Bounds on Mixed Binary/Ternary Codes

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***Abstract*— Upper and lower bounds are presented for the maximal possible size of mixed binary/ternary error-correcting codes. A table up to length 13 is included. The upper bounds are obtained by applying the linear programming bound to the product of two association schemes. The lower bounds arise from a number of different constructions.**

***Index Terms*— Binary codes, clique finding, linear programming bound, mixed codes, tabu search, ternary codes.**

I. Introduction

Let $X = \boldsymbol{F}_2^{n_2}\boldsymbol{F}_3^{n_3}$ be the set of all vectors with n_2 binary and n_3 ternary coordinates (in this order). Let $d(.,.)$ denote Hamming distance on X. We study the existence of large packings in X, i.e., we study the function $N(n_2, n_3, d)$ giving the maximal possible size of a code C in X with $d(c, c') \geq d$ for any two (distinct) codewords $c, c' \in C$. The dual version of this problem, the existence of small coverings in X, has been discussed in [17] and [33]. Both of these problems were originally motivated by the football pool problem (see [16]).

We begin by describing the use of product schemes to get upper bounds on $N(n_2, n_3, d)$, and then discuss various constructions and computer searches that provide lower bounds. Among the codes constructed, there are a few (with $n_2 = 0$) that improve the known lower bounds for ternary codes.

The paper concludes with a table of $N(n_2, n_3, d)$ for $n_2 + n_3 \leq 13$. The first and fourth authors produced a version of this table in 1995 (improving and extending various tables already in the literature, for example, that in [24]). These results were then combined with those of the second and third authors, who had used computer search and various constructions to obtain lower bounds (many of which were tabulated by the second author already in 1991).

II. Products of Association Schemes

Let $(X', \mathcal{R}')$ and $(X'', \mathcal{R}'')$ be two association schemes, with $\mathcal{R}' = \{R_{i'} \mid i' \in I'\}$ and $\mathcal{R}'' = \{R_{i''} \mid i'' \in I''\}$. (For definitions and notation, see [7, ch. 2].) We get a new association scheme $(X, \mathcal{R})$, the *product* of these two, by taking $X = X' \times X''$ for the point set, and $\mathcal{R} = \{R_{i'i''} \mid i' \in I',\ i'' \in I''\}$, where, for $x = (x', x'')$ and $y = (y', y'')$, we have $(x, y) \in R_{i'i''}$ if and only if $(x', y') \in R_{i'}$ and $(x'', y'') \in R_{i''}$.

It is trivial to verify that this product scheme indeed is an association scheme. The intersection numbers are given by $p_{ij}^k = p_{i'j'}^{k'} p_{i''j''}^{k''}$, where $i = (i', i'')$, etc., and the dual intersection numbers by $q_{ij}^k = q_{i'j'}^{k'} q_{i''j''}^{k''}$. The adjacency matrices are given by $A_i = A_{i'} \otimes A_{i''}$, the idempotents by $E_i = E_{i'} \otimes E_{i''}$, and for the eigenmatrix P and dual eigenmatrix Q (defined by $A_j = \sum_i P_{ij} E_i$ and $E_j = \frac{1}{|X|} \sum_i Q_{ij} A_i$) we have $P_{ij} = P_{i'j'} P_{i''j''}$ and $Q_{ij} = Q_{i'j'} Q_{i''j''}$.

Products of more than two schemes can be defined in an analogous way (and the multiplication of association schemes is associative).

Although product schemes are well known, we cannot find an explicit discussion of their properties or applications. There is a short reference in Godsil [15, p. 231] and an only slightly longer one in Dey [11, Sec. 5.10.7]. (We wrote this in 1995. In the meantime several other applications of product schemes have come to our attention. See for example [18], [19], [30], [36], and [37].) Another recent paper dealing with mixed codes is [12].

Our interest in product schemes in the present context stems from the fact that the set of mixed binary/ternary vectors with n_2 binary and n_3 ternary coordinate positions does not, in general, form an association scheme with respect to Hamming distance, and so Delsarte's linear programming bound cannot be directly applied there. This was a source of worry to the fourth author for many years. However, this set does have the structure of a product scheme, and so a version of the linear programming bound can be obtained for both designs (cf. [37]) and codes.

The linear programming bound for codes in an arbitrary association scheme can be briefly described as follows. If C (the code we want to study) is a nonempty subset of an association scheme, we can define its *inner distribution* a by $a_i = |(C \times C) \cap R_i|/|C|$, the average number of codewords at "distance" i from a codeword. Clearly, $a_0 = 1$ (if R_0 is the identity relation), and $\sum a_i = |C|$. A one-line proof[1] shows that one has $aQ \geq 0$ (that is, $(aQ)_j \geq 0$ for all j), and thus we obtain the linear programming bound

$$|C| \leq \max\left\{\sum a_i \mid a_0 = 1 \text{ and } aQ \geq 0\right\}.$$

The upper bound obtained this way will be referred to as the "pure LP" bound. As we shall see, slightly better results can

[1] Let χ be the characteristic vector of C. Then, since E_j is idempotent

$$|C|(aQ)_j = |C| \sum_i a_i Q_{ij} = \sum_i Q_{ij} \chi^\top A_i \chi = \chi^\top E_j \chi = ||E_j \chi||^2 \geq 0.$$

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sometimes be obtained by adding other inequalities that a is known to satisfy.

A. The Hamming Scheme

Of course, the usual Hamming scheme $H(n,q)$ also carries the structure of a product scheme, for $n \geq 2$, and it is sometimes useful to study nonmixed codes using this product scheme setting, getting separate information on the weights in the head and in the tail of the codewords, as in the split weight enumerator of a code (cf. [29, pp. 149–150]).

Consider the Hamming scheme $H(m+n,q)$ as being obtained from the product of $H(m,q)$ and $H(n,q)$ by merging all relations $R_{i',i''}$ with $i'+i''=i$ into one relation R_i. We have

$$A_i = \sum_{i'+i''=i} A_{i'i''}$$
$$E_i = \sum_{i'+i''=i} E_{i'i''}$$

and

$$Q_{ij} = \sum_{j'+j''=j} Q_{i'j'}Q_{i''j''}$$

for any pair (i',i'') with $i'+i''=i$. Indeed, the first holds by definition, the second follows from the third, and the third follows as soon as we have shown that the right-hand side does not depend on the choice of the pair (i',i''). But that follows by viewing all three association schemes involved as merged versions of powers of $H(1,q)$: we must show that

$$Q_{ij} = \sum_{\mathrm{wt}\,(\boldsymbol{j})=j} Q_{i_1j_1}Q_{i_2j_2}\cdots$$

for any 0-1 vector $\boldsymbol{i}$ with $\mathrm{wt}\,(\boldsymbol{i})=i$. However, since such vectors $\boldsymbol{i}$ are equivalent under the symmetric group on the coordinates, the right-hand side is independent of $\boldsymbol{i}$, and the equality follows.

Here we did not need to actually compute the Q_{ij}, but since in $H(1,q)$ we have

$$Q = \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix},$$

it follows immediately that in $H(n,q)$

$$Q_{ij} = \sum_{r+s=j} (-1)^r (q-1)^s \binom{i}{r}\binom{n-i}{s}.$$

The "detailed" linear programming bound obtained in the above manner always implies the "ordinary" linear programming bound: given any solution a of the detailed system

$$\sum a_{i'i''}Q_{i'j'}Q_{i''j''} \geq 0, \qquad \text{for all } j', j''$$

it follows by summing over the pairs (j',j'') with $j'+j''=j$ that $\sum a_iQ_{ij} \geq 0$, where, of course, $a_i = \sum_{i'+i''=i} a_{i'i''}$.

Conversely, given any solution a of the ordinary system $\sum a_iQ_{ij} \geq 0$, we find a solution of the detailed system by letting

$$a_{i'i''} = a_i \binom{m}{i'}\binom{n}{i''} \Big/ \binom{m+n}{i}, \qquad \text{for all } i = i'+i''.$$

Indeed, this follows if we again go "to the bottom," express everything in terms of $H(1,q)$, and use the symmetric group on the coordinates.

Thus the two systems are equivalent over $\boldsymbol{R}$. However, the detailed system can be useful i) if it is known that the a_i are integral, e.g., because C is linear, or ii) when one can add further constraints, e.g., because one has information on a residual code. Jaffe [19] has recently obtained a number of new bounds for binary linear codes by recursive applications of this approach.

III. Comparison with Earlier Results and the Case $d=3$

A. Counting

In the final section we give tables of upper and lower bounds for codes in the mixed binary/ternary scheme. A table with upper bounds was given in Van Lint Jr. and Van Wee [24]. Pure linear programming agrees with or improves all the values in their table with four exceptions, namely the parameter sets $(n_2,\ n_3,\ d) = (3,6,3),\ (5,5,3),\ (7,6,3),\ (8,4,3)$, where [24] gives $345,\ 465,\ 4515,\ 1184$, while the pure LP bound yields the upper bounds $356,\ 469,\ 4560,\ 1209$, respectively. The upper bound used in these cases in [24] is due to Van Wee [43, Theorem 17], and states that if $d=3$, $n_2=b$, and $n_3=t$, then $|C| \leq 2^b3^t/(|\mathrm{ball}|+\varepsilon)$, where $|\mathrm{ball}| = 2t+b+1$, and $\varepsilon = b/(2t+b)$ if b is even or $\varepsilon = 2t/(2t+b-1)$ if b is odd.[2] In fact, a stronger result is true.

Proposition 3.1: If $b>0$, and b is even or $t>0$, then $N(b,t,3) \leq 2^b3^t/(2t+b+2)$.

Proof: Let C be a $(b,t,3)_N$ code. Count paths (u,v,w) with $d(u,v)=d(v,w)=1$, $d(u,C)=2$, $w \in C$, where $v-w$ is nonzero at a binary coordinate position if b is even, and at a ternary coordinate position if b is odd. Put $a := b$ if b is even, and $a := 2t$ if b is odd. For w we have N choices; given w there are a choices for v; given w and v there is at least one choice for u. The number of paths is, therefore, at least Na. On the other hand, there are at most $2^b3^t - N(2t+b+1)$ choices for u, and given u there are at most a choices for (v,w), so the number of paths is at most $(2^b3^t - N(2t+b+1))a$. □

In the four cases mentioned, this yields the bounds $343,\ 457,\ 4443,\ 1152$, respectively. We shall see below (in Proposition 5.10) that the last mentioned bound in fact holds with equality.

B. Linear Programming with Additional Inequalities

The preceding results were obtained by studying what happens close to the code. In general, one should obtain at least as strong results by adding analogous constraints on the $A_{i,j}$ with small $i+j$ to the linear program (note that we change notation here from what is usual in association scheme theory to what is common in coding theory, and write $A_{i,j}$ where the previous section had $a_{i,j}$).

[2] Gerhard van Wee has pointed out to us that there is a typographical error in the statement of this bound in [43, Theorem 17]. The bound given here (which follows at once from [24, Theorem 9]) is the correct version.

What are the obvious inequalities to add when $d = 3$? Well, no two words of weight 3 can agree in two nonzero coordinates, so we have a packing problem for triples in a $(b+2t)$-set, with a prespecified matching of size t, where the triples may not cover any edge of the matching. The extra inequalities are found by counting triples, point-triple incidences, and pair-triple incidences. Starting with the latter, there are

$$\binom{b}{2}$$

pairs in the binary set, $2t(t-1)$ available pairs in the ternary set, and $2bt$ pairs between the two sets. This yields the inequalities

$$\begin{aligned} 3A_{3,0} + A_{2,1} &\le b(b-1)/2 \\ 2A_{2,1} + 2A_{1,2} &\le 2bt \\ A_{1,2} + 3A_{0,3} &\le 2t(t-1). \end{aligned}$$

Next, counting point-triple incidences, we find

$$\begin{aligned} 3A_{3,0} + 2A_{2,1} + A_{1,2} &\le b\lfloor (2t+b-1)/2 \rfloor \\ A_{2,1} + 2A_{1,2} + 3A_{0,3} &\le 2t\lfloor (2t+b-2)/2 \rfloor. \end{aligned}$$

Finally, counting triples, we obtain

$$\begin{aligned} &A_{3,0} + A_{2,1} + A_{1,2} + A_{0,3} \\ &\qquad \le \lfloor ((2t+b)(2t+b-1) - 2t - b\varepsilon_b - 2t\varepsilon_t)/6 \rfloor \end{aligned}$$

where $\varepsilon_b, \varepsilon_t \in \{0,1\}$ with $\varepsilon_b = (2t+b-1) \bmod 2$ and $\varepsilon_t = (2t+b-2) \bmod 2$. Of course, the last three inequalities only contribute when rounding down occurs.

As a test case, let us compute the improved LP bound in the four cases mentioned above. We find 347, 459, 4491, 1178, respectively. This improves the pure LP bound (of course), and three of the four bounds from [24]. However, Proposition 3.1 is stronger—it really encodes information about distance 4, and we would have to add inequalities involving $A_{i,j}$ with $i + j = 4$ to approach or beat it.

Precisely the same ideas work for larger d. We have

$$\begin{aligned} &\sum_{i=0}^{d} \binom{i}{j}\binom{d-i}{r-j} A_{i,d-i} \\ &\qquad \le 2^{r-j}\binom{n_2}{j}\binom{n_3}{r-j} N(n_2-j, n_3-r+j, d-r, d) \end{aligned}$$

for $0 \le r \le d$ and all j, where $N(b,t,w,d)$ is the maximal number of words of constant weight w and mutual distance d with b binary and t ternary coordinates. A bound for $N(b,t,w,d)$ can be computed from the starting values

$$N(b,t,w,d) \le \begin{cases} 1, & \text{if } d > 2w \\ \lfloor (b+t)/w \rfloor, & \text{if } d = 2w \end{cases}$$

and the induction

$$\begin{aligned} N(b,t,w,d) \le \lfloor (&bN(b-1,t,w-1,d) \\ &+ 2tN(b,t-1,w-1,d))/w \rfloor \end{aligned}$$

(if $w > 0$). (The inequalities given earlier for $d = 3$ are special cases of those obtained here.) Occasionally also

$$\begin{aligned} N(b,t,w,d) \le \lfloor (&bN(b-1,t,w,d) \\ &+ tN(b,t-1,w,d))/(b+t-w) \rfloor \end{aligned}$$

(if $w < b + t$) might be useful.

