

POISSON'S EQUATION IN NONLINEAR FILTERING

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Abstract. The aim of this paper is to provide a variational interpretation of the nonlinear filter in continuous time. A time-stepping procedure is introduced, consisting of successive minimization problems in the space of probability densities. The weak form of the nonlinear filter is derived via analysis of the first-order optimality conditions for these problems. The derivation shows the nonlinear filter dynamics may be regarded as a gradient flow, or a steepest descent, for a certain energy functional with respect to the Kullback–Leibler divergence.

The second part of the paper is concerned with derivation of the feedback particle filter algorithm, based again on the analysis of the first variation. The algorithm is shown to be exact. That is, the posterior distribution of the particle matches exactly the true posterior, provided the filter is initialized with the true prior.

1. Introduction. The goal of this paper is to gain insight into the equations arising in nonlinear filtering, as well as into the *feedback particle filter* introduced in recent research. To expose the main ideas, it is useful to restrict our attention to the following special case in which the state evolution is constant:

$$dX_t = 0, \tag{1.1a}$$

$$dZ_t = h(X_t) dt + dW_t, \tag{1.1b}$$

where $X_t \in \mathbb{R}^d$ is the state at time t , $Z_t \in \mathbb{R}^1$ is the observation process, $h(\cdot)$ is a C^1 function, and $\{W_t\}$ is a standard Wiener process. The state is constant, and has initial condition distributed as $X_0 \sim p_0^*$. Unless otherwise noted, the stochastic differential equations (SDEs) are expressed in Itô form. Also, unless noted otherwise, all probability distributions are assumed to be absolutely continuous with respect to the Lebesgue measure, and therefore will be identified with their densities.

The objective of the filtering problem is to estimate the posterior distribution of X_t given the history $Z_t := \sigma(Z_s : s \leq t)$. The posterior is denoted by p^* , so that for any measurable set $A \subset \mathbb{R}^d$,

$$\int_A p^*(x, t) dx = \mathbb{P}\{X_t \in A \mid Z_t\}.$$

The evolution of $p^*(x, t)$ is described by the Kushner–Stratonovich (K-S) partial differential equation

$$dp^* = (h - \hat{h})(dZ_t - \hat{h} dt)p^*, \tag{1.2}$$

with initial condition p_0^* , where $\hat{h}_t = \int h(x)p^*(x, t) dx$. The theory of nonlinear filtering is described in the classic monograph [6].

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Although our analysis is restricted to a particular model with a static state process, it can be extended to broader classes of filtering problems, subject to technical conditions discussed in Remark 1. The main technical condition concerns the existence of a solution and certain a priori bounds for Poisson’s equation that also arises in simulation and optimization theory for Markov models [4, 7]. For the model considered in this paper, bounds are obtained based on a *Poincaré*, or *spectral gap, inequality* (see the bound $\mathbf{PI}(\lambda_0)$ in Assumption A2).

The contributions of this paper are two-fold: One, to show that the dynamics of the K-S equation are a gradient flow for a certain variational problem, with respect to the Kullback–Leibler divergence. Two, the variational problem is used to derive the feedback particle filter, first introduced in [13] (see also [12, 11, 10]).

The first part of the paper concerns the construction of the gradient flow. The analysis is inspired by the optimal transportation literature – in particular, the work of Otto and co-workers on the variational interpretation of the Fokker–Planck–Kolmogorov equation [5]. The construction described in Sec 2 begins with a discrete-time recursion based on the successive solution of minimization problems involving the so-called forward variational representation of the elementary Bayes’ formula (see Mitter and Newton [8]). Lemma 3.1 describes the first order optimality condition for the variational problem at each time-step.

In the continuous-time limit, these first-order conditions yield the nonlinear filter (1.2), as described in the proof of Theorem 3.2. The construction shows that the dynamics of the nonlinear filter may be regarded as a gradient flow, or a steepest descent, for a certain energy functional (“information value of the observation” according to [8]) with respect to the Kullback-Leibler divergence pseudo-metric.

The feedback particle filter algorithm is obtained using similar analysis in Sec 4. This filter is a controlled system, where the control is obtained via consideration of the first order optimality conditions for the variational problem. Theorem 4.2 shows that the filter is exact, i.e., the posterior distribution of the particle matches exactly the true posterior p^* , provided the filter is initialized with the true prior.

The remainder of this paper is organized as follows. The time-stepping procedure is introduced in Sec 2, and properties of its solution established. The gradient flow result – convergence is the solution of the time-stepping procedure to weak solution of the K-S equation (1.2) – appears in Sec 3. The feedback particle filter algorithm appears in Sec 4.

Notation: C^k is used to denote the space of k -times continuously differentiable functions; C_c^k denotes the subspace of functions with compact support. L^∞ is used to denote the space of functions that are bounded a.e. (Lebesgue).

The space of probability densities with finite second moment is denoted

$$\mathcal{P} \doteq \left\{ \rho : \mathbb{R}^d \rightarrow [0, \infty) \text{ measurable} \mid \int_{\mathbb{R}^d} \rho(x) dx = 1, \int_{\mathbb{R}^d} x^2 \rho(x) dx < \infty \right\}. \quad (1.3)$$

$L^2(\mathbb{R}^d; \rho)$ denotes the Hilbert space of functions on \mathbb{R}^d that are square-integrable with respect to density ρ ; $H^k(\mathbb{R}^d; \rho)$ denotes the Hilbert space of functions whose first k derivatives (defined in the weak or distributional sense) are in $L^2(\mathbb{R}^d; \rho)$, and $H_0^1(\mathbb{R}^d; \rho) \doteq \{ \phi \in H^1(\mathbb{R}^d; \rho) \mid \int \phi(x) \rho(x) dx = 0 \}$.

For a function f , $\nabla f = \frac{\partial}{\partial x_i} f$ is used to denote the gradient and $D^2 f = \frac{\partial^2}{\partial x_i \partial x_j} f$ is used to denote the Hessian. The derivatives are interpreted in the weak sense. ■

2. Time-Stepping Procedure. The time-stepping procedure involves a sequence of minimization problems in the space of probability densities \mathcal{P} . We consider a finite time interval $[0, T]$ with an associated discrete-time sequence $\{t_0, t_1, t_2, \dots, t_N\}$ of sampling instants, with $t_0 = 0 < t_1 < \dots < t_N = T$. The corresponding increments are given by $\Delta t_n \doteq t_n - t_{n-1}, n = 1, \dots, N$.

A realization of the stochastic process Z_t , the solution of SDE (1.1b), sampled at discrete times is written as $\{Z_0, Z_1, Z_2, \dots, Z_N\}$. We use $\Delta Z_n \doteq Z_n - Z_{n-1}$ to define the discrete-time observation process, and let

$$Y_n \doteq \frac{\Delta Z_n}{\Delta t_n}.$$

In discrete time, Y_n is viewed as the observation made at time t_n . We eventually let $N \rightarrow \infty$ and simultaneously let $\bar{\Delta}_N \rightarrow 0$, where

$$\bar{\Delta}_N = \max\{\Delta t_n : n \leq N\}. \quad (2.1)$$

The elementary Bayes theorem is used to obtain the posterior distribution, expressed recursively as

$$\rho_0(x) = p_0^*(x), \quad (2.2)$$

$$\rho_n(x) = \frac{\rho_{n-1}(x) \exp(-\phi_n(x))}{\int \rho_{n-1}(y) \exp(-\phi_n(y)) dy}, \quad (2.3)$$

where $\phi_n(x) \doteq \frac{\Delta t_n}{2}(Y_n - h(x))^2$. Note that the $\{\rho_n\}$ are *random* probability measures since they depend on the discrete-time process $\{Z_n\}$. In particular, ρ_n is measurable w.r.t. $\sigma(Z_i : i = 0, \dots, n)$. This observation should be kept in mind when dealing with various parameters associated with the ρ_n , e.g., norm bounds for functions in $L^p(\mathbb{R}^d; \rho_n)$.

The variational formulation of the Bayes recursion is the following *time-stepping procedure*: Set $\rho_0 = p_0^* \in \mathcal{P}$ and inductively define $\{\rho_n\}_{n=1}^N \subset \mathcal{P}$ by taking $\rho_n \in \mathcal{P}$ to minimize the functional

$$I_n(\rho) \doteq D(\rho \mid \rho_{n-1}) + \frac{\Delta t_n}{2} \int \rho(x)(Y_n - h(x))^2 dx, \quad (2.4)$$

where D denotes the relative entropy or Kullback–Leibler divergence,

$$D(\rho \mid \rho_{n-1}) = \int \rho(x) \ln\left(\frac{\rho(x)}{\rho_{n-1}(x)}\right) dx.$$

The proof that ρ_n , as defined in (2.3), is in fact the minimizer is straightforward: By Jensen’s formula, $I_n(\rho) \geq -\ln(\int \rho_{n-1}(y) \exp(-\phi_n(y)) dy)$ with equality if and only if $\rho = \rho_n$. The optimizer ρ_n is in fact the “twisted distribution” that arises in the theory of large deviations for empirical means [2]. Although the optimizer is known, a careful look at the first order optimality equations associated with ρ_n leads to i) the nonlinear filter (1.2) for evolution of the posterior (in Sec 3), and ii) a particle filter algorithm for approximation of the posterior (in Sec 4).

Throughout the paper, the following assumptions are made for the prior distribution p_0^* and for function h :

Assumption A1 The probability density $p_0^* \in \mathcal{P}$ is of the form $p_0^*(x) = e^{-\mathcal{G}_0(x)}$, where $\mathcal{G}_0 \in C^2$, $D^2\mathcal{G}_0 \in L^\infty$, and $|\nabla\mathcal{G}_0|(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Assumption A2 The function $h \in C^2$ with $h, \nabla h, D^2 h \in L^\infty$.

Under assumption A1, the density $\rho_0 = p_0^*$ is known to admit a spectral gap (or Poincaré inequality) [1]: That is, for some $\lambda_0 > 0$, and for all functions $f \in H^1(\mathbb{R}^d; \rho_0)$ with $\int \rho_0 f \, dx = 0$,

$$\int |f(x)|^2 \rho_0(x) \, dx \leq \frac{1}{\lambda_0} \int |\nabla f(x)|^2 \rho_0(x) \, dx. \quad [\mathbf{PI}(\lambda_0)]$$

The following proposition shows that the minimizers all admit a uniform spectral gap. The proof appears in the Appendix 5.1.

PROPOSITION 2.1. *Under Assumption (A1)-(A2),*

(i) *The minimizer ρ_n is of the form $\rho_n = e^{-\mathcal{G}_n(x)}$, where $\mathcal{G}_n \in C^2$. These functions admit the following bounds, uniformly in n : $\nabla \mathcal{G}_n(x) = O(|x|)$, $|\nabla \mathcal{G}_n|(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, and $D^2 \mathcal{G}_n \in L^\infty$.*

(ii) *Suppose $f \in L^2(\mathbb{R}^d; \rho_{n-1})$. Then $f \in L^2(\mathbb{R}^d; \rho_n)$ with*

$$\int \rho_n(x) |f(x)|^2 \, dx \leq C \exp(\alpha |\Delta Z_n|) \int \rho_{n-1}(x) |f(x)|^2 \, dx, \quad (2.5)$$

where the constants C, α are uniformly bounded in n and N .

