On the Explicit Solution of Communication Topology Design for Distributed Control of Large-scale Interconnected Systems

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***Abstract*— Communication networks provide a larger flexibility for the control design of large-scale interconnected systems by allowing the information exchange between the local controllers of the subsystems. This paper presents explicit solutions on communication topology design for interconnected systems with certain class of physical interconnection topology, namely ring, star and line structure based on eigenvalue sensitivity analysis. First, the explicit solutions for the case of scalar subsystems with identical local dynamics and a single communication link are derived. Furthermore, it is investigated how the heterogeneity of the subsystem local dynamics affects the communication topology. Finally it is discussed how the results can be extended to the case of non-scalar subsystems and multiple communication links.**

I. INTRODUCTION

The design of control algorithms for complex dynamical systems has become a vibrant part of research due to the wide applicability and impact with applications ranging from smart power grids, water distribution and traffic systems to large arrays of micro-electro-mechanical systems (MEMS), formation of vehicles, and sensor-actuator networks.

The key challenge for the control of large-scale dynamical systems is the complexity of the overall system in terms of the number of subsystems and their interconnections. First results addressing the complexity of large-scale systems have been achieved within the decentralized control framework developed since the seventies, see, e.g. [1] for a nice overview. Typically, the performance of decentralized control approaches is degraded compared to centralized control approaches as only the local subsystem information is used for the control. Digital communication networks allow the communication between the subsystems and thereby provide a larger flexibility with respect to the control design: Instead of only local subsystem information also neighboring subsystems' states can be used for the control. These novel approaches are also known under the notion of *distributed control* [2]. Using information from the neighboring subsystems results in a better performance [3] and may stabilize the system in the presence of decentralized fixed modes [4].

The optimal distributed controller design with a prespecified controller structure is in general a non-convex problem. Most research has been focussed on characterizing the class of easily solvable problems for which convex solutions exist, e.g. [5]. The introduction of a communication network, on the other hand, also provides an additional degree of freedom for the structural design of the distributed controller in terms of the communication topology. The references [6]–[9] consider the design of distributed controller together with the communication topology such that a certain performance metric is optimized. The incorporation of topology into the design results in a combinatorial problem which becomes intractable for a large network. Most of the related work employ relaxation method such as weighted l_1 minimization to convert the optimization problem into a numerically tractable one. However, all of the work end-up in an optimization formulation without providing an explicit solution.

Having explicit solutions gives the designer more information on the relation between the interconnected system's dynamics, structure and the resulting topology. For example how the heterogeneity of the subsystems, strength of physical interconnection and the number of subsystems influence the topology. This information can be used in designing the interconnected system, given the constraint on the network cost. This motivates us to investigate explicit solutions of topology design for distributed controller of interconnected systems starting with certain class of physical interconnection topology, namely ring, star and line topology. As a main tool in this paper we utilize eigenvalue sensitivity based approach.

The remainder of the paper is organized as follows: After formulating the problem in Section II, eigenvalue sensitivity approach is reviewed in Section III. Explicit solutions on the single communication link design for the distributed controller of interconnected system with interacting scalar subsystems are presented in Section IV. The results are then extended to the case of non-scalar subsystems and multiple communication links in Section V. Due to space limitations, the proof of all the lemmas are omitted.

II. PROBLEM FORMULATION

Consider an interconnected system of N linear time invariant subsystems described as follows

$$\dot{x}_i = A_i x_i + \sum_{j\in\mathcal{N}_i} A_{ij} x_j + B_i u_i,\ x_i(t_0) = x_0^i, \qquad (1)$$

where $i = 1,2,...,N$ denotes the $i-$th subsystem, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$ are the state and the control input to subsystem i, and $A_i, A_{ij} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$. The term $\sum_{j\in\mathcal{N}_i} A_{ij}x_j$ represents the physical interconnection between the subsystems where $\mathcal{N}_i$ is the set of subsystems to which subsystem i is physically connected and $|\mathcal{N}_i|$ denotes the number of physical neighbors of subsystem i. We consider a state feedback controller given by

$$u_i = K_i x_i + \sum_{j\in\mathcal{G}_i} K_{ij} x_j, \qquad (2)$$

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which is known as *distributed* control law since the controller for each subsystem does not only depend on its own states but also the states of the other subsystems. Here $\mathcal{G}_i$ represents a set of subsystems to which controller i communicates, i.e. exchange information. If $K_{ij} = 0, \forall i$ and $\forall j \in \mathcal{G}_i$, then the control law is called a *decentralized* control law. In general, the goal is to design the distributed control (2) such that the performance of the whole system is improved and the stability of the system is guaranteed. Furthermore, the communication topology, i.e. $\mathcal{G}_i, \forall i$ of the distributed control law is also considered as a design parameter. The closed loop expression of the interconnected system (1) with control law (2) can be written as

$$\dot{x} = \bar{A}x, \; x(t_0) = x_0, \tag{3}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{A}_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & \bar{A}_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & \bar{A}_{NN} \end{bmatrix} + \begin{bmatrix} 0 & \bar{A}_{12} & \cdots & \bar{A}_{1N} \\ \bar{A}_{21} & 0 & \cdots & \bar{A}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{N1} & \bar{A}_{N2} & \cdots & 0 \end{bmatrix}, \\ \bar{A} &= A + A_{\text{dist}}, \end{aligned}$$