C. Further Inequalities

The inequalities discussed above described constraints on what happens close to a given codeword. We can also add constraints on the words that differ from a given word in (almost) all binary and/or ternary coordinates. First of all we have

$$\sum_{i+j \le e} A_{n_2-i,j} \le 1 \qquad (L2)$$

where $e = \lfloor (d-1)/2 \rfloor$.

For a property P, let $\delta(P) = 1$ if P holds, and $\delta(P) = 0$ otherwise. If $d = n_3 + 1 > 2$ and $n_2 > 1$, we have

$$\begin{aligned} &A_{n_2-1,n_3} \\ &\le \begin{cases} 1, & \text{if } A_{n_2,n_3} = 1 \\ \min(n_2, 1+\delta(d \le 3)), & \text{if } A_{n_2,n_3-1} = 1 \\ \min(n_2, 2+2\delta(d \le 4)), & \text{if } A_{n_2,n_3} = A_{n_2,n_3-1} = 0. \end{cases} \end{aligned}$$

This can be captured in one inequality:

$$(m_2-1)A_{n_2,n_3} + (m_2-m_1)A_{n_2,n_3-1} + A_{n_2-1,n_3} \le m_2 \qquad (L4)$$

where m_1, m_2 are the above minima.

If $d = n_3 + 1$ and $n_2 > n_3 > 1$, then we have

$$A_{n_2-1,n_3-1} \le \begin{cases} n_3, & \text{if } A_{n_2,n_3} = 1 \\ \min(n_2, 2n_3), & \text{if } A_{n_2,n_3} = 0. \end{cases}$$

This can be captured in one inequality:

$$(m-n_3)A_{n_2,n_3} + A_{n_2-1,n_3-1} \le m \qquad (L5)$$

where m is the above minimum.

Known bounds on $A_2(n,d)$ (the maximal size of a binary code of length n and minimum distance d) can be used:

$$A_{0,n_3} \le A_2(n_3,d) \quad \text{and} \quad A_{n_2,n_3} \le A_2(n_3,d). \qquad (L6)$$

More precise information about A_{0,n_3} and A_{0,n_3-1} can sometimes be obtained using Plotkin's argument (cf. [29, ch. 2], [28], and Proposition 4.2 below). Instead of presenting the somewhat messy general details, we give here only the extra inequality used to show $N(0,11,7) \le 50$, which is

$$A_{0,11} + \frac{4}{11}A_{0,10} \le 4. \qquad (L7)$$

Suppose there are r words of weight $(0, n_3)$ and s words of weight $(0, n_3-1)$, and write $t = r+s$. The sum of all distances between these t words is at least $\binom{t}{2}d$. On the other hand, each column (coordinate position) without a 0 contributes at most

$\lfloor \frac{t}{2} \rfloor \cdot \lceil \frac{t}{2} \rceil$, and each 0 adds at most $\lceil \frac{t}{2} \rceil - 1$ to this (namely, when it is the only 0 in the column). Thus we have

$$\binom{t}{2} d \leq n_3 \cdot \left\lfloor \frac{t}{2} \right\rfloor \cdot \left\lceil \frac{t}{2} \right\rceil + s\left(\left\lceil \frac{t}{2} \right\rceil - 1 \right).$$

For each given s, this yields an upper bound on t (when $2d - n_3 > 0$), and s itself is then bounded by $s \leq t$ (when $2d - n_3 > 2$). We now find an inequality $A_{0,n_3} + \alpha A_{0,n_3-1} \leq \beta$ which is satisfied by all pairs (r, s) found. This argument can be sharpened a little by noticing that if equality holds in this Plotkin bound, then every pair of codewords are at the same distance apart. This is impossible if $r \geq 3$ and d is odd.

Sometimes one can make use of the fact that $|C|A_{i,j}$ must be an integer. For example, pure linear programming gives $N(1,7,3) \leq 243$, with an optimal solution that mentions $A_{0,4} = 34.5$. However, if $|C| = 243$, then either $A_{0,4} \leq 8383/243 = 34.497\cdots$ or $A_{0,4} \geq 8384/243 = 34.502\cdots$. But in both cases adding the extra inequality to the program gives $|C| < 243$. It follows that $N(1,7,3) \leq 242$. See also Lemma 4.6.

IV. Further Bounds

It is easy to determine $N(n_2, n_3, d)$ for very small or very large d. (We shall always assume that n_2 and n_3 are nonnegative, and that d is positive, and all three are integral.)

Proposition 4.1:

i) $$N(b,t,1) = 2^b 3^t.$$

ii) $$N(b,t,2) = \begin{cases} 1, & \text{if } b = t = 0 \\ 2^{b-1}, & \text{if } b > 0, t = 0 \\ 2^b 3^{t-1}, & \text{if } t > 0. \end{cases}$$

iii) If $d > b + t$, then $N(b,t,d) = 1$.

iv) $$N(b,t,b+t) = \begin{cases} 2, & \text{if } b > 0 \\ 3, & \text{if } b = 0, t > 0. \end{cases}$$

v) $N(b,t,b+t-1)$
$$= \begin{cases} 2, & \text{if } b > 3 \\ 3, & \text{if } b \leq 3 \text{ and } b + t/2 > 3 \\ \text{see table,} & \text{otherwise.} \end{cases}$$

$b \backslash t$	0	1	2	3	4	5	6
0	–	–	9	9	9	6	4
1	–	6	6	6	4		
2	4	4	4				
3	4						

It is easy to give an explicit description of the codes achieving these bounds.

Below we shall see that for very small codes the Plotkin bound describes the situation completely. Let us state the Plotkin bound in our case.

Proposition 4.2 ("Plotkin bound"): If $N(b,t,d) \geq M$, then

$$d\binom{M}{2} \leq bM_2^0 M_2^1 + t\big(M_3^0 M_3^1 + M_3^0 M_3^2 + M_3^1 M_3^2\big)$$

where $M_q^i = \lfloor (M+i)/q \rfloor$. When equality holds, any $(b,t,d)_M$ code is equidistant.

We omit the proof, which is analogous to that for the binary case. (A slightly incorrect[3] version of this bound for pure ternary codes was given in [28].)

Given a code, there are various obvious ways of deriving other codes from it.

Proposition 4.3: For nonnegative b and t we have:

i) $N(b,t,d) \leq N(b+1,t,d)$.

ii) $N(b+1,t,d) \leq 2N(b,t,d)$.

iii) $N(b+1,t,d) \leq N(b,t+1,d)$.

iv) $N(b,t+1,d) \leq (3/2)N(b+1,t,d)$.

v) $N(b,t+1,d) \leq N(b+2,t,d)$.

vi) $N(b,t+1,d) \leq N(b,t,d-1)$
and $N(b+1,t,d) \leq N(b,t,d-1)$.

(The inequalities $N(b,t,d) \leq N(b,t+1,d)$ and $N(b,t+1,d) \leq 3N(b,t,d)$ follow from i), iii) and ii), iv), respectively.)

We know precisely where the very small values of $N(b,t,d)$ will occur.

Proposition 4.4:

i) $N(b,t,d) = 1$ precisely when $b + t < d$.

ii) $N(b,t,d) = 2$ precisely when $\frac{2}{3}b + t < d \leq b + t$.

iii) $N(b,t,d) = 3$ precisely when $(4b+5t)/6 < d \leq \frac{2}{3}b + t$.

iv) $N(b,t,d) = 4$ precisely when

$$(3b+4t)/5 < d \leq (4b+5t)/6$$

or

$$(b,t,d) = (2+10j, 0, 1+6j)$$

or

$$(b,t,d) = (2+5j, 1, 2+3j)$$

or

$$(b,t,d) = (9+10j, 2, 7+6j)$$

for some $j \geq 0$.

v) In all other cases, $N(b,t,d) \geq 6$.

Proof: The upper bounds follow from the Plotkin bound, the lower bounds from juxtaposition (see below). All the necessary ingredients for making these codes exist, except in the explicitly listed cases under iv), where we cannot find codes of size 5 or 6, even though the Plotkin bound would permit them. Why are these codes impossible? In the cases $(b,t,d) = (2+5j, 1, 2+3j)$ and $(b,t,d) = (9+10j, 2, 7+6j)$ a code of size 5 or 6 would have equality in the Plotkin bound, hence would be equidistant. Since $\lceil 5 \times 4/9 \rceil = 3$, we can make the ternary coordinate positions binary by selecting a subcode of size at least 3. But in a binary Hamming space, an equilateral triangle has an even side. This eliminates $(b,t,d) = (9+10j, 2, 7+6j)$ and $(b,t,d) = (7+10j, 1, 5+6j)$. The case $(b,t,d) = (2+10j, 1, 2+6j)$ does not occur since shortening would yield a $(b,t,d) = (2+10j, 0, 1+6j)$ code. In this latter code, at most two distances differ from $1+6j$, so we can again throw out two codewords and obtain an equilateral triangle of odd side. □

[3] For example, the bound in [28] gives $N(0,6,5) \leq 3$, whereas in fact $N(0,6,5) = 4$.

Sometimes it is possible to show that a code cannot be obtained by truncation (as in Proposition 4.3 vi)). For example, if a code of minimum distance $d-1$ is obtained by removal of a binary (ternary) coordinate position in a code of minimum distance d, then the distance-$(d-1)$ graph on its codewords does not contain a triangle (K_4, respectively). In the lemma below an integrality argument is used. First we need some preparation.

The following result may be well-known. The proof is almost identical to the proofs for the binary case in [3].

Proposition 4.5: Let $q \mid n$. Any q-ary 1-error-correcting code of length n has size at most $q^n/(n(q-1)+q)$, and the inner distribution of any code meeting this bound is uniquely determined. In particular, this holds for any code with the parameters of the singly shortened perfect q-ary Hamming code.

Proof: For the q-ary Hamming scheme of length n we have $P_{i0}=1$, $P_{i1}=n(q-1)-qi$, and

$$P_{i2}=(q-1)^2\binom{n}{2}-\frac{1}{2}qi[q(2n-1)-2n+2]+\frac{1}{2}i^2q^2.$$

Assume that $q \mid n$. Then $2P_{i2}+2(q-1)P_{i1}+(q-1)nP_{i0}=(qi-(q-1)n-q)(qi-(q-1)n)\geq 0$. Now let C be a q-ary code of length n, minimum distance 3, and size M, and with inner distribution a. Let $u^\top=(n(q-1), 2(q-1), 2, 0, \cdots, 0)$. Then, since $PQ=q^nI$ and $(aQ)_0=M$, we find

$$n(q-1)q^n = aQPu \geq Mn(q-1)(n(q-1)+q)$$

so that $M\leq q^n/(n(q-1)+q)$. If equality holds, then $(aQ)_j=0$ for $j\neq 0, (q-1)n/q-1, (q-1)n/q$, and hence, by [3, Corollary 3.1], the inner distribution a is uniquely determined. □

Lemma 4.6: $N(0,13,4)<3^9$.

Proof: We prove more generally that no ternary code with the parameters of a singly shortened Hamming code of word length n, where $n\equiv \pm 2 \bmod 5$, is the truncation of a distance-4 ternary code.

By the above proposition and [3, Theorem 4.2.5], it follows that the inner distribution of any $(0,13,4)_M$ code with $M=3^9$ is uniquely determined. Doing the computation, we find $A_5=n(n-1)(n-4)(n-5)/30$. But then MA_5 is not an integer, contradiction. □

Finally, the following special upper bound follows from an argument mostly due to Mario Szegedy (personal communication).

Lemma 4.7: $N(0,7,4)\leq 46$.

Proof: Let C be a $(0,7,4)_M$ code, and construct a bipartite graph with two sets of nodes, one set labeled $\{c \mid c\in C\}$ and the other labeled $\{u \mid u\in U=\boldsymbol{F}_3^5\}$. To each node $c\in C$ we associate the set N_c consisting of the 84 vectors in $V=\boldsymbol{F}_3^7$ at distance exactly 2 from c, and to each u we associate the set S_u consisting of the nine vectors $u\alpha\beta\in V$ with $\alpha,\beta\in\boldsymbol{F}_3$. We also define three kinds of edges: $c=v\alpha\beta\in C$ (with $v\in U$ and $\alpha,\beta\in\boldsymbol{F}_3$) is joined to u by a blue edge if $u=v$, by a red edge if $d_H(u,v)=1$, and by a white edge if $d_H(u,v)=2$. Then $|N_c\cap S_u|$ is 4, 4, or 1 in the three cases. It is easy to see that: i) each node c is incident with exactly one blue edge, 10 red edges, and 40 white edges; ii) the possibilities for blue and red edges meeting u are just the following: one blue edge and no red edges (there are exactly M such nodes u), no blue edges and 0, 1, 2, or 3 red edges (we denote the numbers of such nodes by t_0, t_1, t_2, and t_3, respectively). Let there be n_B, n_R, and n_W blue, red, and white edges, respectively.

We will evaluate the sum

$$4n_B+4n_R+n_W=\sum_{c\in C, u\in U}|N_c\cap S_u|$$

in two ways. On the one hand, each $c\in C$ contributes 84, so that the sum is equal to $84M$. On the other hand, let us group the terms in the sum according to the number of blue and red edges meeting u. A small clique-finding program shows that in the five cases mentioned, there are at most 8, 13, 12, 8, and 6 white edges at u. [We may take $u=\mathbf{0}$. Let W be the set of vectors $w\alpha\beta$ where w has weight 2. (These are the words that will get a white edge to u if they are in C.) If u meets a blue edge, we may assume that $\mathbf{0}\in C$. Otherwise, if u meets i red edges, we may assume that the first i of 1000000, 0100011, 0010022 are in C. Now find the largest subset of W with all mutual distances, and all distances to the known codewords at least 4.] This means that in the five cases mentioned, the sum

$$s_u=\sum_{c\in C}|N_c\cap S_u|$$

is at most $4+8$, 13, $4+12$, $4+4+8$, $4+4+4+6$. We can also compute the total contribution of all vectors u incident with a blue edge in a different way. These vectors u contribute $4M$ plus the number of pairs (white edge, blue edge) incident on the points of U. This latter number equals the number of ordered pairs of codewords $(u\alpha\beta, v\gamma\delta)$ with $d_H(u,v)=2$. And this equals the number of unordered pairs of red edges incident on the points of U, that is, t_2+3t_3.

