On the effective dimension and multilevel Monte Carlo

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November 8, 2021

Abstract

I consider the problem of integrating a function f over the d-dimensional unit cube. I describe a multilevel Monte Carlo method that estimates the integral with variance at most ϵ^2 in $O(d + \ln(d)d_t\epsilon^{-2})$ time, for $\epsilon > 0$, where d_t is the truncation dimension of f. In contrast, the standard Monte Carlo method typically achieves such variance in $O(d\epsilon^{-2})$ time. A lower bound of order $d + d_t\epsilon^{-2}$ is described for a class of multilevel Monte Carlo methods.

Keywords: multilevel Monte Carlo, Quasi-Monte Carlo, variance reduction, effective dimension, truncation dimension, time-varying Markov chains

1 Introduction

Monte Carlo simulation is used in a variety of areas including finance, queuing systems, machine learning, and health-care. A drawback of Monte Carlo simulation is its high computation cost. This motivates the need to design efficient simulation tools that optimize the tradeoff between the running time and the statistical error. This need is even stronger for high-dimensional problems, where the time to simulate a single run is typically proportional to the dimension. Variance reduction techniques that improve the efficiency of Monte Carlo simulation have been developed in the previous literature (e.g. (Glasserman 2004, Asmussen and Glynn 2007)).

This paper studies the estimation of $\int_{[0,1]^d} f(x) dx$, where f is a real-valued square-integrable function on $[0,1]^d$. Note that $\int_{[0,1]^d} f(x) dx = E(f(U))$, where $U = (U_1, \ldots, U_d)$ and U_1, \ldots, U_d are independent random variables uniformly distributed on [0,1]. The standard Monte Carlo method estimates E(f(U)) by taking the average of f over n random points uniformly distributed over $[0,1]^d$, and achieves a statistical error of order $n^{-1/2}$. The Quasi-Monte Carlo method (QMC) estimates E(f(U)) by taking the average of f over a predetermined sequence of points in $[0,1]^d$, and achieves an error of order $(\log n)^d/n$ for certain sequences when f has finite Hardy-Krause variation (Glasserman 2004, Ch. 5). Thus, for small values of d, QMC can substantially outperform standard Monte Carlo. Moreover, numerical experiments show that QMC performs well in certain high-dimensional problems where the importance of U_i decreases with i (Glasserman 2004, Ch. 5). Caflisch, Morokoff and Owen (1997) use the ANOVA decomposition, a representation of f as the sum of orthogonal components, to define the effective

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dimension in the truncation sense: the truncation dimension is low when the first variables are important. Sloan and Woźniakowski (1998) prove that QMC is effective for a class of functions where high dimensions have decaying importance. The connection between QMC and various notions of effective dimension is studied in (L'Ecuyer and Lemieux 2000, Owen 2003, Liu and Owen 2006, Wasilkowski 2021). Methods that reduce the effective dimension and improve the performance of QMC are described in (Wang and Sloan 2011, Wang and Tan 2013, Xiao and Wang 2019). Owen (2019) gives a recent survey on the effective dimension. Kahalé (2020b) studies the relationship between the truncation dimension and the randomized dimension reduction method, a recent variance reduction technique applicable to high-dimensional problems.

A major advance in Monte Carlo simulation is the multilevel Monte Carlo method (MLMC), a variance reduction technique introduced by Giles (2008). The MLMC method significantly reduces the time to estimate functionals of a stochastic differential equation, and has many other applications (e.g. (Rosenbaum and Staum 2017, Pisaroni, Nobile and Leyland 2017, Goda, Hironaka and Iwamoto 2020, Kahalé 2020a, Blanchet, Chen, Si and Glynn 2021)). This paper examines the connection between the MLMC method and the truncation dimension. Section 3 describes a MLMC method that, under suitable conditions, estimates E(f(U)) with variance at most ϵ^2 in $O(d + \ln(d)d_t\epsilon^{-2})$ time, for $\epsilon > 0$, where d_t is the truncation dimension of f. In contrast, the standard Monte Carlo method typically achieves variance at most e^2 in $O(de^{-2})$ time. My approach is based on fixing unessential variables and on approximating f(U) by functions of the first components of U. Fixing unessential variables is analysed by Sobol (2001) in the context of the ANOVA decomposition. Section 4 considers a class of MLMC estimators that approximate f(U) by functions of the first components of U. Under general conditions, it gives a lower bound of order $d + d_t e^{-2}$ on the time required by these estimators to evaluate E(f(U)) with variance at most ϵ^2 . Section 5 studies MLMC and the truncation dimension for time-varying Markov chains with d time-steps. Under suitable conditions, it is shown that certain Markov chain functionals can be estimated with variance at most ϵ^2 in $O(d + \epsilon^{-2})$ time, and that the truncation dimension associated with these functionals is upper bounded by a constant independent of d. Randomized MLMC methods for equilibrium expectations of time-homogeneous Markov chains are studied in (Glynn and Rhee 2014).