where $x = [x_1, \cdots, x_N]^T$, $\bar{A}_{ii} = A_i + K_i$ and $\bar{A}_{ij} = d_{ij}B_iK_{ij}$. Here, $d_{ij} \in \{0,1\}$ is a binary number that shows the possibility to perform the state information exchange between controller i and j. Hence $d_{ij} = 1$ means that a communication link is added between the local controllers i and j, i.e. $d_{ij} \sim (i,j), j \in \mathcal{G}_i$ and vice versa. Furthermore, it is assumed that not an arbitrary number of links can be added, i.e. the number is limited by an upper bound induced by the communication constraint

$$\sum_{1 \le i \le j \le N} \gamma_{ij} d_{ij} \le c, \tag{4}$$

where $c > 0$ is the total cost constraint on the communication network, and γ_{ij} represents a cost to establish a link between subsystem i and j, typically related to factors such as the distance between the subsystems. In this paper, as a performance metric, the decay rate of the overall system (3) is considered. It is well known that the solution of (3) is given by $x(t) = e^{\bar{A}(t-t_0)}x_0$ and the state norm satisfies

$$\|x(t)\| \le e^{\text{Re}\{\lambda_{\max}(t-t_0)\}}\|x_0\|, \forall t \ge t_0,$$

where $\text{Re}\{\lambda_{\max}\}$ represents the real part of $\lambda_{\max}(\bar{A})$.

The problem can then be stated as finding the gain and communication topology of the distributed controller such that the overall system is stable and its convergence rate is optimized under a given communication constraint as formulated by the following mixed integer optimization problem.

$$\begin{aligned} \underset{K_i, K_{ij}, d_{ij}}{\text{minimize}} \quad & \text{Re}\{\lambda_{\max}(\bar{A})\} \\ \text{subject to} \quad & \text{Re}\{\lambda_{\max}(\bar{A})\} < 0, \\ & \sum_{1 \le i \le j \le N} \gamma_{ij} d_{ij} \le c, \\ & d_{ij} \in \{0,1\}. \end{aligned} \tag{5}$$

The goal of this paper is to provide an explicit solution for the communication topology design problem for the interconnected system (1). Thus, differ to the works that compute the optimal gain for a given controller structure, in this paper it is assumed that the controller gain K_i, K_{ij} are fixed and the only design parameter is the communication topology $\mathcal{G}_i$.

III. EIGENVALUE SENSITIVITY BASED APPROACH

In this section we review the eigenvalue sensitivity based approach proposed in [6] In general, it is hard to derive the explicit solution to the optimization problem (5). Therefore, in order to analyze the optimal topology design, we constraint ourselves for the remainder of this paper by the following assumptions.

- A1 The subsystems are scalar, i.e. $x_i \in \mathbb{R}$
- A2 The physical interconnection is symmetric, i.e. $A^T = A$ where $\bar{A}_i < 0, A_{ij} > 0$
- A3 The communication is bidirectional, i.e. $A_{\text{dist}}^T = A_{\text{dist}}$
- A4 The distributed controller gains are fixed and equal, i.e. $K_{ij} = K < 0$.

The optimization problem (5) under Assumptions A1-A4 can be solved by relaxing the binary variable into $d_{ij} \in [0,1]$ and reformulating it to a semi-definite programming (SDP) problem as discussed in [9]. However, since we are interested in obtaining the explicit solution, an alternative approach based on eigenvalue sensitivity is utilized to investigate how the structure of the distributed control law affects $\text{Re}\{\lambda_{max}(\bar{A}\}$. Eigenvalue sensitivity gives an insight in the behavior of the eigenvalues of a matrix when the matrix is perturbed, in our case, when the distributed control law is applied to the interconnected system. Moreover, the magnitude of the eigenvalue sensitivity informs about the size of the eigenvalue displacement in the complex plane. The matrix $\bar{A}$ can be seen as the matrix A which is perturbed by the matrix $A_{\text{dist}} = [K_{ij}]$ where $K_{ij} = K$ which is the distributed control gain. Next we present the results on where to add the communication links. The idea is to solve the following

$$\begin{aligned} \underset{d_{ij} \sim (i,j)}{\text{maximize}} \quad & \left| \frac{\partial \lambda_{\max}}{\partial K} \right| \\ \text{subject to} \quad & \frac{\partial \lambda_{\max}}{\partial K} < 0. \\ & \sum_{1 \le i \le j \le N} \gamma_{ij} d_{ij} \le c, \\ & d_{ij} \in \{0,1\}. \end{aligned} \tag{6}$$

For the simplicity of analysis and clarity of the result, for the remainder it is assumed that $\gamma_{ij} = 1, \forall i,j$. Let $v_r = [v_{r_1}, \cdots, v_{r_N}]^T$ be the eigenvector corresponding to $\lambda_{\max}(A)$.

