# Every Finitely Reducible Logic has the Finite Model Property with Respect to the Class of Diamond-Formulae

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# Every Finitely Reducible Logic has the Finite Model Property with Respect to the Class of $\Diamond$-Formulae *

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**Abstract**

In this paper a unified framework for dealing with a broad family of propositional multimodal logics is developed. The key tools for presentation of the logics are the notions of closure relation operation and monotonous relation operation. The two classes of logics: FiRe-logics (finitely reducible logics) and LaFiRe-logics (FiRe-logics with local agreement of accessibility relations) are introduced within the proposed framework. Further classes of logics can be handled indirectly by means of suitable translations. It is shown that the logics from these classes have the finite model property with respect to the class of $\Diamond$-formulae, i.e. each $\Diamond$-formula has a $\mathcal{L}$-model iff it has a finite $\mathcal{L}$-model. Roughly speaking, a $\Diamond$-formula is logically equivalent to a formula in negative normal form without occurrences of modal operators with necessity force. In the proof we introduce a substantial modification of Claudio Cerrato's filtration technique that has been originally designed for graded modal logics. The main core of the proof consists in building adequate restrictions of models while preserving the semantics of the operators used to build terms indexing the modal operators.

Key words: multi-modal logic, epistemic logic, dynamic logic, information logics, algebras of relations, finite model property, filtration

## 1 Introduction

Over the past years a significant amount of research has been invested in the development of modal logics for various computer science applications: knowledge representation, program verification, temporal reasoning, reasoning with incomplete information, modeling concurrency, etc. Usually, these logics differ from the standard modal logics in that they are multimodal and moreover, an algebraic structure is assumed in the families of accessibility relations in their models (see e.g. [2, 3, 7, 11, 12, 13, 18, 16, 22]). The finite model property (fmp) of a logic is one of the most important conditions of its

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applicability. In the paper a class of multimodal logics, referred to as finitely reducible logics (*FiRe-logics*), is defined and characterized. The class includes many applied logics mentioned above. Each FiRe-logic possesses fmp with respect to the family of $\diamond$-formulae. A formal framework is elaborated that enables us to uniformly present the known fmp results as particular cases in the class of FiRe-logics, and to obtain new fmp results that have been open problems until now. For finitely axiomatizable logics our results imply their decidability. Two general methods of proving fmp are presented: a direct method which provides a proof that a logic is in the class of FiRe-logics, or an indirect method that establishes reducibility of the logic to a FiRe-logic. The direct method is inspired by the proof technique presented in [5].

The paper is structured as follows. In Section 2, we define the relation operations used in the sequel. Several examples are given showing the range of our definitions. In Section 3, the notions concerning logics are defined. In Section 4, the class of finitely reducible logics is defined using the notions of the previous sections. In Section 5, we present various logics from the literature being in that class and we give some results that enable us to build new finitely reducible logics from existing ones. In Section 6, we prove that every finitely reducible logic has the finite model property with respect to the class of $\diamond$-formulae. In Section 7, we define a new class of logics such that the relations indexed by modal constants satisfy (in some way to be specified) the local agreement condition [15]. We prove that every logic of that new class has also the finite model property with respect to the class of $\diamond$-formulae. Section 8 is devoted to concluding remarks.

This paper is an expanded and updated version of [9, 10].

## 2 Relation operations

For any set $U$, $\mathcal{P}(U)$ denotes the power set of $U$. For any binary relation $R$ on the set $U$, for all $u \in U$, $R(u)$ is equal to $\{v \in U : (u, v) \in R\}$. We write $Dom_R$ (resp. $Ran_R$) to denote the domain (resp. the codomain or range) of the binary relation $R$.

DEFINITION 2.1. A *relation operation* $\phi$ maps any set $U$ to a mapping $\phi(U) : \mathcal{P}(U \times U)^n \rightarrow \mathcal{P}(U \times U)$. The natural number $n$ is called the *arity* of $\phi$ (also written $ar(\phi)$). $\nabla$

The class of relation operations is written $OR$. A typical example of relation operation is the union operation $\cup$ such that for all sets $U$, for all binary relations $R_1, R_2$ on $U$, $\cup(U)(R_1, R_2) = \{(x, y) \in U \times U : \text{either } (x, y) \in R_1 \text{ or } (x, y) \in R_2\}$ (also written $R_1 \cup R_2$). In the rest of the paper $\phi(U)$ is written $\phi$ when the underlying set $U$ is clear from the context. Let $\phi, \phi'$ be two relation operations of arity $n$. We write $\phi \subseteq \phi'$ iff for all sets $U$ and $R_1, \ldots, R_n \subseteq U \times U$, $\phi(U)(R_1, \ldots, R_n) \subseteq \phi'(U)(R_1, \ldots, R_n)$.

DEFINITION 2.2. A relation operation $\phi$ (of arity $n$) is said to be *monotonous* iff for any $U$ and for any relations $R_1, \ldots, R_n, R \in \mathcal{P}(U \times U)$, for all $i \in \{1, \ldots, n\}$, if $R_i \subseteq R$ then

$$\phi(U)(R_1, \ldots, R_{i-1}, R_i, R_{i+1}, \ldots, R_n) \subseteq \phi(U)(R_1, \ldots, R_{i-1}, R, R_{i+1}, \ldots, R_n)$$

$\nabla$

Let $\phi$ and $\phi'$ be relation operations such that $ar(\phi') = 1$ and $ar(\phi) = n$. The *composition relation operation* $\phi \circ \phi'$ is defined by: for all sets $U$ and for all binary relations on $U$, $R_1, \ldots, R_n \subseteq U \times U$,

$$\phi \circ \phi'(U)(R_1, \ldots, R_n) = \phi'(U)(\phi(U)(R_1, \ldots, R_n))$$

Observe that if $\phi$ and $\phi'$ are monotonous then $\phi \circ \phi'$ is monotonous. For any relation operations $\phi_1, \ldots, \phi_n$ such that $(\ldots(\phi_1 \circ \phi_2) \circ \ldots) \circ \phi_n$ is well-defined $(\ldots(\phi_1 \circ \phi_2) \circ \ldots) \circ \phi_n$ may be written $\phi_1\phi_2 \ldots \phi_n$. The class of monotonous relation operations $\phi$ such that for all sets $U$, $U'$ and all $R_1, \ldots, R_n \subseteq U \times U$

$$\text{if } U \subseteq U', \text{ then } \phi(U)(R_1, \ldots, R_n) \subseteq \phi(U')(R_1, \ldots, R_n)$$

is denoted by $MOR$. When $\phi \in MOR$, $\phi$ is said to be *strongly monotonous*. Standard strongly monotonous relation operations can be found in Figure 2. An example of non monotonous relation operation is the complement relation operation $-$ defined by $-(U)(R) = \{(x, y) \in U : (x, y) \not\in R\}$.

DEFINITION 2.3. A *closure relation operation* $\mathcal{C}$ is a relation operation of arity 1 such that for all sets $U$ and all $R \subseteq U \times U$,

(1) $R \subseteq \mathcal{C}(U)(R)$,

(2) for all $\emptyset \neq U' \subseteq U$, $\mathcal{C}(U')(\mathcal{C}(U)(R) \cap (U' \times U')) = \mathcal{C}(U)(R) \cap (U' \times U')$,

(3) $\mathcal{C}$ is monotonous.

$\nabla$

Any closure relation operation $\mathcal{C}$, satisfies:

$$\forall U, \forall R \subseteq U \times U, \; \mathcal{C}(U)(\mathcal{C}(U)(R)) = \mathcal{C}(U)(R)$$

It is immediate by taking in Definition 2.3(2), $U' = U$. The class of closure relation operations is denoted by $COR$. Examples of closure relation operations can be found in Figure 1. For instance, a binary relation $R$ is Euclidean[1] iff $\mathcal{C}^e((Dom_R \cup Ran_R))(R) = R$ (see also Proposition 2.4 below).

[1]For all $x, y, z$, if $(x, y) \in R$ and $(x, z) \in R$ then $(y, z) \in R$.

PROPOSITION 2.1. For any $\mathcal{C} \in COR$, $\mathcal{C}$ is strongly monotonous, that is for any set $U$, $R \subseteq U \times U$ and $U \subseteq U'$, we have $\mathcal{C}(U)(R) \subseteq \mathcal{C}(U')(R)$.

PROOF: By Definition 2.3(1), $R \subseteq \mathcal{C}(U')(R)$ since $R \subseteq U' \times U'$. As $R \subseteq U \times U$, $R \subseteq \mathcal{C}(U')(R) \cap U \times U$ and by Definition 2.3(3) $\mathcal{C}(U)(R) \subseteq \mathcal{C}(U)(\mathcal{C}(U')(R) \cap U \times U)$ -monotonicity. By Definition 2.3(2), $\mathcal{C}(U)(R) \subseteq \mathcal{C}(U')(R) \cap U \times U$. Hence $\mathcal{C}(U)(R) \subseteq \mathcal{C}(U')(R)$. Q.E.D.

Observe that there exist relation operations satifying the conditions (1) and (2) in Definition 2.3 without being monotonous. Consider the unary relation operation $\phi$ such that for any set $U$, $R \subseteq U \times U$, if $R$ is rectangular[2] then $\phi(U)(R) = R$ otherwise $\phi(U)(R) = U \times U$. It is easy to show that $\phi$ satisfies the conditions (1) and (2) in Definition 2.3. However, consider $R = \{(1,2),(1,4),(3,2)\}$ and $R' = R \cup \{(3,4)\}$. We have $\phi(\{1,2,3,4\})(R) = \{1,2,3,4\} \times \{1,2,3,4\}$ whereas $\phi(\{1,2,3,4\})(R') = \{1,3\} \times \{2,4\}$ (and $R$ is a proper subset of $R'$). So $\phi$ is not monotonous.

In the sequel, by $\phi$ monotonous, we mean $\phi \in MOR$ (i.e. $\phi$ is strongly monotonous) unless otherwise stated.

Let $\mathcal{C}$ be a closure relation operation. A binary relation $R$ is said to be *$\mathcal{C}$-closed* iff $\mathcal{C}(Dom_R \cup Ran_R)(R) = R$. A family $\mathcal{R}$ of binary relations is said to be *$\mathcal{C}$-definable* iff for all binary relations $R$, $R \in \mathcal{R}$ iff $R$ is $\mathcal{C}$-closed. A set $\mathcal{R}$ of binary relations is said to be *closed under restrictions*, iff for all $R \in \mathcal{R}$, for all $\emptyset \neq U' \subseteq (Dom_R \cup Ran_R)$, $R \cap (U' \times U') \in \mathcal{R}$.

PROPOSITION 2.2. If the family of binary relations $\mathcal{R}$ is $\mathcal{C}$-definable for some closure relation operation $\mathcal{C}$ then,

(1) for any binary relation $R$, there exists $R' \in \mathcal{R}$ such that $R \subseteq R'$ and $(Dom_R \cup Ran_R) = (Dom_{R'} \cup Ran_{R'})$.

(2) $\mathcal{R}$ is closed under restrictions.

PROOF: (1) Take a binary relation $R$. By Definition 2.3(1), $R \subseteq \mathcal{C}(Dom_R \cup Ran_R)(R)$. Since $\mathcal{C}(R) = \mathcal{C}(\mathcal{C}(R))$ and $\mathcal{R}$ is $\mathcal{C}$-definable then $\mathcal{C}(R) \in \mathcal{R}$ and $(Dom_{\mathcal{C}(R)} \cup Ran_{\mathcal{C}(R)}) = (Dom_R \cup Ran_R)$.

(2) Take $R \in \mathcal{R}$ and $\emptyset \neq U' \subseteq (Dom_R \cup Ran_R)$. By Definition 2.3(2), $\mathcal{C}(U')(\mathcal{C}(U)(R) \cap (U' \times U')) = \mathcal{C}(U)(R) \cap (U' \times U') = R'$ with $U = (Ran_R \cup Dom_R)$. Since $(Dom_{R'} \cup Ran_{R'}) \subseteq U'$ and by Definition 2.3(1),

$$R' \subseteq \mathcal{C}((Dom_{R'} \cup Ran_{R'}))(R') \subseteq \mathcal{C}(U')(R')$$

Since $R' = \mathcal{C}(U')(R')$ then $\mathcal{C}((Dom_{R'} \cup Ran_{R'}))(R') = R'$. $\mathcal{R}$ being $\mathcal{C}$-definable, $\mathcal{C}(U)(R) = R$. Hence $R \cap (U' \cap U') \in \mathcal{R}$. Q.E.D.