(iii) *The ratio $\frac{\rho_n}{\rho_{n-1}} \in H^1(\mathbb{R}^d; \rho_{n-1})$.*

(iv) *There exists $\bar{\lambda} > 0$, such that ρ_n satisfies $\mathbf{PI}(\bar{\lambda})$ for each n .* ■

The sequence of minimizers $\{\rho_n\}$ is used to construct, via a piecewise-constant interpolation, a density function $\rho^{(N)}(x, t)$ for $t \in [0, T]$: Define $\rho^{(N)}(x, t)$ by setting $\rho^{(N)}(x, t_n) = \rho_n(x)$, and taking $\rho^{(N)}$ to be constant on each time interval $[t_{n-1}, t_n)$ for $n = 1, 2, \dots, N$.

The following section is concerned with convergence analysis for the limit, as $\bar{\Delta}_N \rightarrow 0$. Before describing the analysis, we present a few preliminaries concerning a certain Poisson's equation. This equation is fundamental to both the nonlinear filter (in Sec 3) and the particle filter algorithm (in Sec 4).

2.1. Poisson's Equation. We are interested in obtaining a solution ϕ of Poisson's equation,

$$\begin{aligned} \nabla \cdot (\rho(x) \nabla \phi(x)) &= -(g(x) - \hat{g})\rho(x), \\ \int \phi(x) \rho(x) \, dx &= 0, \end{aligned} \quad (2.6)$$

where $\rho > 0$ is a given density, g is a given function, and $\hat{g} = \int g(x) \rho(x) \, dx$.

The terminology is motivated by Poisson's equation that arises in the theory of Markov processes [4, 7]. Consider the normalized *Smoluchowski equation*, defined as the perturbed gradient flow w.r.t. a potential $U: \mathbb{R}^d \rightarrow \mathbb{R}^d$:

$$d\Phi_t = -\nabla U(\Phi_t) \, dt + \sqrt{2} \, dW_t.$$

Its differential generator is the second-order operator, defined for C^2 functions by $\mathcal{D}\phi = -(\nabla U) \cdot \nabla \phi + \Delta \phi$. On taking $U = -\ln(\rho)$, the first equation in (2.6) becomes the usual Poisson's equation for diffusions,

$$\mathcal{D}\phi(x) = -(g(x) - \hat{g}).$$

This interpretation is appealing, but will not be needed in subsequent analysis. We henceforth consider solutions to (2.6) in a purely analytical setting.

Let $H_0^1(\mathbb{R}^d; \rho) \doteq \{\phi \in H^1(\mathbb{R}^d; \rho) \mid \int \phi(x)\rho(x) dx = 0\}$. A function $\phi \in H_0^1(\mathbb{R}^d; \rho)$ is said to be a weak solution of Poisson's equation (2.6) if

$$\int \nabla \phi(x) \cdot \nabla \psi(x) \rho(x) dx = \int (g(x) - \hat{g})\psi(x)\rho(x) dx, \quad (2.7)$$

for all $\psi \in H^1(\mathbb{R}^d; \rho)$.

The existence-uniqueness result for the weak solution of Poisson's equation is described next; its proof is given in the Appendix 5.3.

THEOREM 2.2. *Suppose $\rho(x) = e^{-\mathcal{G}(x)}$ satisfies **PI**(λ).*

(i) *If $g \in L^2(\mathbb{R}^d; \rho)$, then there exists a unique weak solution $\phi \in H_0^1(\mathbb{R}^d; \rho)$ satisfying (2.7). Moreover, the derivatives of the solution are controlled by the size of the data:*

$$\int |\nabla \phi|^2 \rho(x) dx \leq \frac{1}{\lambda} \int |g - \hat{g}|^2 \rho(x) dx. \quad (2.8)$$

(ii) *If $g \in H^1(\mathbb{R}^d; \rho)$ and $D^2\mathcal{G} \in L^\infty$, then the weak solution has higher regularity: $\phi \in H^2(\mathbb{R}^d; \rho)$ with*

$$\int |D^2\phi|^2 \rho(x) dx \leq C(\lambda; \rho) \int |\nabla g|^2 \rho(x) dx, \quad (2.9)$$

where $C(\lambda; \rho) = \lambda^{-2}(\lambda + \|D^2(\mathcal{G})\|_{L^\infty})$. ■

3. Nonlinear Filter. The analysis proceeds by first obtaining the first variation as described in the following Lemma. The proof appears in the Appendix 5.4.

LEMMA 3.1 (First-order optimality condition). *Consider the minimization problem (2.4) under Assumptions (A1)-(A2). The minimizer ρ_n satisfies the Euler-Lagrange equation*

$$\int \rho_n [-\nabla \mathcal{G}_n \cdot \varsigma + \nabla \mathcal{G}_{n-1} \cdot \varsigma - (\Delta Z_n - h\Delta t_n)\nabla h \cdot \varsigma] dx = 0 \quad (3.1)$$

for each vector field $\varsigma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$. ■

We are now prepared to state the main theorem concerning the limit of the sequence of densities $\{\rho^{(N)}(x, t)\}$. For the purpose of the proof, an alternate form of the E-L equation is more useful. For a given function $g \in L^2(\mathbb{R}^d; \rho_{n-1})$, let $\varsigma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ denote the weak solution (in gradient form) of

$$\nabla \cdot (\rho_{n-1}(x)\varsigma(x)) = -\left(g(x) - \int \rho_{n-1}(x)g(x) dx\right)\rho_{n-1}(x). \quad (3.2)$$

Such a solution exists by Theorem 2.2 (i). The E-L equation (3.1) can then be expressed as

$$\int \rho_n(x)g(x) dx = \int \rho_{n-1}(x)g(x) dx + \int \rho_n(x) [\Delta Z_n - h(x)\Delta t_n] \nabla h(x) \cdot \varsigma(x) dx. \quad (3.3)$$

The derivation of (3.3) from (3.1)-(3.2) appears in Appendix 5.5.

Let us suppose now $\Delta t_n \rightarrow 0$ uniformly, so that $\bar{\Delta}_N \rightarrow 0$ as $N \rightarrow \infty$, where the maximum step size $\bar{\Delta}_N$ was introduced in (2.1). Based on the proof of Prop. 2.1, there exists a limit, denoted as $\rho(x, t)$, such that $\rho^{(N)}(x, t) \rightarrow \rho(x, t)$ pointwise for a fixed sample path, and in the L^2 sense over all sample paths. In fact for the special case of the signal process (1.1a) considered in this paper, the limiting density is given by the following explicit formula:

$$\rho(x, t) \doteq (\text{const.}) \exp \left(h(x) (Z_t - Z_0) - \frac{1}{2} |h(x)|^2 t \right) \rho_0(x). \quad (3.4)$$

The convergence argument appears in Appendix 5.2.

The proof of the following theorem appears in Appendix 5.6. Notationally, $\langle f, \rho_t \rangle \doteq \int f(x) \rho(x, t) dx$ and $\hat{h}_t \doteq \int h(x) \rho(x, t) dx$.

THEOREM 3.2. *The density ρ is a weak solution of the nonlinear filter with prior $\rho_0 = p_0^*$. That is, for any test function $f \in C_c(\mathbb{R}^d)$,*

$$\langle f, \rho_t \rangle = \langle f, \rho_0 \rangle + \int_0^t \langle (h - \hat{h}_s)(dZ_s - \hat{h}_s ds) f, \rho_s \rangle. \quad (3.5)$$

■

REMARK 1. The considerations of this section highlight the variational underpinnings of the nonlinear filter for the special case, $dX_t = 0$.

For a general class of diffusions, the time-stepping procedure is modified as follows: Set $\rho_0 = p_0^* \in \mathcal{P}$ and inductively define $\{\rho_n\}_{n=1}^N \subset \mathcal{P}$ by taking $\rho_n \in \mathcal{P}$ to minimize the functional (2.4),

$$I_n(\rho) \doteq D(\rho | \mathbb{P}[\rho_{n-1}]) + \frac{\Delta t_n}{2} \int \rho(x) (Y_n - h(x))^2 dx,$$

where $\mathbb{P}[\rho_{n-1}]$ is the “push-forward” from time t_{n-1} to t_n , i.e., $\mathbb{P}[\rho_{n-1}]$ is the probability density of X_{t_n} , given ρ_{n-1} as the (initial) density of $X_{t_{n-1}}$. For the special case considered in this section, $\mathbb{P}[\rho_{n-1}] = \rho_{n-1}$.

The proof procedure is easily modified to derive the counterpart of the E-L equation (3.1) and the nonlinear filter (3.5) for a general class of diffusions. The hard part is to establish, in an a priori manner, the spectral bound $\mathbf{PI}(\bar{\lambda})$ in Prop. 2.1. Derivation of the spectral bound for the general case will be a subject of future work. Note that the bound is needed to obtain a unique solution of the Poisson equation.

The following section shows that both the variational analysis and the Poisson equation are also central to construction of a particle filter algorithm in continuous time. ■

4. Feedback Particle Filter. The objective of this section is to employ the time-stepping procedure to construct a particle filter algorithm.

A particle filter is comprised of N stochastic processes $\{X_t^i : 1 \leq i \leq N\}$: The value $X_t^i \in \mathbb{R}^d$ is the state for the i^{th} particle at time t . For each time t , the empirical distribution formed by the “particle population” is used to approximate the posterior distribution. This is defined for any measurable set $A \subset \mathbb{R}^d$ by

$$p^{(N)}(A, t) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}\{X_t^i \in A\}. \quad (4.1)$$

The model for the particle filter is assumed here to be a controlled system,

$$dX_t^i = \underbrace{u(X_t^i, t) dt + K(X_t^i, t) dZ_t}_{dU_t^i}, \quad (4.2)$$

where the functions $K(x, t)$, $u(x, t)$ are \mathbb{R}^d -valued. It is assumed that the initial conditions $\{X_0^i\}_{i=1}^N$ are i.i.d., independent of $\{X_t, Z_t\}$, and drawn from the initial distribution $p^*(x, 0) \equiv p_0^*(x)$ of X_0 .

We impose the following admissibility requirements on the control input U_t^i in (4.2):

DEFINITION 4.1 (Admissible Input). *The control input U_t^i is admissible if the following conditions are met: (i) The random variables $u(x, t)$ and $K(x, t)$ are $\mathcal{Z}_t = \sigma(Z_s : s \leq t)$ measurable for each t . (ii) For each i and t , $\mathbb{E}[|u|] \doteq \mathbb{E}[\sum_l |u_l(X_t^i, t)] < \infty$, and $\mathbb{E}[|K|^2] \doteq \mathbb{E}[\sum_j |K_j(X_t^i, t)|^2] < \infty$. ■*

There are two types of conditional distributions of interest in our analysis:

- (i) $p(x, t)$: Defines the conditional distribution of X_t^i given \mathcal{Z}_t .
- (ii) $p^*(x, t)$: Defines the conditional distribution of X_t given \mathcal{Z}_t .