Proposition 3.1: [6] Consider an interconnected system (3) under assumption A1-A4. The optimal communication topology for a given number c of links to be added can be reformulated as to find c pairs of links between such that the following optimization problem is solved

$$\underset{(i,j),\cdots,(h,l)}{\text{maximize}} \overbrace{|v_{r_i} v_{r_j}| + \cdots + |v_{r_h} v_{r_l}|}^{c \text{ pairs}}. \tag{7}$$

For a single communication link case, i.e. $c = 1$, the optimization problem (7) can be written as

$$\underset{d_{ij} \sim (i,j)}{\text{maximize}} \; |v_{r_i} v_{r_j}|. \tag{8}$$

IV. EXPLICIT SOLUTION ON TOPOLOGY DESIGN

In this section, we present the explicit solution on where to add the communication link for a given controller gain based on the eigenvalue sensitivity analysis. As shown in Section III, the optimization problem (5) can be reformulated as finding the elements of eigenvector corresponding to the

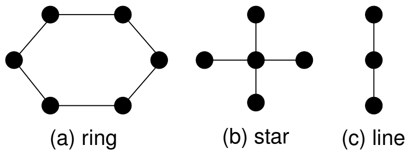

Fig. 1: Physical topology of the interconnected system investigated in the paper: ring, star and line topology. The physical interconnection between the subsystems are identical.

largest eigenvalue for a given controller gain. However, in general the closed form are not available for the generic case. Thus, in this paper as a first step we focus on interconnected system with three different physical topology namely ring, star and line structure as illustrated in Fig. 1 and investigate where to add the communication link when the local dynamics is identical or heterogenous and the physical interconnection between the subsystems are identical. Furthermore, it is assumed that $c = 1$, i.e. we consider the case of a single link. Before proceeding, we introduce the following definitions. Let us represent the structure of the interconnected system, i.e. the structure of matrix A in (3) by a plant graph $G_P = (\mathscr{V}_p, \mathscr{E}_p)$ comprising a set $\mathscr{V}_P = \{1, \cdots, N\}$ of vertices or subsystems and a set $\mathscr{E}_P = \{(j,i) | s_{ij} \neq 0\}$ of edges where $s_{ij} \neq 0$ means that subsystem j is (physically) affecting subsystem i. Note that from Assumption A2, graph G_P is undirected. Moreover, when $s_{ij} \neq 0$, we call vertices i, j are adjacent.

Definition 1: [10] A path of length r from i to j in a graph is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent.

Definition 2: [10] The distance $D_{G_P}(i,j)$ between subsystem i and j in a graph G_P is the length of the shortest path from i to j.

A. Ring topology case

First, we present the explicit solution of communication topology design for interconnected system whose physical interconnection has a ring structure and identical local dynamics.

Proposition 4.1: Consider an interconnected system (3) under assumption A1-A4 with a ring physical topology. In addition we assume that the local dynamics of the subsystems are identical, i.e. $A_i = A_j = a, i \neq j$ and $A_{ij} = b, \forall i,j$. Then the solution of (8) is $d^\star_{ij}$ where $(i^\star, j^\star) = \arg\max D_{G_P}(i,j)$.

Proof: With no loss of generality, we re-order the numbering of subsystems in a clockwise direction as $1, 2, \cdots, N$. The overall dynamics can then be written as

$$A = \begin{bmatrix} a & b & 0 & & b \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ b & & 0 & b & a \end{bmatrix} \tag{9}$$

which is known as circulant matrix. The eigenvalues of the circulant matrix in (9) are given by $\lambda_k = a + b\rho_N^k + b\rho_N^{(N-1)k}$ [11] where $\rho_N = e^{\frac{2i\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right), i^2 = -1$. The eigenvalues are then given by $\lambda_k = a + 2b\cos\left(\frac{2\pi k}{N}\right)$. Furthermore, the corresponding eigenvectors are given by $v_k = [1, \rho_N^k, \rho_N^{2k}, \cdots, \rho_N^{(N-1)k}]^T$. The largest eigenvalue $\lambda_{\max}$ corresponds to λ_N and the corresponding eigenvector can be computed as $v_N = h[1, 1, \cdots, 1]^T, h \in \mathbb{R}$. The optimal communication link is given by the solution of (8). However, the solution of (8) can not be obtained since any combination of (i,j) result in the same value h^2. Note that from trace$(\bar{A}) = \sum \lambda_i(\bar{A})$ we have $\sum \frac{\partial \lambda_i}{\partial K} = 0$. Thus, when $\frac{\partial \lambda_{\max}}{\partial K} < 0$, there exists at least one eigenvalue of A denoted by $\lambda_m(A)$ such that $\frac{\partial \lambda_m}{\partial K} > 0$. Therefore in order to find the optimal communication topology, with no loss of generality we consider the case where only two eigenvalues affected by the perturbation which is the largest eigenvalue $\lambda_{\max}$ and the second largest eigenvalue λ_m where $m = \{1, N-1\}$. From (6), the optimization problem (8) can then be reformulated as

$$\begin{aligned} \underset{i,j}{\text{maximize}} \quad & |v_{m_i} v_{m_j}| \\ \text{subject to} \quad & v_{m_i} v_{m_j} < 0 \end{aligned} \tag{10}$$

where v_m is the eigenvector corresponding to the second largest eigenvalue λ_m. The eigenvector for $m = 1$ is then given by

$$v_1 = \left[1, \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right), \cdots, \cos\left(\frac{2\pi(N-1)}{N}\right) + i\sin\left(\frac{2\pi(N-1)}{N}\right)\right]^T$$