[2] A relation $R$ over $U$ is *rectangular* iff there exist $U', U'' \subseteq U$ such that $R = U' \times U''$.

PROPOSITION 2.3. For any closure relation operation $\mathcal{C}$ there is a unique $\mathcal{C}$-definable set of relations. This set is equal to $\{\mathcal{C}(U)(R) : \exists$ set $U, \exists R \subseteq U \times U\}$.

PROOF: Let $R' = \mathcal{C}(U)(R)$. We show that $\mathcal{C}((Dom_{R'} \cup Ran_{R'}))(R') = R'$. We write $U'$ to denote $(Dom_{R'} \cup Ran_{R'})$. Since $R' \subseteq U' \times U'$ then $\mathcal{C}(U')(R') = \mathcal{C}(U')(R' \cap U' \times U')$. Hence by Definition 2.3(2), $\mathcal{C}(U')(R') = R' \cap U' \times U' = R'$. Q.E.D.

In the sequel we shall use families of binary relations definable by some closure relation operation in order to define the classes of models characterizing the multi-modal logics that are in the scope of this paper.

PROPOSITION 2.4. (1) The set of serial[3] (resp. atomic[4], weakly dense[5], discrete[6]) relations is not $\mathcal{C}$-definable for any closure relation operation $\mathcal{C}$.
(2) The set of reflexive (resp. symmetrical, transitive, equivalence, Euclidean, rectangular, quadratic[7], ideal[8]) relations is $\mathcal{C}^r$-definable (resp. $\mathcal{C}^s$-definable, $\mathcal{C}^t$-definable, $\mathcal{C}^{rst}$-definable, $\mathcal{C}^e$-definable, $\mathcal{C}^{L \times L'}$-definable, $\mathcal{C}^{L \times L}$-definable, $\mathcal{C}^{id.dom}$-definable).

PROOF: (1) This statement can be proved by noting that the class of serial (resp. atomic, weakly dense, discrete) relations is not closed under restrictions. Proposition 2.2 entails the desired result.
(2) By way of example we show that the family of Euclidean relations, say, $\mathcal{R}^e$, is $\mathcal{C}^e$-definable. Take some binary relation $R$. First assume $R \in \mathcal{R}^e$. Assume there exist $x_1, \ldots, x_n, y_1, \ldots, y_m$ such that $n \geq 2$, $m \geq 2$, $x_1 = y_1$, for all $i \in \{1, \ldots, n-1\}$, $(x_i, x_{i+1}) \in R$ and for all $i \in \{1, \ldots, m-1\}$, $(y_i, y_{i+1}) \in R$. Without any loss of generality assume $n \leq m$. By induction on $k$, it can be shown that for all $k \in \{2, \ldots, n\}$, $(x_k, y_k) \in R$ ($R$ is Euclidean). Then, by induction on $k'$, it can be also shown that for all $k' \in \{n, \ldots, m\}$, $(x_n, y_{k'}) \in R$. Hence $(x_n, y_m) \in R$ and by definition of $\mathcal{C}^e$, $\mathcal{C}^e((Dom_R \cup Ran_R))(R) = R$ ($R$ is $\mathcal{C}^e$-closed). Now assume $\mathcal{C}^e((Dom_R \cup Ran_R))(R) = R$. It means in particular that $\{(x_2, y_2) : \exists x_1, y_1 \in (Dom_R \cup Ran_R),\ x_1 = y_1, (x_1, x_2) \in R, (y_1, y_2) \in R\} \subseteq R$. Hence $R \in \mathcal{R}^e$. Q.E.D.

Another way for proving Proposition 2.4(1) is to use Łos-Tarski preservation theorem for first-order logic (fol) -see e.g. [6]. Let $\mathtt{A}_{fol}$ be a *closed* formula

[3]A relation $R$ over $U$ is *serial* iff for all $x \in U$, there is $y \in U$ such that $(x, y) \in R$.

[4]A relation $R$ over $U$ is *atomic* iff for all $x \in U$, there is $y \in U$ such that $(x, y) \in R$ and for all $z \in U$ if $(y, z) \in R$ then $z = y$.

[5]A relation $R$ over $U$ is *weakly dense* iff for all $x, y \in U$, if $(x, y) \in R$ then there is $u \in U$ such that $(x, u) \in R$ and $(u, y) \in R$.

[6]A relation $R$ over $U$ is *discrete* iff for all $x, y \in U$, if $(x, y) \in R$ then there is $z \in U$ such that $(x, z) \in R$ and there is no $u \in U$ such that $(x, u) \in R$ and $(u, z) \in R$.

[7]A relation $R$ over $U$ is *quadratic* iff there exists $U' \subseteq U$ such that $R = U' \times U'$.

[8]A relation $R$ over $U$ is *ideal* iff there exists $U' \subseteq U$ such that $R = U' \times U$.

| $\mathcal{C}$ | Definition of $\mathcal{C}(U)(R)$ |
|---|---|
| $\mathcal{C}^i$ | $R$ |
| $\mathcal{C}^r$ | $R \cup \{(x,x) : x \in U\}$ |
| $\mathcal{C}^{wr}$ | $R \cup \{(x,x) : (x,y) \in R\}$ |
| $\mathcal{C}^{ar}$ | $R \cup \{(y,y) : (x,y) \in R\}$ |
| $\mathcal{C}^s$ | $\{(x,y) :$ either $(x,y) \in R$ or $(y,x) \in R\}$ |
| $\mathcal{C}^t$ | $\{(x,y) : \exists x_1, \ldots, x_n$ such that $n \geq 2, x_1 = x, x_n = y$ and $\forall i \in \{1, \ldots, n-1\}, (x_i, x_{i+1}) \in R\}$ |
| $\mathcal{C}^e$ | $R \cup \{(x,y) : \exists x_1, \ldots, x_n, y_1, \ldots, y_m$ such that $n \geq 2, m \geq 2, x_1 = y_1, x_n = x, y_m = y,$ $\forall i \in \{1, \ldots, n-1\}(x_i, x_{i+1}) \in R, \forall i \in \{1, \ldots, m-1\}(y_i, y_{i+1}) \in R\}$ |
| $\mathcal{C}^{es}$ | $\mathcal{C}^s\mathcal{C}^e(R)$ |
| $\mathcal{C}^{er}$ | $\mathcal{C}^r\mathcal{C}^e(R)$ |
| $\mathcal{C}^{rs}$ | $\mathcal{C}^s\mathcal{C}^r(R)$ |
| $\mathcal{C}^{rt}$ | $\mathcal{C}^t\mathcal{C}^r(R)$ |
| $\mathcal{C}^{rst}$ | $\mathcal{C}^s\mathcal{C}^t\mathcal{C}^r(R)$ |
| $\nu$ | $U \times U$ |
| $\mathcal{C}^{l \times L}$ | $Dom_R \times (Dom_R \cup Ran_R)$ |
| $\mathcal{C}^{L \times l}$ | $(Dom_R \cup Ran_R) \times Ran_R$ |
| $\mathcal{C}^{L \times L}$ | $(Dom_R \cup Ran_R) \times (Dom_R \cup Ran_R)$ |
| $\mathcal{C}^{L \times L'}$ | $Dom_R \times Ran_R$ |
| $\mathcal{C}^{id.dom}$ | $Dom_R \times U$ |
| $\mathcal{C}^{id.ran}$ | $U \times Ran_R$ |

Figure 1: Examples of closure relation operations

of first-order logic such that $\mathtt{A}_{fol}$ has no function symbol and the only predicate symbols occurring in $\mathtt{A}_{fol}$ are the equality symbol = and R (both binary). We can assume without any loss of generality that the only logical connectives are $\wedge, \vee, \neg$. A class $\mathcal{R}$ of binary relations is said to be $\mathtt{A}_{fol}$-*definable* iff for all binary relations $R$ over $W$,

$$R \in \mathcal{R} \text{ iff } ((Dom_R \cup Ran_R), R) \models_{fol} \mathtt{A}_{fol}$$

A closure relation operation $\mathcal{C}$ is said to be *first-order definable* iff there is a closed first-order formula $\mathtt{A}_{fol}$ such that the unique $\mathcal{C}$-definable class $\mathcal{R}$ of binary relations is $\mathtt{A}_{fol}$-definable. Let $\mathcal{R}$ be a $\mathcal{C}$-definable class for some closure relation operation $\mathcal{C}$. If $\mathcal{R}$ is $\mathtt{A}_{fol}$-definable for some non-valid first-order formula $\mathtt{A}_{fol}$ then $\mathcal{R}$ is $\mathtt{A}'_{fol}$-definable for some universal formula $\mathtt{A}'_{fol}$ by the Los-Tarski preservation theorem. $\mathtt{A}'_{fol}$ is of the form $\forall x_1 \ \forall x_2 \ldots \forall x_k \ \mathtt{A}''_{fol}$ where $k \geq 0$, $x_1, \ldots, x_k$ are distinct variables and $\mathtt{A}''_{fol}$ is a quantifier-free formula. Indeed by the *Submodel Preservation Theorem*, $\mathtt{A}_{fol}$ is preserved by taking submodels iff $\mathtt{A}_{fol}$ is logically equivalent to a universal formula. Since $\mathcal{R}$ is closed under restrictions then $\mathtt{A}_{fol}$ is preserved by taking submodels.

Moreover, by Proposition 2.2, there is a *positive* occurrence of the predicate symbol R in $\mathtt{A}'_{fol}$. For suppose otherwise and take $R \notin \mathcal{R}$. Then $((Dom_R \cup Ran_R), R) \not\models_{fol} \forall x_1 \ \forall x_2 \ldots \forall x_k \ \mathtt{A}''_{fol}$ and there exist $w_1, \ldots, w_k \in W$ such that $((Dom_R \cup Ran_R), R) \models_{fol} \neg \mathtt{A}''_{fol} \ [x_1 \leftarrow w_1, \ldots, x_k \leftarrow w_k]$ and $\neg \mathtt{A}''_{fol}$ is logically equivalent to some formula $\bigwedge_{1 \leq i \leq n} (\bigvee_{1 \leq j \leq n_i} L_{i,j})$ in conjunctive normal form where each $L_{i,j}$ has one of the following forms: $x$=$y$, $\neg(x$=$y)$, $\mathtt{R}(x,y)$. Hence, for all $R \subseteq R' \subseteq (Dom_R \cup Ran_R) \times (Dom_R \cup Ran_R)$, $((Dom_R \cup$

$Ran_R), R') \models_{fol} \neg \mathtt{A}''_{fol}[x_1 \leftarrow w_1, \ldots, x_k \leftarrow w_k]$ and $((Dom_R \cup Ran_R), R') \not\models_{fol} \forall x_1\ \forall x_2 \ldots \forall x_k\ \mathtt{A}''_{fol}$, which is in contradiction with Proposition 2.2(1) -it is even possible that $\mathtt{R}$ does not occur in $\mathtt{A}''_{fol}$. Hence for all classes $\mathcal{R}$ of binary relations $\mathtt{A}_{fol}$-definable for some non-valid first-order formula $\mathtt{A}_{fol}$ in prenex normal form, if either $\mathtt{A}_{fol}$ is not *universal* or $\mathtt{A}_{fol}$ does not contain a *positive* occurrence of the predicate symbol $\mathtt{R}$ then there is no closure relation operation $\mathcal{C}$ such that the class $\mathcal{R}$ is $\mathcal{C}$-definable. For instance, although the class of irreflexive relations is closed under restrictions, it is not $\mathcal{C}$-definable for any closure relation operation $\mathcal{C}$.