2 Preliminaries

2.1 The ANOVA decomposition

It is assumed throughout the paper that f is square-integrable with Var(f(U)) > 0. A representation of f in the following form:

$$f = \sum_{Y \subseteq \{1,\dots,d\}} f_Y,\tag{1}$$

is called ANOVA decomposition if, for $Y \subseteq \{1, \dots, d\}$ and $u = (u_1, \dots, u_d) \in [0, 1]^d$,

1. f_Y is a measurable function on $[0,1]^d$ and $f_Y(u)$ depends on u only through $(u_j)_{j\in Y}$.

2. For $j \in Y$,

$$\int_0^1 f_Y(u_1, \dots, u_{j-1}, x, u_{j+1}, \dots, u_d) dx = 0.$$

It can be shown (Sobol 2001, p. 272) that there is a unique ANOVA representation of f, that $f_{\emptyset} = E(f(U))$, and that the f_Y 's are square-integrable. Furthermore, if $Y \neq Y'$,

$$Cov(f_Y(U), f_{Y'}(U)) = 0, (2)$$

and

$$\operatorname{Var}(f(U)) = \sum_{Y \subseteq \{1, \dots, d\}} \sigma_Y^2, \tag{3}$$

where σ_Y is the standard deviation of $f_Y(U)$. For $0 \le i \le d$,

$$E(f(U)|U_1,...,U_i) = \sum_{Y \subseteq \{1,...,i\}} f_Y(U).$$
(4)

Owen (2003) defines the truncation dimension d_t of f as

$$d_t := \frac{\sum_{Y \subseteq \{1,\dots,d\}, Y \neq \varnothing} \max(Y) \sigma_Y^2}{\operatorname{Var}(f(U))}.$$

For $0 \le i \le d$, let

$$D(i) := \sum_{Y \subseteq \{1,\dots,d\}, Y \neq \varnothing, \max(Y) > i} \sigma_Y^2$$

be the total variance corresponding to the last d-i components of f (see (Sobol 2001)). The sequence $(D(i): 0 \le i \le d)$ is decreasing, with D(0) = Var(f(U)) by (3) and D(d) = 0. Proposition 2.1 gives a bound on the variance of f(V) - f(V'), when V and V' are uniformly distributed on $[0,1]^d$ and have the same first i components. It is related to (Sobol 2001, Theorem 3).

Proposition 2.1. Let $i \in \{0, ..., d\}$. Assume that V and V' are uniformly distributed on $[0,1]^d$, and that $V_j = V'_j$ for $1 \le j \le i$. Then $Var(f(V) - f(V')) \le 4D(i)$.

Proof. As $f_Y(V) = f_Y(V')$ for $Y \subseteq \{1, ..., i\}$, we have

$$f(V) - f(V') = \sum_{Y \subseteq \{1,...,d\}, Y \neq \emptyset, \max(Y) > i} f_Y(V) - f_Y(V').$$

By (2),

$$\operatorname{Var}\left(\sum_{Y\subseteq\{1,\ldots,d\},Y\neq\varnothing,\max(Y)>i}f_Y(V)\right)=D(i),$$

and a similar relation holds for V'. Since $Var(Z+Z') \leq 2(Var(Z)+Var(Z'))$ for square-integrable random variables Z and Z', this achieves the proof.

Proposition 2.2 gives a lower bound on the variance of the difference between f(U) and a function of the first i components of U. A similar result is shown in (Sobol 2001, Theorem 1).

Proposition 2.2. Let g be a real-valued square-integrable function on $[0,1]^i$, where $0 \le i \le d$. Then

$$D(i) \leq \operatorname{Var}(f(U) - g(U_1, \dots, U_i)).$$

Proof. Define the random-variable

$$\eta = f(U) - E(f(U)|U_1, \dots, U_i).$$

By properties of the conditional expectation,

$$\operatorname{Var}(\eta) \leq \operatorname{Var}(f(U) - g(U_1, \dots, U_i)).$$

Combining (1) and (4) shows that

$$\eta = \sum_{Y\subseteq \{1,\dots,d\}, Y\not\subseteq \{1,\dots,i\}} f_Y(U).$$

By (2), $Var(\eta) = D(i)$. This completes the proof.

Proposition 2.3 provides an alternative characterisation of the truncation dimension.