Since $\left|\cos\left(\frac{2\pi(N-1)}{N}\right) + i\sin\left(\frac{2\pi(N-1)}{N}\right)\right| = 1$,the optimization problem (10) is equal to

$$\begin{aligned} \underset{i,j}{\text{maximize}} \quad & |\text{Re}\{v_{1_i}\}\text{Re}\{v_{1_j}\}| \\ \text{subject to} \quad & \text{Re}\{v_{1_i}\}\text{Re}\{v_{1_j}\} < 0. \end{aligned} \tag{11}$$

Since $-1 < \cos\left(\frac{2\pi l}{N}\right) < 1$, the solution of (11) is achieved at $i^\star = 1$ and $\cos\left(\frac{2\pi l}{N}\right) = -1$,i.e. $l = \frac{N}{2}$, or $j^\star = \frac{N}{2} + 1$. ■

Next we investigate how the heterogeneity of the local dynamics affects the solution.

Proposition 4.2: Consider an interconnected system (3) under assumption A1-A4 with a ring physical topology. We assume that the local dynamics of the subsystems are identical except for the local dynamics of subsystem m, i.e. $A_m = d, A_i = A_j = a$ where $i, j \neq m$. Furthermore, assumed that $A_{ij} = b, \forall i,j$. Then the solution of (8) is $d^\star_{ij}$ where

- $i^\star = m$ and $D_{G_P}(m, j^\star) = 1$ when $|d| < |a|$
- $D_{G_P}(i^\star, m) > D_{G_P}(q, m)$ and $D_{G_P}(j^\star, m) > D_{G_P}(q, m)$, $\forall q, q \neq i^\star, j^\star$ and $i^\star \neq j^\star$, otherwise.

First we introduce the following Lemmas.

Lemma 4.1: For the following matrix:

$$A = \begin{bmatrix} d & b & 0 & & b \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ b & & 0 & b & a \end{bmatrix}, \tag{12}$$

the elements of the eigenvector corresponding to the largest eigenvalue satisfy $v_{r_{i+1}} = v_{r_{N-(i-1)}}$ where $i = 1, \cdots, \left\lfloor \frac{N+1}{2} \right\rfloor$.

Lemma 4.2: The largest eigenvalue of the matrix:

$$A = \begin{bmatrix} a+\eta & b & 0 & & b \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ b & & 0 & b & a \end{bmatrix} \tag{13}$$

where $|\eta| < |a|$ is given by $\lambda_r(A) = \lambda_r(A_0) + \frac{1}{N}\text{sign}(\eta)$ and $\lambda_r(A_0)$ is the largest eigenvalue of A when $\eta = 0$.

We are now ready to prove Proposition 4.2.

Proof: With no loss of generality, we re-order the numbering of subsystems in a clockwise direction as $1, 2, \cdots, N$ where the susbsystem 1 corresponds to the subsystem m. The overall dynamics of the interconnected system with ring topology can then be written as in (12). As stated in Lemma 4.1, the elements of the eigenvector corresponding to the largest eigenvalue of matrix A in (12), i.e. v_r has the following pattern: $v_{r_{i+1}} = v_{r_{N-(i-1)}}$. Next we will show that the following holds.

$$v_{r_1} \geq v_{r_2} \geq \cdots \geq v_{r_{\lfloor \frac{N+1}{2} \rfloor}} \text{ or } v_{r_1} \leq v_{r_2} \leq \cdots \leq v_{r_{\lfloor \frac{N+1}{2} \rfloor}}.$$

From definition and using Lemma 4.1, we can write

$$\begin{bmatrix} d & b & 0 & & & b \\ b & a & b & & & \\ & \ddots & \ddots & \ddots & & \\ & & b & a & b \\ b & & 0 & b & a \end{bmatrix} \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ \vdots \\ v_{r_3} \\ v_{r_2} \end{bmatrix} = \lambda_{\max} \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ \vdots \\ v_{r_3} \\ v_{r_2} \end{bmatrix}. \tag{14}$$

Equation (14) can then be described by $\lfloor \frac{N}{2} \rfloor + 1$ equations where each equation is given by $dv_{r_1} + bv_{r_2} + bv_{r_2} = \lambda_r v_{r_1}$ for $i = 1$ and $bv_{r_{i-1}} + av_{r_i} + bv_{r_{i+1}} = \lambda_r v_{r_i}$ for $i = 2, \cdots, \lfloor \frac{N}{2} \rfloor + 1$. For $i = 1$ we have

$$v_{r_1} = \frac{2b}{\lambda_r - d} v_{r_2}. \tag{15}$$

With no loss of generality, for the remainder of the proof we assume that $|d| < |a|$. From Lemma 4.2 we have $\lambda_r(A) = \lambda_r(A_0) + \lambda' = 2b + a + \frac{1}{N}$. Equation (15) can then be written as $v_{r_1} = \frac{2b}{2b+a-d+\frac{1}{N}} v_{r_2}$. Since $|d| < |a|$, we have $a - d < 0$. Thus $\frac{2b}{2b+a-d+\frac{1}{N}} \geq 1$, i.e. $v_{r_1} \geq v_{r_2}$. Next, when $i = \lfloor \frac{N}{2} \rfloor + 1$ we have

$$v_{r_{\lfloor \frac{N}{2} \rfloor}} = \frac{\lambda_r - a}{2b} v_{r_{\lfloor \frac{N}{2} \rfloor + 1}} = \frac{2b + \frac{1}{N}}{2b} v_{r_{\lfloor \frac{N}{2} \rfloor + 1}}. \tag{16}$$