Altogether we have found the following system of (in)equalities:

$$\begin{aligned} M+t_0+t_1+t_2+t_3&=243\\ t_1+2t_2+3t_3&=10M\\ 4M+t_2+3t_3&\leq 12M\\ 4M+13t_0+16t_1+17t_2+21t_3&\geq 84M. \end{aligned}$$

Combining these with coefficients -18, 2, -3, and 1 yields

$$-5t_0\geq 94M-4374$$

so that $M\leq 4374/94<47$. □

V. Constructions

A. Juxtaposition

Let $\pi_C=\{C_i \mid 1\leq i\leq m\}$ be a partition of a code C, and $\pi_D=\{D_j \mid 1\leq j\leq n\}$ a partition of a code D. Let $(C/\pi_C)\mid(D/\pi_D)$ denote the code consisting of all codewords $(c_i \mid d_i)$ with $c_i\in C_i$ and $d_i\in D_i$, for $i=1,\cdots,\min(m,n)$, where here $\mid$ denotes concatenation (juxtaposition). The size of this code is $\sum_{i=1}^{\min(m,n)} |C_i|\cdot|D_i|$ (which is equal to $|C|\cdot|D_j|$ if all D_j have the same size and $n \geq m$). Its minimum distance is at least $\min\{d_C + d_D, d_{C_i}, d_{D_i} \mid 1 \leq i \leq \min(m,n)\}$ (where, of course, d_Z denotes the minimum distance of a code Z). This construction is indicated by jb in the tables below. (It is essentially Construction X4 of [29, p. 584].)

The partition π_C of C will often be a partition into translates of a subcode E of C. In this case we write C/E instead of C/π_C.

Let $(n_2, n_3, d)_M$ denote a mixed code with n_2 binary and n_3 ternary coordinates, minimum distance d, and M codewords. Let $[n,k,d]_q$ denote a q-ary linear code of length n, dimension k, and minimum distance d, where we omit q if $q = 2$.

The following are examples of the juxtaposition construction:

i) $N(3,3,3) \geq 18$ because of

$$([3,3,1]/[3,1,3]) \mid ([3,2,2]_3/[3,1,3]_3).$$

ii) $N(4,2,4) \geq 6$ from

$$([4,3,2]/[4,1,4]) \mid ([2,1,2]_3/[2,0,\infty]_3).$$

iii) $N(8,4,4) \geq 384$ from

$$([8,7,2]/[8,4,4]) \mid ([4,3,2]_3/[4,1,4]_3).$$

iv) $N(9,4,4) \geq 540$ from

$$([4,3,2]_3/[4,1,4]_3) \mid (D/\pi_D))$$

where $\pi_D = \{D_0 + u \mid u \in U\}$ for an even weight $(9,0,4)_{20}$ code[4] D_0, where U is a set of nine even-weight vectors with all pairwise distances 2 such as $\{0\} \cup \{e_1 + e_j \mid 2 \leq j \leq 9\}$.

v) $N(8,6,4) \geq 2304$ (and hence $N(7,6,4) \geq 1152$). Indeed, we construct $([8,7,2]/[8,4,4]) \mid (D/\pi_D)$ where $\pi_D = \{D_0 + u \mid u \in U\}$ for some $(0,6,4)_{18}$ code D_0 contained in the zero-sum $[6,5,2]_3$ ternary code E, where U is a set of at least eight vectors in E with pairwise distance at most 3. In fact, it is easy to find a U of size 11: take $\{0\} \cup \{\pm(e_1 - e_j) \mid 2 \leq j \leq 6\}$. One way to construct D_0 is to take the 18 words of weight 6 in the hexacode given in [9, p. 82, eq. (64)], and rename the symbols (from $1, \omega, \bar{\omega}$ to $0, 1, 2$). Then D_0 contains the three multiples of $\mathbf{1}$, and 15 words with symbol distribution $0^2 1^2 2^2$, so that it is contained in E. For later use we remark that D_0 is invariant under translation by $\mathbf{1}$ (since the hexacode is invariant under multiplication by ω), so that we have a partition $(0,6,4)_{18}/(0,6,6)_3$.

For further examples, see also Section V-B below.

If π_C is the partition into singletons, we write C instead of C/π_C. This construction is indicated by jc in the tables. (It is [29, p. 581, Construction X].) Examples:

i) $[3,2,2] \mid ((0,6,4)_{18}/(0,6,6)_3) = (3,6,6)_{12}$.
ii) $[4,3,2] \mid ((0,6,4)_{18}/(0,6,6)_3) = (4,6,6)_{18}$.
iii) $(1,3,3)_6 \mid ([8,4,4]/[8,1,8]) = (9,3,7)_{12}$.
iv) $(0,4,3)_9 \mid ([8,4,4]/[8,1,8]) = (8,4,7)_{16}$.
v) $(0,4,3)_9 \mid ((9,0,4)_{18}/(9,0,8)_2) = (9,4,7)_{18}$. (There is a binary constant weight code with length 9, weight 4, minimum distance 4, and size 18 [8]; but any such code has distance distribution $0^1 4^{12} 6^4 8^1$, and so can be partitioned into $(9,0,8)_2$ codes.)
vi) $(0,2,2)_3 \mid ([12,4,6]/[12,2,8]) = (12,2,8)_{12}$.
vii) $(2,4,4)_8 \mid ([8,4,4]/[8,1,8]) = (10,4,8)_{16}$.
viii) $(0,5,4)_6 \mid ([8,4,4]/[8,1,8]) = (8,5,8)_{12}$.

If π_D is also the partition into singletons, we have ordinary juxtaposition (pasting two codes side by side, as in [29, p. 49]). This construction is indicated by j in the tables. Its main use is the construction of codes of size 4 or 6 and large minimum distance (cf. Proposition 4.4). We do not list all applications. Examples are:

$(3,0,2)_4 \mid (3,0,2)_4 \mid (2,2,3)_4 = (8,2,7)_4$.
$(1,3,3)_6 \mid (4,2,4)_6 = (5,5,7)_6$.
$(10,0,6)_6 \mid (1,2,2)_6 = (11,2,8)_6$.
$(0,5,4)_6 \mid (2,6,6)_6 = (2,11,10)_6$.

Sometimes it is possible to adjoin further words after performing the juxtaposition construction. The Steiner system $S(5,6,12)$ is a binary code of length 12, constant weight 6, minimum distance 4, and size 132. It has a partition into six $(12,0,6)_{22}$ Hadamard codes. So

$$(1,2,2)_6 \mid ((12,0,4)_{132}/(12,0,6)_{22}) = (13,2,6)_{132}.$$

Adding $\mathbf{0}$ and $\mathbf{1}$ shows $N(13,2,6) \geq 134$. If we shorten once or twice before adding $\mathbf{0}$ and $\mathbf{1}$, we find $N(12,2,6) \geq 68$ (hence $N(12,1,5) \geq 68$) and $N(11,2,6) \geq 38$. (The $(1,2,2)_6$ code must not contain 000 or 111.)

To see why this partition of the Steiner system exists, we remark that the extended ternary Golay code has 24 words of weight 12, and if we normalize so that $\mathbf{1}$ and $\mathbf{2}$ are in the code, then the other 22 words form a Hadamard code. Adding $\mathbf{1}$ we see that the places where these 22 words take a fixed value are the supports of codewords of weight 6, that is, belong to $S(5,6,12)$. This produces one Hadamard code inside $S(5,6,12)$. Its stabilizer in M_{12} is $2 \times M_{11}$, of index 6, so we find six pairwise disjoint copies.

B. Partitions of Zero-Sum Codes

As a special case of the juxtaposition construction, suppose C is an $(n_2, n_3, 2)_M$ code with a partition π_C into eight parts, each with minimum distance (at least) 3. Then $(C/\pi_C) \mid ([7,7,1]/[7,4,3])$ is an $(n_2+7, n_3, 3)_{16M}$ code. In this way we find $N(7,5,3) \geq 1296$, $N(9,4,3) \geq 1728$, $N(8,5,3) \geq 2544$, and $N(7,6,3) \geq 3792$ using $(0,5,2)_{81}$, $(2,4,2)_{108}$, $(1,5,2)_{159}$, and $(0,6,2)_{237}$ codes C with appropriate partitions.

Motivated by this construction, we investigate distance-2 codes and their partitions into distance-3 codes. As a consequence, we will show that $N(1,6,4) = 33$, $N(2,6,3) \geq 134$, $N(2,7,3) \geq 396$, and $N(4,6,3) \geq 486$.

Lemma 5.1: Let $t \geq 1$. There is (up to translation and sign change at some coordinate positions) a unique $(0,t,2)_M$ code C with $M = N(0,t,2) = 3^{t-1}$, namely, the code consisting of the words with zero coordinate sum. (In other words, any

[4] See [2], [29, p. 57], and [9, p. 140].

$(0,t,2)_M$ code C is of the form $C=\{u\in \boldsymbol{F}_3^t \mid u\cdot c=\gamma\}$ for some "parity-check" vector $c\in\{1,2\}^t$ and $\gamma\in\boldsymbol{F}_3$.)

Proof: Induction on t. For $t=1$ the statement is obvious. Assume $t>1$. By induction we may assume that the subcode of C consisting of the words ending in 0 is a zero-sum code. If $a1\in C$ and $d_H(a,b)=1$ (for some $a,b\in\boldsymbol{F}_3^{t-1}$), then precisely one of $b0$, $b1$, $b2$ occurs in C, but not $b1$, and $b0$ only if $\sum b=0$. Consequently, if $\sum a\neq 0$ and $\sum b\neq 0$ and $d_H(a,b)=1$ then $a1\in C$ if and only if $b2\in C$. But the distance-1 graph on the nonzero-sum vectors in $\boldsymbol{F}_3^{t-1}$ is connected, so a single choice determines all of C. By a sign change in the last coordinate, if necessary, we can force one zero-sum vector that ends in 1 or 2, and then C is the zero-sum code. □

For optimal $(b,t,2)$ codes with $b>0$ the classification is much more messy, and we shall not try to write down the details.

Parts of a partition of an optimal $(0,t,2)$ code into distance-3 codes are often smaller than arbitrary $(0,t,3)$ codes. Indeed, contrast $N(0,4,3)=9$, $N(0,5,3)=18$, $N(0,6,3)\geq 38$ with the following result.

Lemma 5.2:

i) Any zero-sum $(0,4,3)_M$ code has $M\leq 4$, and there is (up to coordinate permutation and translation by a zero-sum vector) a unique optimal code, namely $\{0000,\ 0111,\ 2220,\ 1122\}$.

ii) Any zero-sum $(0,5,3)_M$ code has $M\leq 11$, and there is a unique optimal code, namely $\{00000,\ (01221)_5,\ (02112)_5\}$, where $(u)_5$ denotes the five codewords obtained from u by cyclic coordinate permutations.

iii) Any zero-sum $(0,6,3)_M$ code has $M\leq 33$, and there is a unique optimal code, namely $\{aaaaaa, a(abccb)_5 \mid \{a,b,c\}=\{0,1,2\}\}$.

These codes can be found inside the ternary Golay code G: let a be a codeword of weight 5 with support A. Pick the 33 words of G that have weight at most 1 on A and discard the coordinate positions in A to get a zero-sum $(0,6,3)_{33}$ code. If $\bar{G}$ is the extended (self-dual) ternary Golay code, and $1a\in\bar{G}$, then taking the 33 words of $\bar{G}$ that have weight at most 1 on A and deleting the coordinate positions in A we get a $(1,6,4)_{33}$ code after arbitrarily changing the check position in the three codewords where it is 0. Consequently, $N(1,6,4)\geq 33$. (In fact, $N(1,6,4)=33$, since exhaustive search shows that $N(2,4,3)\leq 22$.)

Lemma 5.3: $N(2,7,3)\geq 396$.

Proof: Let $U=\{000000,\ 2(00001)_5\}$. Let C be a zero-sum $(0,6,3)_{33}$ code. Then the six translates $C+u$ for $u\in U$ are pairwise-disjoint. Thus we can construct

$$((2,1,1)_{12}/(2,1,3)_2)\mid((0,6,2)_{198}/(0,6,3)_{33}) = (2,7,3)_{396}. \quad \square$$

Lemma 5.4: $N(2,6,3)\geq 134$.

Proof: Let $U=\{00000,\ (11112)_5\}$. Let C be a zero-sum $(0,5,3)_{11}$ code. Then the six translates $C+u$ for $u\in U$ are pairwise-disjoint. We can add the (nonzero-sum) word 11111 to C and obtain a partition π of a $(0,5,2)_{67}$ code into six distance-3 codes. Now the required code is constructed as

$$((2,1,1)_{12}/(2,1,3)_2)\mid((0,5,2)_{67}/\pi)=(2,6,3)_{134}. \quad \square$$

Lemma 5.5: The $(0,5,2)_{81}$ zero-sum code has a partition into eight distance-3 codes.

Proof: Take $C_0=\{00000,\ (01221)_5,\ (02112)_5\}$ of size 11, and seven codes of size 10, namely

$$\begin{aligned} C_1&=\{(00012)_5,(02211)_5\}\\ C_2&=\{(00021)_5,\ (01122)_5\}\\ C_3&=\{00111,\ 00222,\ 11010,\\ &\quad 22020,\ 11121,\ 22212,\ 12102,\ 21201, 10200,\ 02010\}\\ C_{3+j}&=\sigma^j C_3\ (j=0,1,2,3,4)\end{aligned}$$

where σ is the cyclic coordinate permutation. □

As a consequence we find $N(7,5,3)\geq 1296$, as announced.

Lemma 5.6: $N(4,6,3)\geq 486$.

Proof: Let π be the partition constructed in the previous lemma. Then

$$((4,1,1)_{48}/(4,1,3)_6)\mid((0,5,2)_{81}/\pi)=(4,6,3)_{486}.$$

Note that the former ingredient exists: we can construct $(4,1,3)_6$ as $([4,3,2]/[4,1,4])\mid(0,1,1)_3$, and then use translates by the eight coset leaders of $[4,1,4]$. □

We have not found a partition of the $(0,6,2)_{243}$ zero-sum code into eight distance-3 codes, but can come close. First notice that a partition of a zero-sum $(0,6,2)_{3M}$ code into distance-3 codes that are invariant under translation by $\mathbf{1}$ is equivalent to a partition of a zero-sum $(0,5,2)_M$ code into codes in which the distances 2 and 5 do not occur.