From a given collection of monotonous relation operations (resp. closure relation operations), it is possible to define new monotonous relation operations (resp. closure relation operations). By way of example, consider the mapping $t_c : COR \rightarrow OR$ such that for all $\mathcal{C} \in COR$, for all sets $U$ and $R \subseteq U \times U$,

$$t_c(\mathcal{C})(U)(R) = \bigcup\{\mathcal{C}(C_x^R)(R \cap (C_x^R \times C_x^R)) : x \in U\}$$

where $C_x^R$ is the set $\{x\} \cup \{y \in U : \exists\ \mathcal{C}^s(R)\text{−path between } x \text{ and } y\}$. For any

| $\phi$ (arity) | Definition of $\phi(U)(R_1, \ldots, R_n)$ |
|---|---|
| $\cup$ (2) | $\{(x, y) :$ either $(x, y) \in R_1$ or $(x, y) \in R_2\}$ |
| $\cap$ (2) | $\{(x, y) : (x, y) \in R_1$ and $(x, y) \in R_2\}$ |
| $\cup^*$ (2) | $\cup \mathcal{C}^{rt}(R_1, R_2)$ |
| ; (2) | $\{(x, y) : \exists z, (x, z) \in R_1$ and $(z, y) \in R_2\}$ |
| $\|\|$ (2) | $\{(x, y) :$ either $(x, y) \in R_1$ or $(x, y) \in R_2$, and $R_1(x) \times R_2(y) \neq \emptyset\}$ |

Figure 2: Examples of monotonous relation operations

set $U$, $R \subseteq R' \subseteq U \times U$ and $x \in U$,

- $C_x^R \subseteq C_x^{R'}$, $R \cap (C_x^R \times C_x^R) \subseteq R' \cap (C_x^{R'} \times C_x^{R'})$,
- $\bigcup_{x \in U} R \cap (C_x^R \times C_x^R) = R$, $\{C_x^R : x \in U\}$ is a partition of $U$.

PROPOSITION 2.5. For any $\mathcal{C} \in COR$,

(1) for any set $U$, $R \subseteq U \times U$, $x \in U$, $C_x^R = C_x^{t_c(\mathcal{C})(U)(R)}$,

(2) for any set $U$, $R \subseteq R' \subseteq U \times U$, $t_c(\mathcal{C})(U)(R) \subseteq t_c(\mathcal{C})(U)(R')$ (monotonicity),

(3) for any set $U$, $R \subseteq U \times U$, $R \subseteq t_c(\mathcal{C})(U)(R)$,

(4) $t_c(\mathcal{C})$ is a closure relation operation.

For instance, the family of difunctional[9] relations is $t_c(\mathcal{C}^{L \times L'})$-definable. Moreover, although $t_c(\mathcal{C}^r) = \mathcal{C}^r$ we have $t_c(\nu) = \mathcal{C}^{rst}$.

[9] A relation $R$ over the set $U$ is *difunctional* iff for all $x \in U$, there exist $C_x^{R,1}$ and $C_x^{R,2}$ subsets of $C_x^R$ such that $R \cap (C_x^R \times C_x^R) = C_x^{R,1} \times C_x^{R,2}$.

The relation operations involve in the definition of the FiRe-logics shall be monotonous and most of them are also finitely reducible in the sense of Definition 2.4 below.

DEFINITION 2.4. A relation operation $\phi$ (with $n = ar(\phi)$) is said to be *finitely reducible* iff for any set $U$, $R_1, \ldots, R_n \subseteq U \times U$,

- if $(x, y) \in \phi(U)(R_1, \ldots, R_n)$
- then there is a *finite* set $U' \subseteq U$ and there exist $R'_1, \ldots, R'_n \subseteq U' \times U'$ such that
  - $(x, y) \in \phi(U')(R'_1, \ldots, R'_n)$ and,
  - for all $i \in \{1, \ldots, n\}$, $R'_i \subseteq R_i \cap U' \times U'$.

$\nabla$

## 3 Multi-modal logics

The various relation operations and $\mathcal{C}$-definable sets of binary relations defined in Section 2 shall be used to define classes of *modal frames*. In the forthcoming sections, we shall show how the properties of the relation operations induce properties of logics characterized by modal frames involving those operations. We believe that the relationships between relation operations and logics are the main original part of the present work. Before defining the (semantical) notion of logic we shall use, some (rather standard) preliminary definitions are needed.

A (propositional) *modal language* $\mathtt{L}$ is determined by four sets which are supposed to be pairwise disjoint:

- a non-empty countable set $\mathtt{F}_0$ of *propositional variables*,
- a non-empty countable set $\mathtt{M}_0$ of *modal constants*,
- a countable set $\mathtt{OP}$ of *propositional operators*,
- and a countable set $\mathtt{OM}$ of *modal operators* (which can be empty).

The set $\mathtt{M}$ of *modal expressions* is the smallest set that satisfies the following conditions: $\mathtt{M}_0 \subseteq \mathtt{M}$ and if $\oplus$ is any $n$-ary modal operator and $\mathsf{a}_0, \ldots, \mathsf{a}_{n-1} \in \mathtt{M}$ then $\oplus(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}) \in \mathtt{M}$. The set $\mathtt{F}_\mathtt{L}$ of $\mathtt{L}$-*formulae* is the smallest set that satisfies the following conditions: $\mathtt{F}_0 \subseteq \mathtt{F}_\mathtt{L}$, if $\ominus$ is any $n$-ary propositional operator and $\mathtt{A}_1, \ldots, \mathtt{A}_n \in \mathtt{F}_\mathtt{L}$ then $\ominus(\mathtt{A}_1, \ldots, \mathtt{A}_n) \in \mathtt{F}_\mathtt{L}$ and if $\mathsf{a} \in \mathtt{M}$ and $\mathtt{A} \in \mathtt{F}_\mathtt{L}$ then $\{[\mathsf{a}]\mathtt{A}, \langle\mathsf{a}\rangle\mathtt{A}\} \subseteq \mathtt{F}_\mathtt{L}$ -for the sake of simplicity $[\mathsf{a}]$ and $\langle\mathsf{a}\rangle$ are also called *modal operators*. We assume throughout the paper that any modal language used in the sequel satisfies the following conditions: $\mathtt{F}_0$ is a fixed countable set of propositional variables and the propositional operators are the unary

$\neg$ (negation) and the binary $\wedge$ (conjunction) and $\vee$ (disjunction). For any syntactic category $\mathtt{X}$, and for any syntactic object $\mathtt{O}$ we write $\mathtt{X}(\mathtt{O})$ to denote the set composed of elements of $\mathtt{X}$ occurring in $\mathtt{O}$. For example $\mathtt{F}_{\mathtt{L}}(\mathtt{A})$ is the set of subformulas of $\mathtt{A}$.

Let $\mathtt{L}$ be a modal language. We write $mw(\mathtt{A})$ to denote the *modal weight* of $\mathtt{A}$, i.e. the number of occurrences of modal operators in $\mathtt{A}$ -of the form $[\mathsf{a}]$ or $\langle \mathsf{a} \rangle$. We also write $md(\mathtt{A})$ to denote the *modal degree* of $\mathtt{A}$, i.e. the maximal number of modal operators that appear in front of a propositional variable in $\mathtt{A}$. We write $\overline{\mathtt{M}}(\mathtt{A})$ to denote the set $\{\mathsf{a} \in \mathtt{M} : \exists [\mathsf{a}]\mathtt{B} \in \mathtt{F}_{\mathtt{L}}(\mathtt{A}) \text{ or } \exists \langle \mathsf{a} \rangle \mathtt{B} \in \mathtt{F}_{\mathtt{L}}(\mathtt{A})\}$. Let $\mathtt{t}_{nnf}$ be the mapping $\mathtt{F}_{\mathtt{L}} \rightarrow \mathtt{F}_{\mathtt{L}}$ such that:

- $\mathtt{t}_{nnf}(\mathtt{p}) = \mathtt{p}$ , $\mathtt{t}_{nnf}(\neg \mathtt{p}) = \neg \mathtt{p}$ for all $\mathtt{p} \in \mathtt{F}_0$,
- $\mathtt{t}_{nnf}(\mathtt{A} \wedge \mathtt{B}) = \mathtt{t}_{nnf}(\mathtt{A}) \wedge \mathtt{t}_{nnf}(\mathtt{B})$, $\mathtt{t}_{nnf}(\mathtt{A} \vee \mathtt{B}) = \mathtt{t}_{nnf}(\mathtt{A}) \vee \mathtt{t}_{nnf}(\mathtt{B})$,
- $\mathtt{t}_{nnf}([\mathsf{a}]\mathtt{A}) = [\mathsf{a}]\mathtt{t}_{nnf}(\mathtt{A})$ , $\mathtt{t}_{nnf}(\langle \mathsf{a} \rangle \mathtt{A}) = \langle \mathsf{a} \rangle \mathtt{t}_{nnf}(\mathtt{A})$,
- $\mathtt{t}_{nnf}(\neg(\mathtt{A} \wedge \mathtt{B})) = \mathtt{t}_{nnf}(\neg \mathtt{A}) \vee \mathtt{t}_{nnf}(\neg \mathtt{B})$, $\mathtt{t}_{nnf}(\neg(\mathtt{A} \vee \mathtt{B})) = \mathtt{t}_{nnf}(\neg \mathtt{A}) \wedge \mathtt{t}_{nnf}(\neg \mathtt{B})$,
- $\mathtt{t}_{nnf}(\neg \langle \mathsf{a} \rangle \mathtt{A}) = [\mathsf{a}]\mathtt{t}_{nnf}(\neg \mathtt{A})$, $\mathtt{t}_{nnf}(\neg [\mathsf{a}]\mathtt{A}) = \langle \mathsf{a} \rangle \mathtt{t}_{nnf}(\neg \mathtt{A})$,
- $\mathtt{t}_{nnf}(\neg\neg \mathtt{A}) = \mathtt{t}_{nnf}(\mathtt{A})$.

We write $\mathtt{F}_{\mathtt{L}}^{\Diamond}$ (resp. $\mathtt{F}_{\mathtt{L}}^{\Box}$) to denote the set of formulae $\mathtt{A}$ such that $\mathtt{t}_{nnf}(\mathtt{A})$ does not contain any necessity modal operator (of the form $[\mathsf{a}]$) (resp. $\mathtt{t}_{nnf}(\mathtt{A})$ does not contain any possibility modal operator). The formulae of $\mathtt{F}_{\mathtt{L}}^{\Diamond}$ (resp. $\mathtt{F}_{\mathtt{L}}^{\Box}$) are said to be *$\Diamond$-formulae* (resp. *$\Box$-formulae*). The mapping $\mathtt{t}_{nnf}$ can be viewed as a procedure transforming any formula into a formula in negative normal form (nnf). It is easy to show that $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}^{\Diamond}$ (resp. $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}^{\Box}$) iff $\neg \mathtt{A} \in \mathtt{F}_{\mathtt{L}}^{\Box}$ (resp. $\neg \mathtt{A} \in \mathtt{F}_{\mathtt{L}}^{\Diamond}$). The set of $\Diamond$-formulae has numerous interesting properties (see e.g. [1]).

As usual, by an $\mathtt{L}$-*model* we understand a triple $(U, \{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}, V)$ such that

- $U$ is a non-empty set,
- for all $\mathsf{a} \in \mathtt{M}$, $\mathcal{R}_{\mathsf{a}}$ is a binary relation on $U$ and,
- $V$ is a mapping $\mathtt{F}_0 \rightarrow \mathcal{P}(U)$.

The set of $\mathtt{L}$-models is written $\mathcal{M}od_{\mathtt{L}}$. Let $\mathcal{M} = (U, \{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}, V)$ be an $\mathtt{L}$-model. As usual, we say that a formula $\mathtt{A}$ is *satisfied by the object $u \in U$ in $\mathcal{M}$* (written $\mathcal{M}, u \models \mathtt{A}$) when the following conditions are satisfied.

- $\mathcal{M}, u \models \mathtt{p}$ iff $u \in V(\mathtt{p})$, for all $\mathtt{p} \in \mathtt{F}_0$,
- $\mathcal{M}, u \models \neg \mathtt{A}$ iff not $\mathcal{M}, u \models \mathtt{A}$,
- $\mathcal{M}, u \models \mathtt{A} \wedge \mathtt{B}$ iff $\mathcal{M}, u \models \mathtt{A}$ and $\mathcal{M}, u \models \mathtt{B}$,

- $\mathcal{M}, u \models \mathtt{A} \vee \mathtt{B}$ iff either $\mathcal{M}, u \models \mathtt{A}$ or $\mathcal{M}, u \models \mathtt{B}$,
- $\mathcal{M}, u \models [\mathsf{a}]\mathtt{A}$ iff for all $v \in \mathcal{R}_{\mathsf{a}}(u)$, $\mathcal{M}, v \models \mathtt{A}$,
- $\mathcal{M}, u \models \langle\mathsf{a}\rangle\mathtt{A}$ iff there is $v \in \mathcal{R}_{\mathsf{a}}(u)$ such that $\mathcal{M}, v \models \mathtt{A}$.