Since $\frac{2b+\frac{1}{N}}{2b} \geq 1$, we have $v_{r_{\lfloor \frac{N}{2} \rfloor}} \geq v_{r_{\lfloor \frac{N}{2} \rfloor + 1}}$. In addition, when $i = \lfloor \frac{N}{2} \rfloor$ we have

$$bv_{r_{\lfloor \frac{N}{2} \rfloor - 1}} + av_{r_{\lfloor \frac{N}{2} \rfloor}} + bv_{r_{\lfloor \frac{N}{2} \rfloor + 1}} = \lambda_r v_{r_{\lfloor \frac{N}{2} \rfloor}}. \tag{17}$$

Substituting (16), Equation (17) can be written as

$$v_{r_{\lfloor \frac{N}{2} \rfloor - 1}} = \underbrace{\frac{1}{b}\left[\lambda_r - a - \frac{2b^2}{\lambda_r - a}\right]}_{= g_{\lfloor \frac{N}{2} \rfloor}} v_{r_{\lfloor \frac{N}{2} \rfloor}}.$$

The term $g_{\lfloor \frac{N}{2} \rfloor} = 2 + \frac{1}{bN} - \frac{2b^2}{2b^2 + \frac{b}{N}}$. Taking the derivative of $g_{\lfloor \frac{N}{2} \rfloor}$ w.r.t. N we have $\frac{\partial g_{\lfloor \frac{N}{2} \rfloor}}{\partial N} = -\frac{1}{bN^2} - \frac{b}{N^2\left(2b^2 + \frac{b}{N}\right)^2} < 0$, i.e. $g_{\lfloor \frac{N}{2} \rfloor}$ is a decreasing function of N. Furthermore, when $N \to \infty$ we have $g_{\lfloor \frac{N}{2} \rfloor} \to 1$. Thus it can be concluded that $g_{\lfloor \frac{N}{2} \rfloor} \geq 1$, i.e. $v_{r_{\lfloor \frac{N}{2} \rfloor - 1}} \geq v_{r_{\lfloor \frac{N}{2} \rfloor}}$. Using the similar procedure as above, when $i = \lfloor \frac{N}{2} \rfloor - 1$ we have $\frac{\partial g_{\lfloor \frac{N}{2} \rfloor - 1}}{\partial N} = -\frac{1}{bN^2} - b\frac{\partial g^{-1}_{\lfloor \frac{N}{2} \rfloor}}{\partial N} < 0$. Thus $g_{\lfloor \frac{N}{2} \rfloor - 1} \to 1$ when $N \to \infty$. It can be concluded that $g_{\lfloor \frac{N}{2} \rfloor - 1} \geq 1$, i.e. $v_{r_{\lfloor \frac{N}{2} \rfloor - 2}} \geq v_{r_{\lfloor \frac{N}{2} \rfloor - 1}}$. Finally, we can write

$$v_{r_{j-1}} = \underbrace{\frac{1}{b}\left[\lambda_r - a - \frac{b}{g_{j+1}}\right]}_{= g_j} v_{r_j} \tag{18}$$

for $3 \leq j \leq \lfloor \frac{N}{2} \rfloor - 2$. Furthermore, it can be proven in the similar way that $g_j \geq 1$ which results in $v_{r_{j-1}} \geq v_{r_j}$. Thus, by collecting all results it can be concluded that $v_{r_1} \geq v_{r_2} \geq \cdots \geq v_{r_{\lfloor \frac{N+1}{2} \rfloor}}$. The optimal communication link which is the solution of (8) is then given by $i^\star = 1 = m$ and $j^\star = 2$ or $j^\star = N$. Furthermore, for the case $|d| > |a|$, it can also be proven that $v_{r_1} \leq v_{r_2} \leq \cdots \leq v_{r_{\lfloor \frac{N+1}{2} \rfloor}}$. Thus the solution of (8) is given by $i^\star = \lfloor \frac{N+1}{2} \rfloor$ and $j^\star = \lfloor \frac{N+1}{2} \rfloor - 1$ or $j^\star = \lfloor \frac{N+1}{2} \rfloor + 1$. This completes the proof. ■

B. Star topology case

Next, we present the explicit solution of communication topology design for the interconnected system whose physical interconnection has a star structure.

Proposition 4.3: Consider an interconnected system (3) under assumption A1-A4 with a star physical topology. Under the assumption that the local dynamics of the subsystems are identical except for subsystem m with the largest degree, i.e. $A_m = d$, where $\deg(m) = |\mathcal{N}_m| = N - 1$ and $A_i = A_j = a$ where $i, j \neq m$. Furthermore, we assume that $A_{ij} = b, \forall i, j$. Then the solution of (8) is $d^\star_{ij}$ where

- $i^\star = m$ and $D_{G_P}(i^\star, j^\star) = 1$ when $d - a > b(2 - N)$,
- $D_{G_P}(i^\star, j^\star) = 2$, otherwise.