Proposition 2.3.

$$\sum_{i=0}^{d} D(i) = d_t \operatorname{Var}(f(U)).$$

Proof.

$$\sum_{i=0}^{d} D(i) = \sum_{i=0}^{d} \sum_{Y \subseteq \{1,\dots,d\}, Y \neq \emptyset} \mathbf{1}\{i < \max(Y)\} \sigma_{Y}^{2}$$

$$= \sum_{Y \subseteq \{1,\dots,d\}, Y \neq \emptyset} \max(Y) \sigma_{Y}^{2}$$

$$= d_{t} \operatorname{Var}(f(U)).$$

2.2 Work-normalized variance

Let μ be a real number and let ψ be a square-integrable random variable with positive variance and expected running time τ . Assume that ψ is an unbiased estimator of μ , i.e., $E(\psi) = \mu$. The work-normalized variance $\tau \text{Var}(\psi)$ is a standard measure of the performance of ψ (Glynn and Whitt 1992): asymptotically efficient unbiased estimators have low work-normalized variance. For $\epsilon > 0$, let n_{ϵ} be the smallest integer such that the variance of the average of n_{ϵ} independent copies of ψ is at most ϵ^2 . Thus, $n_{\epsilon} = \lceil \text{Var}(\psi) \epsilon^{-2} \rceil$. As $(x+1)/2 \leq \lceil x \rceil \leq x+1$ for x>0,

$$\frac{\tau + \tau \operatorname{Var}(\psi)\epsilon^{-2}}{2} \le T(\psi, \epsilon) \le \tau + \tau \operatorname{Var}(\psi)\epsilon^{-2},\tag{5}$$

where $T(\psi, \epsilon) := n_{\epsilon}\tau$ is the total expected time required to estimate μ with variance at most ϵ^2 by taking the average of independent runs of ψ .

2.3 Reminder on MLMC

Let ϕ be a square-integrable random variable that is approximated with increasing accuracy by square-integrable random variables ϕ_l , $0 \le l \le L$, where L is a positive integer, with $\phi_L = \phi$ and $\phi_0 = 0$. For $1 \le l \le L$, let $\hat{\phi}_l$ be the average of n_l independent copies of $\phi_l - \phi_{l-1}$, where n_l is an arbitrary positive integer. Suppose that $\hat{\phi}_1, \ldots, \hat{\phi}_L$ are independent. Since

$$E(\phi) = \sum_{l=1}^{L} E(\phi_l - \phi_{l-1}),$$

 $\hat{\phi} := \sum_{l=1}^L \hat{\phi}_l$ is an unbiased estimator of $E(\phi),$ i.e.,

$$E(\hat{\phi}) = E(\phi). \tag{6}$$

As observed in (Giles 2008),

$$\operatorname{Var}(\hat{\phi}) = \sum_{l=1}^{L} \frac{V_l}{n_l},\tag{7}$$

where $V_l := \text{Var}(\phi_l - \phi_{l-1})$ for $1 \leq l \leq L$. The expected time required to simulate $\hat{\phi}$ is

$$\hat{T} := \sum_{l=1}^{L} n_l \hat{t}_l, \tag{8}$$

where \hat{t}_l is the expected time to simulate $\phi_l - \phi_{l-1}$. The analysis in (Giles 2008) shows that

$$\left(\sum_{l=1}^{L} \sqrt{V_l \hat{t}_l}\right)^2 \le \hat{T} \operatorname{Var}(\hat{\phi}), \tag{9}$$

with equality when the n_l 's are proportional to $\sqrt{V_l/\hat{t}_l}$ (ignoring integrality constraints).

3 The MLMC algorithm

Let $L = \lceil \log_2(d) \rceil$ and, for $0 \le l \le L - 1$, let $m_l = 2^l - 1$, with $m_L = d$. For $1 \le l \le L$ and $u, u' \in [0, 1]^d$, let

$$h_l(u, u') := f(u_1, \dots, u_{m_l}, u'_{m_l+1}, \dots, u'_d),$$

with $h_0(u, u') := 0$. Note that $h_L(u, u') = f(u)$. Let U' be a copy of U and, for $1 \le l \le L$, let $(U^{l,j}, 1 \le j \le n_l)$ be n_l copies of U, where $n_l := \lceil (d/L)2^{-l} \rceil$. Assume that the random variables $(U', U^{l,j}, 1 \le l \le L, 1 \le j \le n_l)$ are independent. For $1 \le l \le L$, set

$$\tilde{\phi}_l := \frac{1}{n_l} \sum_{i=1}^{n_l} (h_l(U^{l,j}, U') - h_{l-1}(U^{l,j}, U')), \tag{10}$$