As usual, the modal operators $\langle\mathsf{a}\rangle$ and $[\mathsf{a}]$ are not independent, each of them can be defined in terms of the other. Moreover, $\mathcal{M}, u \models \mathtt{A}$ iff $\mathcal{M}, u \models \mathtt{t}_{nnf}(\mathtt{A})$ for any $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}$. In the sequel, when it is possible, only the operators of the form $\langle\mathsf{a}\rangle$ are used. A formula $\mathtt{A}$ is said to be *true* in the $\mathtt{L}$-model $\mathcal{M} = (U, \{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}, V)$ iff for all $u \in U$, $\mathcal{M}, u \models \mathtt{A}$.

By a *logic* $\mathcal{L}$, we understand a triple[10] $\langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ such that $\mathtt{L}$ is a modal language, $\mathcal{S} \subseteq \mathcal{M}od_{\mathtt{L}}$ and $\models_{\mathcal{L}}$ is the restriction of $\models$ to the sets $\mathcal{S}$ and $\mathtt{L}$ (satisfiability relation). For all models $\mathcal{M} \in \mathcal{S}$, $\mathcal{M}$ is said to be a *model for* $\mathcal{L}$. An $\mathtt{L}$-formula $\mathtt{A}$ is said to be $\mathcal{L}$*-valid* iff $\mathtt{A}$ is true in all $\mathtt{L}$-models of $\mathcal{S}$. An $\mathtt{L}$-formula $\mathtt{A}$ is said to be $\mathcal{L}$*-satisfiable* iff there is $\mathcal{M} = (U, \{\mathcal{R}_a : a \in \mathtt{M}\}, V) \in \mathcal{S}$, $u \in U$ such that $\mathcal{M}, u \models_{\mathcal{L}} \mathtt{A}$. A logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ has the *finite model property* with respect to the set $X \subseteq \mathtt{F}_{\mathtt{L}}$ iff for every $\mathcal{L}$-satisfiable formula $\mathtt{A} \in X$, there exist $\mathcal{M} = (U, \{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}, V) \in \mathcal{S}$, $w \in U$ such that $U$ is finite and $\mathcal{M}, w \models_{\mathcal{L}} \mathtt{A}$. We shall use in the sequel the following notion of translation between two logics.

Definition 3.1. A *translation* from $\mathcal{L}_1 = \langle \mathtt{L}_1, \mathcal{S}_1, \models_1 \rangle$ to $\mathcal{L}_2 = \langle \mathtt{L}_2, \mathcal{S}_2, \models_2 \rangle$ is a computable mapping $t : \mathtt{F}_{\mathtt{L}_1} \rightarrow \mathtt{F}_{\mathtt{L}_2}$ such that

(1) for all $\mathtt{A} \in \mathtt{F}_{\mathtt{L}_1}$, $t(\neg\mathtt{A}) = \neg t(\mathtt{A})$,

(2) for all $\mathcal{M}_1 = (U, \{\mathcal{R}^1_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}_1\}, V) \in \mathcal{S}_1$, there is $\mathcal{M}_2 = (U, \{\mathcal{R}^2_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}_2\}, V) \in \mathcal{S}_2$ such that $(\star)$ for all $\mathtt{A} \in \mathtt{F}_{\mathtt{L}_1}$, $u \in U$, $\mathcal{M}_1, u \models \mathtt{A}$ iff $\mathcal{M}_2, u \models t(\mathtt{A})$.

(3) for all $\mathcal{M}_2 = (U, \{\mathcal{R}^2_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}_2\}, V) \in \mathcal{S}_2$, there is $\mathcal{M}_1 = (U, \{\mathcal{R}^1_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}_1\}, V) \in \mathcal{S}_1$ such that $(\star)$.

The translation $t$ is said to be $\Diamond$*-preserving* iff for all $\mathtt{A} \in \mathtt{F}_{\mathtt{L}_1}$, $\mathtt{A} \in \mathtt{F}^{\Diamond}_{\mathtt{L}_1}$ iff $t(\mathtt{A}) \in \mathtt{F}^{\Diamond}_{\mathtt{L}_2}$. $\nabla$

Such translations are semantically stronger than the ones usually found in the literature which is shown in the proposition below.

Proposition 3.1. Assume there is a translation $t$ (in the sense of Definition 3.1) from $\mathcal{L}_1$ to $\mathcal{L}_2$. Then,

(1) For all $\mathtt{L}_1$-formulae $\mathtt{A}$,

(a) $\mathtt{A}$ is $\mathcal{L}_1$-valid iff $t(\mathtt{A})$ is $\mathcal{L}_2$-valid.

[10] It is also possible to define a logic in terms of *frames* (structures of the form $(U, \{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\})$) but the definition of logic used in the paper shall be sufficient for our needs.

(b) $\mathtt{A}$ is $\mathcal{L}_1$-satisfiable iff $t(\mathtt{A})$ is $\mathcal{L}_2$-satisfiable.

(2) If $\mathcal{L}_2$ has the finite model property with respect to $X$ (resp. a decidable validity problem, a decidable satisfiability problem) then $\mathcal{L}_1$ has the finite model property with respect to $\{\mathtt{A} \in \mathtt{F}_{\mathtt{L}_1} : \exists \mathtt{B} \in X, t(\mathtt{A}) = \mathtt{B}\}$ (resp. a decidable validity problem, a decidable satisfiability problem).

# 4 Finitely reducible logics

In this section we apply the formal framework developed in Section 2 to define classes of modal logics. We show that a broad family of logics can be uniformly presented within this framework.

An *operator interpretation function* $\mathcal{I}$ for the language $\mathtt{L}$ such that $\mathtt{OM} \neq \emptyset$, is a mapping $\mathcal{I} : \mathtt{OM} \to MOR$ such that the arities of $\mathcal{I}(\oplus)$ and $\oplus$ are equal for all $\oplus \in \mathtt{OM}$. If $\mathtt{OM} = \emptyset$, by definition the unique *operator interpretation function* $\mathcal{I}$ for $\mathtt{L}$ is the empty set. For a given language $\mathtt{L}$, and an operator interpretation function $\mathcal{I}$, the family $\{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ of binary relations over a set $U$ is said to *respect* $\mathcal{I}$ iff

$$\text{for all } \mathtt{a}_1, \ldots, \mathtt{a}_n \in \mathtt{M}, \oplus^n \in \mathtt{OM}, R_{\oplus^n(\mathtt{a}_1,\ldots,\mathtt{a}_n)} = \mathcal{I}(\oplus^n)(U)(\mathcal{R}_{\mathtt{a}_1}, \ldots, \mathcal{R}_{\mathtt{a}_n})$$

Observe that if $\mathcal{I} = \emptyset$ then every $\mathtt{L}$-model respects $\mathcal{I}$. A family $\{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ respecting $\mathcal{I}$ is completely determined by the family $\{\mathcal{R}_{\mathtt{c}} : \mathtt{c} \in \mathtt{M}_0\}$ and $U$. By extension, we say that the $\mathtt{L}$-model $(U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ respects $\mathcal{I}$ iff $\{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ respects $\mathcal{I}$. For a given language $\mathtt{L}$, and an operator interpretation function $\mathcal{I}$, we write $\mathcal{M}od_{\mathtt{L}}^{\mathcal{I}}$ to denote the set of $\mathtt{L}$-models respecting $\mathcal{I}$. For a given language $\mathtt{L}$ and a mapping $\mathcal{C} : \mathtt{M}_0 \to COR$, we write $\mathcal{M}od_{\mathtt{L}}^{\mathcal{C}}$ to denote the set of $\mathtt{L}$-models $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ such that

$$\text{for all } \mathtt{c} \in \mathtt{M}_0, \mathcal{C}(\mathtt{c})(U)(\mathcal{R}_{\mathtt{c}}) = \mathcal{R}_{\mathtt{c}}.$$

As a consequence, for all $\mathtt{c} \in \mathtt{M}_0$, $\mathcal{C}(\mathtt{c})((Dom_{\mathcal{R}_{\mathtt{c}}} \cup Ran_{\mathcal{R}_{\mathtt{c}}}))(\mathcal{R}_{\mathtt{c}}) = \mathcal{R}_{\mathtt{c}}$. We use the symbol $\mathcal{C}$ both for this mapping and for any member of $COR$ because for each $\mathtt{c}$, $\mathcal{C}(\mathtt{c}) \in COR$.

Let $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ be an $\mathtt{L}$-model respecting $\mathcal{I}$. It can be shown that for all $\mathtt{a} \in \mathtt{M}$, $\mathtt{a}' \in \mathtt{M}(\mathtt{a})$, $\mathtt{b}, \mathtt{b}' \in \mathtt{M}$ such that $\mathtt{b}$ is obtained from the modal expression $\mathtt{a}$ by replacing a given occurrence of $\mathtt{a}'$ in $\mathtt{a}$ by $\mathtt{b}'$ and $\mathcal{R}_{\mathtt{a}'} \subseteq \mathcal{R}_{\mathtt{b}'}$, we have $\mathcal{R}_{\mathtt{a}} \subseteq \mathcal{R}_{\mathtt{b}}$. This is due to the monotonicity of the relation operations in $\{\mathcal{I}(\oplus) : \oplus \in \mathtt{OM}\}$ when $\mathtt{OM} \neq \emptyset$.

We shall introduce a particular class of logics, the *FiRe-logics*. First, some preliminary definitions are needed.

Definition 4.1. A logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ is said to be a *preFiRe-logic* iff there exist an operator interpretation function $\mathcal{I}$ for $\mathtt{L}$ and a mapping $\mathcal{C} : \mathtt{M}_0 \rightarrow COR$ such that $\mathcal{S}_{\mathcal{L}} = \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}$. $\nabla$

An example of preFiRe-logic is the propositional dynamic logic PDL (see, e.g. [20]) that can be seen as a structure $\langle \mathtt{L}, \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}}, \models_{PDL} \rangle$ with $\mathtt{OM} = \{^*, \cup, \circ\}$, $\mathtt{M}_0$ is a countable set of program constants, $\mathcal{C} : \mathtt{M}_0 \rightarrow \{\mathcal{C}^i\}$, $\mathcal{I}(^*) = \mathcal{C}^{rt}$, $\mathcal{I}(\circ) =;$ and $\mathcal{I}(\cup) = \cup$. It is of particular interest to notice that the canonical model for PDL (see [21]) does not belong to $\mathcal{M}od^{\mathcal{C}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}}$.

Definition 4.2. For a given modal language $\mathtt{L}$, and an operator interpretation function $\mathcal{I}$, the $\mathtt{L}$-model $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ respecting $\mathcal{I}$ is said to be *well-founded with respect to* $\mathcal{I}$ iff for all $x, y \in U$, $\mathtt{a} \in \mathtt{M}$ such that $(x, y) \in \mathcal{R}_{\mathtt{a}}$ there exist a *finite* subset $\{x, y\} \subseteq X^{\mathtt{a}}_{x,y} \subseteq U$ and a family $\{P_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ of binary relations over $X^{\mathtt{a}}_{x,y}$ respecting $\mathcal{I}$ such that

(1) $(x, y) \in P_{\mathtt{a}}$ and

(2) for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $P_{\mathtt{c}} \subseteq \mathcal{R}_{\mathtt{c}}$.

$\nabla$

Definition 4.3. A preFiRe-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ is said to be a *FiRe-logic* (*finitely reducible logic*) iff every $\mathtt{L}$-model in $\mathcal{S}_{\mathcal{L}}$ is well-founded with respect to $\mathcal{I}$. $\nabla$

The condition of well-foundness states that for each model of the logic, if $(x, y) \in \mathcal{R}_{\mathtt{a}}$ then there exist a *finite* subset $X$ of worlds and a family $\{P_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ of binary relations over $X$ respecting $\mathcal{I}$ such that for each modal constant $\mathtt{c}$ occuring in $\mathtt{a}$, $P_{\mathtt{c}} \subseteq \mathcal{R}_{\mathtt{c}}$. In short, everytime $(x, y) \in \mathcal{R}_{\mathtt{a}}$ holds, we can select a finite number of *worlds* and *arrows* in order to preserve this relationship in the reduced model. Observe that we do not necessarily require that

$$\text{for all } \mathtt{c}' \in \mathtt{M}_0, \mathcal{C}(\mathtt{c}')(X^{\mathtt{a}}_{x,y})(P_{\mathtt{c}'}) = P_{\mathtt{c}'}.$$

Indeed, once $\{P_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ exists, the *unique* family $\{P'_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ respecting $\mathcal{I}$ such that for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$ $P'_{\mathtt{c}} = \mathcal{C}(\mathtt{c})(X^{\mathtt{a}}_{x,y})(P'_{\mathtt{c}})$ also satisfies: (1) $(x, y) \in P'_{\mathtt{a}}$ and (2) for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $P'_{\mathtt{c}} \subseteq \mathcal{R}_{\mathtt{c}}$. The constraints for the modal constants in the definition of PreFire-logics do not interfer with the well-foundness of the models but they allow to capture a larger class of logics. The logic PDL previously mentioned is a FiRe-logic (see in Section 5 a formal proof of this fact).