The functions $\{u(x, t), K(x, t)\}$ are said to be *optimal* if $p \equiv p^*$. That is, given $p^*(\cdot, 0) = p(\cdot, 0)$, our goal is to choose $\{u, K\}$ in the feedback particle filter so that the evolution equations of these conditional distributions coincide.

The optimal functions are obtained from the time-stepping procedure introduced in Sec 2. Recall that at step n of the procedure, the distribution ρ_n is obtained upon minimizing the functional (2.4), repeated below:

$$I_n(\rho) \doteq D(\rho \mid \rho_{n-1}) + \frac{\Delta t_n}{2} \int \rho(x) (Y_n - h(x))^2 dx.$$

The optimizer has an explicit representation given in (2.3).

The key is to construct a diffeomorphism $x \mapsto s_n(x)$ such that $\rho = s_n^\#(\rho_{n-1})$, where $s_n^\#$ denotes the push-forward operator. The push-forward of a probability density ρ by a smooth map s is defined through the change-of-variables formula

$$\int g(x) [s^\#(\rho)](x) dx = \int g(s(x)) \rho(x) dx,$$

for all continuous and bounded test functions g .

The particle filter equations are obtained from the first-order optimality conditions for s_n . For this purpose, we look at the cumulative objective function, defined for $N \geq 1$ by

$$J^{(N)}(\underline{s}) \doteq \sum_{n=1}^N \left(I_n(s_n^\#(\rho_{n-1})) - \frac{\Delta t_n}{2} Y_n^2 \right), \quad (4.3)$$

where $\underline{s} \doteq (s_1, s_2, \dots, s_N)$ denotes a sequence of diffeomorphisms. The objective is to construct a minimizer, denoted as $\underline{\chi} \doteq (\chi_1, \chi_2, \dots, \chi_N)$, and consider the limit as $N \rightarrow \infty$, $\bar{\Delta}_N \rightarrow 0$. Note the sequence $\{\rho_{n-1}(x)\}_{n=1}^N$ is assumed given here (see (2.3)). Its limit, which we denote as $\rho(x, t)$, see (3.4), is equal to $p^*(x, t)$, the posterior distribution of X_t given \mathcal{Z}_t , by Theorem 3.2.

The calculations in Appendix 5.7 provide the following characterization of the optimal functions $\{u, K\}$:

(i) The function K is a solution to

$$\nabla \cdot (\rho K) = -(h - \hat{h})\rho, \quad (4.4)$$

(ii) The function u is obtained as

$$u(x, t) = -\frac{1}{2}K(x, t)(h(x) + \hat{h}_t) + \Omega(x, t), \quad (4.5)$$

where $\hat{h}_t \doteq \int h(x)\rho(x, t) dx$ and $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_d)$ is a \mathbb{R}^d -valued function with

$$\Omega_l(x, t) := \frac{1}{2} \sum_{k=1}^d K_k(x, t) \frac{\partial K_l}{\partial x_k}(x, t).$$

This in particular yields the following feedback particle filter algorithm – obtained upon substituting ρ by p , the posterior distribution of X_t^i given \mathcal{Z}_t :

Feedback particle filter (in Stratonovich form) is given by

$$dX_t^i = K(X_t^i, t) \circ dI_t^i, \quad (4.6)$$

where

$$dI_t^i \doteq dZ_t - \frac{1}{2}(h(X_t^i) + \hat{h}_t) dt, \quad \hat{h}_t := \mathbb{E}[h(X_t^i) | \mathcal{Z}_t].$$

The gain function is expressed as

$$K(x, t) = \nabla \phi(x, t),$$

and it is obtained at each time t as a solution of Poisson's equation:

$$\begin{aligned} \nabla \cdot (p(x, t) \nabla \phi(x, t)) &= -(h(x) - \hat{h})p(x, t), \\ \int \phi(x, t) p(x, t) dx &= 0, \end{aligned}$$

where p denotes the conditional distribution of X_t^i given \mathcal{Z}_t . ■

This algorithm requires approximations in numerical implementation since both the gain K and the conditional mean \hat{h} depend upon the density p to be estimated. This is resolved by replacing p by the empirical distribution (4.1) to obtain $\hat{h}_t \approx \frac{1}{N} \sum_{i=1}^N h(X_t^i) =: \hat{h}_t^{(N)}$. Likewise, a Galerkin algorithm is used to obtain a finite-dimensional approximation of the gain function K ; cf., [10].

The following theorem shows that, in absence of these approximations, the feedback particle filter is exact. Its proof appears in the Appendix 5.9.

THEOREM 4.2. *Under Assumptions (A1)-(A2), the feedback particle filter (4.6) is exact. That is, provided $p(\cdot, 0) = p^*(\cdot, 0)$, we have for all $t \geq 0$,*

$$p(\cdot, t) = p^*(\cdot, t).$$

■

REMARK 2. The extension of the feedback particle filter to the general nonlinear filtering problem is straightforward. In particular, consider the filtering problem

$$\begin{aligned} dX_t &= a(X_t) dt + dB_t, \\ dZ_t &= h(X_t) dt + dW_t, \end{aligned}$$

where $X_t \in \mathbb{R}^d$ is the state at time t , $Z_t \in \mathbb{R}$ is the observation, $a(\cdot)$, $h(\cdot)$ are C^1 functions, and $\{B_t\}$, $\{W_t\}$ are mutually independent standard Wiener processes.

For the solution to this problem, the feedback particle filter is given by

$$dX_t^i = a(X_t^i) dt + dB_t^i + K(X_t^i, t) \circ dI_t^i,$$

where the formulae for K and I^i are as before. The extension of the Theorem 4.2 to this more general case requires a well-posedness analysis of the solution of Poisson's equation. The key is to obtain a priori spectral bounds (see also Remark 1) which will be a subject of future publication. ■

5. Appendix. The convergence proofs here require bounds in the almost-sure and L^2 senses.

Recall that we consider a finite time interval $[0, T]$, and for each N we consider a discrete-time sequence $\{0, t_1, t_2, \dots, t_N\}$ with $0 \leq t_1 \leq \dots \leq t_N = T$, and denote $\Delta t_n \doteq t_n - t_{n-1}$. We let $\bar{\Delta}_N = \max_n \Delta t_n$, which is assumed to vanish as $N \rightarrow \infty$.

We use $C > 0$ to denote a constant that may depend on N and on the process path $\{Z_t\}$, but is uniformly bounded in L^2 . Recall that the densities ρ_0, \dots, ρ_N are random objects that depend on the samples Z_0, \dots, Z_N . In particular, the observation process has continuous sample paths, so there exists such a C for which $|Z_t| \leq C$ for all $t \in [0, T]$.

5.1. Proof of Prop. 2.1. (i) Using (2.3), $\rho_n(x) = c_n \exp(-\sum_{k=1}^n \phi_k(x)) \rho_0(x)$, where c_n is a normalizing constant and $\phi_k(x) = \frac{\Delta t_k}{2} (Y_k - h(x))^2$. Therefore,

$$\mathcal{G}_n(x) \doteq -\ln \rho_n(x) = \mathcal{G}_0(x) + \sum_{k=1}^n \frac{\Delta t_k}{2} (Y_k - h(x))^2 - \ln(c_n).$$

Differentiating,

$$\begin{aligned} \nabla \mathcal{G}_n(x) &= \nabla \mathcal{G}_0(x) - \sum_{k=1}^n \Delta t_k (Y_k - h(x)) \nabla h(x) \\ &= \nabla \mathcal{G}_0(x) - (Z_{t_n} - Z_{t_0}) \nabla h(x) + t_n h(x) \nabla h(x), \end{aligned}$$

and similarly,

$$\frac{\partial^2 \mathcal{G}_n}{\partial x_i \partial x_j} = \frac{\partial^2 \mathcal{G}_0}{\partial x_i \partial x_j} - (Z_{t_n} - Z_{t_0}) \frac{\partial^2 h}{\partial x_i \partial x_j} + t_n \left(\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + h \frac{\partial^2 h}{\partial x_i \partial x_j} \right).$$

From the assumption (A2) on h , it follows that, if \mathcal{G}_0 satisfies the properties listed in assumption (A1), then so does \mathcal{G}_n . This is because the sample paths of Z_t are a.s. continuous and thus bounded on $[0, T]$.

(ii) Using (2.3),

$$\rho_n(x) = \rho_{n-1}(x) \frac{\exp(-\frac{\Delta t_n}{2} Y_n^2) \exp(h(x) \Delta Z_n - \frac{\Delta t_n}{2} |h(x)|^2)}{\exp(-\frac{\Delta t_n}{2} Y_n^2) \int \rho_{n-1}(y) \exp(h(y) \Delta Z_n - \frac{\Delta t_n}{2} |h(y)|^2) dy}. \quad (5.1)$$

On canceling the common term $\exp(-\frac{\Delta t_n}{2} Y_n^2)$ from both the numerator and denominator, we can write $\rho_n(x) = \rho_n(x) \exp(H_n(x)) / \int \rho_{n-1}(y) \exp(H_n(y)) dy$, where we have defined $H_n(x) = h(x)\Delta Z_n - \frac{\Delta t_n}{2}|h(x)|^2$. Since ρ_{n-1} is a probability density, we have

$$\left\| \frac{\rho_n}{\rho_{n-1}} \right\|_{\infty} \leq \exp(\text{osc}(H_n)), \quad \text{osc}(H_n) \doteq \sup H_n - \inf H_n.$$

Because $\frac{\Delta t_n}{2}|h(x)|^2 \geq 0$, $\sup H_n \leq |\Delta Z_n| \|h\|_{\infty}$, whereas $\inf H_n \geq -|\Delta Z_n| \|h\|_{\infty} - \frac{\Delta t_n}{2} \|h\|_{\infty}^2$. Combining these estimates, we get the bound

$$\left\| \frac{\rho_n}{\rho_{n-1}} \right\|_{\infty} \leq \exp\left(2|\Delta Z_n| \|h\|_{\infty} + \frac{\Delta t_n}{2} \|h\|_{\infty}^2\right). \quad (5.2)$$

It follows that

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^d; \rho_n)}^2 &= \int \rho_n(x) |f(x)|^2 dx \\ &\leq \exp\left(2|\Delta Z_n| \|h\|_{\infty} + \frac{\Delta t_n}{2} \|h\|_{\infty}^2\right) \int \rho_{n-1}(x) |f(x)|^2 dx. \end{aligned}$$

The second equation provides the bound (2.5) in part (ii) of the proposition with $C = \exp(\frac{\Delta N}{2} \|h\|_{\infty}^2)$ and $\alpha = 2\|h\|_{\infty}$.

Based on this and the definition (1.3), we see that the minimizer $\rho_n \in \mathcal{P}$ if $\rho_{n-1} \in \mathcal{P}$ (take $f(x) = x$ to establish a bounded second moment). By induction, $\rho_n \in \mathcal{P}$ if $\rho_0 \in \mathcal{P}$.

