggregation operator.

D. Threshold as Transition Costs: Łukasiewicz t-norm

Another specific full instantiation is obtained as a variant of F_{κ_Z}: also starting from a complete last row, each row is obtained from the following one by removing the first subdiagonal value, instead of the last one.

Formally, the parameters are then defined as:

$$\kappa_\alpha^\gamma = \tau_{M-1-(\alpha-\gamma)} \text{ for } \gamma \leq \alpha \tag{13}$$

It corresponds to the matrix:

$$\kappa_L = \begin{bmatrix} \tau_{M-1} & & & & & \\ \tau_{M-2} & \tau_{M-1} & & & & \\ \tau_{M-3} & \tau_{M-2} & \tau_{M-1} & & & \\ \tau_{M-4} & \tau_{M-3} & \tau_{M-2} & \tau_{M-1} & & \\ \vdots & \vdots & \vdots & & \ddots & \\ \tau_0 & \tau_1 & \tau_2 & \cdots & \cdots & \tau_{M-1} \end{bmatrix}$$

and the graphical representation given in Fig. 5. Like F_{κ_Z} discussed in the previous section, it satisfies all constraints established in Section III.

This parameter configuration can be interpreted in terms of transition costs, where the transition threshold between τ_α and τ_γ depends on their difference: in the global abating behaviour of the operator, where τ_β defines the decreasing factor from τ_α, this configuration defines a linear abating behaviour.

On top of this interpretation, this configuration can be considered as a reference case because, as stated in the next property, it is the AbOp instantiation that encodes the classical Łukasiewicz t-norm:

Property 6:

$$\forall \tau_\alpha, \tau_\beta \in \mathcal{L}_M, \quad F_{\kappa_L}(\tau_\alpha, \tau_\beta) = \tau_{\max(\alpha+\beta-(M-1),0)}$$

Proof: For any $\tau_\alpha, \tau_\beta \in \mathcal{L}_M$,

$$\begin{aligned} F_{\kappa_L}(\tau_\alpha, \tau_\beta) &= \min\{\tau_\gamma \in \mathcal{L}_M | \tau_\beta \leq \kappa_\alpha^\gamma\} \\ &= \min\{\tau_\gamma \in \mathcal{L}_M | \gamma \leq \alpha \text{ and } \tau_\beta \leq \kappa_\alpha^\gamma\} \\ &= \min\{\tau_\gamma \in \mathcal{L}_M | \gamma \leq \alpha \text{ and} \\ &\qquad \tau_\beta \leq \tau_{M-1-(\alpha-\gamma)}\} \end{aligned}$$

$\tau_\beta \leq \tau_{M-1-\alpha+\gamma}$ is equivalent to $\gamma \geq \alpha + \beta - (M-1)$, by definition, which leads to the minimal τ_γ value $\tau_{\alpha+\beta-(M-1)}$ when $\alpha + \beta \geq (M-1)$. Otherwise, the minimal value is τ_0. ■

This instantiation of AbOp shows that it can encode another classic t-norm as a specific implementation.

E. Intermediate Examples on $\mathcal{L}_5$

The examples commented previously are either extreme cases, with little expressivity, or classic t-norms. This section illustrates the flexibility of the general AbOp operator, which offers many other configurations.

The considered configurations, in the case of $\mathcal{L}_5$, are graphically given in Figure 6, they only differ by the value of κ_4^2. Their matrices are respectively left and right, for the top and bottom parts of the figure (note that the second and fourth columns are empty):

$$\kappa_{I_1} = \begin{bmatrix} \tau_4 & & & & \\ \tau_4 & & & & \\ \tau_3 & & \tau_4 & & \\ & & \tau_4 & & \\ & & \tau_3 & & \tau_4 \end{bmatrix} \quad \kappa_{I_2} = \begin{bmatrix} \tau_4 & & & & \\ \tau_4 & & & & \\ \tau_3 & & \tau_4 & & \\ & & \tau_4 & & \\ & & \tau_1 & & \tau_4 \end{bmatrix}$$

These two operators first have the specifity of excluding τ_1 and τ_3 from the set of possible output values, which reduces to $\{\tau_0, \tau_2, \tau_4\}$. The aggregation they perform offers a scale simplification, from a 5-valued to a 3-valued domain.