Thus we find a partition of a zero-sum $(0,6,2)_{237}$ code from

$$\begin{aligned}\{&\{00111,\ 00222,\ 01002,\ 02010,\ 10200,\ 11211,\\ &12021,\ 12102,\ 20001,\ 21120,\ 22212\},\ \{00102,\\ &00210,\ 01020,\ 02121,\ 10011,\ 11112,\ 12000,\\ &12222,\ 20022,\ 21201,\ 22110\},\ \{00120,\ 01200,\\ &02001,\ 02112,\ 10101,\ 11022,\ 12210,\ 20010,\\ &20202,\ 21111,\ 22221\},\ \{00000,\ 01122,\ 02211,\\ &10221,\ 12012,\ 12120,\ 20112,\ 21021,\ 21210,\\ &22101\},\ \{00021,\ 01110,\ 02202,\ 10122,\ 11001,\\ &11220,\ 12111,\ 20100,\ 21012,\ 22020\},\ \{01212,\\ &02022,\ 02100,\ 10002,\ 11010,\ 11121,\ 12201,\\ &20220,\ 21102,\ 22011\},\ \{00201,\ 01011,\ 10020,\\ &10212,\ 11100,\ 20121,\ 21222,\ 22002\},\ \{00012,\\ &01101,\ 02220,\ 10110,\ 11202,\ 20211,\ 21000,\ 22122\}\}.\end{aligned}$$

As a consequence we find $N(7,6,3)\geq 3792$.

Shortening this yields a good partition of a $(1,5,2)_{158}$ code. An explicit code does slightly better. The code below is a good partition of a $(1,5,2)_{159}$ code, and if we delete the words that have a 2 at the second position, we obtain a good partition of a $(2,4,2)_{108}$ code. (This last one is optimal: $N(2,4,2)=108$.) As a consequence we find $N(8,5,3)\geq$

2544 and $N(9,4,3) \geq 1728$.

$\{\{000122,\ 001100,\ 001211,\ 002021,\ 010010,\ 011222,\ 012101,\ 020201,\ 021002,\ 022112,\ 022220,\ 100001,\ 100220,\ 101012,\ 102110,\ 110102,\ 111200,\ 112022,\ 112211,\ 120212,\ 121121,\ 122000\},\ \{000002,\ 000110,\ 001121,\ 002201,\ 010220,\ 011000,\ 012011,\ 012122,\ 020021,\ 021212,\ 022100,\ 100211,\ 101102,\ 102020,\ 111110,\ 111221,\ 112202,\ 120122,\ 120200,\ 121001,\ 122111\},\ \{000101,\ 001112,\ 002000,\ 002222,\ 010202,\ 011021,\ 012110,\ 020012,\ 020120,\ 021200,\ 022211,\ 100022,\ 101120,\ 101201,\ 102011,\ 110000,\ 110111,\ 111212,\ 112220,\ 120221,\ 121010\},\ \{000020,\ 000212,\ 001001,\ 010121,\ 011012,\ 012200,\ 020102,\ 021110,\ 021221,\ 022022,\ 100100,\ 101210,\ 102002,\ 102221,\ 110222,\ 111020,\ 111101,\ 112112,\ 120011,\ 121202,\ 122120\},\ \{000011,\ 001220,\ 002102,\ 010022,\ 010100,\ 011111,\ 012212,\ 020210,\ 021122,\ 022001,\ 100112,\ 101021,\ 102200,\ 110201,\ 111002,\ 112010,\ 112121,\ 120020,\ 121100,\ 122222\},\ \{000221,\ 001022,\ 002210,\ 010112,\ 011120,\ 011201,\ 012002,\ 021011,\ 022121,\ 100202,\ 101000,\ 101111,\ 102122,\ 110021,\ 110210,\ 112100,\ 120101,\ 121220,\ 122012\},\ \{000200,\ 002012,\ 002120,\ 010001,\ 011102,\ 011210,\ 012221,\ 020111,\ 021020,\ 022202,\ 100010,\ 101222,\ 102101,\ 110120,\ 111011,\ 120002,\ 121112,\ 122021,\ 122210\},\ \{001010,\ 001202,\ 002111,\ 010211,\ 012020,\ 020000,\ 020222,\ 021101,\ 100121,\ 102212,\ 110012,\ 111122,\ 112001,\ 120110,\ 121211,\ 122102\}\}.$

C. The $(u, u+v)$ Construction

Given two codes U and V, consider the code C whose codewords are $(u, u+v)$, for $u \in U$, $v \in V$. (Here $+$ must act coordinatewise, and satisfy $a+0=0+a=a$, and $a+b \neq a+c$ when $b \neq c$. In particular, there is no problem adding binary and ternary coordinates—just view them all as ternary coordinates.) By calculating the parameters of the resulting code (cf. [29, p. 76]) we obtain

Proposition 5.7: $N(b,t,d) \geq N(b_1,t_1,d_1)N(b_2,t_2,d_2)$ for

$$d = \min(2d_1, d_2)$$
$$b+t = b_1+t_1+\max(b_1+t_1, b_2+t_2)$$

and

$$t = t_1 + \max(t_1, t_2).$$

For example,

$$N(1,12,4) \geq N(0,6,2)N(1,6,4) \geq 3^5 \cdot 33 = 8019.$$

There are similar constructions that combine more than two codes. For example, given three mixed binary/ternary codes U, V, and W, consider the code C whose codewords are $(u, u-v, (u+v)+w)$, where the coordinate positions that U and V have in common are interpreted as ternary coordinates. We obtain

Proposition 5.8:

$$N(b,t,d) \geq N(b_1,t_1,d_1)N(b_2,t_2,d_2)N(b_3,t_3,d_3)$$

for

$$d = \min(3d_1, 2d_2, d_3)$$
$$b+t = b_1+t_1+m+\max(m, b_3+t_3)$$

and

$$t = t_1 + \max(t_1,t_2) + \max(t_1,t_2,t_3)$$

where $m = \max(b_1+t_1, b_2+t_2, b_1+b_2+\min(t_1,t_2))$.

If d is not even, or not a multiple of 3, then small modifications are slightly more efficient. For example, we find

$$N(0,16,3) \geq N(0,5,1)N(0,6,2)N(0,5,3) = 3^5 \cdot 3^5 \cdot 18 = 1062882$$

by taking $(u, u-v, u+v+w, v_0)$ for $u \in U$, $vv_0 \in V$, $w \in W$. (Giving the two copies of V a common coordinate position results in $2d_2 - 1$ instead of $2d_2$ in the expression for the minimum distance.)

D. Constructions from the Binary Hamming Code

The following result is useful for constructing codes with minimum distance 3 or 4.

Proposition 5.9: Let C be an $(n_2, n_3, 3)_M$ code and assume that u is a vector of weight 3 on the binary coordinates and weight 0 on the ternary coordinates such that $C = C + u$. Then we can construct an $(n_2-3, n_3+1, 3)_N$ code D with $N \geq \frac{3}{8}M$ by taking three of the four patterns 000, 001, 010, 100 on the support of u, and replacing them by ternary symbols 0, 1, 2.

Since the binary $[15,11,3]$ Hamming code is perfect (and linear), it is a good starting point for constructions.

The picture below shows the distribution of codewords in the $[15,11,3]$ binary Hamming code H with respect to five codewords u_j $(1 \leq j \leq 5)$ of weight 3 with pairwise-disjoint supports. The rows are numbered from 0 to 5. Each left son in row j gives the number of codewords counted by its father that have the pattern 000 on the support of u_j, the right son gives the number of codewords with one of the patterns 001, 010, or 100. Thus the entry for the father is twice the sum of the entries for the sons. For rows 1, 2, 3 the entries are determined because the positions are independent, and all patterns occur equally often. For rows 4 and 5 the zero entries are caused by the fact that H has minimum distance 3, and they determine the remaining entries.

2048
256 768
32 96 288
4 12 36 108
2 0 6 12 42
1 0 0 3 3 18

We now start with H and repeatedly apply Proposition 5.9, obtaining codes showing that $N(12,1,3) \geq 768$, $N(9,2,3) \geq$

288, $N(6,3,3) \geq 108$, $N(3,4,3) \geq 42$, and $N(0,5,3) \geq 18$. That $N(3,4,3) \geq 42$ was shown earlier by Karl-Göran Välivaara.

A similar construction can be used starting from the $[16,11,4]$ extended binary Hamming code $\bar{H}$. Let u be a codeword of weight 4, and replace the four binary coordinate positions in its support by two ternary positions, replacing $1000, 0100, 0010, 1001, 0101, 0011$ by $00, 11, 22, 01, 12, 20$, respectively. We see that $N(12,2,4) \geq \frac{6}{16} \times 2048 = 768$ so that $N(11,2,4) \geq 384$. (In fact, equality holds in both cases since $N(12,0,3) = 256$.)

Proposition 5.10: $N(8,4,3) = 1152$.

Proof: The upper bound follows from Proposition 3.1. The lower bound can be established using the juxtaposition construction $(C/\pi_C) \mid ([4,4,1]_3/[4,2,3]_3)$, where π_C is a partition of the zero-sum $(8,0,2)_{128}$ code into eight distance-3 codes. (Such a partition is equivalent to a perfect one-error-correcting code with one octal coordinate, eight binary coordinates, and 128 codewords.) The partition π_C can be obtained using the construction of Section V-B in reverse. Let S be the set of coordinate positions of H. Fix a codeword a of weight 8 in $H^{\perp}$, and let A be its support. We find the $(8,0,2)_{128}$ code C by discarding the coordinate positions in $S \backslash A$. The discarded tails of a codeword $c \in C$ form a coset of the $[7,4,3]$ code D formed by the codewords of H with support in $S \setminus A$, and we can define the partition π_C by letting the eight parts correspond to the eight cosets of D. □

E. Constructions from the Ternary Golay Code

Most of the lower bounds for $d = 5$ and $d = 6$ are derived from the $[12,6,6]_3$ extended ternary Golay code. We saw in Section V-B how to obtain $N(1,6,4) \geq 33$ using this code.

Lemma 5.11: $N(6,6,6) \geq 66$, $N(8,4,6) \geq 32$, and $N(9,3,6) \geq 26$.

Proof: Take the $[12,6,6]_3$ extended ternary Golay code $\bar{G}$ and let A be the support of some codeword of weight 6. There are 3, 0, 6, 6, 18, 30, and 66 codewords whose support meets A in a given subset of size $0, 1, 2, 3, 4, 5$, and 6 (respectively). Taking the 66 codewords that do not vanish on A, we see that $N(6,6,6) \geq 66$. If we pick two or three more positions outside A, and require that the codewords do not vanish there either, we find $N(8,4,6) \geq 32$ and $N(9,3,6) \geq 26$. □

Lemma 5.12: $N(6,7,5) \geq 342$.

Proof: Take the $[11,6,5]_3$ ternary Golay code G. In the last two coordinates replace each ternary digit by two binary digits (e.g., replace $0, 1, 2$ by $00, 01, 10$). Discard all words that have a 0 in either of the first two coordinates. We now have a $(6,7,5)_{324}$ code G'. There are precisely 36 words at distance 5 from G' (all ending in $\cdots 1111$), forming four cosets of the subcode Z of G consisting of the codewords that are 0 on the first two and the last two coordinates. The four cosets are permuted transitively by the four-group generated by multiplication by -1 and the element σ of M_{11} that interchanges the first two coordinates and fixes the last two, so that there are three different ways of taking the union of two cosets. These unions have minimum distances $1, 4, 5$ (corresponding to $-\sigma$, -1, σ, respectively). Taking this latter union together with G' yields the required code of size $342 = 324 + 18$. □

Lemma 5.13: $N(7,6,5) \geq 234$.

Proof: Take the $[11,6,5]_3$ ternary Golay code G, chosen in such a way that $c = 00011111100$ is a codeword. In the last two coordinates replace each ternary digit by two binary digits (e.g., replace $0, 1, 2$ by $00, 01, 10$). Discard all words that have a 0 in one of the first three coordinates. We now have a $(7,6,5)_{216}$ code G''. There are precisely 54 words at distance 5 from G'' (all ending in $\cdots 1111$). Inspection using a small clique-finding program shows that there is a set of 18 (invariant under negation and under translation by c) among these 54 that have mutual distances at least 5. Adding these to G'' yields the required code. □

F. Some Cyclic Codes

Let C_1 be the smallest length 13 ternary code which is invariant under translation by $\mathbf{1}$ and under cyclic coordinate permutations, and which contains the vector 0121122221121. Let $C_2 = -C_1$. Both are equidistant $(0,13,8)_{39}$ codes, and $C_1 \cup C_2$ is a $(0,13,7)_{78}$ code. If we now replace the first coordinate of all words in C_1 by 0 and in C_2 by 1, we get a code C with minimum distance 7, so $N(1,12,7) \geq 78$.

Let C_3 be the smallest code which is invariant under negation and under translation by $\mathbf{1}$, and contains

$$\{\mathbf{0},\ 001212010122,\ 001221122100, 010120201212,\ 012120012120\}.$$

Then C_3 is a $(0,12,7)_{27}$ code.

We construct a code proving that $N(0,13,7) \geq 105$ by taking C and adding the words from C_3, each prefixed by a 2.

The code C_1 already improves the old bound $N(0,13,8) \geq 36$, but we can do even better. Indeed, we find $N(0,13,8) \geq 42$ by taking the smallest cyclic code containing

$$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, 0000122121221,\ 0021102220112,\ 0120220210121\}.$$

Similarly, we get $N(0,12,7) \geq 51$ from the smallest cyclic code containing

$$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, 000011202121,\ 001222102211,\ 002020121122,\ 002201110101\}.$$

Finally, one obtains $N(0,14,9) \geq 31$ from the smallest cyclic code containing

$$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, 00001122102121,\ 00222011102012\}.$$

G. Constructions Using a Union of Cosets

We give the codes in humanly readable format, and the coset leaders in compressed format: the binary part in hexadecimal, the ternary part in base 9, both right-justified. If the linear code is the direct sum of a binary part of dimension b and a ternary part of dimension t, then we arrange the code generators so that there is a binary identity matrix of order b in front, and a ternary identity matrix of order t at the back, and the coset leaders are zero on these $b + t$ positions, so that we need give only the remaining $n_2 - b + n_3 - t$ coordinates.

(i) $N(11,2,3) \geq 832$. Use 52 cosets of a 4-dimensional binary code.

matrix	cosets
1000110000000	190, 240, 3F0, 420, 001, 271, 2A1, 311, 5D1, 6C1, 0B2, 172, 1C2, 2D2, 502,
0100101100000	612, 662, 7A2, 033, 143, 293, 2E3, 583, 653, 064, 0D4, 1A4, 3C4, 414, 6B4,
0010101010000	704, 774, 205, 365, 3B5, 475, 4A5, 086, 326, 356, 4F6, 7E6, 1F7, 447, 537,
0001101001000	627, 797, 058, 0E8, 238, 388, 498.