Proposition 4.1. If the preFiRe-logic $\langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}} \rangle$ is a FiRe-logic then for all $\mathcal{C}' : \mathtt{M}_0 \rightarrow COR$, $(\mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}'}_{\mathtt{L}}, \models_{\mathcal{L}})$ is also a FiRe-logic.

Although it is obvious that every FiRe-logic is a preFiRe-logic, it is not straightforward to see that there are preFiRe-logics that are not FiRe-logics. Consider the relation operation $\phi^{\infty}$ defined as follows: for any set $U$, $R \subseteq U \times U$, $\phi^{\infty}(U)(R) = \{(x,y) \in R : \exists$ an infinite $R$-path between $x$ and $y$ without cycles$\}$. For example $\phi^{\infty}(\mathtt{Re})(<) =<$ (resp. $\phi^{\infty}(\omega)(<) = \emptyset$) where $\mathtt{Re}$ is the set of real numbers (resp. $\omega$ is the set of natural numbers) and $<$ is the usual strict greater-than relation. Moreover, for any *finite* set $U$, $R \subseteq U \times U$, $\phi^{\infty}(U)(R) = \emptyset$. It is easy to show that $\phi^{\infty}$ is monotonous

Now consider the language $\mathtt{L}_0$ such that $\mathtt{M}_0 = \{\mathtt{1}\}$, $\mathtt{OM} = \{\mathtt{op}\}$ and the operator interpretation function $\mathcal{I}_0 : \{\mathtt{op}\} \rightarrow \{\phi^{\infty}\}$. We shall show that the preFiRe-logic $\mathcal{L}_0 = \langle \mathtt{L}_0, \mathcal{M}od^{\mathcal{I}_0}_{\mathtt{L}_0} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}_0}, \models_{\mathcal{L}_0}\rangle$ is not a FiRe-logic with $\mathcal{C}(\mathtt{1}) = \mathcal{C}^i$. Actually, take the $\mathtt{L}_0$-model $(\mathtt{Re}, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ such that $\mathcal{R}_{\mathtt{1}} =<$ and for all $\mathtt{p} \in \mathtt{F}_0$, $V(\mathtt{p}) = \emptyset$. It is clear that $(0,1) \in \mathcal{R}_{\mathtt{op(1)}} = \mathcal{I}_0(\mathtt{op})(\mathtt{Re})(\mathcal{R}_{\mathtt{1}}) = \phi^{\infty}(\mathtt{Re})(<)$. However, for any finite subset $\{0,1\} \subseteq X^{\mathtt{op(1)}}_{0,1} \subseteq \mathtt{Re}$ and for any relation $P_{\mathtt{1}} \subseteq X^{\mathtt{op(1)}}_{0,1} \times X^{\mathtt{op(1)}}_{0,1}$, $(0,1) \notin \phi^{\infty}(X^{\mathtt{op(1)}}_{0,1})(P_{\mathtt{1}}) = \emptyset$. Hence $(\mathtt{Re}, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ is not well-founded with respect to $\mathcal{I}_0$ and $\mathcal{L}_0$ is not a FiRe-logic although it is a preFiRe-logic.

## 5 Some properties of FiRe-logics

The class of well-founded models can be partly characterized by properties of monotonous relation operations involved in the operator interpretation functions. Proposition 5.1 below relates Definition 4.2 with Definition 2.4: a sufficient condition for the well-foundedness of models is to admit only finitely reducible monotonous relation operations.

PROPOSITION 5.1. Let $\mathcal{L} = (\mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}})$ be a preFiRe-logic. If $\{\mathcal{I}(\oplus) : \oplus \in \mathtt{OM}\}$ contains only finitely reducible relation operations then $\mathcal{L}$ is a FiRe-logic.

PROOF: Let $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ be an $\mathcal{L}$-model and $x, y \in U$. We show by structural induction on $\mathtt{a}$ that if $(x,y) \in \mathcal{R}_{\mathtt{a}}$ then there exists a family $\{r_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ over a finite set $X \subseteq U$, respecting $\mathcal{I}$, such that for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $r_{\mathtt{c}} \subseteq \mathcal{R}_{\mathtt{c}}$ and $(x,y) \in r_{\mathtt{a}}$.
*Base case*: assume $(x,y) \in \mathcal{R}_{\mathtt{c}}$ for some $\mathtt{c} \in \mathtt{M}_0$. Define for all $\mathtt{c}' \in \mathtt{M}_0$, $r_{\mathtt{c}'} = \{(x,y)\}$ and $X = \{x,y\}$. Let $\{r_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ be the *unique* family over $X$ respecting $\mathcal{I}$ that extends $\{r_{\mathtt{c}'} : \mathtt{c}' \in \mathtt{M}_0\}$.
*Induction step*: assume $(x,y) \in \mathcal{R}_{\oplus(\mathtt{a}_1,\ldots,\mathtt{a}_n)}$ with $n = ar(\mathcal{I}(\oplus))$. Hence $(x,y) \in \mathcal{I}(\oplus)(U)(\mathcal{R}_{\mathtt{a}_1}, \ldots, \mathcal{R}_{\mathtt{a}_n})$. Since $\mathcal{I}(\oplus)$ is finitely reducible there exists a finite set $U' \subseteq U$, $R_1, \ldots, R_n \subseteq U' \times U'$ such that

- $(x,y) \in \mathcal{I}(\oplus)(U')(R_1, \ldots, R_n)$ and,

- for all $i \in \{1, \ldots, n\}$, $R_i \subseteq \mathcal{R}_{\mathtt{a}_i}$.

We write $\bigcup_{i \in \{1,\ldots,n\}} R_i = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ ($U'$ is finite) and we write $I$ to denote the set $I = \{\langle i, j\rangle : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}, (x_j, y_j) \in R_i\}$. By induction hypothesis, for $\langle i, j\rangle \in I$, there exists a family $\{r^{i,j}_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ over a finite set $U_{i,j} \subseteq U$ respecting $\mathcal{I}$ such that for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a}_j)$, $r^{i,j}_{\mathtt{c}} \subseteq \mathcal{R}_{\mathtt{c}}$ and $(x_j, y_j) \in r^{i,j}_{\mathtt{a}_i}$. Define $X = (\bigcup\{U_{i,j} : \langle i, j\rangle \in I\}) \cup U'$ and for all $\mathtt{c} \in \mathtt{M}_0$, $r_{\mathtt{c}} = \bigcup\{r^{i,j}_{\mathtt{c}} : \langle i, j\rangle \in I\}$. Let $\{r_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ be the unique family over $X$ respecting $\mathcal{I}$ that extends $\{r_{\mathtt{c}} : \mathtt{c} \in \mathtt{M}_0\}$. By way of example, we show that $(x, y) \in r_{\mathtt{a}}$. By structural induction on $\mathtt{a}$, one can show that for $\langle i, j\rangle \in I$, $r^{i,j}_{\mathtt{a}} \subseteq r_{\mathtt{a}}$. Hence for $i_0 \in \{1, \ldots, n\}$,

$$R_{i_0} \subseteq \bigcup_{\langle i_0, j\rangle \in I} r^{i_0,j}_{\mathtt{a}_{i_0}} \subseteq r_{\mathtt{a}_{i_0}}$$

Since $\mathcal{I}(\oplus)$ is strongly monotonous,

$$\mathcal{I}(\oplus)(U')(R_1, \ldots, R_n) \subseteq \mathcal{I}(\oplus)(X)(R_1, \ldots, R_n) \subseteq \mathcal{I}(\oplus)(X)(r_{\mathtt{a}_1}, \ldots, r_{\mathtt{a}_n})$$

So $(x, y) \in r_{\mathtt{a}}$. Q.E.D.

The converse of Proposition 5.1 is an open problem. The following proposition can be easily proved.

PROPOSITION 5.2. The following operations are finitely reducible relation operations: $\{\mathcal{C}^i, \mathcal{C}^r, \mathcal{C}^{wr}, \mathcal{C}^s, \mathcal{C}^t, \mathcal{C}^{ar}, \mathcal{C}^e, \cup, \cap, ;, ||, \nu, \mathcal{C}^{id.dom}, \mathcal{C}^{id.ran}, \mathcal{C}^{l\times L}, \mathcal{C}^{L\times l}, \mathcal{C}^{L\times L}, \mathcal{C}^{L\times L'}\}$.

Proposition 5.2 is restricted to a given class of monotonous relation operations although a natural question arises: how to define new finitely reducible monotonous relation operations from a *basic* set of monotonous relation operations? This is the purpose of Proposition 5.3. Before stating it, a preliminary definition is needed.

DEFINITION 5.1. Let $\mathtt{L}$ (resp. $\mathtt{L}'$) be a modal language and $\mathcal{I}$ (resp. $\mathcal{I}'$) be an operator interpretation function on $\mathtt{L}$ (resp. on $\mathtt{L}'$). We say that $\mathcal{I}'$ is *less expressive than* $\mathcal{I}$ iff there exists a 1-1 mapping $t : \mathtt{M}'_0 \to \mathtt{M}_0$ and an extending mapping $t : \mathtt{M}' \to \mathtt{M}$ such that

- for all $\mathtt{a} \in \mathtt{M}'$, $\mathtt{M}_0(t(\mathtt{a})) = \{t(\mathtt{c}) : \mathtt{c} \in \mathtt{M}'_0(\mathtt{a})\}$,
- for any family $\{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$ of binary relations over $U$ respecting $\mathcal{I}$ and $\{\mathcal{R}'_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}'\}$ of binary relations over $U$ respecting $\mathcal{I}'$ such that for all $\mathtt{c} \in \mathtt{M}'_0$, $\mathcal{R}'_{\mathtt{c}} = \mathcal{R}_{t(\mathtt{c})}$, we have for all $\mathtt{a} \in \mathtt{M}'$, $\mathcal{R}'_{\mathtt{a}} = \mathcal{R}_{t(\mathtt{a})}$.

$\nabla$

With such a definition we have the possibility to express that a monotonous relation operation can be defined from other relation operations by means of the interpretations of the languages $\mathtt{L}$ and $\mathtt{L}'$. The relation "less expressive than" is transitive.

PROPOSITION 5.3. Let $\mathtt{L}$ (resp. $\mathtt{L}'$) be a modal language and $\mathcal{I}$ (resp. $\mathcal{I}'$) be an operator interpretation function on $\mathtt{L}$ (resp. on $\mathtt{L}'$) such that every $\mathtt{L}$-model respecting $\mathcal{I}$ is well-founded with respect to $\mathcal{I}$ and $\mathcal{I}'$ is less expressive than $\mathcal{I}$. Then,

(1) every $\mathtt{L}'$-model respecting $\mathcal{I}'$ is well-founded with respect to $\mathcal{I}'$,

(2) For any $\mathcal{C} : \mathtt{M}'_0 \rightarrow COR$, every logic $\langle \mathtt{L}', \mathcal{M}od^{\mathcal{I}'}_{\mathtt{L}'} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}'}, \models \rangle$ is a FiRe-logic.

PROOF: (1) Let $\{\mathcal{R}'_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}'\}$ be a family of binary relations over $U$ respecting $\mathcal{I}'$. Define the unique family $\{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}$ of binary relations over $U$ respecting $\mathcal{I}$ such that for all $\mathsf{c} \in \mathtt{M}_0$, $\mathcal{R}_{\mathsf{c}} = \mathcal{R}'_{t^{-1}(\mathsf{c})}$.