(iii) Denoting the quantity on the right-hand side of (5.2) by \mathcal{E} , we conclude that the ratio $\rho_n/\rho_{n-1} \in L^2(\mathbb{R}^d; \rho_{n-1})$, with

$$\int \left(\frac{\rho_n}{\rho_{n-1}}\right)^2 \rho_{n-1} dx = \int \left(\frac{\rho_n}{\rho_{n-1}}\right) \rho_n dx \leq \mathcal{E}.$$

By a direct calculation,

$$\nabla \left(\frac{\rho_n}{\rho_{n-1}}\right) = (-\nabla \mathcal{G}_n + \nabla \mathcal{G}_{n-1}) \frac{\rho_n}{\rho_{n-1}} = (\Delta Z_n - \Delta t_n h) \nabla h \frac{\rho_n}{\rho_{n-1}}.$$

The gradient is in $L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ because,

$$\begin{aligned} \int \left| \nabla \left(\frac{\rho_n}{\rho_{n-1}}\right) \right|^2 \rho_{n-1} dx &= \int (\Delta Z_n - \Delta t_n h)^2 |\nabla h|^2 \frac{\rho_n}{\rho_{n-1}} \rho_n dx \\ &\leq 2(|\Delta Z_n|^2 + |\Delta t_n|^2 \|h\|_{\infty}^2) \|\nabla h\|_{\infty}^2 \mathcal{E}. \end{aligned}$$

(iv) We claim that $\rho_n(x) = e^{-v_n(x)} \rho_0(x)$ where $v_n(x)$ is uniformly bounded. Then ρ_n satisfies **PI**(λ_n) with

$$\lambda_n = \exp(-\text{osc}(v_n)) \lambda_0. \quad (5.3)$$

This is because, for any f satisfying $\int \rho_n f dx = 0$,

$$\begin{aligned} \int |f|^2 \rho_n(x) dx &= \int |f|^2 e^{-v_n(x)} \rho_0(x) dx \leq e^{-\inf v_n} \int |f|^2 \rho_0(x) dx \\ &\leq e^{-\inf v_n} \frac{1}{\lambda_0} \int |\nabla f|^2 \rho_0(x) dx \leq e^{\sup v_n - \inf v_n} \frac{1}{\lambda_0} \int |\nabla f|^2 \rho_n(x) dx. \end{aligned}$$

A uniform bound on v_n yields a uniform bound on λ_n .

We now prove the claim that v_n is uniformly bounded. Using (5.1) iteratively, we can write $\rho_n(x) = e^{-v_n(x)}\rho_0(x)$ with

$$v_n(x) = -\psi_n(x) + \ln\left(\int \rho_0 \exp(\psi_n) dx\right),$$

where $\psi_n(x) = (Z_{t_n} - Z_{t_0})h(x) - \frac{t_n}{2}|h(x)|^2$. It then follows that

$$\text{osc}(v_n) \leq 2\|\psi_n\|_\infty \leq C\|h\|_\infty + T\|h\|_\infty^2,$$

where C depends upon the sample path Z_t for $t \in [0, T]$ but is independent of N .

Using (5.3), ρ_n satisfies $\mathbf{PI}(\bar{\lambda})$ with $\bar{\lambda} = \lambda_0 \exp(-(C\|h\|_\infty + T\|h\|_\infty^2))$.

5.2. Convergence of $\{\rho^{(N)}\}$. Now we explain in what sense $\rho^{(N)}$ converges to ρ as $N \rightarrow \infty$. Recalling formula (3.4), we have $\rho(x, t) = e^{-v(x, t)}\rho_0(x)$ with

$$v(x, t) = -\psi(x, t) + \ln\left(\int \rho_0 \exp(\psi) dx\right),$$

where $\psi(x, t) = (Z_t - Z_0)h(x) - \frac{t}{2}|h(x)|^2$. Define $v^{(N)}(x, t) = v_n(x)$ whenever $t \in [t_{n-1}, t_n]$. Assuming the maximum step size $\bar{\Delta}_N \rightarrow 0$ as $N \rightarrow \infty$, we deduce that

$$v^{(N)} - v \rightarrow 0$$

uniformly with respect to $x \in \mathbb{R}^d, t \in [0, T]$, due to the boundedness of h and (uniform) continuity of the sample path $t \mapsto Z_t$. Hence

$$\frac{\rho^{(N)}}{\rho} = \exp(v - v^{(N)}) \rightarrow 1$$

uniformly with respect to x and t . In particular, $\rho^{(N)} \rightarrow \rho$ pointwise.

5.3. Proof of Theorem 2.2. A density ρ is assumed to satisfy $\mathbf{PI}(\lambda)$: That is, for all functions $\phi \in H_0^1(\mathbb{R}^d; \rho)$,

$$\int |\phi(x)|^2 \rho(x) dx \leq \frac{1}{\lambda} \int |\nabla \phi(x)|^2 \rho(x) dx. \quad (5.4)$$

Consider the inner product

$$\langle \phi, \psi \rangle \doteq \int \nabla \phi(x) \cdot \nabla \psi(x) \rho(x) dx.$$

On account of (5.4), the norm defined by using the inner product $\langle \cdot, \cdot \rangle$ is equivalent to the standard norm in $H_0^1(\mathbb{R}^d; \rho)$.

(i) Consider the BVP in its weak form (2.7). The integral on the right hand side is a bounded linear functional on H_0^1 , since

$$\begin{aligned} \left| \int (g(x) - \hat{g})\psi(x)\rho(x) dx \right|^2 &\leq \left(\int |g(x) - \hat{g}|^2 \rho(x) dx \right) \left(\int |\psi(x)|^2 \rho(x) dx \right) \\ &\leq k_g \int |\nabla \psi(x)|^2 \rho(x) dx, \end{aligned}$$

where (5.4) is used to obtain the second inequality, with $k_g = \lambda^{-1} \int |g(x) - \hat{g}|^2 \rho(x) dx$.

It follows from the Hilbert-space form of the Riesz representation theorem that there exists a unique $\phi \in H_0^1$ such that

$$\langle \phi, \psi \rangle = \int (g(x) - \hat{g}) \psi(x) \rho(x) dx$$

holds for all $\psi \in H_0^1(\mathbb{R}^d; \rho)$. It trivially also holds for all constant functions ($\psi \equiv \text{const.}$). Hence, it holds for all $\psi \in H^1(\mathbb{R}^d; \rho)$ and ϕ is a weak solution of the BVP, satisfying (2.7).

The estimate (2.8) follows by substituting $\psi = \phi$ in (2.7) and using Cauchy-Schwarz.

(ii) For the estimate (2.9), we first establish the following bound:

$$\int |D^2 \phi|^2 \rho dx \leq \int \nabla \phi \cdot G \rho dx, \quad (5.5)$$

where the vector function $G \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho)$ is defined by

$$G = D^2(\log \rho) \nabla \phi + \nabla g$$

and where $|D^2 \phi|^2 = \sum_{j,k} (\frac{\partial^2 \phi}{\partial x_j \partial x_k})^2$.

Since each entry of the Hessian matrix $D^2(\log \rho)$ is bounded and $\nabla g \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho)$, we have $G \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho)$. The elliptic regularity theory [3, Section 6.3] applied to the weak solution $\phi \in H^1(\mathbb{R}^d; \rho)$ says that $\phi \in H_{\text{loc}}^3(\mathbb{R}^d)$. Hence the partial differential equation holds pointwise:

$$-\nabla \cdot (\rho \nabla \phi) = (g - \hat{g}) \rho. \quad (5.6)$$

Differentiating with respect to x_k gives

$$-\nabla \cdot \left(\rho \nabla \frac{\partial \phi}{\partial x_k} \right) - \nabla \left(\frac{\partial \log \rho}{\partial x_k} \right) \cdot (\rho \nabla \phi) - \frac{\partial \log \rho}{\partial x_k} \rho \nabla \cdot (\rho \nabla \phi) = \frac{\partial g}{\partial x_k} \rho + (g - \hat{g}) \frac{\partial \log \rho}{\partial x_k} \rho.$$

The final terms on the left and right sides cancel, by equation (5.6). Thus the preceding formula becomes

$$-\nabla \cdot \left(\rho \nabla \frac{\partial \phi}{\partial x_k} \right) = G_k \rho, \quad (5.7)$$

Let $\beta(x) \geq 0$ be a smooth, compactly supported ‘‘bump’’ function, meaning $\beta(x)$ is radially decreasing with $\beta(0) = 1$. Let $s > 0$, and multiply (5.7) by $\beta(sx)^2 \frac{\partial \phi}{\partial x_k}$. Integrate by parts on the left side (noting the boundary terms vanish because β has compact support) to obtain

$$\int \nabla \left[\beta(sx)^2 \frac{\partial \phi}{\partial x_k} \right] \cdot \left(\nabla \frac{\partial \phi}{\partial x_k} \right) \rho dx = \int \beta(sx)^2 \frac{\partial \phi}{\partial x_k} G_k \rho dx. \quad (5.8)$$

The left side of (5.8) can be expressed as

$$\int \beta(sx)^2 \left| \nabla \frac{\partial \phi}{\partial x_k} \right|^2 \rho dx + 2s \int \frac{\partial \phi}{\partial x_k} \beta(sx) (\nabla \beta)(sx) \cdot \left(\nabla \frac{\partial \phi}{\partial x_k} \right) \rho dx.$$

The second term is bounded by

$$\begin{aligned} & 2s\|\nabla\beta\|_{L^\infty(\mathbb{R}^d)} \int \left| \frac{\partial\phi}{\partial x_k} \right| \beta(sx) \left| \nabla \frac{\partial\phi}{\partial x_k} \right| \rho \, dx \\ & \leq s\|\nabla\beta\|_\infty \int \left[\left(\frac{\partial\phi}{\partial x_k} \right)^2 + \beta(sx)^2 \left| \nabla \frac{\partial\phi}{\partial x_k} \right|^2 \right] \rho \, dx \end{aligned}$$

and so the left side of (5.8) is bounded from below by

$$(1 - s\|\nabla\beta\|_{L^\infty(\mathbb{R}^d)}) \int \beta(sx)^2 \left| \nabla \frac{\partial\phi}{\partial x_k} \right|^2 \rho \, dx - s\|\nabla\beta\|_\infty \int \left(\frac{\partial\phi}{\partial x_k} \right)^2 \, dx.$$

The right hand side of (5.8) tends to $\int \frac{\partial\phi}{\partial x_k} G_k \rho \, dx$, as $s \rightarrow 0$, by dominated convergence, and since $\beta(x)$ is radially decreasing, with $\beta(0) = 1$.

Letting $s \rightarrow 0$ in (5.8), we conclude from the monotone convergence theorem that

$$\int \left| \nabla \frac{\partial\phi}{\partial x_k} \right|^2 \rho \, dx \leq \int \frac{\partial\phi}{\partial x_k} G_k \rho \, dx.$$

Summing over k establishes the bound (5.5).