(ii) $N(4,5,3) \geq 186$. This improves the bound $N(4,5,3) \geq 178$ found by Seppo Rankinen. Use 62 cosets of a 1-dimensional ternary code.

matrix	cosets
	900, 201, E02, A03, 104, D05, 406, F07, 010, B11, 712, 313, 814, 415, D16,
	F20, 523, 826, 627, 328, 330, 831, 432, 033, B34, 735, E36, 537, 938, C40,
000011001	F43, 146, A47, 648, E51, 252, 653, D54, 155, B56, 057, C58, 560, C63, 367,
	A68, A70, 171, D72, 973, 274, E75, 776, C77, 078, 781, 882, 484, B85, 286,
	987, 588.

(iii) $N(5,5,3) \geq 342$. Use 57 cosets of the direct sum of a 1-dimensional ternary code and a 1-dimensional binary code.

matrix	cosets
	700, A02, B04, 605, 006, D08, 410, 311, F12, E13, 814, 515, 916, 218, D20,
0000011001	022, 124, C25, 626, B28, 830, F31, 332, 233, 434, 536, 937, E38, B40, 041,
1111000000	C42, 143, D44, A47, 651, 955, F56, 458, 160, C61, D63, 065, A66, 768, 672,
	773, A75, C76, 178, E80, 981, 582, 483, 284, F85, 386, 888.

(iv) $N(3,7,3) \geq 684$. Use 76 cosets of a 2-dimensional ternary code.

matrix	cosets
	1000, 4005, 5012, 7014, 3016, 0021, 2023, 6028, 6032, 2034, 5036, 4041, 1043, 2048, 3050,
	1057, 2060, 7065, 3072, 0074, 6076, 7081, 4083, 0088, 6101, 0103, 2108, 6115, 5117, 4122,
0001100010	3124, 7126, 3132, 1134, 6136, 2141, 4143, 1148, 5150, 0155, 7155, 4157, 0162, 4164, 3166,
0002111101	1171, 7173, 4178, 6180, 2187, 3201, 6203, 1208, 0215, 6217, 2222, 5224, 4226, 5231, 3233,
	4238, 0240, 7240, 5245, 3247, 1252, 6254, 2256, 4260, 2265, 0267, 7267, 6272, 5276, 1283,
	3288.

(v) $N(4,7,3) \geq 1332$. Use 74 cosets of the direct sum of a 2-dimensional ternary code and a 1-dimensional binary code.

matrix	cosets
	6001, 3003, 0008, 1010, 2015, 3017, 4022, 5024, 6026, 3031, 6038, 7040, 4045, 1052, 2053,
00001100010	7057, 4060, 7065, 5067, 5072, 6074, 0076, 0081, 1083, 3088, 2100, 1105, 0114, 5116, 7122,
00002111101	4123, 2127, 5132, 7134, 4136, 4141, 6143, 7148, 0150, 3155, 1157, 1160, 2165, 0167, 0172,
11110000000	5174, 3176, 6181, 7183, 5188, 5201, 6203, 3208, 4210, 7215, 6217, 2222, 3224, 0226, 0232,
	2234, 1236, 1241, 3243, 2248, 6250, 5255, 4257, 4264, 7266, 2271, 1278, 5280, 6285.

(vi) $N(7,2,4) \geq 26$. Use 13 cosets of a 1-dimensional binary code.

matrix	cosets
111100000	0F0, 120, 240, 291, 3E1, 3B3, 0C5, 175, 225, 0A7, 147, 277, 3D8.

(vii) $N(10,3,4) \geq 400$. Use 25 cosets of a 4-dimensional binary code.

matrix	cosets
1000111000000	
0100110100000	0C00, 1A01, 3101, 2B04, 1505, 2606, 3D06, 0308, 0210, 2412, 3F12, 3013, 0E14, 1914, 1717,
0010101100000	0D18, 2A18, 2721, 1622, 2922, 0123, 3E23, 3325, 1B26, 1C27.
0001011100000	

(viii) $N(5,5,4) \geq 108$. Use 54 cosets of a 1-dimensional binary code.

matrix	cosets
1111000000	4003, 1008, E015, D016, 9021, 2022, 5031, 3033, A040, 7048, C054, 0056, A068, 1070, 6071, 5085, F087, E101, 4112, 2113, 9115, F123, 5127, 9136, 2137, 5143, 6150, 1152, A155, 7162, B164, C165, D171, E176, 0184, 3186, B202, D204, 6208, 7210, 0217, A226, 8231, E233, B247, C248, D250, 7254, 2260, 5266, 8273, 3275, E282, 9288.

(ix) $N(6,5,4) \geq 208$. Use 52 cosets of a 2-dimensional binary code.

matrix	cosets
10111000000 01110100000	D005, 1016, A017, 6022, B030, C031, 3044, 4045, E056, 9057, F067, 8068, 7070, 0071, 2083, 5084, 2100, 5101, B113, C114, 8126, F128, 7136, 0137, F141, 8142, D153, 6154, E163, 9164, 3178, 4176, 1180, A181, 3207, 4208, E210, 9211, 7223, 0224, 1233, A234, 2246, 5247, 4250, 3252, D260, 6261, 8273, F275, B286, C287.

(x) $N(2,6,4) \geq 51$. Use 17 cosets of a 1-dimensional ternary code.

matrix	cosets
00111111	1017, 3022, 2044, 1053, 0061, 3066, 0106, 3148, 2150, 1170, 3184, 3213, 2227, 3231, 0235, 2272, 1288.

(xi) $N(3,6,4) \geq 87$. Use 29 cosets of a 1-dimensional ternary code.

matrix	cosets
000111111	7002, 4011, 1026, 5033, 3040, 0045, 2057, 0060, 7074, 4088, 0108, 5115, 3117, 2120, 3135, 4146, 7151, 4164, 7166, 1182, 5207, 4223, 4232, 7248, 1254, 3261, 5270, 0277, 2285.

(xii) $N(4,7,4) \geq 360$. Use 40 cosets of a 2-dimensional ternary code.

matrix	cosets
00001110010 00001001101	3004, 6006, 8007, D014, 4035, D036, 6074, 8075, F078, 4080, B080, 1115, E117, 7120, 9121, 0126, 2133, B137, A142, C143, 5158, 9163, 7165, C168, 5171, 3176, 2181, F201, 5203, 2218, A224, 0244, 7247, 9248, 8250, 6252, F253, 0262, A266, D282.

(xiii) $N(5,7,4) \geq 612$. Use 34 cosets of the direct sum of a 2-dimensional ternary code and a 1-dimensional binary code.

matrix	cosets
000001110010 000002101101 111100000000	9001, 4013, D026, 0030, B036, 8054, F055, 4068, B074, 7076, E081, 6105, 9113, 4121, 3131, 6140, A145, 9158, 5160, 2166, 7184, 0185, F200, 8206, C212, 1217, 2224, 5225, E238, 3243, A250, D264, E273, 3288.

(xiv) $N(4,8,4) \geq 891$. Use 33 cosets of a 3-dimensional ternary code.

matrix	cosets
000011100100 000021010010 000020101001	A002, 1011, 4020, 4038, A047, 1056, 7065, D074, A083, F100, 6107, 9107, C116, 3128, 3134, 6143, 9143, F148, C155, C161, 3170, 6182, 9182, 0184, 5203, 2215, E224, 8230, 5242, B251, 2266, 8278, 5287.

(xv) $N(2,11,4) \geq 5589$ and $N(5,8,4) \geq 1674$. Use 23 cosets of a 5-dimensional ternary code to find $N(2,11,4) \geq 5589$. Discarding all codewords u with $u_8 = 0$ or $u_{12} = 0$ or $u_{13} = 0$, we find $N(5,8,4) \geq 1674$.

matrix	cosets
0011100010000 0011010001000 0011001000100 0011000100010 0020111100001	1085, 2202, 1276, 3323, 3418, 2437, 3460, 1514, 2550, 0561, 2614, 0628, 3635, 1650, 3687, 1702, 3741, 0754, 2776, 0803, 1837, 0842, 2885.

(xvi) $N(6,4,5) \geq 24$. Use 12 cosets of a 1-dimensional binary code.

matrix	cosets
1111110000	1E08, 1515, 0921, 0726, 0A30, 0437, 0F44, 1352, 1963, 1C71, 1276, 0085.

(xvii) $N(8,4,5) \geq 82$. Use 41 cosets of a 1-dimensional binary code.

matrix	cosets
111111110000	0000, 2514, 6215, 4917, 3B18, 3824, 1625, 6E27, 7528, 7634, 4535, 3D37, 6838, 2A41, 3442, 7943, 1344, 6746, 5E48, 6151, 1952, 4A53, 2F55, 3256, 0457, 6B64, 3165, 1A67, 2668, 4671, 6D72, 3E73, 0875, 1576, 7077, 3781, 7A82, 6483, 5D84, 2986, 4388.

H. Explicitly Presented Codes

In this section we give a number of codes for which we have no better description than to simply list all the words. These were found by a variety of techniques: by hand, by exhaustive search, by clique-finding using a number of different programs, or by heuristic search procedures like those described in Section VI.

The first few codes are given explicitly.

$N(3,4,5)=6$	$N(6,3,6)=6$	$N(8,2,6)=7$	$N(6,6,8)=7$	$N(7,2,5)=9$	$N(4,4,5)=9$
001 0000	100 100 100	0000000000	000000000000	000000000	00000000
010 2101	010 010 010	0000111111	000011111111	000011111	00011111
100 2220	001 001 001	0011001122	000101222222	001100112	00101222
110 0012	111 111 111	0101110022	011110001122	010101021	01012022
101 1111	000 111 222	1011010011	101110112200	011010120	01102101
011 1222	111 000 222	1101011100	110110220011	100110022	10112210
		1110100101	111001002211	101111100	11001120
				110001102	11010201
				111000011	11100012

$N(6,5,7)=9$	$N(2,8,7)=9$	$N(9,4,8)=9$	$N(4,3,4)=11$	$N(6,4,6)=12$
00000000000	0000000000	0000000000000	0000000	0000000000
00001111111	0001111111	0000011111111	0001111	0000111111
00110011222	0010122222	0001100112222	0010122	0001012222
01010122012	0112200112	0110101001122	0101022	0110001122
01101022120	0121212020	0111010002211	0110011	0110112200
10100122201	1012211200	1010111012200	0111200	0111010011
11001100222	1102002221	1011011100022	1001202	1010100212
11110000111	1111120001	1101101010011	1010210	1011002101
11111111000	1120021110	1101110101100	1011021	1011111020
			1100101	1100102021
			1111112	1101001210
				1101110102

$N(5,6,7)=12$	$N(1,9,7)=12$	$N(4,8,8)=12$	$N(3,9,8)=13$	$N(5,4,5)=14$
00000000000	0000000000	000000000000	000000000000	000000000
00001111111	0001111112	000011111111	000011111111	000011111
00110011222	0120012222	001100112222	000101222222	000101222
00111222001	0122200111	001122220011	001222001122	001012022
01010122120	0211212001	010111222200	011012022201	001102101
01011200212	0212021120	010122001122	011121110002	001110210
10100202112	1011020211	101012021202	011200211210	010110021
10101120220	1012102022	101021202120	100222112200	011000112
11000221201	1100122101	110002122021	101110202011	100110102
11001012022	1101201220	110020210212	110020221102	101000221
11110110011	1220221012	111101010101	110102011021	110001120
11111001100	1222110200	111110101010	110211000212	110012201
			111001102120	110102012
				111111000

$N(0,10,7) \geq 14$
0000000000
0001111111
0010122222
0112200112
0121212020
0222001221
1012211200
1121120201
1202110022
1220102110
2102022011
2120011102
2201202202
2211020120

$N(2,6,5) = 15$
00000000
00011111
00101222
00222012
01012202
01102110
01121001
01220220
10020122
10122200
10200211
10211020
11002021
11110012
11201102

$N(2,10,8) = 18$
000000000000
000011111111
001100112222
001122220011
002211222200
010202021212
010220202121
012112010120
012121101002
101212100102
101221011020
102002201221
102020022112
110011220022
110022112200
111100221100
111111002211
112200110011

$N(5,3,4) = 20$
00000000
00001111
00010122
00100212
00111001
01000221
01011010
01101100
01110111
01111222
10001222
10010011
10011100
10100101
10101010
10110220
11000110
11001001
11010202
11100022

$N(5,2,3) = 22$
0000000
0000111
0001022
0010012
0010120
0011001
0100021
0101100
0110102
0111010
0111121
1000102
1001010
1001121
1010021
1011100
1100012
1100120
1101001
1110000
1110111
1111022

The remaining codes are given in the compressed format where the binary portion is in hexadecimal and the ternary portion in base 9, both right-justified.