Remark 1: Note that the result can be extended in a straightforward manner for the case of $\deg(m) = 1$.

First we introduce the following Lemma.

Lemma 4.3: The eigenvalues of the matrix

$$A = \begin{bmatrix} d & b & b & \cdots & b \\ b & a & 0 & & 0 \\ & & & & \vdots \\ b & 0 & a & & \\ \vdots & \vdots & & \ddots & 0 \\ b & 0 & \cdots & 0 & a \end{bmatrix} \tag{19}$$

where $a, d < 0$ and $b > 0$ is given by

$$\begin{aligned} \lambda_{1,2} &= \frac{a + d \pm \sqrt{(a+d)^2 - 4\left(ad - (N-1)b^2\right)}}{2} \\ \lambda_3 &= \cdots = \lambda_N = a. \end{aligned}$$

We are now ready to prove Proposition 4.3.

Proof: With no loss of generality, we re-order the numbering of subsystems where the subsystem with the largest degree, i.e. subsystem m as subsystem 1 and the others in clockwise direction as subsystem $2, \cdots, N$. The overall dynamics of the interconnected system with star topology can then be written as in (19). The eigenvector corresponding to the largest eigenvalue λ_r can be written as

$$\begin{aligned} dv_{r_1} + bv_{r_2} + \cdots + bv_{r_N} &= \lambda_r v_{r_1} \\ bv_{r_1} + av_{r_2} &= \lambda_r v_{r_2} \\ &\vdots \\ bv_{r_1} + av_{r_N} &= \lambda_r v_{r_N}. \end{aligned}$$

Then, it can be computed that $v_{r_2} = \cdots = v_{r_N} = \frac{b}{\lambda_r - a} v_{r_1}$. From Lemma 4.3, $\lambda_r = \lambda_1$, thus we have

$$v_{r_2} = \cdots = v_{r_N} = \frac{2b}{d-a+\sqrt{(a+d)^2 - 4(ad-(N-1)b^2)}} v_{r_1}.$$

With no loss of generality, taking $v_{r_2} = \cdots = v_{r_N} = 1$,

$$v_{r_1} = \frac{d-a+\sqrt{(a+d)^2 - 4(ad-(N-1)b^2)}}{2b}. \tag{20}$$

The optimal communication link is formulated as the optimization problem (8) whose solution is given by $i^\star = 1$, $j^\star \neq 1$ when $v_{r_1} > 1$ and $i^\star, j^\star \neq 1$ when $v_{r_1} < 1$. Next, the condition for $v_{r_1} > 1$ can be computed as follows

$$\begin{aligned} v_{r_1} &> 1 \\ \sqrt{(a+d)^2 - 4(ad-(N-1)b^2)} &> 2b-(d-a) \\ d-a &> b(2-N). \end{aligned}$$

This completes the proof. ∎

In addition, we have the following Corollary for the case of interconnected system with identical local dynamics.

Corollary 4.1: Consider an interconnected system (3) under assumption A1-A4 with a star physical topology. In addition we assume that the local dynamics of the subsystems are identical, i.e. $A_i = A_j = a, i \neq j$ and $A_{ij} = b, \forall i, j$. Then the solution of (8) is $d^\star_{ij}$ where $i^\star = m$ and $D_{G_P}(i^\star, j^\star) = 1$.

C. Line topology case

Finally we present the explicit solution for the interconnected system with a line structure.

Proposition 4.4: Consider an interconnected system (3) under assumption A1-A4 with a line physical topology. We assume that the local dynamics of the subsystems are identical, i.e. $A_i = A_j = a, i \neq j$ and $A_{ij} = b, \forall i, j$. Furthermore, with no loss of generality, the numbering of subsystems is re-ordered from left to right or up to down as $1, 2, \cdots, N$. Then the solution of (8) is $d^\star_{ij}$ where

- $j^\star = \frac{N+1}{2}$ and $i^\star = \frac{N+1}{2} + 1$ or $i^\star = \frac{N+1}{2} - 1$ if N is odd
- $j^\star = \frac{N}{2}$ and $i^\star = \frac{N}{2} + 1$ if N is even.

Proof: The overall dynamics of the interconnected system with line topology can then be written as

$$A = \begin{bmatrix} a & b & & & \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ & & & b & a \end{bmatrix}. \tag{21}$$

In order to prove the Proposition, first we need to compute the largest eigenvalue and the corresponding eigenvector of (21). In general, the eigenvalue of A in (21) is given by [12]

$$\lambda_j = a + 2|b| \cos\left(\frac{j}{N+1}\pi\right), j = 1, \cdots, N, \tag{22}$$

while the corresponding eigenvector is given by $v_j = y_j[\Delta u_j], j = 1, \cdots, N$ where

$$u_j = \left(\frac{2}{N+1}\right)^{\frac{1}{2}} \left[\sin\left(\frac{j\pi}{N+1}\right), \cdots, \sin\left(\frac{Nj\pi}{N+1}\right)\right]^T,$$

$$y_j = \left(\frac{2}{N+1} \sum_{j=1}^{N} \sin^2\left(\frac{ij\pi}{N+1}\right)\right)^{-\frac{1}{2}}, \ \Delta = \mathrm{diag}(1, \cdots, 1).$$