and let $\tilde{\phi} := \sum_{l=1}^{L} \tilde{\phi}_l$. The estimator $\tilde{\phi}$ does not fall, stricto sensu, in the category of MLMC estimators described in Section 2.3. This is because the n_l summands in the right-hand side of (10) are dependent random variables, in general. Note that $h_l(u, u')$ depends on u only

through its first m_l components. Thus, once U' is simulated, $h_l(U^{l,j}, U')$ and $h_{l-1}(U^{l,j}, U')$ can be calculated by simulating only the first m_l components of $U^{l,j}$. For $1 \leq l \leq L$, let \tilde{t}_l be the expected time to simulate the first m_l components of U and calculate $h_l(U, U')$, once U' is simulated and f(U') is calculated. In other words, \tilde{t}_l is the expected time to redraw U_1, \ldots, U_{m_l} and recalculate f(U), without modifying the last $d-m_l$ components of U. In particular, \tilde{t}_L is the expected time to simulate U and calculate f(U). Let \tilde{T} be the expected time to simulate $\tilde{\phi}$.

Theorem 3.1 below shows that $\tilde{\phi}$ is an unbiased estimator of E(f(U)). Also, when \hat{t}_l is linear in m_l , the work-normalized variance of $\tilde{\phi}$ satisfies the bound $\tilde{T}\operatorname{Var}(\tilde{\phi}) = O(\ln(d)d_t\operatorname{Var}(f(U)))$, that depends on d only through $\ln(d)$. By (14), E(f(U)) can be estimated via $\tilde{\phi}$ with variance at most ϵ^2 in expected time that depends asymptotically (as ϵ goes to 0) on $\ln(d)$. In contrast, assuming the expected time to simulate f(U) is of order d, the work-normalized variance of the standard Monte Carlo estimator is of order $d\operatorname{Var}(f(U))$ and, by (5), the standard Monte Carlo algorithm achieves variance at most ϵ^2 in $O(d + d\operatorname{Var}(f(U))\epsilon^{-2})$ expected time.

Theorem 3.1. We have

$$E(\tilde{\phi}) = E(\tilde{\phi}|U') = E(f(U)), \tag{11}$$

 $\operatorname{Var}(\tilde{\phi}) = E(\operatorname{Var}(\tilde{\phi})|U'), \text{ and }$

$$\operatorname{Var}(\tilde{\phi}) \le 16 \frac{\lceil \log_2(d) \rceil}{d} d_t \operatorname{Var}(f(U)).$$
 (12)

If, for some constant \tilde{c} and $1 \leq l \leq L$,

$$\tilde{t}_l \le \tilde{c}m_l,\tag{13}$$

then $\tilde{T} \leq 9\tilde{c}d$ and, for $\epsilon > 0$,

$$T(\tilde{\phi}, \epsilon) = O(d + \ln(d)d_t \operatorname{Var}(f(U))\epsilon^{-2}). \tag{14}$$

Proof. By the definition of $\tilde{\phi}_l$,

$$E(\tilde{\phi}_l|U') = E(\Delta_l|U'),$$

where $\Delta_l := h_l(U, U') - h_{l-1}(U, U')$. Summing over l implies that $E(\tilde{\phi}|U') = E(f(U))$. Taking expectations and using the tower law implies (11). Conditional on U', the n_l summands in the right-hand side of (10) are independent and have the same distribution as Δ_l . Thus, for $1 \leq l \leq L$,

$$\operatorname{Var}(\tilde{\phi}_l|U') = \frac{\operatorname{Var}(\Delta_l|U')}{n_l}.$$

Furthermore, conditional on U', the random variables $\tilde{\phi}_l$, $1 \leq l \leq L$, are independent. Hence,

$$\operatorname{Var}(\tilde{\phi}|U') = \sum_{l=1}^{L} \frac{\operatorname{Var}(\Delta_{l}|U')}{n_{l}}.$$
(15)

As $\operatorname{Var}(Z) = \operatorname{Var}(E(Z|U')) + E(\operatorname{Var}(Z|U'))$ for any square-integrable random variable Z, using (11) shows that $\operatorname{Var}(\tilde{\phi}) = E(\operatorname{Var}(\tilde{\phi}|U'))$. Similarly, $E(\operatorname{Var}(\Delta_l|U')) \leq \operatorname{Var}(\Delta_l)$. Conse-

quently, taking expectations in (15) implies that

$$\operatorname{Var}(\tilde{\phi}) \leq \sum_{l=1}^{L} \frac{\operatorname{Var}(\Delta_{l})}{n_{l}}$$
$$\leq \frac{L}{d} \sum_{l=1}^{L} 2^{l} \operatorname{Var}(\Delta_{l}).$$