(i) $N(8,1,3) \geq 50$. This improves the bound $N(8,1,3) \geq 49$ found by Mario Szegedy.

$$000,\ 0B1,\ 0C1,\ 162,\ 192,\ 1F0,\ 232,\ 260,\ 2D0,\ 310,\ 371,\ 3A0,\ 3C2,\ 442,\ 471,\ 490,\ 4E0,\ 501,\ 530,\ 5A2,\ 5D1,\ 621,\ 682,\ 6F2,\ 752,\ 7E1,\ 870,\ 8A2,\ 8D2,\ 932,\ 951,\ 980,\ A02,\ A91,\ AE1,\ B21,\ BF2,\ C12,\ C20,\ C81,\ D61,\ DB1,\ DC2,\ E51,\ E62,\ EB0,\ EC0,\ F00,\ F70,\ F92.$$

(ii) $N(6,2,3) \geq 38$.

$$070,\ 1D0,\ 220,\ 2C0,\ 041,\ 0A1,\ 291,\ 351,\ 012,\ 1E2,\ 2F2,\ 332,\ 382,\ 003,\ 0E3,\ 313,\ 3F3,\ 0D4,\ 274,\ 324,\ 3C4,\ 175,\ 195,\ 245,\ 2A5,\ 096,\ 136,\ 146,\ 256,\ 3A6,\ 187,\ 1F7,\ 207,\ 2E7,\ 028,\ 0C8,\ 368,\ 3D8.$$

(iii) $N(2,5,3) \geq 52$.

$$0006,\ 0035,\ 0050,\ 0072,\ 0115,\ 0121,\ 0137,\ 0160,\ 0202,\ 0223,\ 0241,\ 0258,\ 0264,\ 0276,\ 1001,\ 1028,\ 1046,\ 1084,\ 1132,\ 1144,\ 1186,\ 1210,\ 1233,\ 1282,\ 2010,\ 2025,\ 2031,\ 2048,\ 2063,\ 2087,\ 2104,\ 2126,\ 2143,\ 2152,\ 2168,\ 2217,\ 2236,\ 2254,\ 2275,\ 2280,\ 3014,\ 3053,\ 3062,\ 3100,\ 3118,\ 3157,\ 3171,\ 3185,\ 3205,\ 3221,\ 3242,\ 3267.$$

(iv) $N(3,5,3) \geq 98$. A code proving $N(3,5,3) \geq 98$ has been published in Norway (see [41]) as a football pool system.

$$0000,\ 7000,\ 0014,\ 5015,\ 6017,\ 3018,\ 3024,\ 6025,\ 5027,\ 0028,\ 4034,\ 2035,\ 7037,\ 1038,\ 2041,\ 4042,\ 6043,\ 5046,\ 1051,\ 7052,\ 0053,\ 3056,\ 1064,\ 7065,\ 2067,\ 4068,\ 7071,\ 1072,\ 3073,\ 0076,\ 4081,\ 2082,\ 5083,\ 6086,\ 5104,\ 3105,\ 0107,\ 6108,\ 1111,\ 7112,\ 4113,\ 2116,\ 2121,\ 4122,\ 1123,\ 7126,\ 3131,\ 5132,\ 7133,\ 4136,\ 0140,\ 1145,\ 6150,\ 2158,\ 6161,\ 0162,\ 2163,\ 1166,\ 5170,\ 6175,\ 3177,\ 4177,\ 3180,\ 0184,\ 7184,\ 5188,\ 6204,\ 0205,\ 3207,\ 5208,\ 4211,\ 2212,\ 7213,\ 1216,\ 7221,\ 1222,\ 2223,\ 4226,\ 0231,\ 6232,\ 1233,\ 2236,\ 3240,\ 5244,\ 0248,\ 7248,\ 5250,\ 3255,\ 4255,\ 6257,\ 5261,\ 3262,\ 4263,\ 7266,\ 6270,\ 2274,\ 0280,\ 1287.$$

(v) $N(0,6,3) \geq 38$. This improves an earlier bound $N(0,6,3) \geq 37$ found by Lohinen [27], and by Vaessens *et al*. [42].

848, 774, 570, 358, 404, 000, 526, 072,
757, 716, 321, 811, 608, 468, 265, 453,
732, 145, 640, 213, 250, 584, 502, 181,
377, 136, 128, 866, 315, 661, 834, 882,
623, 441, 086, 054, 363, 207.

(vi) $N(1,6,3) \geq 71$. This improves an earlier bound $N(1,6,3) \geq 69$ by Seppo Rankinen (personal communication).

0020, 0042, 0077, 0125, 0130, 0144,
0186, 0204, 0218, 0251, 0262, 0273,
0328, 0336, 0354, 0365, 0370, 0413,
0452, 0461, 0478, 0500, 0545, 0587,
0614, 0631, 0682, 0707, 0712, 0735,
0746, 0784, 0823, 0858, 0866, 0871,
1005, 1057, 1060, 1116, 1138, 1153,
1171, 1222, 1240, 1267, 1317, 1332,
1343, 1381, 1420, 1434, 1466, 1485,
1508, 1524, 1556, 1572, 1606, 1655,
1664, 1678, 1728, 1751, 1762, 1773,
1801, 1815, 1833, 1847, 1880.

(vii) $N(6,3,4) \geq 34$ (found by Mario Szegedy, personal communication).

0006, 0304, 0523, 0610, 0821, 0B16,
0C15, 0D07, 1128, 1202, 1417, 1721,
1914, 1A23, 1C00, 1F05, 2025, 2117,
2320, 2401, 2708, 2903, 2A14, 2D22,
2E26, 3010, 3227, 3315, 3603, 3808,
3B01, 3C24, 3D16, 3E12.

(viii) $N(4,4,4) \geq 28$.

000, 044, 118, 181, 225, 267, 332, 373,
427, 465, 536, 650, 711, 788, 858, 904,
940, A33, A72, B26, C13, C31, D22,
D77, E08, E84, F45, F60.

(ix) $N(7,3,5) \geq 20$.

0900, 7000, 3E02, 0603, 5D05, 6305,
1511, 2211, 3B13, 4814, 6516, 0F18,
5218, 5B21, 4422, 2D24, 7624, 1C26,
6A26, 3128.

(x) $N(10,3,5) \geq 128$ and $N(11,2,5) \geq 96$. The former is given explicitly, the latter follows by discarding the words ending in 1.

00500, 04A00, 1BA00, 1F500, 27600,
29F00, 32000, 3C900, 03C02, 0E002,
15F02, 18302, 22B02, 25102, 39402,
3EE02, 0A603, 0DC03, 11903, 16303,
26D03, 2B103, 30E03, 3D203, 01205,
0FB05, 14405, 1AD05, 28805, 2C705,
33705, 37805, 07910, 1C610, 2AC10,
31310, 02211, 05411, 19D11, 1EB11,
24F11, 28111, 33E11, 3F011, 0B712,
10812, 2DA12, 36512, 08B13, 13413,
24013, 3FF13, 0B814, 0E514, 10714,
15A14, 27314, 29614, 32914, 3CC14,
06E15, 1D115, 21D15, 3A215, 01E16,
0F216, 14D16, 1A116, 22716, 2D516,
36A16, 39816, 04318, 08418, 13B18,
1FC18, 23018, 2E918, 35618, 38F18,
09020, 12F20, 2E320, 35C20, 01B21,
0FE21, 14121, 1A421, 23521, 26821,
38A21, 3D721, 0CD22, 17222, 20622,
3B922, 05723, 1E823, 23A23, 38523,
00C24, 0C224, 17D24, 1B324, 2AF24,
2D924, 31024, 36624, 02125, 19E25,
2F425, 34B25, 06426, 0BD26, 10226,
1DB26, 20926, 2CE26, 37126, 3B626,
05828, 0AA28, 11528, 1E728, 27F28,
29328, 32C28, 3C028.

(xi) $N(9,4,5) \geq 136$.

02200, 0DD00, 16F00, 19000, 01E02,
0E102, 13502, 1CA02, 05303, 0AC03,
17803, 18703, 00905, 0F605, 14405,
1BB05, 15610, 1A910, 01111, 0EE11,
16011, 19F11, 07B12, 08412, 06513,
09A13, 05C14, 0A314, 14B14, 1B414,
12E15, 1D115, 00F16, 0F016, 13316,
1CC16, 04218, 0BD18, 11818, 1E718,
04820, 0B720, 02D21, 0D221, 13A21,
1C521, 10322, 1FC22, 11D23, 1E223,
00624, 0F924, 17724, 18824, 03025,
0CF25, 07E26, 08126, 12426, 1DB26,
05528, 0AA28, 16928, 19628, 0FA30,
11B30, 1E430, 0AF32, 12832, 04E33,
0B133, 13633, 16335, 09737, 0C837,
10D37, 15237, 03C40, 0F541, 12741,
1FF43, 13944, 0A646, 16A46, 02148,
0DE48, 17448, 18E50, 05F51, 0A051,

1EB51, 06652, 02B53, 0BE54, 16C54, 1A555, 0ED56, 14756, 1B856, 0F358, 13F58, 00560, 05062, 1D762, 1C963, 07D65, 08265, 19C65, 03867, 06767, 1A167, 1FE67, 0C370, 00A71, 1D871, 14D72, 1B272, 10073, 07274, 08D74, 1C674, 01775, 0E875, 05976, 19576, 18B78, 17180, 11481, 09982, 0D483, 04184, 19384, 15A85, 01286, 00C88, 1C088.

(xii) $N(7,4,6) \geq 18$.

0000, 0344, 0588, 1848, 1B80, 1D04, 2884, 2B08, 2D40, 3611, 4E15, 5267, 5453, 6252, 6763, 7115, 7C62, 7F57.

(xiii) $N(9,4,6) \geq 48$.

00703, 0AA04, 12906, 13607, 1D211, 04B12, 1A413, 03D14, 0B316, 00017, 18F17, 15C18, 16521, 18822, 0D927, 0C628, 17330, 0B932, 1D833, 07E35, 0CC41, 10642, 05747, 1E148, 13C50, 08351, 05052, 1DF52, 0F553, 1B255, 16A57, 00D58, 05D60, 1F462, 03063, 1BF63, 10C64, 0D365, 18266, 0A567, 08973, 12374, 02E76, 1B877, 0FA80, 03782, 15683, 17985.

(xiv) $N(7,6,6) \geq 99$.

30000, 77013, 1D024, 55032, 42040, 21045, 7C047, 2E053, 3F068, 0C072, 4B085, 09108, 1E116, 6F127, 22137, 4D143, 68152, 56154, 35156, 71164, 2B170, 00183, 63202, 50215, 0A221, 2D231, 1B233, 4E238, 36242, 41257, 14267, 5F271, 72286, 41303, 13312, 4E314, 2D316, 36327, 72335, 14343, 5F356, 0A366, 63377, 58381, 2E402, 55417, 21421, 7C423, 42428, 4B431, 3F444, 0C457, 1D460, 38478, 77482, 56500, 35505, 68507, 22513, 4D522, 00532, 71540, 2B558, 6F563, 09574, 1E585, 2B604, 00617, 71628, 1E631, 6F642, 09650, 4D667, 35671, 68673, 56678, 22682, 5F705, 72711, 14722, 58736, 0A745, 63753, 36763, 41772, 4E780, 2D785, 1B787, 0C803, 4B816, 3F820, 77837, 1D848, 38854, 7C862, 42864, 21866, 2E877, 55883.

(xv) $N(8,5,7) \geq 20$.

F4018, 4F025, 21035, 97056, EA060, 1C071, 82112, BE124, 7D137, CC143, 1B163, D1181, 24186, 41216, 56232, B9240, 0A257, 8D268, E7274, 78285.

(xvi) $N(6,6,7) \geq 17$.

24006, 12022, 39084, 27175, 3E231, 01240, 0F360, 29438, 16446, 33503, 0C515, 15635, 2A643, 3D710, 04781, 0B827, 30878.

(xvii) $N(7,6,7) \geq 24$.

21001, 1B026, 6C046, 0F141, 55188, 6F225, 7A261, 08275, 43374, 10437, 5E453, 26465, 69480, 44521, 33552, 3D577, 77637, 59642, 2A654, 16670, 4A708, 3C722, 70813, 05833.

(xviii) $N(4,7,7) \geq 14$.

00000, 10444, 21158, 31861, 40878, 62463, 72017, 82284, 92708, B0386, C1337, D1170, E0721, F0235.

(xix) $N(5,7,7) \geq 20$.

121214, 032028, 0E2362, 120782, 051272, 1F0135, 1C2718, 031463, 172867, 081587, 142083, 141431, 1F1520, 002835, 192304, 0B2741, 050724, 111646, 080100, 060546.

(xx) $N(6,7,7) \geq 28$.

1A0017, 200052, 3D0185, 0C0236, 310410, 320564, 0B0582, 2A0723, 140821, 0F1073, 2B1131, 101186, 021300, 191354, 2C1417, 161548, 331628, 251853, 381872, 352047, 262168, 112205, 3E2350, 292366, 072424, 002674, 0D2742, 1F2867.

(xxi) $N(3,8,7) \geq 16$.

30242, 40634, 51020, 21887, 72368, 02513, 03382, 53876, 64530, 65174, 35653, 26006, 76524, 17347, 48258, 18702.

(xxii) $N(4,8,7) \geq 22$.

81356, F1718, 21862, 12010, 62155, 72367, A2514, D2851, 53162, E3538, 93607, 63620, 34345, 44584, B5286, 86222, D6443, 36784, E7041, 77203, 08431, C8665.

(xxiii) $N(5,8,7) \geq 33$.

070342, 0D0467, 110847, 051253, 141321, 092115, 1F2206, 122788, 173087, 183224, 023270, 0D3620, 0A3735, 104038, 1D4272, 174405, 0B4557, 0E5411, 045664, 1B5673, 046128, 116163, 1C6346, 1E6861, 037001, 1A7150, 0E7385, 007444, 077776, 087806, 168245, 018582, 1D8754.

(xxiv) $N(3,9,7) \geq 26$.

600028, 500562, 701847, 302310, 003153, 404675, 205881, 207706, 408441, 112458, 412823, 014001, 715115, 616270, 116514, 516636, 218365, 220742, 621164, 021576, 323077, 523721, 624350, 425238, 327255, 128160.

(xxv) $N(4,9,7) \geq 39$.

000000, 001444, 002888, 113045, 114406, 116581, 128623, 213627, 214262, 218146, 226405, 303770, 307358, 312834, 321121, 414853, 417312, 423137, 505211, 507165, 520376, 605363, 620842, 815184, 825332, 827217, 906836, 912712, A08520, A11673, C10504, C26780, D01627, D22255, E02108, E06054, F13452, F17286, F28474.

(xxvi) $N(3,10,7) \geq 48$.

000000, 001444, 002888, 043045, 044406, 046581, 083627, 084262, 088113, 113731, 115383, 117017, 126505, 135122, 136776, 151253, 172542, 215204, 216156, 224640, 235377, 237605, 250412, 270873, 322038, 327784, 361321, 363536, 417332, 423576, 434811, 438137, 452663, 471725, 522171, 530358, 563064, 578806, 608520, 634183, 661248, 685355, 686768, 703715, 711567, 742750, 748275, 786251.

(xxvii) $N(2,11,7) \geq 58$.

0000000, 0001444, 0002888, 0043045, 0044406, 0046581, 0058604, 0078740, 0084087, 0113722, 0115237, 0117365, 0126176, 0133863, 0137212, 0138356, 0170317, 0181530, 0225513, 0242162, 0250757, 1014270, 1025332, 1026225, 1035121, 1063478, 1067837, 1070683, 1108671, 1151713, 1171158, 1204626, 1208133, 1286401, 2023841, 2037183, 2050368, 2065205, 2122055, 2127527, 2145474, 2164648, 2211681, 2241246, 2262420, 2276855, 3010707, 3031552, 3058148, 3077014, 3101264, 3142630, 3176566, 3183020, 3200315, 3233037, 3257370, 3282877.

(xxviii) $N(6,7,8) \geq 14$.

000000, 011444, 021888, 0D2058, 0F0804, 162115, 1B0372, 272480, 2A2643, 2C1512, 312832, 360337, 380284, 3B1106.

(xxix) $N(5,8,8) \geq 18$.

000000, 004444, 008888, 070148, 074621, 0B1283, 0B8332, 0D3567, 0D8713, 135516, 157065, 162834, 166480, 190751, 1A3675, 1A7107, 1C1512, 1C5056.