From (22), the largest eigenvalue, i.e. $j = 1$ is given by

$$\lambda_{\max} = a + 2|b| \cos\left(\frac{1}{N+1}\pi\right) \tag{23}$$

and the corresponding eigenvector is

$$v_1 = \left(\sum_{j=1}^{N} \sin^2\left(\frac{i\pi}{N+1}\right)\right)^{-\frac{1}{2}} \mathrm{diag}(1, \cdots, 1) \left[\sin\left(\frac{1\pi}{N+1}\right), \cdots, \sin\left(\frac{N\pi}{N+1}\right)\right]^T.$$

From $\sum_{i=1}^{N} \sin^2(ix) = \frac{1}{4}\{1 + 2N - \csc(x)\sin[x(1+2N)]\}$ and taking $x = \frac{\pi}{N+1}$, we have $\sum_{i=1}^{N} \sin^2\left(\frac{i\pi}{N+1}\right) = \frac{N+1}{2}$. Thus the eigenvector corresponding to $\lambda_{\max}$ is given by

$$v_1 = \sqrt{\frac{2}{N+1}} \left[\sin\left(\frac{1\pi}{N+1}\right), \cdots, \sin\left(\frac{N\pi}{N+1}\right)\right]^T.$$

The maximum value of the element of v_1 is equal to 1 which occurs at

$$\sin\left(\frac{j\pi}{N+1}\right) = \sin\left(\frac{\pi}{2}\right) \Leftrightarrow j = \frac{N+1}{2}.$$

The optimal communication link is formulated as the optimization problem (7) whose solution is then given by

- $j^\star = \frac{N+1}{2}$ and $i^\star = \frac{N+1}{2} + 1$ or $i^\star = \frac{N+1}{2} - 1$ if N is odd
- $j^\star = \frac{N}{2}$ and $i^\star = \frac{N}{2} + 1$ if N is even.

This completes the proof. ∎

V. EXTENSION TO NON-SCALAR CASE AND MULTIPLE COMMUNICATION LINKS

Next the results in Section IV are extended into the case of non-scalar subsystems and multiple communication links.

A. Non-scalar subsystems

First we investigate the following question: for which class of interconnected system do the results for the scalar case still hold? We consider an interconnected system given by the following assumptions.

V1 The state of subsystem i, i.e. $x_i \in \mathbb{R}^n$

V2 A_i is real symmetric, i.e. $A_i = A_i^T$ and $\lambda_{\max}(A_i) < 0$.

V3 The physical interconnections are identical, i.e. $A_{ij} = A_{qs}$, $i \neq j, q \neq s$ and $A_{ij} = A_{ij}^T$, $A_{ij} = lA_i, l < 0 \in \mathbb{R}$

V4 The communication is bidirectional and the controller gains are fixed and equal. Moreover $B_i K_{ij} = kJ_n$ where $k \in \mathbb{R}, k < 0$ and J is a unit matrix of size n.

Let us consider an interconnected system (3) under Assumption V1-V4 where the local dynamics of the subsystems are identical, i.e. $A_i = A_j = \hat{A}, i \neq j$. Next we compute the change of the largest eigenvalue of the matrix A when it is perturbed by the matrix $K = [B_i K_{ij}] \in \mathbb{R}^{Nn \times Nn}$ where $B_i K_{ij} \in \mathbb{R}^{n \times n}$ is given by Assumption V4. Without loss of generality, we assume that the physical interconnection topology is given by a ring and the communication link is added between the local controller of subsystem i and j. Then we have $\frac{\partial \lambda_{\max}}{\partial k} = v_r^T D v_r$ where $v_r = [v_{r_1}, v_{r_2}, \cdots, v_{r_n}, v_{r_{n+1}}, \cdots, v_{r_{Nn}}]^T$ and the block matrices $D_{ij} = D_{ji} = -J_n$ and zero otherwise. After straightforward computation we have $\frac{\partial \lambda_{\max}}{\partial k} = -2v_r^i v_r^j$ where

$$v_r^i = \sum_{p=(i-1)n+1}^{in} v_{r_p}. \tag{24}$$

Next we compute the eigenvector of A corresponding to the largest eigenvalue $\lambda_{\max}$. The matrix A can be written as

$A = C \otimes -\hat{A}$ where $\otimes$ denotes the Kronecker product and the matrix $C \in \mathbb{R}^{N\times N}$ is given by

$$C = \begin{bmatrix} -1 & h & 0 & & h \\ h & -1 & h & & \\ & \ddots & \ddots & \ddots & \\ & & h & -1 & h \\ h & & 0 & h & -1 \end{bmatrix}$$

with $h = -l$.The Nn eigenvalues of $C \otimes -\hat{A}$ are given by [13]

$$\lambda_1(C)\lambda_1(-\hat{A}), \cdots, \lambda_1(C)\lambda_n(-\hat{A}), \lambda_2(C)\lambda_1(-\hat{A}), \cdots, \lambda_N(C)\lambda_n(-\hat{A}).$$