Next we prove (2.9). First,

$$\int |\nabla\phi|^2 \rho \, dx \leq \lambda^{-1} \int |g - \hat{g}|^2 \rho \, dx \leq \lambda^{-2} \int |\nabla g|^2 \rho \, dx \quad (5.9)$$

by (2.8) followed by (5.4) applied to the function $g - \hat{g} \in H_0^1(\mathbb{R}^d; \rho)$. Second, by the definition of G , the L^2 -triangle inequality, and (5.9), we show that

$$\begin{aligned} \left(\int |G|^2 \rho \, dx \right)^{1/2} & \leq \|D^2(\log \rho)\|_\infty \left(\int |\nabla\phi|^2 \rho \, dx \right)^{1/2} + \left(\int |\nabla g|^2 \rho \, dx \right)^{1/2} \\ & \leq \left(\frac{\|D^2(\log \rho)\|_\infty}{\lambda} + 1 \right) \left(\int |\nabla g|^2 \rho \, dx \right)^{1/2}. \end{aligned} \quad (5.10)$$

Now we take (5.5) and apply Cauchy–Schwarz, followed by (5.9) and (5.10), to find:

$$\begin{aligned} \int |D^2\phi|^2 \rho \, dx & \leq \left(\int |\nabla\phi|^2 \rho \, dx \right)^{1/2} \left(\int |G|^2 \rho \, dx \right)^{1/2} \\ & \leq (\lambda^{-2} \int |\nabla g|^2 \rho \, dx)^{1/2} \left(\frac{\|D^2(\log \rho)\|_\infty}{\lambda} + 1 \right) \left(\int |\nabla g|^2 \rho \, dx \right)^{1/2} \\ & = \lambda^{-2} (\lambda + \|D^2(\log \rho)\|_\infty) \int |\nabla g|^2 \rho \, dx, \end{aligned}$$

which proves (2.9).

5.4. Proof of Lemma 3.1. We compute the first variation of the functional (2.4), which we reproduce here for reference:

$$I_n(\rho) := \int \rho(x) \ln \rho(x) \, dx - \int \rho(x) \ln \rho_{n-1}(x) \, dx + \int \rho(x) \frac{(\Delta Z_n - h(x)\Delta t_n)^2}{2\Delta t_n} \, dx. \quad (5.11)$$

Following the methodology of [5], a vector field ς is used to generate the first variation; we initially assume that $\varsigma \in C_c^1$. Let $\Phi_\tau(x)$ be the solution of

$$\frac{d}{d\tau}\Phi = \varsigma(\Phi), \quad \Phi_0(x) = x.$$

For small τ , define $\rho_\tau = \Phi_\tau^\# \rho_n$ to be the push-forward of the minimizer ρ_n . We have

$$\det(\nabla\Phi_\tau(x)) \rho_\tau(\Phi_\tau(x)) = \rho_n(x),$$

and $i(\tau) = I_n(\rho_\tau)$ has a minimum at $\tau = 0$.

The three terms in the E-L equation (3.1) are obtained by explicitly evaluating the derivative $\frac{d}{d\tau}i(\tau)$, at $\tau = 0$, of the three terms in (5.11):

(i) The first term is the negative entropy

$$\begin{aligned} \int \rho_\tau(z) \ln \rho_\tau(z) dz &= \int \rho_n(x) \ln [\rho_\tau(\Phi_\tau(x))] dx \\ &= \int \rho_n(x) \ln [\rho_n(x) (\det(\nabla\Phi_\tau(x)))^{-1}] dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{d\tau} \int \rho_\tau(z) \ln \rho_\tau(z) dz \Big|_{\tau=0} &= - \int \rho_n(x) \frac{d}{d\tau} \ln[\det(\nabla\Phi_\tau(x))] \Big|_{\tau=0} dx \\ &= - \int \rho_n(x) \nabla \cdot \varsigma(x) dx = - \int \rho_n(x) \nabla \mathcal{G}_n(x) \cdot \varsigma(x) dx, \end{aligned}$$

where the final equality is obtained by using integration by parts. The interchange of the order of the differentiation and the integration is justified because the difference quotient

$$\frac{1}{\tau} (\ln[\det(\nabla\Phi_\tau(x))] - \ln[\det(\nabla\Phi_0(x))])$$

converges uniformly to $\frac{d}{d\tau} \ln[\det(\nabla\Phi_\tau(x))] \Big|_{\tau=0} = \nabla \cdot \varsigma(x)$. This is because ς is assumed to have a compact support and $\Phi_\tau(x) = \Phi_0(x) = x$ outside this compact set.

(ii) The second term is given by

$$\int \rho_\tau(z) \mathcal{G}_{n-1}(z) dz = \int \rho_n(x) \mathcal{G}_{n-1}(\Phi_\tau(x)) dx,$$

and

$$\begin{aligned} \frac{d}{d\tau} \int \rho_n(x) \mathcal{G}_{n-1}(\Phi_\tau(x)) dx \Big|_{\tau=0} &= \int \rho_n(x) \frac{d}{d\tau} \mathcal{G}_{n-1}(\Phi_\tau(x)) \Big|_{\tau=0} dx \\ &= \int \rho_n(x) \nabla \mathcal{G}_{n-1}(x) \cdot \varsigma(x) dx, \end{aligned}$$

which is justified again because ς has compact support.

(iii) For the third term, similarly,

$$\begin{aligned} \frac{d}{d\tau} [\dots] \Big|_{\tau=0} &= \int \rho_n(x) \frac{d}{d\tau} \frac{(\Delta Z_n - h(\Phi_\tau(x)) \Delta t_n)^2}{2\Delta t_n} \Big|_{\tau=0} dx \\ &= - \int \rho_n [\Delta Z_n - h(x) \Delta t_n] \nabla h(x) \cdot \varsigma(x) dx. \end{aligned}$$

Extension of the E-L equation to an arbitrary vector field in $L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ requires a standard approximation argument. Suppose $\varsigma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$. Using Prop. 2.1 (ii), $\varsigma \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_n)$. It then suffices to approximate ς by a sequence of smooth, compactly supported vector fields, noting that $|\nabla \mathcal{G}_k|(x) = O(|x|)$ as $x \rightarrow \infty$, and that $h, \nabla h$ are bounded by assumption (A2). Recall here that \mathcal{P} is the space of probability densities with finite second moment.

REMARK 3. *Although the proof given here stresses the variational aspect, the Euler-Lagrange equation can be obtained directly from manipulating the solution (2.3): Taking the logarithm of (2.3) gives*

$$-\mathcal{G}_n + \mathcal{G}_{n-1} + \phi_n = \text{const.}$$

and applying the gradient operator yields:

$$-\nabla \mathcal{G}_n + \nabla \mathcal{G}_{n-1} - (\Delta Z_n - h \Delta t_n) \nabla h = 0.$$

Multiplying by $\rho_n \varsigma$ and integrating gives (3.1). ■

5.5. Derivation of (3.3). Suppose ς is a weak solution of (3.2). Then for any test function $\psi \in H^1(\mathbb{R}^d; \rho_{n-1})$,

$$\int \nabla \psi(x) \cdot \varsigma(x) \rho_{n-1}(x) dx = \int g(x) \psi(x) \rho_{n-1}(x) dx - \int g \rho_{n-1} dx \int \psi \rho_{n-1} dx. \quad (5.12)$$

Take $\psi(x) = \frac{\rho_n(x)}{\rho_{n-1}(x)}$ – the ratio is known to be an element of $H^1(\mathbb{R}^d; \rho_{n-1})$ by Prop. 2.1 (iii). The gradient of the ratio is obtained as

$$\nabla \left(\frac{\rho_n}{\rho_{n-1}} \right) = (-\nabla \mathcal{G}_n + \nabla \mathcal{G}_{n-1}) \frac{\rho_n}{\rho_{n-1}}.$$

Substituting this in (5.12),

$$\int (-\nabla \mathcal{G}_n + \nabla \mathcal{G}_{n-1}) \cdot \varsigma(x) \rho_n(x) dx = \int g(x) \rho_n(x) dx - \int g(x) \rho_{n-1}(x) dx. \quad (5.13)$$

Combining (5.13) with (3.1) gives the equation (3.3).

5.6. Proof of Theorem 3.2. We are given a test function $f \in C_c$. So, $f \in L^2(\mathbb{R}^d; \rho_n)$ for all $n \in \{1, 2, \dots, N\}$. Furthermore, there exists a uniform bound,

$$\|f\|_{L^2(\mathbb{R}^d; \rho_n)} < \|f\|_{L^\infty} < C \quad \forall n. \quad (5.14)$$

Denote $\hat{f}_n \doteq \int \rho_n(x) f(x) dx$.

Let $\xi_n \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ be the weak solution of

$$\nabla \cdot (\rho_{n-1}(x) \xi_n(x)) = - \left(f(x) - \hat{f}_{n-1} \right) \rho_{n-1}(x). \quad (5.15)$$

Such a solution exists by Theorem 2.2, and moreover,

$$\int \rho_{n-1} |\xi_n|^2 dx < (\text{const.}) \int \rho_{n-1} |f - \hat{f}_{n-1}|^2 dx < C, \quad (5.16)$$

where the (const.) is independent of n (by Prop. 2.1 (iv)), and using Prop. 2.1 (ii),

$$\int \rho_n(x) |\xi_n(x)|^2 dx \leq C \exp(\alpha |\Delta Z_n|) \int \rho_{n-1}(x) |\xi_n(x)|^2 dx. \quad (5.17)$$

Using the E-L equation (3.3) with $g = f$ and $\varsigma = \xi_n$, for $n = 1, 2, \dots, N$:

$$\hat{f}_n - \hat{f}_{n-1} = \int \rho_n(x) [\Delta Z_n - h(x) \Delta t_n] \nabla h(x) \cdot \xi_n(x) dx,$$

and, upon summing,

$$\hat{f}_N = \hat{f}_0 + \sum_{n=1}^N \int \rho_n(x) [\Delta Z_n - h(x) \Delta t_n] \nabla h(x) \cdot \xi_n(x) dx. \quad (5.18)$$

The remainder of the proof thus is to show that, as $\Delta t_n \rightarrow 0$, the summation converges to the Itô integral in (3.5), where the convergence is in L^2 .

We fix n , and express the summand as

$$S_n := \int \rho_n(x) \nabla h(x) \cdot \xi_n(x) dx \Delta Z_n - \int \rho_n(x) h(x) \nabla h(x) \cdot \xi_n(x) dx \Delta t_n \quad (5.19)$$

Each of these terms is well-defined because $\xi_n \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_n)$ (see (5.17)), and $h, \nabla h \in L^\infty$.

The two terms are simplified separately in the following two steps:

Step 1. Consider the second term $-\left(\int \rho_n(x) h(x) \nabla h(x) \cdot \xi_n(x) dx\right) \Delta t_n$. Let $\eta_n \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ denote the weak solution of

$$\nabla \cdot (\rho_{n-1}(x) \eta_n(x)) = - \left(h(x) \nabla h(x) \cdot \xi_n(x) - \int \rho_{n-1} h \nabla h \cdot \xi_n dx \right) \rho_{n-1}(x).$$

Repeating the earlier argument, using (5.16) and the fact that $h, \nabla h \in L^\infty$,

$$\int \rho_{n-1}(x) |\eta_n(x)|^2 dx < C,$$

and

$$\int \rho_n(x) |\eta_n(x)|^2 dx \leq C \exp(\alpha |\Delta Z_n|) \int \rho_{n-1}(x) |\eta_n(x)|^2 dx. \quad (5.20)$$

Using the E-L equation (3.3) with $g = h \nabla h \cdot \xi_n$ and $\varsigma = \eta_n$,

$$\int \rho_n h \nabla h \cdot \xi_n dx \Delta t_n = \int \rho_{n-1} h \nabla h \cdot \xi_n dx \Delta t_n + \mathcal{E}_n^{(1)}, \quad (5.21)$$

where

$$\mathcal{E}_n^{(1)} = \int \rho_n \nabla h \cdot \eta_n dx (\Delta Z_n \Delta t_n) - \int \rho_n h \nabla h \cdot \eta_n dx (\Delta t_n)^2.$$

In order to establish convergence, we will require bounds for the two integrals. Since $h, \nabla h \in L^\infty$, using (5.20),

$$|\mathcal{E}_n^{(1)}| < C \exp\left(\frac{\alpha}{2} |\Delta Z_n|\right) \left(\int \rho_{n-1}(x) |\eta_n(x)|^2 dx \right)^{\frac{1}{2}} (|\Delta Z_n \Delta t_n| + (\Delta t_n)^2). \quad (5.22)$$

Given the uniform L^2 bound on C , it follows that $\mathbb{E}[|\mathcal{E}_n^{(1)}|^2]^{\frac{1}{2}} = O(\bar{\Delta}_N^{3/2})$, uniformly in n .