So for all $\mathsf{a} \in \mathtt{M}'$, $\mathcal{R}'_{\mathsf{a}} = \mathcal{R}_{t(\mathsf{a})}$. Assume $(x, y) \in \mathcal{R}'_{\mathsf{a}}$. By hypothesis,

- $(x, y) \in \mathcal{R}_{t(\mathsf{a})}$ and,
- $\{\mathcal{R}_{\mathsf{a}} : \mathsf{a} \in \mathtt{M}\}$ is well-founded with respect to $\mathcal{I}$.

So there is a family $\{r_{\mathsf{b}} : \mathsf{b} \in \mathtt{M}\}$ over $\{x, y\} \subseteq X^{t(\mathsf{a})}_{x,y} \subseteq U$, respecting $\mathcal{I}$ such that

- $(x, y) \in r_{t(\mathsf{a})}$ and,
- for all $\mathsf{c}' \in \mathtt{M}_0(t(\mathsf{a}))$, $r_{\mathsf{c}'} \subseteq \mathcal{R}_{\mathsf{c}'}$.

Define the family $\{r'_{\mathsf{b}} : \mathsf{b} \in \mathtt{M}'\}$ respecting $\mathcal{I}'$ such that for all $\mathsf{c} \in \mathtt{M}'_0$, $r'_{\mathsf{c}} = r_{t(\mathsf{c})}$ and take $X^{\mathsf{a}}_{x,y} = X^{t(\mathsf{a})}_{x,y}$. It follows that $r'_{\mathsf{a}} = r_{t(\mathsf{a})}$ ($\mathcal{I}'$ is less expressive than $\mathcal{I}$) and for all $\mathsf{c}' \in \mathtt{M}'_0(\mathsf{a})$, $r'_{\mathsf{c}'} = r_{t(\mathsf{c}')} \subseteq \mathcal{R}_{t(\mathsf{c}')} = \mathcal{R}'_{\mathsf{c}'}$.
(2) Direct consequence of (1) (see Definition 4.2 and Proposition 4.1). Q.E.D.

Let $\Phi$ be a non-empty countable subset of $MOR$ and $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}} \rangle$ be a preFiRe-logic for some $\mathcal{C} : \mathtt{M}_0 \rightarrow COR$ and for some operator interpretation function $\mathcal{I}$ such that $\{\mathcal{I}(\oplus) : \oplus \in \mathtt{OM}\} \subseteq \Phi$. $\mathcal{L}$ is said to be a $\Phi$-*logic*. A $\Phi$-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}} \rangle$ is $\Phi$-*complete* iff $\{\mathcal{I}(\oplus) : \oplus \in \mathtt{OM}\} = \Phi$. A preFiRe-logic $\mathcal{L}' = (\mathtt{L}', \mathcal{M}od^{\mathcal{I}'}_{\mathtt{L}'} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}'}, \models')$ is said to be $\Phi$-*composed* for some $\Phi \subseteq MOR$ iff there is a $\Phi$-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models \rangle$ such that $\mathcal{I}'$ is less expressive than $\mathcal{I}$.

PROPOSITION 5.4. For any non-empty countable subset $\Phi$ of $MOR$, there is a $\Phi$-complete logic being a FiRe-logic iff every $\Phi$-composed logic is a FiRe-logic.

PROOF: Since $\Phi$ is non-empty if every $\Phi$-composed logic is an FiRe-logic, in particular every $\Phi$-complete logic satisfies this property. Since there is a $\Phi$-complete logic (simply take one modal operator for each element of $\Phi$), one way is proved. Now assume there is a $\Phi$-complete logic being a FiRe-logic, say $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models \rangle$. For any $\Phi$-composed logic, $\mathcal{L}' = \langle \mathtt{L}', \mathcal{M}od^{\mathcal{I}'}_{\mathtt{L}'} \cap$

$\mathcal{M}od^{\mathcal{C}'}_{\mathtt{L}'}, \models'\rangle$, there is a $\Phi$-logic $\mathcal{L}'' = \langle \mathtt{L}'', \mathcal{M}od^{\mathcal{I}''}_{\mathtt{L}''} \cap \mathcal{M}od^{\mathcal{C}''}_{\mathtt{L}''}, \models''\rangle$ such that $\mathcal{I}'$ is less expressive than $\mathcal{I}''$. Since $\mathcal{I}''$ is less expressive than $\mathcal{I}$ ($\{\mathcal{I}''(\oplus) : \oplus \in \mathtt{OM}''\} \subseteq \{\mathcal{I}(\oplus) : \oplus \in \mathtt{OM}\} = \Phi$), by transitivity $\mathcal{I}'$ is less expressive than $\mathcal{I}$. By Proposition 5.3, $\mathcal{L}'$ is a FiRe-logic. Q.E.D.

By Proposition 5.4, we have therefore defined a process of introducing new FiRe-logics.

Corollary 5.5. Let $\Phi_0$ be the set below

$$\{\mathcal{C}^i, \mathcal{C}^r, \mathcal{C}^{wr}, \mathcal{C}^{ar}, \mathcal{C}^s, \mathcal{C}^t, \mathcal{C}^e, \cup, \cap, ;, ||, \nu, \mathcal{C}^{id.dom}, \mathcal{C}^{id.ran}, \mathcal{C}^{l\times L}, \mathcal{C}^{L\times l}, \mathcal{C}^{L\times L}, \mathcal{C}^{L\times L'}\}$$

Every $\Phi_0$-composed preFiRe-logic is a FiRe-logic.

By a *FiRe-reducible logic*, we understand a logic $\mathcal{L}$ such that there exist a FiRe-logic $\mathcal{L}'$ and a translation from $\mathcal{L}$ to $\mathcal{L}'$. Consider the logic $\mathcal{L}' = \langle \mathtt{L}', \mathcal{S}', \models'\rangle$ such that, $\mathtt{M}'_0 = \{\mathtt{1}, \mathtt{2}, \mathtt{3}\}$, $\mathtt{OM}' = \emptyset$, and for all $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}'\}, V) \in \mathcal{M}od_{\mathtt{L}'}$, $\mathcal{M} \in \mathcal{S}'$ iff

- $R_1$ is Euclidean,
- $R_2$ is symmetric and,
- $R_3 = R_1 \cap R_2$.

It can be easily shown that $\mathcal{L}'$ is FiRe-reducible although not being a preFiRe-logic. Consider the preFiRe-logic $\mathcal{L} = (\mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}})$ such that $\mathtt{M}_0 = \{\mathtt{1}, \mathtt{2}\}$, $\mathtt{OM} = \{\mathtt{inter}\}$, $\mathcal{C}(\mathtt{1}) = \mathcal{C}^e$, $\mathcal{C}(\mathtt{2}) = \mathcal{C}^s$ and $\mathcal{I}(\mathtt{inter}) = \cap$. $\mathcal{L}$ is a FiRe-logic by Proposition 5.4. Let $t : \mathtt{F}_{\mathtt{L}'} \to \mathtt{F}_{\mathtt{L}}$ be the mapping, homomorphic for the propositional operators $\wedge, \vee, \neg$ such that $t(\langle \mathtt{1}\rangle \mathtt{A}) = \langle \mathtt{1}\rangle t(\mathtt{A})$, $t(\langle \mathtt{2}\rangle \mathtt{A}) = \langle \mathtt{2}\rangle t(\mathtt{A})$ and $t(\langle \mathtt{3}\rangle \mathtt{A}) = \langle \mathtt{1\ inter\ 2}\rangle t(\mathtt{A})$. It is straightforward that $t$ is a $\diamond$-preserving translation, in the sense of Definition 3.1, from $\mathcal{L}'$ to $\mathcal{L}$.

In the forthcoming sections we shall show that every FiRe-reducible logic with a $\diamond$-preserving translation (and *a fortiori* every FiRe-logic) and every LaFiRe-reducible logic with a $\diamond$-preserving translation (and *a fortiori* every LaFiRe-logic) -see Definition 7.1- has the finite model property with respect to the set of $\diamond$-formulae. Figure 3 presents logics belonging to one of the classes previously defined (the references are given in order to provide a reader with useful pointers to the literature although they are not supposed to contain the original utterance of the logics).

In figure 3, the logic PDL with intersection does not admit the operator '?'.

# 6 Finite model property with respect to $\mathtt{F}^{\diamond}_{\mathtt{L}}$

Throughout this section $\mathcal{L}$ is a FiRe-logic $\langle \mathtt{L}, \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}, \models_{\mathcal{L}}\rangle$ for some operator interpretation function $\mathcal{I}$ and for some mapping $\mathcal{C} : \mathtt{M}_0 \to COR$. The

| Name | Extended name | Reference | Class |
|---|---|---|---|
| S4+S5 | | [14] | FiRe-logic |
| S4+deontic | | [14] | FiRe-reducible |
| PDL | Propositional Dynamic Logic | [21] | FiRe-logic |
| PDL+$\cap$ | PDL with intersection | [7] | FiRe-logic |
| DAL | Data Analysis Logic | [12] | FiRe-logic |
| DALLA | DAL with Local Agreement | [15] | LaFiRe-logic |
| K5$_n$ | Multi K5 | [4] | FiRe-logic |
| T$_n$, S4$_n$, S5$_n$ | simple multimodal logics | [16] | FiRe-logic |
| T$_n^C$, S4$_n^C$, S5$_n^C$ | multimodal logics with common knowledge | [16] | FiRe-reducible |
| T$_n^D$, S4$_n^D$, S5$_n^D$ | multimodal logics with distributed knowledge | [16] | FiRe-reducible |
| S4+5 | | [23] | FiRe-reducible |
| LA-logics | logics with local agreement | [8] | LaFiRe-reducible |

Figure 3: Some known logics from the literature

main goal of this section is to show that $\mathcal{L}$ has the finite model property with respect to $\mathtt{F}_{\mathtt{L}}^{\diamond}$. To do so, we prove that when an $\mathcal{L}$-model $\mathcal{M}'$ satisfies $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}$, we can build a finite $\mathcal{L}$-model $\mathcal{M}^*$ from $\mathcal{M}'$ that satisfies $\mathtt{A}$. The construction of $\mathcal{M}^*$ uses a significant variant of the technique introduced in [5].

Definition 6.1. Let $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}$. The set $LS(\mathtt{A})$ of formulae is the smallest set satisfying: (1) $\mathtt{A} \in LS(\mathtt{A})$, (2) if $\mathtt{B} \in LS(\mathtt{A})$ and $\mathtt{B} = \mathtt{C} \wedge \mathtt{C}'$ or $\mathtt{B} = \mathtt{C} \vee \mathtt{C}'$ then $\{\mathtt{C}, \mathtt{C}'\} \subseteq LS(\mathtt{A})$ and if $\mathtt{B} \in LS(\mathtt{A})$ and $\mathtt{B} = \neg\mathtt{C}$ then $\mathtt{C} \in LS(\mathtt{A})$. The sets of formulae $P(\mathtt{A})$, $M(\mathtt{A})$ and $I(\mathtt{A})$ are defined as follows:

- $P(\mathtt{A}) = \mathtt{F}_0 \cap LS(\mathtt{A})$,
- $M(\mathtt{A}) = \{\langle\mathtt{a}\rangle\mathtt{B} : \langle\mathtt{a}\rangle\mathtt{B} \in LS(\mathtt{A})\}$,
- $I(\mathtt{A}) = \{\mathtt{B} : \langle\mathtt{a}\rangle\mathtt{B} \in LS(\mathtt{A})\}$.