VI. Heuristic Searches

Given parameters n_2, n_3, M, and d, we search for mixed binary/ternary codes of size M, with n_2 binary and n_3 ternary coordinate positions, and minimum Hamming distance d. For very small parameters an exhaustive search is possible. For slightly larger parameters we employed tabu search [14]. Let $\mathcal{S}$ be the set of all codes C satisfying the requirements except possibly that on the minimum distance, that is, the set of all M-subsets of $X := \boldsymbol{F}_2^{n_2}\boldsymbol{F}_3^{n_3}$. Starting with an arbitrary $C_0 \in \mathcal{S}$ we do a walk on $\mathcal{S}$ in the hope of encountering a $C \in \mathcal{S}$ with minimum distance d. Each step goes from a code C to a neighbor C', that is, to a code C' obtained from C by replacing a single codeword by one that is at Hamming distance 1. We choose the best neighbor, where the badness of a code C is

TABLE I
VALUES OF $A_3(n, d)$

$n \backslash d$	3	4	5	6	7	8	9	10	11	12	13	14	15
4	9												
5	18	6											
6	48 38*	18	4										
7	144 99	46* 33	10	3									
8	340 243	138* 99	27*	9	3								
9	937 729	340 243	81*	27	6	3							
10	2811 2187	937 729	243	81	18 14*	6	3						
11	7029 6561	2561* 1458	729	243	50* 36	12	4	3					
12	19683	7029 4374	1562 729	729	138 51*	36	9	4	3				
13	59049	19682* 8019	4163 2187	1562 729	363 105*	103 42*	27	6	3	3			
14	153527 118098	59046* 24057	10736 6561*	3885 2187	836 243	237 81	66* 31*	15 12	6	3	3		
15	434815 354294	153527 72171	29524 6561	10736 2187	2268 729*	711 243	166 81	45 27*	10	6	3	3	
16	1304445 1062882*	434815 216513	77217 19683	29524 6561	6643 729	2079 297	451 243	127 54	30 18	9	4	3	3

measured either by

$$\sum_{c \in C} \sum_{c' \in C \setminus \{c\}} \max(0, d - d(c, c'))$$

(this worked well for large d and small M) or by

$$\sum_{x \in X} \max(0, c(x) - 1)$$

where $c(x)$ measures the number of codewords close to x and is chosen in such a way that the code has minimum distance d if and only if $c(x) \leq 1$ for all x (this worked better for small d). There is some freedom in the choice of the function $c(x)$. For odd d, say $d = 2e + 1$, we took $c(x)$ to be the number of codewords at distance at most e from x. For even d, say $d = 2e$, we took $c(x)$ to be the number of codewords at distance at most $e - 1$ from x plus K^{-1} times the number of codewords at distance e from x, where $K \geq \lfloor (n_2 + n_3)/e \rfloor$. (We took $K = 10$.)

In order to avoid looping, a so-called tabu list—after which this search method is called tabu search—containing (attributes of) reverses of recent moves is maintained. Moves in the tabu list are not allowed within a given number L of steps.

Almost the same methods and programs were used earlier for finding covering codes [32, 33].

A. Searching for Codes with a Given Structure

Searching for codes by these methods becomes ineffective if the codes are too large (for $d = 3$, when there are more than about 100 codewords, for example). However, imposing some structure on the code allows us to search for larger codes.

A method used by Kamps and Van Lint [21] and Blokhuis and Lam [6] leads to codes that are unions of cosets of linear codes. This method was originally developed for covering codes. An analogous method that works for error-correcting codes was presented in [34]. Let us formulate it here for the case of mixed binary/ternary error-correcting codes. (See also [10] and [33], where the method is applied to mixed binary/ternary covering codes.)

Let A be an $n_2 \times m_2$ binary matrix of rank n_2 and let B be an $n_3 \times m_3$ ternary matrix of rank n_3. For two words $x = (x_2,\ x_3)$, $y = (y_2,\ y_3)$ with $x_2,\ y_2 \in \boldsymbol{F}_2^{n_2}$, $x_3,\ y_3 \in \boldsymbol{F}_3^{n_3}$, we define the *distance between x and y using A and B* to be

$$d_{A,B}(x, y) = \min\{\mathrm{wt}\,(t_2) + \mathrm{wt}\,(t_3) \mid At_2 = x_2 - y_2,\ Bt_3 = x_3 - y_3\}$$

with $t_2 \in \boldsymbol{F}_2^{m_2}$ and $t_3 \in \boldsymbol{F}_3^{m_3}$. As the matrices A and B have full rank, the distance $d_{A,B}(x, y)$ is always defined. For a set of words $C \subseteq \boldsymbol{F}_2^{n_2}\boldsymbol{F}_3^{n_3}$ we further define

$$d_{A,B}(C) = \min_{c, c' \in C, c \neq c'} d_{A,B}(c, c').$$

Proposition 6.1: Let A be a parity-check matrix for a binary linear code with minimum distance d_2, let B be a parity-check matrix for a ternary linear code with minimum distance d_3, and let C be a subset of $\boldsymbol{F}_2^{n_2}\boldsymbol{F}_3^{n_3}$. Then the code

$$W = \{(w_2, w_3) \in \boldsymbol{F}_2^{n_2}\boldsymbol{F}_3^{n_3} \mid (Aw_2, Bw_3) \in C\}$$

has minimum distance $\min\{d_{A,B}(C), d_2, d_3\}$ and $|W| = 2^{m_2 - n_2} 3^{m_3 - n_3} |C|$.

In searching for codes using this approach, the following idea from [31] was used. First, we construct a family of inequivalent matrices A and B with given parameters. Then the computer search is carried out separately for all possible combinations of these matrices.

Most of the codes given in Section V-G were found in this way.

TABLE II-A
VALUES OF $N(n_2, n_3, d)$ FOR $d = 2$

$d=2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
1	1	2	6	18	54	162	486	1458	4374	13122	39366	118098	354294	
2	2	4	12	36	108	324	972	2916	8748	26244	78732	236196		
3	4	8	24	72	216	648	1944	5832	17496	52488	157464			
4	8	16	48	144	432	1296	3888	11664	34992	104976				
5	16	32	96	288	864	2592	7776	23328	69984					
6	32	64	192	576	1728	5184	15552	46656						
7	64	128	384	1152	3456	10368	31104							
8	128	256	768	2304	6912	20736								
9	256	512	1536	4608	13824									
10	512	1024	3072	9216										
11	1024	2048	6144											
12	2048	4096												
13	4096													

TABLE II-B
VALUES OF $N(n_2, n_3, d)$ FOR $d = 3$

$d=3$	0	1	2	3	4	5	6
0	1	1	1	3	9	${}^{d}18$	${}^{x}38$–48^{L}
1	1	1	2	6	12^{c}	${}^{d}33$	${}^{x}71$–96
2	1	2	4	9	22^{e}	${}^{x}52$–66	${}^{jb}134$–178^{L}
3	2	3	6^{e}	${}^{jb}18$	${}^{H}42$–44	${}^{x}98$–126^{L2}	264–343^{s}
4	2	${}^{d}6$	12	28–33	72–88	${}^{xc}186$–243^{s}	${}^{jb}486$–631^{L2}
5	4	8^{e}	${}^{x}22^{c}$	54–65^{L}	144–167^{L2}	${}^{xc}342$–457^{s}	948–1227^{s}
6	8	16	${}^{x}38$–44	${}^{H}108$–123^{s}	288–322^{LX}	648–863^{LX}	1896–2332^{s}
7	${}^{d}16$	${}^{d}26$–30	72–85^{L2}	192–230^{s}	576–609^{s}	${}^{jb}1296$–1612^{L2}	${}^{jb}3792$–4443^{s}
8	20^{B5}	${}^{x}50$–60	144–160^{L5}	384–417^{L2}	${}^{H}1152^{s}$	${}^{jb}2544$–3110^{s}	
9	${}^{d}40$	96–109^{L2}	${}^{H}288$–293^{L4}	768–806^{L}	${}^{jb}1728$–2131^{L2}		
10	72–76^{K}	192–213^{L}	512–556^{LX}	1152–1536^{s}			
11	${}^{d}144$–152	384	${}^{xc}832$–1049^{L2}				
12	256^{BB}	${}^{H}768$					
13	512						

$d=3$	7	8	9	10	11	12	13
0	99–144	243–340^{L}	729–937^{L}	2187–2811	6561–7029^{L}	19683^{L}	${}^{TH}59049$
1	198–242^{LZ}	486–680	1458–1874	4374–4920^{L}	13122–14058	39366	
2	${}^{jb}396$–484	972–1284^{L}	2916–3514^{L}	8748–9840	26244–26790^{L}		
3	${}^{xc}684$–902^{L}	1944–2464^{L2}	5832–6846^{s}	17496–18589^{L}			
4	${}^{xc}1332$–1749^{s}	3888–4767^{L1}	11664–12887^{L2}				
5	2592–3259^{LX}	7776–9128^{s}					
6	5184–6362^{s}						

VII. TABLES

Tables of bounds on binary codes can be found in many places—see, e.g., Conway and Sloane [9, Table 9.1, p. 248]. An improvement was given in [22].

An early table of bounds on $A_3(n, d)$, the maximal size of a ternary code of length n and minimum distance d, was given in [28]. Another table was given in Vaessens, Aarts, and van Lint [42]. We know of 19 improvements to the latter table, and give an updated version in Table I. (We explain only the entries that have changed, indicated by an asterisk.) We omit the trivial entries ($A_3(n, 1) = 3^n$, and if $n > 0$ then $A_3(n, 2) = 3^{n-1}$ and $A_3(n, n) = 3$).

The differences between Table I and the table in [42] are as follows.

Since ternary linear [14, 8, 5] and [15, 6, 7] codes exist ([23], [26]), we have $A_3(14, 5) \geq 6561$ and $A_3(15, 7) \geq 729$.

In [34] it was shown that $A_3(16, 3) \geq 1062882$ (using a variation on Proposition 5.8).

Svanström [38] showed that $A_3(15, 10) \geq 24$, and Bitan and Etzion [5] improved this to $A_3(15, 10) \geq 27$.

In this paper we find that $A_3(6, 3) \geq 38$, $A_3(10, 7) \geq 14$, $A_3(12, 7) \geq 51$, $A_3(13, 7) \geq 105$, $A_3(13, 8) \geq 42$, and $A_3(14, 9) \geq 31$ (see Section V-F).

Concerning upper bounds, Mario Szegedy (personal communication) proved that $A_3(7, 4) \leq 47$ (cf. Lemma 4.7) and Antti Perttula [35] showed that $A_3(11, 7) \leq 52$.

In this paper we find $A_3(7, 4) \leq 46$, $A_3(8, 4) \leq 138$, $A_3(11, 4) \leq 2561$, $A_3(13, 4) < 3^9$, $A_3(14, 4) < 3^{10}$ (see Lemma 4.6), $A_3(8, 5) = 27$, $A_3(9, 5) = 81$, $A_3(11, 7) \leq 50$, and $A_3(14, 9) \leq 66$ (by the linear programming bound, using the analog of ($L7$) for this case).

Table II gives lower and upper bounds for $N(n_2, n_3, d)$. We vary n_2 vertically and n_3 horizontally.

TABLE II-C
VALUES OF $N(n_2, n_3, d)$ FOR $d = 4$

$d = 4$	0	1	2	3	4	5	6
0	1	1	1	1	3	6	$^{He}18$
1	1	1	1	2	4	12	$^{G}33$
2	1	1	2	3	8	22	$^{xc}51$–66
3	1	2	3	6	15^{e}	34–44	$^{xc}87$–124^{L}
4	2	2	$^{jb}6$	$^{x}11^{e}$	$^{x}28$–30	58–86^{L}	144–242^{L}
5	2	4	8^{d}	$^{x}20^{e}$	48–60	$^{xc}108$–167^{d}	288–454^{L}
6	4	8	16	$^{x}34$–40	96–120	$^{xc}208$–319^{L6}	576–863^{d}
7	8	16	$^{xc}26$–30	64–80	192–230^{L2}	384–609^{d}	$^{jb}1152$–1612^{d}
8	$^{u}16$	20^{d}	48–60	128–160	$^{jb}384$–417^{d}	768–1120^{L2}	
9	20	40	96–109^{d}	256–293^{d}	$^{jb}540$–782^{L}		
10	$^{B}40$	72–76^{d}	192–213^{d}	$^{xc}400$–556^{d}			
11	72–76	144–152	$^{H}384$				
12	$^{J}144$–152	256^{d}					
13	$^{u}256$						

$d = 4$	7	8	9	10	11	12	13
0	33–46^{Sz}	$^{V}99$–138	243–340^{d}	729–937^{d}	1458–2561^{L1}	$^{u}4374$–7029^{d}	8019–19682^{t}
1	66 92	162–242^{d}	486–680	972–1749^{L1}	2916–4920^{d}	$^{u}8019$–14058	
2	108–178^{d}	324–484	729–1272^{L1}	1944 3498	$^{xc}5589$–9777^{L1}		
3	216–343^{L1}	486–902^{d}	1296–2464^{d}	3726–6791^{L1}			
4	$^{xc}360$–631^{d}	$^{xc}891$–1749^{d}	2484 4767^{d}				
5	$^{xc}612$–1223^{L}	$^{xc}1674$–3259^{d}					
6	1152–2332^{d}						

TABLE II-D
VALUES OF $N(n_2, n_3, d)$ FOR $d = 5$

$d = 5$	0	1	2	3	4	5	6
0	1	1	1	1	1	3	4
1	1	1	1	1	2	3	8
2	1	1	1	2	3	6	$^{x}15^{e}$
3	1	1	2	3	$^{x}6$	$^{d}12$	24–27
4	1	2	2	4	$^{x}9^{e}$	18^{e}	48–54
5	2	2	4	6^{e}	$^{x}14^{e}$	33–36	96–108
6	2	3	6	12	$^{xc}24$–28	66–72	144–216
7	2	4	$^{x}9^{e}$	$^{x}20$–24	44–56	$^{d}99$ 144	$^{G}234$–432
8	4	7^{e}	$^{d}16^{e}$	32–48	$^{xc}82$–112	156–288	
9	6^{L}	12^{e}	26–32	64–91^{L}	$^{x}136$–224		
10	12	24	48–64	$^{x}128$–170^{L2}			
11	24	$^{d}38$–48	$^{x}96$–121^{L}				
12	32^{B5}	$^{jx}68$–86^{L2}					
13	$^{NR}64$						

$d = 5$	7	8	9	10	11	12	13
0	$^{V}10^{e}$	27	81	243	$^{d}729$	729–1562^{L}	$^{l}2187$–4163^{L}
1	18^{e}	54	162	486	729–1145^{L}	1458–2984^{L}	
2	36	108	324	729–867^{L}	972–2157^{L}		
3	72	216	486–633^{L6}	729–1567^{L}			
4	144	324–432	729–1153^{L1}				
5	216–288	486–850^{L1}					
6	$^{G}342$–576						

A. Notes on Tables II-A to II-H

All unmarked upper bounds follow from Propositions 4.1, 4.3, or 4.4.