Step 2. The calculation for the first term in (5.19), $(\int \rho_n(x) \nabla h(x) \cdot \xi_n(x) dx) \Delta Z_n$, is similar. Let $\zeta_n \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ denote the weak solution of

$$\nabla \cdot (\rho_{n-1}(x) \zeta_n(x)) = - \left(\nabla h(x) \cdot \xi_n(x) - \int \rho_{n-1} \nabla h \cdot \xi_n dx \right) \rho_{n-1}(x). \quad (5.23)$$

As before, $\int \rho_{n-1} |\zeta_n|^2 dx < C$, and using the E-L equation (3.3) with $g = \nabla h \cdot \xi_n$ and $\varsigma = \zeta_n$,

$$\int \rho_n \nabla h \cdot \xi_n dx \Delta Z_n = \int \rho_{n-1} \nabla h \cdot \xi_n dx \Delta Z_n + \int \rho_n \nabla h \cdot \zeta_n dx (\Delta Z_n)^2 + \mathcal{E}_n^{(2)}, \quad (5.24)$$

where

$$\mathcal{E}_n^{(2)} = - \int \rho_n h \nabla h \cdot \zeta_n dx (\Delta Z_n \Delta t_n),$$

and using the a priori bound for ζ_n ,

$$|\mathcal{E}_n^{(2)}| < C \exp\left(\frac{\alpha}{2} |\Delta Z_n|\right) \left(\int \rho_{n-1}(x) |\zeta_n(x)|^2 dx \right)^{\frac{1}{2}} |(\Delta Z_n \Delta t_n)|. \quad (5.25)$$

Using the two formulae (5.21) and (5.24) from Steps 1 and 2, the summand (5.19) is given by

$$\begin{aligned} S_n &= \int \rho_n(x) \nabla h(x) \cdot \xi_n(x) dx \Delta Z_n - \int \rho_n(x) h(x) \nabla h(x) \cdot \xi_n(x) dx \Delta t_n \\ &= \int \rho_{n-1} \nabla h \cdot \xi_n dx \Delta Z_n + \int \rho_n \nabla h \cdot \zeta_n dx (\Delta Z_n)^2 - \int \rho_{n-1} h \nabla h \cdot \xi_n dx \Delta t_n \\ &\quad + \mathcal{E}_n^{(1)} + \mathcal{E}_n^{(2)}. \end{aligned} \quad (5.26)$$

Both error terms satisfy $\mathbb{E}[|\mathcal{E}_n^{(i)}|^2]^{\frac{1}{2}} = O(\bar{\Delta}_N^{3/2})$ for $i = 1, 2$.

In the following step, the first two integrals in (5.26) are further simplified.

Step 3. For the first integral, integration by parts gives

$$\begin{aligned} \int \rho_{n-1} \nabla h \cdot \xi_n dx \Delta Z_n &= - \int h \nabla \cdot (\rho_{n-1} \xi_n) dx \Delta Z_n \\ &= \int \rho_{n-1}(x) h(x) (f(x) - \hat{f}_{n-1}) dx \Delta Z_n, \end{aligned} \quad (5.27)$$

where the second equality follows from (5.15).

For simplifying the second integral, the E-L equation (3.3) is used once more. As before, let $\varphi_n \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ denote the weak solution of

$$\nabla \cdot (\rho_{n-1}(x) \varphi_n(x)) = - \left(\nabla h(x) \cdot \zeta_n(x) - \int \rho_{n-1} \nabla h \cdot \zeta_n dx \right) \rho_{n-1}(x),$$

together with an a priori bound $\int \rho_{n-1} |\varphi_n|^2 dx < C$.

The E-L equation (3.3) then gives

$$\int \rho_n \nabla h \cdot \zeta_n \, dx (\Delta Z_n)^2 = \int \rho_{n-1} \nabla h \cdot \zeta_n \, dx (\Delta Z_n)^2 + \mathcal{E}_n^{(3)}, \quad (5.28)$$

where

$$\mathcal{E}_n^{(3)} = \int \rho_n \nabla h \cdot \varphi_n \, dx (\Delta Z_n)^3 - \int \rho_n h \nabla h \cdot \varphi_n \, dx (\Delta Z_n)^2 \Delta t_n,$$

and using the a priori bound for φ_n ,

$$|\mathcal{E}_n^{(3)}| < C \exp\left(\frac{\alpha}{2} |\Delta Z_n|\right) \left(\int \rho_{n-1}(x) |\varphi_n(x)|^2 \, dx \right)^{\frac{1}{2}} (|(\Delta Z_n)^3| + |(\Delta Z_n)^2 \Delta t_n|). \quad (5.29)$$

Hence this third error term is also uniformly bounded, $\mathbb{E}[|\mathcal{E}_n^{(3)}|^2]^{\frac{1}{2}} = O(\bar{\Delta}_N^{3/2})$.

Substituting the formulae (5.27)-(5.28) in (5.26), the summand is given by

$$\begin{aligned} S_n &= \int \rho_{n-1} h (f - \hat{f}_{n-1}) \, dx \Delta Z_n \\ &\quad + \int \rho_{n-1} \nabla h \cdot \zeta_n \, dx (\Delta Z_n)^2 - \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx \Delta t_n \\ &\quad + \mathcal{E}_n^{(1)} + \mathcal{E}_n^{(2)} + \mathcal{E}_n^{(3)}, \end{aligned} \quad (5.30)$$

where recall ξ_n is defined by (5.15) and ζ_n by (5.23).

Now, using integration by parts together with (5.15) and (5.23),

$$\begin{aligned} \int \rho_{n-1} \nabla h \cdot \zeta_n \, dx &= - \int h \nabla \cdot (\rho_{n-1} \zeta_n) \, dx \\ &= \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx - \int \rho_{n-1} h \, dx \int \rho_{n-1} \nabla h \cdot \xi_n \, dx \\ &= \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx + \int \rho_{n-1} h \, dx \int h \nabla \cdot (\rho_{n-1} \xi_n) \, dx \\ &= \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx - \int \rho_{n-1} h \, dx \int \rho_{n-1} h (f - \hat{f}_{n-1}) \, dx. \end{aligned}$$

Substituting the result of this calculation in (5.30), the summand is given by

$$\begin{aligned} S_n &= \int \rho_{n-1} h (f - \hat{f}_{n-1}) \, dx \Delta Z_n \\ &\quad - \int \rho_{n-1} h \, dx \int \rho_{n-1} h (f - \hat{f}_{n-1}) \, dx (\Delta Z_n)^2 + \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx ((\Delta Z_n)^2 - \Delta t_n) \\ &\quad + \mathcal{E}_n^{(1)} + \mathcal{E}_n^{(2)} + \mathcal{E}_n^{(3)}. \end{aligned} \quad (5.31)$$

Step 4. Substituting the summand (5.31) in the series (5.18) and letting $\Delta t_n \rightarrow 0$, we arrive at the Itô integral:

$$\begin{aligned} \hat{f}_t &= \hat{f}_0 + \int_0^t \int \rho(x, s) h(x) (f(x) - \hat{f}_s) \, dx \, dZ_s \\ &\quad + \int_0^t \int \rho(x, s) h(x) \, dx \int \rho(x, s) h(x) (f(x) - \hat{f}_s) \, dx \, ds \\ &= \hat{f}_0 + \int_0^t \int \rho(x, s) (h(x) - \hat{h}_s) f(x) \, dx \, (dZ_s - \hat{h}_s \, ds). \end{aligned} \quad (5.32)$$

Convergence is obtained on applying the following L^2 limits.

(i) Since ξ_n is a weak solution of the Poisson's equation (5.15),

$$\sum_{n=1}^N \left(\int \rho_{n-1} h \nabla h \cdot \xi_n \, dx \right) ((\Delta Z_n)^2 - \Delta t_n) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

The proof of this limit is based on the following result for the summand. Fix $s \in \mathbb{R}_+$, and let n and N tend to infinity in such a way that $t_n \rightarrow s$ as $n, N \rightarrow \infty$. We then have

$$\begin{aligned} \lim_{n, N \rightarrow \infty} \int \rho_{n-1} h \nabla h \cdot \xi_n \, dx &= \lim_{n, N \rightarrow \infty} \frac{1}{2} \int h^2 (f - \hat{f}_{n-1}) \rho_{n-1} \, dx \\ &= \frac{1}{2} \int h^2 (f - \hat{f}_s) \rho \, dx. \end{aligned}$$

(ii) The apriori bounds (5.22), (5.25) and (5.29) are used to show that

$$\mathcal{E} =: \sum_{n=1}^N |\mathcal{E}_n^{(1)}| + |\mathcal{E}_n^{(2)}| + |\mathcal{E}_n^{(3)}| \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

where the convergence is in L^2 . This follows because we have the bound $\mathbb{E}[\mathcal{E}^2] = O(\bar{\Delta}^{1/2})$.

5.7. Derivation of the feedback particle filter. We consider the cumulative objective function (4.3), repeated below:

$$J^{(N)}(\underline{s}) \doteq \sum_{n=1}^N \left(I_n(s_n^\#(\rho_{n-1})) - \frac{\Delta t_n}{2} Y_n^2 \right), \quad (5.33)$$

where $\underline{s} \doteq (s_1, s_2, \dots, s_N)$ denotes a sequence of diffeomorphisms. The sequence $\{\rho_{n-1}(x)\}_{n=1}^N$ is assumed given here (see (2.3)). The objective is to construct a minimizer, denoted as $\underline{\chi} \doteq (\chi_1, \chi_2, \dots, \chi_N)$, and consider the limit as $N \rightarrow \infty$, $\bar{\Delta}_N \rightarrow 0$.

The calculations in this section are strictly formal. Generally, the technicalities are downplayed in the interest of succinctly describing the main calculations. The Einstein tensor notation is employed for some of the more laborious calculations.