For $2 \leq l \leq L$, we have $\Delta_l = f(V) - f(V')$, where $V = (U_1, \dots, U_{m_l}, U'_{m_l+1}, \dots, U'_d)$, and $V' = (U_1, \dots, U_{m_{l-1}}, U'_{m_{l-1}+1}, \dots, U'_d)$. Applying Proposition 2.1 with $i = m_{l-1}$ yields

$$Var(\Delta_l) \le 4D(m_{l-1}). \tag{16}$$

Since $\Delta_1 = f(U')$, (16) also holds for l = 1. For $1 \le l \le L$, we have $2^l \le 4(m_{l-1} - m_{l-2})$, where $m_{-1} := -1$. Hence, because the sequence D is decreasing,

$$2^{l}D(m_{l-1}) \le 4 \sum_{i=m_{l-2}+1}^{m_{l-1}} D(i).$$

Thus,

$$\sum_{l=1}^{L} 2^{l} \operatorname{Var}(\Delta_{l}) \leq 4 \sum_{l=1}^{L} 2^{l} D(m_{l-1})$$

$$\leq 16 \sum_{l=1}^{L} \sum_{i=m_{l-2}+1}^{m_{l-1}} D(i)$$

$$= 16 \sum_{i=0}^{m_{L-1}} D(i)$$

$$\leq 16 d_{t} \operatorname{Var}(f(U)),$$

where the last equation follows from Proposition 2.3. This implies (12).

Assume now that (13) holds. Simulating $\tilde{\phi}$ requires to draw U' and calculate f(U') once and to simulate $h_l(U,U') - h_{l-1}(U,U')$ for n_l independent copies of U, $1 \leq l \leq L$. As $m_l \leq 2^l$, given U', simulating $h_l(U,U')$ (resp. $h_{l-1}(U,U')$) takes at most $\tilde{c}2^l$ (resp. $\tilde{c}2^{l-1}$) expected time. Thus the expected time to simulate $h_l(U,U') - h_{l-1}(U,U')$ is at most $3\tilde{c}2^{l-1}$, and

$$\tilde{T} \leq \tilde{c}d + 3\tilde{c} \sum_{l=1}^{L} n_l 2^{l-1}$$

$$\leq \tilde{c}d + 3\tilde{c} \sum_{l=1}^{L} (1 + \frac{d}{L2^l}) 2^{l-1}$$

$$\leq \tilde{c}d + 3\tilde{c} 2^L + 3\tilde{c} \frac{d}{2}$$

$$\leq 9\tilde{c}d,$$

where the second equation follows from the inequality $n_l \leq 1 + d/(L2^l)$. (14) follows immediately

from (5).

Remark 5.1 in Section 5 shows that (13) holds for a class of Markov chain functionals.

3.1 Deterministic fixing of unessential variables

The estimator $\tilde{\phi}$ uses U' to fix the unessential variables. This section studies the replacement of U' by a deterministic vector. For $v \in [0,1]^d$ and $1 \le l \le L$, set

$$\tilde{\phi}_{l,v} := \frac{1}{n_l} \sum_{j=1}^{n_l} (h_l(U^{l,j}, v) - h_{l-1}(U^{l,j}, v)),$$

 $\tilde{\phi}_v := \sum_{l=1}^L \tilde{\phi}_{l,v}$. In other words, the random variable $\tilde{\phi}_v$ is obtained from $\tilde{\phi}$ by substituting U' with v. Let $\tilde{T}(v)$ be the expected running time of $\tilde{\phi}_v$. For any $v \in [0,1]^d$, the estimator $\tilde{\phi}_v$ falls in the class of MLMC estimators described in Section 2.3, with $\phi = f(U)$ and $\phi_l = h_l(U,v)$ for $0 \le l \le L$. Corollary 3.1 shows that $\tilde{\phi}_v$ is an unbiased estimator of E(f(U)) and that there is $v^* \in [0,1]^d$ such that the variance of $\tilde{\phi}_{v^*}$ and its running time are no worse, up to a constant, than those of $\tilde{\phi}$.