The entries in Table II-A are all given by Proposition 4.1(ii). Concerning Table II-D, the rows of a well-known orthogonal array (L_{18} in [40, p. 1153]) form a $(1,\ 7,\ 5)_{18}$ code. In the second part of Table II-E *all* lower bounds follow directly from the extended ternary Golay code. For $d \geq 9$ the exact values are known. For $d = 10$ we have $N(0,\ 13,\ 10) \leq 6$ from the Plotkin bound, so all entries in the table are at most 6, and follow directly from Proposition 4.4.

Key to Table II. Lower bounds:

- B Best code, see [2], [25].
- G From the ternary Golay code.
- GH Generalized Hadamard matrix, see [28].
- H From the binary $[15, 11, 3]$ Hamming code, see Section V-D.
- Ha From the Hadamard matrix of order 12.
- He Words of weight 6 in the quaternary $[6, 3, 4]_4$ hexacode.
- J Julin code, see [20] or [29, ch. 2, Sec. 7].

TABLE II-E

VALUES OF $N(n_2, n_3, d)$ FOR $d = 6$

$d=6$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	3
1	1	1	1	1	1	2	3
2	1	1	1	1	2	3	6
3	1	1	1	2	3	4	$^{jc}12$
4	1	1	2	2	4	8	$^{jc}18$
5	1	2	2	3	6^d	12^e	33–36
6	2	2	3	$^{x}6$	$^{x}12$	22–24	$^{G}66$–72
7	2	2	4	8^e	$^{x}18$–24	44–48	$^{x}99$–144
8	2	4	$^{x}7^d$	16	$^{G}32$–43^L	66–96	
9	4	6^P	12^d	$^{G}26$–32	$^{x}48$–77^L		
10	6	12	24	38–61^L			
11	12	24	$^{jx}38$–48				
12	$^{Ha}24$	32^d					
13	32						

$d=6$	7	8	9	10	11	12	13
0	3	9	27	81	243	$^{G}729$	729–1562^d
1	6	18	54	162	486	729–1145^d	
2	12	36	108	324	729–867^d		
3	24	72	216	486–614^L			
4	48	144	324–425^L				
5	96	216–288					
6	144–192						

TABLE II-F

VALUES OF $N(n_2, n_3, d)$ FOR $d = 7$

$d=7$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	2
2	1	1	1	1	1	2	3
3	1	1	1	1	2	3	4
4	1	1	1	2	2	3	6^P
5	1	1	2	2	3	$^{j}6$	$^{x}12$
6	1	2	2	3	4	$^{x}9^e$	$^{x}17$–24
7	2	2	2	4	8	16^e	$^{x}24$–45^{L4}
8	2	2	4	6	$^{jc}16$	$^{x}20$–32	
9	2	3	4^e	$^{jc}12$	$^{jc}18$–26^L		
10	2	4	8	16–18			
11	4	6	12^e				
12	4^e	$^{jc}12$					
13	8						

$d=7$	7	8	9	10	11	12	13
0	3	3	6^P	$^{x}14$ 18	$^{d}36$–50^{L7}	$^{cy}51$–138^L	$^{cy}105$–363^L
1	3	$^{j}6$	$^{x}12$	24–36	36–100	$^{cy}78$ 251^L	
2	4	$^{x}9^P$	$^{d}18$–24	$^{d}36$ 72	$^{x}58$–182^L		
3	7^P	$^{x}16$–18	$^{x}26$–48	$^{x}48$–134^L			
4	$^{x}14$	$^{x}22$–36	$^{x}39$–96				
5	$^{x}20$–28	$^{x}33$–72					
6	$^{x}28$–56						

- NR From the binary $(15,\ 256,\ 5)$ Nordstrom–Robinson code.
- TH Ternary $[13, 10, 3]_3$ Hamming code.
- V From [42].
- cy From a cyclic code, see Section V-F.
- d Follows from lower bound for larger d (Proposition 4.3(vi)).
- j Juxtaposition, see Section V-A.
- jb Juxtaposition, using two partitioned codes, see Section V-A.
- jc Juxtaposition, using one partitioned code, see Section V-A.
- jx Juxtaposition plus additional words, see Section V-A.
- l Linear code.
- u From the $(u, u+v)$ construction (Proposition 5.7).
- x Explicit construction, see Section V-H.
- xc Explicit construction by taking a union of cosets, see Section V-G.

Key to Table II. Upper bounds:

- $B5$ See [4].
- BB See [3]. (This also falls under $L2$.)
- K See [22].
- L Pure LP bound, using only the Delsarte inequalities.
- $L1$ LP bound, with additional inequalities for words of weight d, cf. Section III.

TABLE II-G
VALUES OF $N(n_2, n_3, d)$ FOR $d = 8$

$d=8$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	1	3	3	${}^{j}6$	12^{P}	${}^{GH}36$	${}^{cy}42$–103^{L}
1	1	1	1	1	1	1	1	2	3	4	9^{P}	24	36–72	
2	1	1	1	1	1	1	2	3	4	7^{P}	${}^{x}18$	36–48		
3	1	1	1	1	1	2	3	3	6^{P}	${}^{x}13^{e}$	24–36			
4	1	1	1	1	2	2	3	${}^{j}6$	${}^{x}12$	18–26				
5	1	1	1	2	2	3	4	8^{c}	${}^{x}18$–24					
6	1	1	2	2	3	4	${}^{x}7^{P}$	${}^{x}14$–16						
7	1	2	2	2	4	6^{P}	12^{e}							
8	2	2	2	3	${}^{j}6$	${}^{jc}12$								
9	2	2	3	4^{d}	${}^{x}9$									
10	2	2	4	6^{e}										
11	2	4	${}^{j}6$											
12	4	4^{d}												
13	4													

TABLE II-H
VALUES OF $N(n_2, n_3, d)$ FOR $d = 9$

$d=9$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	1	1	3	3	4	9	${}^{l}27$
1	1	1	1	1	1	1	1	1	2	3	4	6^{P}	18	
2	1	1	1	1	1	1	1	2	3	3	6	12		
3	1	1	1	1	1	1	2	3	3	${}^{j}6$	${}^{x}10^{P}$			
4	1	1	1	1	1	2	2	3	4	${}^{x}8^{P}$				
5	1	1	1	1	2	2	3	4	6^{P}					
6	1	1	1	2	2	3	4	6						
7	1	1	2	2	2	3	${}^{j}6$							
8	1	2	2	2	3	4								
9	2	2	2	3	4									
10	2	2	2	4										
11	2	2	4											
12	2	3												
13	2													

$L\alpha$ LP bound, with the additional inequality $(L\alpha)$, $(\alpha = 2, 4, 5, 6, 7)$.
LX LP bound, with several of the above mentioned additional inequalities.
LZ LP bound plus integrality, see Section III.
P From the Plotkin bound.
Sz From Lemma 4.7.
d Follows from upper bound for smaller d (Proposition 4.3(vi)).
e Exhaustive search.
s By Lemma 3.1.
t By Lemma 4.6.

Any improvements to the tables should be sent to the authors by electronic mail, to aeb@cwi.nl, PatricOstergard@hut.fi, or njas@research.att.com.

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REFERENCES

[1] D. Applegate and D. S. Johnson, personal communication, 1993. [Online]. Available from ftp://dimacs.rutgers.edu/pub/challenge/graph/solvers.
[2] M. R. Best, "Binary codes with a minimum distance of four," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 738–742, 1980.
[3] M. R. Best and A. E. Brouwer, "The triply shortened binary Hamming code is optimal," *Discr. Math.*, vol. 17, pp. 235–245, 1977.
[4] M. R. Best, A. E. Brouwer, F. J. MacWilliams, A. M. Odlyzko, and N. J. A. Sloane, "Bounds for binary codes of length less than 25," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 81–93, 1978.
[5] S. Bitan and T. Etzion, "Constructions for optimal constant weight cyclically permutable codes and difference families," *IEEE Trans. Inform. Theory*, vol. 41, pp. 77–87, 1995.
[6] A. Blokhuis and C. W. H. Lam, "More coverings by rook domains," *J. Combin. Theory Ser. A*, vol. 36, pp. 240–244, 1984.
[7] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Ergebnisse der Mathematik 3.18. Heidelberg, Germany: Springer-Verlag, 1989.
[8] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, "A new table of constant weight codes," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1334–1380, 1990.
[9] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 2nd ed., Grundlehren der math. Wissenschaften 290. New York: Springer-Verlag, 1993.
[10] R. Davies and G. F. Royle, "Graph domination, tabu search and the football pool problem," *Discr. Appl. Math.*, vol. 10, pp. 217–228, 1997.
[11] A. Dey, *Theory of Block Designs*. New Delhi, India: Wiley, 1986.
[12] T. Etzion and G. Greenberg, "Constructions of mixed perfect codes and other covering codes," *IEEE Trans. Inform. Theory*, vol. 39, pp. 209–214, 1993.
[13] T. A. Feo, M. G. C. Resende, and S. H. Smith, "A greedy randomized adaptive search procedure for maximum independent set," *Oper. Res.*, vol. 42, pp. 860–878, 1994.
[14] F. Glover, "Tabu search—Part I," *ORSA J. Comput.*, vol. 1, pp. 190–206, 1989.
[15] C. D. Godsil, *Algebraic Combinatorics*. New York: Chapman & Hall, 1993.
[16] H. Hämäläinen, I. Honkala, S. Litsyn, and P. Östergård, "Football pools—A game for mathematicians," *Amer. Math. Monthly*, vol. 102, pp. 579–588, 1995.
[17] H. Hämäläinen and S. Rankinen, "Upper bounds for football pool problems and mixed covering codes," *J. Combin. Theory Ser. A*, vol. 56, pp. 84–95, 1991.
[18] P. W. Heijnen, "There exists no binary $[33, 9, 13]$-code" (in Dutch), Master's thesis, Delft Univ. of Technol., Delft, The Netherlands, 1993.
[19] D. B. Jaffe, "Binary linear codes: New results on nonexistence," Tech. Rep., Dept. Math. Statist., Univ. of Nebraska, Lincoln, Oct. 1996 [Online]. Available from http://math.unl.edu/~djaffe/codes/code.ps.gz.
[20] D. Julin, "Two improved block codes," *IEEE Trans. Inform. Theory*, vol. IT-11, p. 459, 1965.
[21] H. J. L. Kamps and J. H. van Lint, "A covering problem," *Colloq. Math. Soc. János Bolyai*, vol. 4, pp. 679–685, 1970.
[22] Y. Klein, S. Litsyn, and A. Vardy, "Two new bounds on the size of binary codes with a minimum distance of three," *Des. Codes Cryptogr.*, vol. 6, pp. 219–227, 1995.
[23] F. R. Kschischang and S. Pasupathy, "Some ternary and quaternary codes and associated sphere packings," *IEEE Trans. Inform. Theory*, vol. 38, pp. 227–246, 1992.
[24] J. H. van Lint, Jr., and G. J. M. van Wee, "Generalized bounds on binary/ternary mixed packing and covering codes," *J. Combin. Theory Ser. A*, vol. 57, pp. 130–143, 1991.

[25] S. Litsyn and A. Vardy, "The uniqueness of the Best code," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1693–1698, 1994.

[26] P. Lizak, "Minimum distance bounds for linear codes over GF(3) and GF(4)," Master's thesis, Math. and Comput. Sci. Dept., Univ. Salford, July 1992.

[27] Lohinen (pseudonym of Olof Laxén), in the Finnish weekly *Veikkaus-Lotto*, nos. 41 $(N(0, 6, 3) \geq 37)$ and 49 $(N(0, 7, 3) \geq 99)$, 1981.

[28] C. Mackenzie and J. Seberry, "Maximal ternary codes and Plotkin's bound," *Ars Combin.*, vol. 17A, pp. 251–270, 1984.

[29] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1977.

[30] W. J. Martin, "Designs in product association schemes," May 1996, preprint.

[31] P. R. J. Östergård, "New upper bounds for the football pool problem for 11 and 12 matches," *J. Combin. Theory Ser. A*, vol. 67, pp. 161–168, 1994.

[32] ——, "Constructing covering codes by tabu search," *J. Combin. Des.*, vol. 5, pp. 71–80, 1997.

[33] P. R. J. Östergård and H. O. Hämäläinen, "A new table of binary/ternary mixed covering codes," *Des. Codes Cryptogr.*, vol. 11, pp. 151–178, 1997.

[34] P. R. J. Östergård and M. K. Kaikkonen, "New single-error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1261–1262, 1996.

[35] A. Perttula, personal communication, 1995.

[36] J. Simonis, "MacWilliams identities and coordinate partitions," *Lin. Alg. Appl.*, vol. 216, pp. 81–91, 1995.

[37] N. J. A. Sloane and J. Stufken, "A linear programming bound for orthogonal arrays with mixed levels," *J. Statist. Plann. Inference*, vol. 56, pp. 295–306, 1996.

[38] M. Svanström, "A ternary code from orthogonal vectors over the complex numbers," in *Proc. 5th Int. Workshop on Algebraic and Combinatorial Coding Theory* (Sozopol, Bulgaria, June 1–7, 1996). Shumen, Bulgaria: Unicorn, 1996, pp. 266–269.

[39] M. Szegedy, "CKit: A toolkit for generating C programs," in preparation.

[40] G. Taguchi, *System of Experimental Design*. White Plains, NY: UNIPUB/Krause, 1987.

[41] Thor (pseudonym), in *Tippernes Systembok 16*. Oslo, Norway: Tippebladet Tips, 1992.

[42] R. J. M. Vaessens, E. H. L. Aarts, and J. H. van Lint, "Genetic algorithms in coding theory—A table for $A_3(n, d)$," *Discr. Appl. Math.*, vol. 45, pp. 71–87, 1993.

[43] G. J. M. van Wee, "Bounds on packings and coverings by spheres in q-ary and mixed Hamming spaces," *J. Combin. Theory Ser. A*, vol. 57, pp. 117–129, 1991.