Thus under Assumption V2, the largest eigenvalue of A can be computed as $\lambda_{\max}(A) = \lambda_{\max}(C)\lambda_{\max}(-\hat{A})$ or $\lambda_{\max}(A) = \lambda_{\max}(C)\lambda_{\min}(-\hat{A})$. Furthermore, if $z_1, \cdots, z_N$ are linearly independent right eigenvectors of C corresponding to $\lambda_1(C), \cdots, \lambda_N(C)$ and $w_1, \cdots, w_n$ are linearly independent right eigenvectors of $-\hat{A}$ corresponding to $\lambda_1(-\hat{A}), \cdots, \lambda_n(-\hat{A})$, then $z_i \otimes w_j \in \mathbb{R}^{Nn}$ are linearly independent right eigenvectors of $C \otimes -\hat{A}$ corresponding to $\lambda_i(C)\lambda_j(-\hat{A})$ [13]. Thus the right eigenvectors of A corresponding to the largest eigenvalue $\lambda_{\max}$ is given by $v_r = z_r \otimes w_r$ or $v_r = z_r \otimes w_1$. Then Eq. (24) can be re-written as

$$v_r^i = z_{r_i} \sum_{j=1}^{n} w_{r_j} \text{ or } v_r^i = z_{r_i} \sum_{j=1}^{n} w_{1_j}$$

where $w_r = [w_{r_1}, \cdots, w_{r_n}]^T$, $w_1 = [w_{1_1}, \cdots, w_{1_n}]^T$. Then

$$\frac{\partial \lambda_{\max}}{\partial k} = -2 z_{r_i} z_{r_j} \underbrace{\left| \sum_{j=1}^{n} w_{r_j} \right|^2}_{\text{constant}} \text{ or } \frac{\partial \lambda_{\max}}{\partial k} = -2 z_{r_i} z_{r_j} \underbrace{\left| \sum_{j=1}^{n} w_{1_j} \right|^2}_{\text{constant}}.$$

Thus $\frac{\partial \lambda_{\max}}{\partial k} \sim -2 z_{r_i} z_{r_j}$. It is clear that the optimization problem is reduced to the case of the scalar case by finding a pair of elements of eigenvector corresponding to $\lambda_{\max}(C)$.

B. Multiple communication links

Next we discuss the case where multiple communication links are going to be added, i.e. $\gamma_{ij} = 1, c > 1$. Without loss of generality we consider the scalar subsystems. The results also hold for non-scalar case in the previous subsection. We have the following Lemmas.

Lemma 5.1: Consider an interconnected system (3) under assumption A1-A4 with identical subsystems and a star physical topology. In addition, we assume that the subsystem with degree $N-1$ as subsystem 1 and the others in clockwise direction as subsystem $2, \cdots, N$. Then $v_{r_i} > 0$ is the concave function of i and $\arg\max_i v_{r_i} = 1$.

Lemma 5.2: Consider an interconnected system (3) under assumption A1-A4 with identical subsystems and a line physical topology. In addition we re-order the numbering of subsystems from left to right or up to down as $1, 2, \cdots, N$. Then $v_{r_i} > 0$ is the concave function of i and

$$\arg\max_i v_{r_i} = \begin{cases} \frac{N+1}{2} & \text{if } N \text{ is odd} \\ \frac{N}{2}, \frac{N+1}{2} & \text{if } N \text{ is even.} \end{cases}$$

Lemma 5.3: Consider an interconnected system (3) under assumption A1-A4 with ring physical topology. Furthermore, let us assume that the local dynamics of the subsystems are identical except for subsystem 1 and the subsystems are numbered in a clockwise direction as $1, 2, \cdots, N$. Then $v_{r_i} > 0$ is a concave (resp. convex) function of i if $|A_1| > |A_j|, j \neq 1$ (resp. $|A_1| < |A_i|, j \neq 1$) and $\arg\max_i v_{r_i} = \left\lfloor \frac{N}{2} \right\rfloor + 1$ (resp. $\arg\max_i v_{r_i} = 1$).

Lemma 5.4: Consider an interconnected system (3) under assumption A1-A4 with star physical topology. Furthermore, let us assume that the local dynamics of the subsystems are identical except for subsystem with degree $N-1$ and the subsystems are numbered such that subsystem with degree $N-1$ as subsystem 1 and the others in clockwise direction as subsystem $2, \cdots, N$. Then v_{r_i} is a concave (resp. convex) function of i if $A_1 - A_j < A_{ij}(2-N), j \neq 1$ (resp. $A_1 - A_j > A_{ij}(2-N), j \neq 1$) and $\arg\max_i v_{r_i} = i \neq 1$ (resp. $\arg\max_i v_{r_i} = 1$).

The topology for multiple communication links can be computed efficiently by using Lemmas 5.1-5.4 and solving (7).

VI. CONCLUSIONS

This paper presents explicit solutions of communication topology design for distributed controller of interconnected systems with certain class of physical interconnection topology: ring, star and line structure. As can be observed, for the class of systems considered with homogeneous subsystems and a single link case, the ring structure results in a communication topology with the highest cost w.r.t. the distance between the controllers. Furthermore, it is shown that for the heterogeneous subsystems with star topology, the number of subsystems also plays a role in the resulting topology.

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