Corollary 3.1. For $v \in [0, 1]^d$,

$$E(\tilde{\phi}_v) = E(f(U)). \tag{17}$$

Moreover, there is $v^* \in [0,1]^d$ such that $\operatorname{Var}(\tilde{\phi}_{v^*}) \leq 3\operatorname{Var}(\tilde{\phi})$ and $\tilde{T}(v^*) \leq 3\tilde{T}$. For $\epsilon > 0$,

$$T(\tilde{\phi}_{v^*}, \epsilon) = O(d + \ln(d)d_t \operatorname{Var}(f(U))\epsilon^{-2}).$$
(18)

Proof. (17) is a special case of (6). For $v \in [0,1]^d$, let $\xi(v) := \operatorname{Var}(\tilde{\phi}_v)$. As $\xi(U') = \operatorname{Var}(\tilde{\phi}|U')$, it follows from Theorem 3.1 that $E(\xi(U')) = \operatorname{Var}(\tilde{\phi})$. Thus $\xi(U') \leq 3\operatorname{Var}(\tilde{\phi})$ with probability at least 2/3. Similarly, $\tilde{T}(U') \leq 3\tilde{T}$ with probability at least 2/3. Hence, there is $v^* \in [0,1]^d$ such that $\operatorname{Var}(\tilde{\phi}_{v^*}) \leq 3\operatorname{Var}(\tilde{\phi})$ and $\tilde{T}(v^*) \leq 3\tilde{T}$. Using (5) yields (18).

The MLMC estimator $\tilde{\phi}_v$ is obtained by approximating f with functions of its first components. A lower bound on the performance of such estimators is given in Section 4.

4 The lower bound

This section considers a class of MLMC unbiased estimators of E(f(U)) based on successive approximations of f by deterministic functions of its first components. In (Kahalé 2020b), a lower bound on the work-normalized variance of such estimators is given in terms of that of the randomized dimension reduction estimator. This section provides a lower bound on the work-normalized variance of these estimators in terms of the truncation dimension.

Using the notation in Section 2.3 with $\phi = f(U)$, consider a MLMC estimator $\hat{\phi}$ of E(f(U)) obtained by summing the averages on independent copies of $\phi_l - \phi_{l-1}$, $1 \le l \le L$, where L is a positive integer and the ϕ_l 's satisfy the following assumption:

Assumption 1 (A1). For $0 \le l \le L$, ϕ_l is a square-integrable random variable equal to a deterministic measurable function of U_1, \ldots, U_{m_l} , with $\phi_0 = 0$ and $\phi_L = f(U)$, where $(m_l : 0 \le l \le L)$ is a strictly increasing sequence of integers, with $m_0 = 0$ and $m_L = d$.

The proof of the lower bound is based on the following lemma.

Lemma 4.1. Let $(\nu_i : 0 \le i \le d)$ be a decreasing sequence such that $\nu_{m_l} \le \text{Var}(f(U) - \phi_l)$ for $0 \le l \le L$, with $\nu_d = 0$. Then

$$\sum_{i=0}^{d} \nu_i \le \left(\sum_{l=1}^{L} \sqrt{m_l V_l}\right)^2.$$

Proof. An integration by parts argument (Kahalé 2020b, Lemma EC.4) shows that

$$\sum_{l=0}^{L-1} (\sqrt{m_{l+1}} - \sqrt{m_l}) \sqrt{\nu_{m_l}} \le \sum_{l=1}^{L} \sqrt{m_l V_l}.$$

On the other hand, for $0 \le l \le L - 1$, we have

$$(\sqrt{m_{l+1}} - \sqrt{m_l})\sqrt{\nu_{m_l}} = \sum_{i=m_l}^{m_{l+1}-1} (\sqrt{i+1} - \sqrt{i})\sqrt{\nu_{m_l}}$$

$$\geq \sum_{i=m_l}^{m_{l+1}-1} \alpha_i,$$

where $\alpha_i = (\sqrt{i+1} - \sqrt{i})\sqrt{\nu_i}$. Summing over $l \in \{0, \dots, L-1\}$ implies that

$$\sum_{i=0}^{d} \alpha_i \le \sum_{l=1}^{L} \sqrt{m_l V_l}.$$

On the other hand,

$$\left(\sum_{i=0}^{d} \alpha_i\right)^2 = \sum_{i=0}^{d} \alpha_i \left(\alpha_i + 2\sum_{j=0}^{i-1} \alpha_j\right)$$

$$\geq \sum_{i=0}^{d} \alpha_i \left(\alpha_i + 2\sum_{j=0}^{i-1} (\sqrt{j+1} - \sqrt{j})\sqrt{\nu_i}\right)$$

$$= \sum_{i=0}^{d} \alpha_i (\alpha_i + 2\sqrt{i\nu_i})$$

$$= \sum_{i=0}^{d} \nu_i.$$

This concludes the proof.

Theorem 4.1 provides a lower bound the work-normalized variance of $\hat{\phi}$ that matches, up to a logarithmic factor, the upper bound in Theorem 3.1.