The optimization problem (5.33) can be considered term-by-term since $\{\rho_{n-1}\}$ is fixed for fixed N and Δt_n . With these parameters fixed, and attention focused to the n th summand, we recast the optimization problem as one over s_n as follows:

$$\begin{aligned} I_n(s_n) \doteq & - \int \rho_{n-1}(x) \ln(\det(Ds_n(x))) \, dx - \int \rho_{n-1}(x) \ln \frac{\rho_{n-1}(s_n(x))}{\rho_{n-1}(x)} \, dx \\ & + \frac{\Delta t_n}{2} \int \rho_{n-1}(x) (Y_n - h(s_n(x)))^2 \, dx, \end{aligned} \quad (5.34)$$

where we have used the identity $\rho_n(s_n(x)) \det(Ds_n(x)) = \rho_{n-1}(x)$. As in the initial problem formulation, the minimizer is denoted as χ_n . The minimal value exists because the functional $I_n(\cdot)$ is bounded from below – see the discussion following the introduction of the functional $I_n(\rho)$ in (2.4). In fact, a minimizer may be obtained in closed form by considering the transport problem

$$s_n^\#(\rho_{n-1}) = \rho_n.$$

Existence of solutions to such problems have been extensively investigated in the optimal transportation literature; cf., [9]. As with the derivation of the nonlinear filter, we proceed via analysis of the first variation. Such an approach is more tractable and leads to the elegant form of the feedback particle filter. Once the filter has been derived, its optimality is established by showing the filter to be exact; cf., Proof of Theorem 4.2 in Sec. 5.9.

The first-order conditions for optimization problem (5.34) appear in the following Lemma. Given $\nu \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$, the directional derivative is denoted

$$\delta I_n(\chi_n) \cdot \nu \doteq \left. \frac{d}{d\varepsilon} I_n(\chi_n + \varepsilon\nu) \right|_{\varepsilon=0}.$$

LEMMA 5.1 (First-Order Optimality Conditions). *Consider the minimization problem (5.34) under Assumptions (A1)-(A2). The first-order optimality condition for the minimizer $\chi_n(x)$ is given by*

$$\begin{aligned} 0 = \delta I_n(\chi_n) \cdot \nu &= \int \rho_{n-1}(x) \text{tr}(D\chi_n^{-1}(x)D\nu(x)) \, dx \\ &+ \int \rho_{n-1}(x) \frac{1}{\rho_{n-1}(\chi_n(x))} \nabla \rho_{n-1}(\chi_n(x)) \cdot \nu(x) \, dx \\ &+ \int \rho_{n-1}(x) (\Delta Z_n - h(\chi_n(x))\Delta t_n) \nabla h(\chi_n(x)) \cdot \nu(x) \, dx, \end{aligned} \quad (5.35)$$

where $\nu \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ is an arbitrary perturbation of χ_n . \blacksquare

Proof. The three terms in (5.35) are obtained by explicitly evaluating the derivative $\frac{d}{d\varepsilon} I_n(\chi_n + \varepsilon\nu)$, at $\varepsilon = 0$, for the three terms in (5.34):

(i) The first term is given by

$$- \int \rho_{n-1}(x) [\ln(\det(D\chi_n(x))) + \ln(\det(I + \varepsilon D\chi_n^{-1}(x)D\nu(x)))] \, dx.$$

Therefore, for the first term,

$$\begin{aligned} \left. \frac{d}{d\varepsilon} [\dots] \right|_{\varepsilon=0} &= - \int \rho_{n-1}(x) \left. \frac{d}{d\varepsilon} \ln(\det(I + \varepsilon D\chi_n^{-1}(x)D\nu(x))) \right|_{\varepsilon=0} \, dx \\ &= - \int \rho_{n-1}(x) \text{tr}(D\chi_n^{-1}(x)D\nu(x)) \, dx. \end{aligned}$$

(ii) The second term is obtained by a direct calculation

$$\begin{aligned} \left. \frac{d}{d\varepsilon} [\dots] \right|_{\varepsilon=0} &= - \int \rho_{n-1}(x) \left. \frac{d}{d\varepsilon} \ln \rho_{n-1}(\chi_n(x) + \varepsilon\nu(x)) \right|_{\varepsilon=0} \, dx \\ &= - \int \rho_{n-1}(x) \frac{1}{\rho_{n-1}(\chi_n(x))} \nabla \rho_{n-1}(\chi_n(x)) \cdot \nu(x) \, dx. \end{aligned}$$

(iii) Similarly for the third term,

$$\begin{aligned} \left. \frac{d}{d\varepsilon} [\dots] \right|_{\varepsilon=0} &= \frac{\Delta t_n}{2} \int \rho_{n-1}(x) \left. \frac{d}{d\varepsilon} (Y_n - h(\chi_n(x) + \varepsilon\nu(x)))^2 \right|_{\varepsilon=0} \, dx \\ &= - \int \rho_{n-1}(x) (\Delta Z_n - h(\chi_n(x))\Delta t_n) \nabla h(\chi_n(x)) \cdot \nu(x) \, dx. \end{aligned}$$

□

Since our interest is in the limit as $\Delta t_n \rightarrow 0$ and $N \rightarrow \infty$, we now restrict to diffeomorphisms of the form $\chi_n(x) = x + \mathbf{K}(x, n)\Delta Z_n + u(x, n)\Delta t_n$, where the appropriate function spaces are: $\mathbf{K} \in H^1(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$ and $u \in H^1(\mathbb{R}^d \rightarrow \mathbb{R}^d; \rho_{n-1})$. Starting from (5.35), the following is established in Appendix 5.8:

$$\delta I_n(\chi_n) \cdot \nu = E_z(n) \Delta Z_n + E_\Delta(n) \Delta t_n + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3), \quad (5.36)$$

where, denoting $\mathbf{K}(x, n) \doteq (\mathbf{K}_1(x, n), \dots, \mathbf{K}_d(x, n))$, $u(x, n) \doteq (u_1(x, n), \dots, u_d(x, n))$ and expressing $\nu(x) = \nu(x, n) \doteq (\nu_1(x), \dots, \nu_d(x))$, the following equations give expressions for E_z and E_Δ (expressed using Einstein's tensor notation):

$$\begin{aligned} E_z &= - \int \frac{\partial}{\partial x_j} \left(\rho_{n-1} \frac{\partial \mathbf{K}_j}{\partial x_i} \right) \nu_i \, dx - \int \rho_{n-1} \frac{\partial^2 \ln \rho_{n-1}}{\partial x_i \partial x_j} \mathbf{K}_j \nu_i \, dx - \int \rho_{n-1} \frac{\partial h}{\partial x_i} \nu_i \, dx \\ &= - \int \rho_{n-1} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho_{n-1}} \frac{\partial}{\partial x_j} (\rho_{n-1} \mathbf{K}_j) \right) \nu_i \, dx - \int \rho_{n-1} \frac{\partial h}{\partial x_i} \nu_i \, dx, \end{aligned} \quad (5.37)$$

$$\begin{aligned} E_\Delta &= - \int \rho_{n-1} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho_{n-1}} \frac{\partial}{\partial x_j} (\rho_{n-1} u_j) \right) \nu_i \, dx + \int \rho_{n-1} \left(h \frac{\partial h}{\partial x_i} - \frac{\partial^2 h}{\partial x_i \partial x_j} \mathbf{K}_j \right) \nu_i \, dx \\ &\quad - \frac{1}{2} \int \rho_{n-1} \frac{\partial^3 \ln \rho_{n-1}}{\partial x_i \partial x_j \partial x_k} \mathbf{K}_j \mathbf{K}_k \nu_i(x) \, dx + \int \frac{\partial}{\partial x_j} \left(\rho_{n-1} \frac{\partial \mathbf{K}_j}{\partial x_k} \frac{\partial \mathbf{K}_k}{\partial x_i} \right) \nu_i \, dx. \end{aligned} \quad (5.38)$$

We now return to the objective function $J^{(N)}(s)$ defined in (5.33). For any fixed N , the first order optimality condition for the minimizer $\underline{\chi} \doteq (\chi_1, \chi_2, \dots, \chi_N)$ is now immediate:

$$0 = \delta J^{(N)}(\underline{\chi}) \cdot \underline{\nu} = \sum_{n=1}^N E_z(n) \Delta Z_n + E_\Delta(n) \Delta t_n + \sum_{n=1}^N \left[O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3) \right], \quad (5.39)$$

where $\underline{\nu}(x) \doteq (\nu(x, 1), \dots, \nu(x, N))$ and $\nu(\cdot, n) \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ is an arbitrary perturbation. Recall now, $\chi_n(x) \doteq x + \mathbf{K}(x, n)\Delta Z_n + u(x, n)\Delta t_n$. The sequence $\{\rho_n\}$, $\{\mathbf{K}(x, n)\}$, $\{u(x, n)\}$ and $\{\nu(x, n)\}$ are used to construct, via interpolation, $\rho^N(x, t)$, $\mathbf{K}^{(N)}(x, t)$, $u^N(x, t)$ and $\nu^N(x, t)$, respectively. Recall $\rho^{(N)} \rightarrow \rho(x, t)$, given in (3.4). Likewise we formally denote the limit of $\mathbf{K}^{(N)}(x, t)$, $u^N(x, t)$ and $\nu^N(x, t)$ as $\mathbf{K}(x, t)$, $u(x, t)$ and $\nu(x, t)$, respectively.

With this notation, the right-hand side of (5.39), as $N \rightarrow \infty$, is expressed as an Itô integral,

$$\begin{aligned} &- \int_0^T \int \rho(x, s) \left(\frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho \mathbf{K}_j) \right) + \frac{\partial h}{\partial x_i} \right) \nu_i(x, s) \, dx \, dZ_s \\ &- \int_0^T \int \rho(x, s) \left(\frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho u_j) \right) - h \frac{\partial h}{\partial x_i} + \frac{\partial^2 h}{\partial x_i \partial x_j} \mathbf{K}_j \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^3 \ln \rho}{\partial x_i \partial x_j \partial x_k} \mathbf{K}_j \mathbf{K}_k - \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\rho \frac{\partial \mathbf{K}_j}{\partial x_k} \frac{\partial \mathbf{K}_k}{\partial x_i} \right) \right) \nu_i(x, s) \, dx \, ds. \end{aligned}$$

Since $\delta J^{(N)}(\underline{\chi}) \cdot \underline{\nu} = 0$ by optimality, and $\underline{\nu}$ is arbitrary, we obtain weak-sense differential equations for \mathbf{K} and u . The following two equations follow, also defined in

the weak sense:

$$\frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho \mathbf{K}_j) \right) = -\frac{\partial h}{\partial x_i}, \quad (5.40)$$

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho u_j) \right) &= h \frac{\partial h}{\partial x_i} - \frac{\partial^2 h}{\partial x_i \partial x_j} \mathbf{K}_j \\ &\quad - \frac{1}{2} \frac{\partial^3 \ln \rho}{\partial x_i \partial x_j \partial x_k} \mathbf{K}_j \mathbf{K}_k + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\rho \frac{\partial \mathbf{K}_j}{\partial x_k} \frac{\partial \mathbf{K}_k}{\partial x_i} \right). \end{aligned} \quad (5.41)$$

The BVP (4.4) is obtained by integrating (5.40) once:

$$\frac{\partial}{\partial x_j} (\rho \mathbf{K}_j) = -(h - \hat{h})\rho,$$

where $\hat{h} \doteq \int h(x)\rho(x) dx$. Using this the righthand-side of (5.41) is simplified, and the resulting equation is given by

$$\frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho u_j) \right) = \frac{\partial h}{\partial x_i} \hat{h} + \frac{1}{2} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial^2}{\partial x_j \partial x_k} (\rho \mathbf{K}_j \mathbf{K}_k) \right). \quad (5.42)$$

The simplification is obtained by first expressing the two terms involving h in the righthand-side of (5.41) as,

$$h \frac{\partial h}{\partial x_i} - \frac{\partial^2 h}{\partial x_i \partial x_j} \mathbf{K}_j = \frac{\partial h}{\partial x_i} \hat{h} + \frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho \mathbf{K}_j) \frac{\partial}{\partial x_i} \left(\frac{1}{\rho} \frac{\partial}{\partial x_k} (\rho \mathbf{K}_k) \right) + \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{\rho} \frac{\partial}{\partial x_k} (\rho \mathbf{K}_k) \right) \mathbf{K}_j.$$

Substituting this in the righthand-side of (5.41) gives the first term $\frac{\partial h}{\partial x_i} \hat{h}$ in the righthand-side of (5.42), and four terms involving only ρ and \mathbf{K} . It is a straightforward but tedious calculation to simplify these four terms into the form expressed as the second term in the righthand-side of (5.42).