$\nabla$

For any set $\Gamma \subseteq \mathtt{F}_{\mathtt{L}}$, $P(\Gamma)$, $LS(\Gamma)$, $M(\Gamma)$ and $I(\Gamma)$ are defined in the natural way. For instance $P(\Gamma) = \bigcup_{\mathtt{A}\in\Gamma} P(\mathtt{A})$. Now assume that $\mathcal{M}', v \models \mathtt{A}$ for some $\mathtt{A} \in \mathtt{F}_{\mathtt{L}}$, $\mathcal{M}' = (U', \{\mathcal{R}'_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V')$ is an $\mathcal{L}$-model, $v \in U'$ and $m = md(\mathtt{A})$. Now we set:

$$LS_0 = LS(\mathtt{A}),\ P_0 = P(\mathtt{A}),\ M_0 = M(\mathtt{A}),\ I_0 = I(\mathtt{A})$$

$$\ldots$$

$$LS_{k+1} = LS(I_k),\ P_{k+1} = P(I_k),\ M_{k+1} = M(I_k),\ I_{k+1} = I(I_k)$$

$$\ldots$$

The modal degree of $LS_i$ decreases by 1 at each step. So after $m$ steps $md(LS_m) = 0$ and $M_m = I_m = \emptyset$. Given $w \in U'$, we define $M(w) = \bigcup\{M_j : j \in \{0, \ldots, m\}\}$. For all $\mathtt{a} \in \overline{\mathtt{M}}(\mathtt{A})$, we partition $\mathcal{R}'_{\mathtt{a}}(w)$ according to the values of the formulae in $\{\mathtt{B} : \langle\mathtt{a}\rangle\mathtt{B} \in M(w)\}$, i.e. $\forall u, u' \in \mathcal{R}'_{\mathtt{a}}(w)$, $u \equiv_{w,\mathtt{a}} u'$ iff

$$\forall \mathtt{B} \in \{\mathtt{C} : \langle\mathtt{a}\rangle\mathtt{C} \in M(w)\},\ \mathcal{M}', u \models \mathtt{B} \text{ iff } \mathcal{M}', u' \models \mathtt{B}$$

The equivalence class of $u \in \mathcal{R}'_{\mathtt{a}}(w)$ with respect to the relation $\equiv_{w,\mathtt{a}}$ is denoted by $|u|_{w,\mathtt{a}}$. Since $\{\mathtt{B} : \langle \mathtt{a} \rangle \mathtt{B} \in M(w)\}$ is finite, $\{|u|_{w,\mathtt{a}} : u \in \mathcal{R}'_{\mathtt{a}}(w)\}$ is finite and

$$card(\{|u|_{w,\mathtt{a}} : u \in \mathcal{R}'_{\mathtt{a}}(w)\}) \leq 2^{card(M(w))} \leq 2^{card(mw(\mathtt{A}))}$$

We define the *reduction*[11] of $\{\mathcal{R}'_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}$, namely $\{\mathcal{R}''_{\mathtt{c}} : \mathtt{c} \in \mathtt{M}_0\}$, as follows. For all $w \in U'$, $\mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A})$, $Y \in \{|u|_{w,\mathtt{b}} : u \in \mathcal{R}'_{\mathtt{b}}(w)\}$, take any $x \in Y$ (the *representative element* of the class $Y$). By definition $(w,x) \in \mathcal{R}'_{\mathtt{b}}$. Since $\mathcal{L}$ is a FiRe-logic, there exist a finite subset $\{w,x\} \subseteq X^{\mathtt{b}}_{w,x} \subseteq U'$ and a family $\{r^{(w,x,\mathtt{b})}_{\mathtt{a}'} : \mathtt{a}' \in \mathtt{M}\}$ over $X^{\mathtt{b}}_{w,x}$ respecting $\mathcal{I}$ such that

- $(w,x) \in r^{(w,x,\mathtt{b})}_{\mathtt{b}}$ and,
- for all $\mathtt{c}' \in \mathtt{M}_0(\mathtt{b})$, $r^{(w,x,\mathtt{b})}_{\mathtt{c}'} \subseteq \mathcal{R}'_{\mathtt{c}'}$.

For all $\mathtt{c} \in \mathtt{M}_0(\mathtt{A})$, we define

$$\begin{aligned}\mathcal{R}''_{\mathtt{c}} = \bigcup\{r^{(w,x,\mathtt{b})}_{\mathtt{c}} : \exists w \in U', \exists \mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A}) \text{ such that } \mathtt{c} \in \mathtt{M}_0(\mathtt{b}),\\ \exists Y \in \{|u|_{w,\mathtt{b}} : u \in \mathcal{R}'_{\mathtt{b}}(w)\}, x \text{ repr. elt. of } Y\}\end{aligned}$$

For all $\mathtt{c} \in \mathtt{M}_0 \setminus \mathtt{M}_0(\mathtt{A})$, $\mathcal{R}''_{\mathtt{c}} = \emptyset$ (arbitrary value). Now we build the model $\mathcal{M}^*$ as follows. First $U^*_0 = \{v\}$. Then,

$$U^*_1 = \bigcup\{X^{\mathtt{b}}_{v,x} : \mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A}), u \in \mathcal{R}'_{\mathtt{b}}(v), x \text{ repr. elt. of } |u|_{v,\mathtt{b}}\}$$

$\ldots$

$$U^*_{n+1} = \bigcup\{X^{\mathtt{b}}_{w',x} : w' \in U^*_n, \mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A}), u \in \mathcal{R}'_{\mathtt{b}}(w'), x \text{ repr. elt. of } |u|_{w',\mathtt{b}}\}$$

$\ldots$

$$U^*_m = \bigcup\{X^{\mathtt{b}}_{w',x} : w' \in U^*_{m-1}, \mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A}), u \in \mathcal{R}'_{\mathtt{b}}(w'), x \text{ repr. elt. of } |u|_{w',\mathtt{b}}\}.$$

and $U^* = \bigcup\{U^*_i : i \in \{0,\ldots,m\}\}$.

Let $\mathcal{M}^* = (U^*, \{\mathcal{R}^*_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V^*)$ be the $\mathcal{L}$-model such that

- for all $\mathtt{c} \in \mathtt{M}_0$, $\mathcal{R}^*_{\mathtt{c}} = \mathcal{C}(\mathtt{c})(U^*)(\mathcal{R}''_{\mathtt{c}} \cap U^* \times U^*)$,
- for all $\oplus \in \mathtt{OM}$ (of arity $n$), $\mathtt{a}_1, \ldots, \mathtt{a}_n \in \mathtt{M}$,

$$\mathcal{R}^*_{\oplus(\mathtt{a}_1,\ldots,\mathtt{a}_n)} = \mathcal{I}(\oplus)(U^*)(\mathcal{R}^*_{\mathtt{a}_1}, \ldots, \mathcal{R}^*_{\mathtt{a}_n})$$

- for all $\mathtt{p} \in \mathtt{F}_0$, $V^*(\mathtt{p}) = V'(\mathtt{p}) \cap U^*$.

$U^*$ is finite but it may not exist a computable map $g : \mathtt{F_L} \to \omega$ such that $card(U^*) \leq g(\mathtt{A})$. It is easy to check that $\mathcal{M}^*$ is really an $\mathcal{L}$-model. The idea of using different closure operations while keeping a unique schema for building $\mathcal{M}^*$ was also used in [5] for graded modal logics.

[11] We use the term 'reduction' since for all $\mathtt{a} \in \mathtt{M}$, $\mathcal{R}''_{\mathtt{a}}(w) \subseteq \mathcal{R}'_{\mathtt{a}}(w)$.

PROPOSITION 6.1. If there is $\mathtt{C} \in \mathtt{F}_{\mathtt{L}}^{\diamond}$ such that $\mathtt{t}_{nnf}(\mathtt{C}) = \mathtt{A}$ then for all $i \in \{0, \ldots, m\}$, for all $w \in U_i^*$, for all $\mathtt{B} \in (LS_i \cup P_i \cup M_i)$, if $\mathcal{M}', w \models \mathtt{B}$ then $\mathcal{M}^*, w \models \mathtt{B}$.

PROOF: The proof is by induction on $i$. By definition, for all $w \in U^*$, $\mathtt{p} \in \mathtt{F}_0$, $\mathcal{M}', w \models \mathtt{p}$ iff $\mathcal{M}^*, w \models \mathtt{p}$. Hence $\mathcal{M}', w \models \neg\mathtt{p}$ iff $\mathcal{M}^*, w \models \neg\mathtt{p}$.
*Base case*: assume $w \in U_m^*$. The set $LS_m$ is composed of Boolean combinations of propositional variables from $P_m$ where the occurrences of $\neg$ appear only in front of propositional variables. It follows that for all $\mathtt{B} \in LS_m \cup P_m$, $\mathcal{M}', w \models \mathtt{B}$ iff $\mathcal{M}^*, w \models \mathtt{B}$. Moreover $M_m = I_m = \emptyset$.
*Induction step*: assume $w \in U_i^*$ $(i < m)$ and $\mathcal{M}', w \models \langle \mathtt{a} \rangle \mathtt{B}$ with $\langle \mathtt{a} \rangle \mathtt{B} \in M_i$. There is $w' \in \mathcal{R}'_{\mathtt{a}}(w)$ such that $\mathcal{M}', w' \models \mathtt{B}$. Let $w''$ be the representative element of $|w'|_{w,\mathtt{a}}$. So $w'' \in U_{i+1}^*$. Consider the *unique* family $\{r_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}\}$ over $\bigcup_{\mathtt{c} \in \mathtt{M}_0(\mathtt{a})}(Dom_{r_{\mathtt{c}}^{(w,w'',\mathtt{a})}} \cup Ran_{r_{\mathtt{c}}^{(w,w'',\mathtt{a})}})$ respecting $\mathcal{I}$ such that

- for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $r_{\mathtt{c}} = r_{\mathtt{c}}^{(w,w'',\mathtt{a})}$,
- for all $\mathtt{c} \in \mathtt{M}_0 \setminus \mathtt{M}_0(\mathtt{a})$, $r_{\mathtt{c}} = \emptyset$.

By construction,

- for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $r_{\mathtt{c}} \subseteq X_{w,w''}^{\mathtt{a}} \times X_{w,w''}^{\mathtt{a}} \subseteq U^* \times U^*$,
- $(w, w'') \in r_{\mathtt{a}}$,
- $r_{\mathtt{a}} \cap U^* \times U^* = r_{\mathtt{a}} \cap X_{w,w''}^{\mathtt{a}} \times X_{w,w''}^{\mathtt{a}}$.

Since by construction, for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{a})$, $r_{\mathtt{c}} \subseteq \mathcal{R}''_{\mathtt{c}}$ then $r_{\mathtt{c}} \subseteq \mathcal{R}''_{\mathtt{c}} \cap U^* \times U^* \subseteq \mathcal{R}^*_{\mathtt{c}}$ (remember $\mathcal{C}(\mathtt{c})$ is a closure relation operation). By monotonocity of the relation operations and closure operations, $r_{\mathtt{a}} \subseteq \mathcal{R}^*_{\mathtt{a}}$. So $(w, w'') \in \mathcal{R}^*_{\mathtt{a}}$. By the induction hypothesis, $\mathcal{M}^*, w'' \models \mathtt{B}$ since $\mathcal{M}', w'' \models \mathtt{B}$ $(w'' \equiv_{w,\mathtt{a}} w')$, $\mathtt{B} \in L_{i+1}$ and $w'' \in U_{i+1}^*$. Hence $\mathcal{M}^*, w \models \langle \mathtt{a} \rangle \mathtt{B}$. Since every formula in $LS_i$ is a Boolean combination of formulae from $M_i \cup P_i$ (with the symbol $\neg$ occurring only in front of propositional variables), the proof is completed. Q.E.D.

Hence $\mathcal{M}^*, v \models \mathtt{A}$ since $v \in U_0^*$ and $\mathtt{A} \in LS_0$. It is worth observing that the applied technique is a hybrid construction based on filtration and restriction.

COROLLARY 6.2. $\mathcal{L}$ has the finite model property with respect to $\mathtt{F}_{\mathtt{L}}^{\diamond}$.

PDL with intersection (admitting the operator '?') has not the finite model property [17]. Because of the operator '?', this logic is not a FiRe-logic. However PDL with intersection and converse but without '?' is a FiRe-logic and has not the finite model property. Indeed, the formula A below is only satisfiable in infinite models:

$$\mathtt{A} = [\mathtt{c}^*](\langle \mathtt{c} \rangle (\mathtt{p} \vee \neg\mathtt{p}) \wedge \neg\langle \mathtt{c}\mathtt{c}^* \cap \mathtt{c}^{-1}(\mathtt{c}^{-1})^* \rangle (\mathtt{p} \vee \neg\mathtt{p}))$$

Hence it is not possible to extend Corollary 6.2 to the set $\mathtt{F_L}$ and it is an open problem to characterize the conditions under which Corollary 6.2 can be extended to $\mathtt{F_L}$. Although no finite axiomatization of $\mathcal{L}$ is assumed, a partial decidability result can be established when the finite sets involved in the well-foundedness of models have a bounded cardinality depending on the modal expressions.