Theorem 4.1. If Assumption A1 holds and there is a positive constant \hat{c} such that $\hat{t}_l \geq \hat{c}m_l$ for $1 \leq l \leq L$, then $\hat{c}d_t \text{Var}(f(U)) \leq \hat{T} \text{Var}(\hat{\phi})$ and, for $\epsilon > 0$,

$$T(\tilde{\phi}, \epsilon) = \Omega(d + d_t \operatorname{Var}(f(U))\epsilon^{-2}). \tag{19}$$

Proof. It follows from (9) that

$$\hat{c}\left(\sum_{l=1}^{L}\sqrt{m_{l}V_{l}}\right)^{2} \leq \hat{T}\operatorname{Var}(\hat{\phi}).$$

By Proposition 2.2 and Assumption A1, $D(m_l) \leq \text{Var}(f(U) - \phi_l)$ for $0 \leq l \leq L$. Applying Lemma 4.1 with $\nu_i = D(i)$ for $0 \leq i \leq d$ yields

$$\left(\sum_{l=1}^{L} \sqrt{m_l V_l}\right)^2 \geq \sum_{i=0}^{d} D(i)$$

$$= d_t \operatorname{Var}(f(U)),$$

where the second equation follows from Proposition 2.3. This shows that $\hat{c}d_t \operatorname{Var}(f(U)) \leq \hat{T}\operatorname{Var}(\hat{\phi})$. By (8), the expected running time of $\hat{\phi}$ is lower-bound by $\hat{t}_L \geq \hat{c}d$. Together with (5), this implies (19).

5 Time-varying Markov chains

This section shows that, under certain conditions, the expectation of functionals of time-varying Markov chains with d time-steps can be estimated efficiently via MLMC, and that the associated truncation dimension is upper bounded by a constant independent of d.

Let d be a positive integer and let $(X_i:0\leq i\leq d)$ be a time-varying Markov chain with state-space F and deterministic initial value X_0 . Assume that there are independent random variables Y_i , $0\leq i\leq d-1$, uniformly distributed in [0,1], and measurable functions g_i from $F\times[0,1]$ to F such that $X_{i+1}=g_i(X_i,Y_i)$ for $0\leq i\leq d-1$. Our goal is to estimate $E(g(X_d))$ where g is a deterministic real-valued measurable function on F such that $g(X_d)$ is square-integrable. It is assumed that g and the g_i 's can be calculated in constant time. For $1\leq i\leq d$, set $U_i=Y_{d-i}$. An inductive argument shows that there is a real-valued measurable function f on $[0,1]^d$ such that $g(X_d)=f(U)$, where $U=(U_1,\ldots,U_d)$. When X_d is mainly determined by the last Y_i 's, the first U_i 's are the most important arguments of f.

Remark 5.1. Redrawing U_1, \ldots, U_i while keeping U_{i+1}, \ldots, U_d unchanged amounts to keeping X_0, \ldots, X_{d-i} unchanged and redrawing X_{d-i+1}, \ldots, X_d . This can be achieved in O(i) time. Thus (13) holds for f.

Given $i \in \{0, \ldots, d\}$, define the time-varying Markov chain $(X_j^{(i)}: d-i \leq j \leq d)$ by setting $X_{d-i}^{(i)}:=X_0$ and $X_{j+1}^{(i)}=g_j(X_j^{(i)},Y_j)$ for $d-i \leq j \leq d-1$. Thus, $X_d^{(i)}$ is the state of the original Markov chain X at time-step d if the chain is at state X_0 at time-step d-i. Note that $g(X_d^{(i)})$ can be calculated in O(i) time and is a deterministic function of U_1,\ldots,U_i . Roughly speaking,

if X_d is determined to a large extent by the last Y_j 's, then $X_d^{(i)}$ should be "close" to X_d for large values of i. This motivates the following assumption:

Assumption 2 (A2). There are constants c' and $\gamma < -1$ independent of d such that, for $0 \le i \le d$, we have $E((g(X_d) - g(X_d^{(i)}))^2) \le c'(i+1)^{\gamma}$.