F_{I_1} (top part of the figure) implements a severe abating effect as, except in the rare cases where $\tau_\alpha \in \{\tau_2, \tau_4\}$ and $\tau_\beta = \tau_4$, the result of the aggregation is strictly lower than the first argument τ_α. The single case for which $F_{I_1}(\tau_\alpha, \tau_\beta) = \tau_4$ is observed if $\tau_\alpha = \tau_\beta = \tau_4$. If $\tau_\beta \neq \tau_4$, $F_{I_1}(\tau_4, \tau_\beta)$ dramatically decreases to τ_2. Therefore, it can be considered as more severe than the Zadeh and Łukasiewicz t-norms, which have τ_{M-1} as neutral element. However, it is less severe than the drastic operator F_{κ_D}, offering a semantically richer behaviour.

F_{I_2} (bottom part of the figure) is significantly less severe: when $\tau_\alpha = \tau_4$, the output value is also equal to τ_4 except if τ_β is really low, more precisely τ_1 or τ_0. F_{I_2} allows to implement a subtle abating effect, which has the particularity of being irregular: a middle value of τ_β has a neutral effect on a high initial τ_α, whereas it triggers a severe decrease if τ_α is lower than τ_2. Its abating behaviour is neither constant nor linear. F_{I_2} can be characterised as offering an accelerated abating effect when F_{I_1} offers a regularly severe one.

These examples illustrate that the general operator makes it possible to precisely parametrise the desired abating effect.

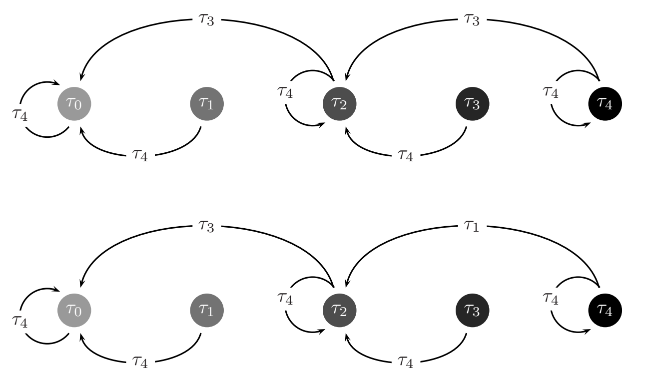

Fig. 6. Graphical representations of the intermediate cases F_{I_1} (upper graph) and F_{I_2} (lower graph) discussed in Section IV-E.

V. CONCLUSION AND FUTURE WORKS

This paper showed the flexibility of the AbOp operator and its parameter configurations, which allow to compactly represent a many-valued aggregation operator: it generalises existing operators and offers rich behaviours that can be finely tuned through a matrix representation. It also offers a graphical representation that eases its interpretation. It thus constitutes a valuable tool to perform aggregation when the criteria to aggregate are measured on discrete ordered scales, beyond pure logical or arithmetical approaches.

Ongoing works aim at further developing formal characterisations of the AbOp configurations that offer desirable theoretical properties, such as associativity or idempotence. Future works will include the definition of other characteristics, among which a possible notion of degree of severity. A refined typology based on the abating behaviours, e.g. distinguishing between regular, linear or accelerated ones, is also a relevant research direction.

REFERENCES

[1] C. Batini and M. Scannapieco. *Data and Information Quality*. Springer, 2016.
[2] T. Calvo, G. Mayor, and R. Mesiar, editors. *Aggregation Operators: New Trends and Applications*, volume 97. Springer, 2002.
[3] P. Capet and T. Delavallade, editors. *Information Evaluation*. Wiley, 2014.
[4] S. Gottwald. *A treatise on many-valued logics*, volume 9 of *Studies in logic and computation*. Research Studies Press, Baldock, 2001.
[5] M. Grabisch, J. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Number 127 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.
[6] N. Rescher. *Many-valued Logic*. McGraw Hill, 1969.
[7] A. Revault d'Allonnes and M.-J. Lesot. Formalising information scoring in a multivalued logic framework. In *Proc. of the Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU'14*, pages 314–323. Springer, 2014.
[8] H. Seridi and H. Akdag. Approximate Reasoning for Processing Uncertainty. *Journal of Advanced Computational Intelligence and Intelligent Informatics (JACIII)*, 5(2):110–118, 2001.