It is readily verified, by direct substitution, that (5.42) admits a closed-form solution:

$$u_j = -\mathbf{K}_j \frac{(h + \hat{h})}{2} + \frac{1}{2} \frac{\partial \mathbf{K}_j}{\partial x_k} \mathbf{K}_k.$$

This gives (4.5).

5.8. Derivation of Equation (5.36). We substitute $\chi_n(x) = x + \mathbf{K}(x, n)\Delta Z_n + u(x, n)\Delta t_n$ in (5.35) and obtain explicit expressions for terms up to order $O(\Delta Z_n)$, $O(\Delta t_n)$. Since we are eventually interested in the limit as $\Delta t_n \rightarrow 0$, we use the Itô's rule $(\Delta Z_n)^2 = \Delta t_n$ to simplify the calculations. The calculations for the three terms appearing in (5.35) are as follows:

(i), The third term is expressed as

$$- \int \rho_{n-1}(x) (\Delta Z_n - h(x + \mathbf{K}\Delta Z_n + u\Delta t_n)\Delta t_n) \nabla h(x + \mathbf{K}\Delta Z_n + u\Delta t_n) \cdot \nu(x) dx.$$

Using Taylor series,

$$\begin{aligned} h(x + \mathbf{K}\Delta Z_n + u\Delta t_n) &= h(x) + O(\Delta Z_n, \Delta t_n), \\ \frac{\partial h}{\partial x_i}(x + \mathbf{K}\Delta Z_n + u\Delta t_n) &= \frac{\partial h}{\partial x_i}(x) + \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \mathbf{K}_j(x) \Delta Z_n + O(\Delta t_n, \Delta Z_n \Delta t_n, \Delta Z_n^2), \end{aligned}$$

the third term is simplified as

$$= \left(- \int \rho_{n-1}(x) \frac{\partial h}{\partial x_i} \nu_i(x) dx \right) \Delta Z_n + \left(\int \rho_{n-1}(x) \left(h \frac{\partial h}{\partial x_i} - \frac{\partial^2 h}{\partial x_i \partial x_j} \mathbf{K}_j \right) \nu_i(x) dx \right) \Delta t_n + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3).$$

(ii) The second term in (5.35) is similarly simplified as

$$\begin{aligned} & - \int \rho_{n-1}(x) \nabla \ln(\rho_{n-1}(x + \mathbf{K}(x) \Delta Z_n + u(x) \Delta t_n)) \cdot \nu(x) dx \\ &= - \int \rho_{n-1}(x) \frac{\partial}{\partial x_i} \ln(\rho_{n-1}) \nu_i(x) dx + \left(- \int \rho_{n-1}(x) \frac{\partial^2 \ln \rho_{n-1}}{\partial x_i \partial x_j} \mathbf{K}_j \nu_i(x) dx \right) \Delta Z_n \\ & \quad + \left(- \int \rho_{n-1}(x) \left(\frac{\partial^2 \ln \rho_{n-1}}{\partial x_i \partial x_j} u_j + \frac{1}{2} \frac{\partial^3 \ln \rho_{n-1}}{\partial x_i \partial x_j \partial x_k} \mathbf{K}_j \mathbf{K}_k \right) \nu_i(x) dx \right) \Delta t_n \\ & \quad + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3). \end{aligned}$$

(iii) Finally, for the remaining term in (5.35),

$$\begin{aligned} & - \int \rho_{n-1}(x) \text{tr}(D\chi_n^{-1}(x) D\nu(x)) dx \\ &= - \int \rho_{n-1}(x) \text{tr}((I + D\mathbf{K}(x) \Delta Z_n + Du(x) \Delta t_n)^{-1} D\nu(x)) dx \\ &= \int \nabla \rho_{n-1}(x) \cdot \nu(x) dx + \left(\int \rho_{n-1}(x) \text{tr}(D\mathbf{K}(x) D\nu(x)) dx \right) \Delta Z_n \\ & \quad + \left(\int \rho_{n-1}(x) (\text{tr}(Du(x) D\nu(x)) - \text{tr}((D\mathbf{K})^2(x) D\nu(x))) dx \right) \Delta t_n \\ & \quad + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3). \end{aligned}$$

Now, the terms with trace are simplified by using integration by parts, e.g.,

$$\begin{aligned} \int \rho_{n-1}(x) \text{tr}(D\mathbf{K}(x) D\nu(x)) dx &= \int \rho_{n-1}(x) \frac{\partial \mathbf{K}_j}{\partial x_i} \frac{\partial \nu_i}{\partial x_j} dx \\ &= - \int \frac{\partial}{\partial x_j} \left(\rho_{n-1}(x) \frac{\partial \mathbf{K}_j}{\partial x_i} \right) \nu_i(x) dx. \end{aligned}$$

As a result, the final term is given by

$$\begin{aligned} &= \int \frac{\partial \rho_{n-1}}{\partial x_i}(x) \nu_i(x) dx + \left(- \int \frac{\partial}{\partial x_j} \left(\rho_{n-1}(x) \frac{\partial \mathbf{K}_j}{\partial x_i} \right) \nu_i(x) dx \right) \Delta Z_n \\ & \quad + \left(- \int \frac{\partial}{\partial x_j} \left(\rho_{n-1}(x) \frac{\partial u_j}{\partial x_i} \right) \nu_i(x) dx + \int \frac{\partial}{\partial x_j} \left(\rho_{n-1}(x) \frac{\partial \mathbf{K}_j}{\partial x_k} \frac{\partial \mathbf{K}_k}{\partial x_i} \right) \nu_i(x) dx \right) \Delta t_n \\ & \quad + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3) \end{aligned}$$

Collecting the three terms, the E-L equation (5.35) is given by

$$\delta I_n(\chi_n) \cdot \nu = E_1 + E_z \Delta Z_n + E_\Delta \Delta t_n + O(\Delta t_n^2, \Delta Z_n \Delta t_n, \Delta Z_n^3),$$

where E_1 is the $O(1)$ term given by

$$E_1 = - \int \rho_{n-1}(x) \frac{\partial}{\partial x_i} \ln(\rho_{n-1}) \nu_i(x) dx + \int \frac{\partial \rho_{n-1}}{\partial x_i}(x) \nu_i(x) dx = 0,$$

E_z is the $O(\Delta Z_n)$ term given in (5.37), and E_Δ is the $O(\Delta t_n)$ term given in (5.38).

5.9. Proof of Theorem 4.2. We first assume that U_t^i is admissible. In this case, the evolution of $p(x, t)$ is according to the forward equation:

$$dp = -\nabla \cdot (p\mathbf{K}) dZ_t - \nabla \cdot (pu) dt + \frac{1}{2} \sum_{l,k=1}^d \frac{\partial^2}{\partial x_l \partial x_k} (p\mathbf{K}_l \mathbf{K}_k) dt. \quad (5.43)$$

To prove that the filter is exact, one needs to show that with the choice of $\{u, \mathbf{K}\}$ given by (4.4)-(4.5), we have $dp(x, t) = dp^*(x, t)$, for all x and t , in the sense that they are defined by identical stochastic differential equations. Recall dp^* is defined according to the K-S equation (1.2). The strong form of evolution equations is used for notational convenience. The proof with the weak form is entirely analogous, by using integration by parts.

Recall that the gain function \mathbf{K} is a solution of Poisson's equation,

$$\nabla \cdot (p\mathbf{K}) = -p(h - \hat{h}). \quad (5.44)$$

On multiplying both sides of (4.5) by $-p$, we obtain

$$\begin{aligned} -up &= \frac{1}{2} \mathbf{K}(h - \hat{h})p - \Omega p + p\mathbf{K}\hat{h} \\ &= -\frac{1}{2} \mathbf{K} \nabla \cdot (p\mathbf{K}) - \Omega p + p\hat{h}\mathbf{K} \end{aligned} \quad (5.45)$$

where (5.44) is used to obtain the second equality. Denoting $E := \frac{1}{2} \mathbf{K} \nabla \cdot (p\mathbf{K})$, a direct calculation shows that

$$E_l + \Omega_l p = \frac{1}{2} \sum_{k=1}^d \frac{\partial}{\partial x_k} (p[\mathbf{K}\mathbf{K}^T]_{lk}).$$

Substituting this in (5.45), on taking the divergence of both sides, we obtain

$$-\nabla \cdot (pu) + \frac{1}{2} \sum_{l,k=1}^d \frac{\partial^2}{\partial x_l \partial x_k} (p\mathbf{K}_l \mathbf{K}_k) = \nabla \cdot (p\mathbf{K})\hat{h}. \quad (5.46)$$

Using (5.44) and (5.46) in the forward equation (5.43),

$$dp = (h - \hat{h})(dZ_t - \hat{h} dt)p.$$

This is precisely the K-S equation (1.2), as desired.

Finally, we show that U_t^i is admissible. This follows from Prop. 2.1 and Theorem 2.2. The posterior distribution p^* is the limit of the minimizer sequence $\{\rho_n\}$, where ρ_n satisfies $\mathbf{PI}(\bar{\lambda})$ and $\bar{\lambda} > 0$ for all n . By Theorem 2.2, a unique solution $\mathbf{K}(x, t) = \nabla \phi(x, t)$ exists for each $p(x, t) = p^*(x, t)$. The a priori bounds (2.8)-(2.9) are used to show that

$$\mathbb{E}[|\mathbf{K}|^2] \leq \mathbb{E} \left[\frac{1}{\bar{\lambda}} \int |h(x)|^2 p(x, t) dx \right] < \infty,$$

$$\mathbb{E}[|u|] \leq \mathbb{E} \left[\left(\frac{1}{\bar{\lambda}} + C(\bar{\lambda}; p)^{1/2} \right) \int (|h(x)|^2 + |\nabla h|^2) p(x, t) dx \right] < \infty,$$

where the expression for $C(\bar{\lambda}; p)$ appears in Theorem 2.2, and we have used the fact that $h, \nabla h \in L^\infty$. That is, the resulting control input in the feedback particle filter is admissible. \blacksquare

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