PROPOSITION 6.3. Let $\mathcal{L}$ be a FiRe-logic such that $(\star)$ there exists a computable mapping $\mathtt{bound} : \mathtt{M} \rightarrow \omega$ such that for all $\mathcal{L}$-models $(U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$, for all $\mathtt{a} \in \mathtt{M}$, the set $X^{\mathtt{a}}_{x,y}$ in the condition of well-foundness satisfies $card(X^{\mathtt{a}}_{x,y}) \leq \mathtt{bound}(\mathtt{a})$. Then the validity problem for $\mathcal{L}$ restricted to $\mathtt{F}^{\Box}_{\mathtt{L}}$ is decidable.

Under the hypothesis of Proposition 6.3, $card(U^*) \leq \frac{\alpha^{(1+md(\mathtt{A}))}-1}{\alpha-1}$ where $\alpha = (\Sigma_{\mathtt{a}\in\overline{\mathtt{M}}(\mathtt{A})}\mathtt{bound}(\mathtt{a})) \times 2^{card(mw(\mathtt{A}))}$ since for all $0 \leq i \leq m-1$,

$$U^*_{i+1} \leq U^*_i \times \Sigma_{\mathtt{a}\in\overline{\mathtt{M}}(\mathtt{A})}(2^{card(mw(\mathtt{A}))} \times \mathtt{bound}(\mathtt{a}))$$

COROLLARY 6.4. Let $\Phi_1$ be the set below

$$\{\mathcal{C}^i, \mathcal{C}^{wr}, \mathcal{C}^{ar}, \mathcal{C}^r, \mathcal{C}^s, \cup, \cap, ;, ||, \nu, \mathcal{C}^{id.dom}, \mathcal{C}^{id.ran}, \mathcal{C}^{l\times L}, \mathcal{C}^{L\times l}, \mathcal{C}^{L\times L}, \mathcal{C}^{L\times L'}\}$$

Any composed $\Phi_1$-logic has the finite model property with respect to $\mathtt{F}^{\Diamond}_{\mathtt{L}}$ and the validity problem restricted to $\mathtt{F}^{\Box}_{\mathtt{L}}$ is decidable.

## 7 Class of LaFiRe-logics

We introduce a third class of logics in Definition 7.1 that contains logics from the literature (see for instance [15]). Unlike the FiRe-logics, the peculiarity of the class is the interdependence of the modal operators indexed by constants.

Let $\mathtt{L}$ be a modal language and $lo$ be a set of linear[12] orders over $\mathtt{M_0}$. The set $\mathcal{M}od^{lo}_{\mathtt{L}}$ of $\mathtt{L}$-models is defined as the set of $\mathtt{L}$-models $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V)$ such that for all $u \in U$, there is $\rhd \in lo$ such that for all $\mathtt{c}, \mathtt{c}' \in \mathtt{M_0}$, if $\mathtt{c} \rhd \mathtt{c}'$ then $\mathcal{R}_{\mathtt{c}}(u) \subseteq \mathcal{R}_{\mathtt{c}'}(u)$.

DEFINITION 7.1. A logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ is said to be a *LaFiRe-logic* iff there exist an operator interpretation function $\mathcal{I}$, a mapping $\mathcal{C} : \mathtt{M_0} \rightarrow COR$ and a set $lo(\mathcal{L})$ of linear orders over $\mathtt{M_0}$ such that

(1) $\mathcal{S}_{\mathcal{L}} = \mathcal{M}od^{\mathcal{I}}_{\mathtt{L}} \cap \mathcal{M}od^{lo(\mathcal{L})}_{\mathtt{L}} \cap \mathcal{M}od^{\mathcal{C}}_{\mathtt{L}}$,

(2) every $\mathtt{L}$-model in $\mathcal{S}_{\mathcal{L}}$ is well-founded with respect to $\mathcal{I}$,

[12] A binary relation $\rhd$ over $U$ is said to be *linear* iff $\rhd$ is reflexive, transitive, totally connected (for all $x, y \in U$ either $(x, y) \in \rhd$ or $(y, x) \in \rhd$) and antisymmetric (for all $x, y \in U$ if $(x, y) \in \rhd$ and $(y, x) \in \rhd$ then $x = y$).

(3) for all $\mathtt{c}, \mathtt{c}' \in \mathtt{M}_0$, for all $\rhd \in lo(\mathcal{L})$, if $\mathtt{c} \rhd \mathtt{c}'$ then $\mathcal{C}(\mathtt{c}) \subseteq \mathcal{C}(\mathtt{c}')$.

$\nabla$

The notion of LaFiRe-reducible logics (some examples can be found in [8]) is defined as for the FiRe-reducible logics. Binary relations $R$ and $R'$ on a set $U$ are said to be in *local agreement* (LA) iff

for all $u \in U$ either $R(u) \subseteq R'(u)$ or $R'(u) \subseteq R(u)$ [15].

It is easy to show that for any LaFiRe-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$, for any model $\mathcal{M} = (U, \{\mathcal{R}_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V) \in \mathcal{S}$, for all $\mathtt{a}, \mathtt{b} \in \mathtt{M}_0$, $\mathcal{R}_{\mathtt{a}}$ and $\mathcal{R}_{\mathtt{b}}$ are in local agreement. Various logics from the literature can be translated into LaFiRe-logics (see for instance [15, 19]) whereas S5 is a LaFiRe-logic.

In the rest of this section $\mathcal{L}$ denotes an LaFiRe-logic (we use the notations from Definition 7.1). Now assume that $\mathcal{M}', v \models \mathtt{A}$ for some $\mathtt{L}$-formula $\mathtt{A}$, $\mathcal{M}' = (U', \{\mathcal{R}'_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V')$ is a model for $\mathcal{L}$, $v \in U'$ and $m = md(\mathtt{A})$.

The construction of the family $\{\mathcal{R}''_{\mathtt{c}} : \mathtt{c} \in \mathtt{M}_0\}$ is modified as follows. As done in Section 6, for all $w \in U'$, $\mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A})$, $Y \in \{|u|_{w,\mathtt{b}} : u \in \mathcal{R}'_{\mathtt{b}}(w)\}$, take any $x \in Y$ (the representative element of the class $Y$). By definition $(w, x) \in \mathcal{R}'_{\mathtt{b}}$. Since every $\mathcal{L}$-model is well-founded with respect to $\mathcal{I}$, there exist a finite subset $\{w, x\} \subseteq X^{\mathtt{b}}_{w,x} \subseteq U$ and a family $\{r_{\mathtt{b}'} : \mathtt{b}' \in \mathtt{M}\}$ over $X^{\mathtt{b}}_{w,x}$ respecting $\mathcal{I}$ such that $(w, x) \in r_{\mathtt{b}}$, and for all $\mathtt{c} \in \mathtt{M}_0(\mathtt{b})$, $r_{\mathtt{c}} \subseteq \mathcal{R}'_{\mathtt{c}}$. For all $\mathtt{c} \in \mathtt{M}_0(\mathtt{A})$, we define

$$\mathcal{R}''_{\mathtt{c}} = \bigcup \{ r^{(w,x,\mathtt{b})}_{\mathtt{c}'} : \exists w \in U', \exists \mathtt{b} \in \overline{\mathtt{M}}(\mathtt{A}), \exists \mathtt{c}' \in \mathtt{M}_0(\mathtt{b}),$$
$$\exists Y \in \{|u|_{w,\mathtt{b}} : u \in \mathcal{R}'_{\mathtt{b}}(w)\}, x \text{ repr. elt. of } Y, \text{ and } \mathcal{R}'_{\mathtt{c}'}(w) \subseteq \mathcal{R}'_{\mathtt{c}}(w)\}$$

For all $w \in U'$, there exists a linear order $\rhd \in lo(\mathcal{L})$ such that $\mathtt{M}_0(\mathtt{A}) = \{\mathtt{c}_1, \ldots, \mathtt{c}_n\}$, $\mathtt{c}_1 \rhd \ldots \rhd \mathtt{c}_n$ and for all $i \in \{1, \ldots, n-1\}$, $\mathcal{R}'_{\mathtt{c}_i}(w) \subseteq \mathcal{R}'_{\mathtt{c}_{i+1}}(w)$. By construction, for all $1 \leq i < j \leq n$, $\mathcal{R}''_{\mathtt{c}_i}(w) \subseteq \mathcal{R}''_{\mathtt{c}_j}(w)$. For all $\mathtt{c} \in \mathtt{M}_0$, we fix $\mathcal{R}''_{\mathtt{c}}(w) = \mathcal{R}''_{\mathtt{c}_n}(w)$. In the construction of the model $\mathcal{M}^* = (U^*, \{\mathcal{R}^*_{\mathtt{a}} : \mathtt{a} \in \mathtt{M}\}, V^*)$, $U^*$ is defined as in Section 6 and

- for all $\mathtt{c} \in \mathtt{M}_0$, $\mathcal{R}^*_{\mathtt{c}} = \mathcal{C}(\mathtt{c})(U^*)(\mathcal{R}''_{\mathtt{c}} \cap U^* \times U^*)$ ,
- for all $\oplus \in \mathtt{OM}$ (arity $n$), $\mathtt{a}_1, \ldots, \mathtt{a}_n \in \mathtt{M}$, $\mathcal{R}^*_{\oplus(\mathtt{a}_1,\ldots,\mathtt{a}_n)} = \mathcal{I}(\oplus)(U^*)(\mathcal{R}^*_{\mathtt{a}_1}, \ldots, \mathcal{R}^*_{\mathtt{a}_n})$,
- for all $\mathtt{p} \in \mathtt{F}_0$, $V^*(\mathtt{p}) = V'(\mathtt{p}) \cap U^*$.

$\mathcal{M}^*$ is indeed a model for $\mathcal{L}$. The only condition that requires some work is the restricted local agreement condition. By definition, there exists $\rhd_v \in lo(\mathcal{L})$ such that for all $\mathtt{c}, \mathtt{c}' \in \mathtt{M}_0$, if $\mathtt{c} \rhd_v \mathtt{c}'$ then $\mathcal{R}_{\mathtt{c}}(v) \subseteq \mathcal{R}_{\mathtt{c}'}(v)$. Hence if $\mathtt{c} \rhd_v \mathtt{c}'$ then $\mathcal{R}''_{\mathtt{c}} = \mathcal{R}''_{\mathtt{c}'}$, $\mathcal{C}(\mathtt{c}) \subseteq \mathcal{C}(\mathtt{c}')$ and therefore $\mathcal{R}^*_{\mathtt{c}} \subseteq \mathcal{R}^*_{\mathtt{c}'}$. So for all $u \in U^*$, for all $\mathtt{c}, \mathtt{c}' \in \mathtt{M}_0$, if $\mathtt{c} \rhd_v \mathtt{c}'$ then $\mathcal{R}^*_{\mathtt{c}}(u) \subseteq \mathcal{R}^*_{\mathtt{c}'}(u)$. As in Proposition 6.1, if there is $\mathtt{C} \in \mathtt{F}^{\diamond}_{\mathtt{L}}$ such that $\mathtt{t}_{nnf}(\mathtt{C}) = \mathtt{A}$ then for all $i \in \{0, \ldots, m\}$, for all $w \in U^*_i$, for all $\mathtt{B} \in (LS_i \cup P_i \cup M_i)$, if $\mathcal{M}', w \models \mathtt{B}$ then $\mathcal{M}^*, w \models \mathtt{B}$.

Corollary 7.1. Let $\mathcal{L}$ be an LaFiRe-logic satisfying ($\star$) of Proposition 6.3. Then, $\mathcal{L}$ has the finite model property with respect to $\mathtt{F}_{\mathtt{L}}^{\Diamond}$ and the validity problem restricted to $\mathtt{F}_{\mathtt{L}}^{\Box}$ is decidable.

## 8 Concluding remarks

In the paper we have defined classes of multi-modal logics that contain numerous logics from the literature. They are characterized by closure relation operations, monotonous closure relation operations and finitely reducible relation operations. The classes contain knowledge logics, dynamic logics and also various information logics (see for instance [12]).

For each logic of the classes we have proved the finite model property with respect to the class of $\Diamond$-formulae using a substantial variant of Cerrato's filtration technique. Although the class of $\Diamond$-formulae plays a special role in modal logic theory, it is an open problem whether, for instance, every FiRe-logic such that all the relation operations involved in the semantics are first-order definable closure relation operations, has a decidable satisfiability problem. Moreover, there is at least one FiRe-logic without the finite model property: PDL with intersection, converse but without the test operator '?'.

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