I now describe a multilevel estimator of $E(\phi)$, where $\phi = g(X_d)$, using the notation in Section 2.3. Let $L = \lceil \log_2(d) \rceil$ and, for $1 \le l \le L - 1$, let $m_l = 2^l - 1$. Let $\phi_0 = 0$, $\phi_L = d$ and, for $1 \le l \le L - 1$, let $\phi_l = g(X_d^{(m_l)})$. For $1 \le l \le L$, let $\hat{\phi}_l$ be the average of n_l independent copies of $\phi_l - \phi_{l-1}$, where $n_l = \lceil d2^{l(\gamma-1)/2} \rceil$. Suppose that $\hat{\phi}_1, \ldots, \hat{\phi}_L$ are independent. Set $\hat{\phi} := \sum_{l=1}^L \hat{\phi}_l$. By (6), $E(\hat{\phi}) = E(\phi)$. Let \hat{T} (resp. \hat{t}_l) be the expected time to simulate $\hat{\phi}$ is (resp. $\phi_l - \phi_{l-1}$). Proposition 5.1 shows that, under Assumption A2, $\hat{\phi}$ can be used to estimate $E(\phi)$ with precision ϵ in $O(d + \epsilon^{-2})$ time and, if $Var(g(X_d))$ is lower-bounded by a constant independent of d, the truncation dimension d_t associated with $g(X_d)$ is upper-bounded by a constant independent of d. In contrast, the standard Monte Carlo method typically achieves precision ϵ in $O(d\epsilon^{-2})$ time.

Proposition 5.1. Suppose that Assumption A2 holds. Then there are constants c_1 , c_2 and c_3 independent of d such that $\hat{T} \leq c_1 d$, $Var(\hat{\phi}) \leq c_2 / d$, and $T(\hat{\phi}, \epsilon) \leq c_3 (d + \epsilon^{-2})$. Moreover,

$$d_t \operatorname{Var}(g(X_d)) \le c' \frac{\gamma}{\gamma + 1}.$$

Proof. By construction, $\hat{t}_l \leq c2^l$ for some constant c independent of d. By (8),

$$\hat{T} \leq c \sum_{l=1}^{L} (1 + d2^{l(\gamma - 1)/2}) 2^{l}$$

$$\leq cd(4 + \frac{1}{1 - 2(\gamma + 1)/2}).$$

By Assumption A2, for $0 \le l \le L$,

$$\operatorname{Var}(g(X_d) - \phi_l) \le c' 2^{l\gamma}.$$

Since $\text{Var}(Z+Z') \leq 2(\text{Var}(Z)+\text{Var}(Z'))$ for square-integrable random variables Z and Z', it follows that $V_l \leq 4c'2^{(l-1)\gamma}$ for $1 \leq l \leq L$. Together with (7), this shows that

$$\operatorname{Var}(\hat{\phi}) \leq 4c' \sum_{l=1}^{L} \frac{2^{(l-1)\gamma}}{d2^{l(\gamma-1)/2}} \\ \leq \frac{4c' 2^{(1-\gamma)/2}}{d(1-2^{(\gamma+1)/2})}.$$

Using (5) implies the desired bound on $T(\hat{\phi}, \epsilon)$.

By Proposition 2.2, for $0 \le i \le d$,

$$D(i) \le E((g(X_d) - g(X_d^{(i)}))^2)$$

 $< c'(i+1)^{\gamma}.$

Thus, using Proposition 2.3,

$$d_t \operatorname{Var}(f(U)) \leq c' \sum_{i=1}^d i^{\gamma}$$

$$\leq c' (1 + \int_1^d x^{\gamma} dx)$$

$$\leq c' \frac{\gamma}{\gamma + 1}.$$

5.1 A Lindley recursion example

In this example, $F = \mathbb{R}$ and $(X_i : 0 \le i \le d)$ satisfies the time-varying Lindley equation

$$X_{i+1} = (X_i + \zeta_i(Y_i))^+,$$

with $X_0 = 0$, where ζ_i , $0 \le i \le d - 1$, is a real-valued function on [0, 1]. Our goal is to estimate $E(X_d)$. Thus g is the identity function and $g_i(x, y) = (x + \zeta_i(y))^+$ for $(x, y) \in \mathbb{R} \times [0, 1]$. Lindley equations often arise in queuing theory (Asmussen and Glynn 2007).

Proposition 5.2. If there are constants $\theta > 0$ and $\kappa < 1$ independent of d such that

$$E(e^{\theta\zeta_i(Y_i)}) \le \kappa \tag{20}$$

for $0 \le i \le d-1$, then $E((X_d - X_d^{(i)})^2) \le \theta' \kappa^i$ for $0 \le i \le d-1$, where θ' is a constant independent of d.

Proposition 5.2 shows that, if (20) holds, then so does Assumption A2, hence the conclusions of Proposition 5.1 hold as well. The proof of Proposition 5.2 is essentially the same as that of (Kahalé 2020b, Proposition 10), and is therefore omitted. A justification of (20) for time-varying queues and numerical examples showing the efficiency of MLMC for estimating Markov chain functionals are given in (Kahalé 2020b).

Acknowledgments

This work was achieved through the Laboratory of Excellence on Financial Regulation (Labex ReFi) under the reference ANR-10-LABX-0095. It benefitted from a French government support managed by the National Research Agency (ANR). The author thanks Art Owen for helpful comments.

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