Chordality of locally semicomplete and weakly quasi-transitive digraphs

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Abstract

Chordal graphs are important in the structural and algorithmic graph theory. A digraph analogue of chordal graphs was introduced by Haskin and Rose in 1973 but has not been a subject of active studies until recently when a characterization of semicomplete chordal digraphs in terms of forbidden subdigraphs was found by Meister and Telle.

Locally semicomplete digraphs, quasi-transitive digraphs, and extended semicomplete digraphs are amongst the most popular generalizations of semicomplete digraphs. We extend the forbidden subdigraph characterization of semicomplete chordal digraphs to locally semicomplete chordal digraphs. We introduce a new class of digraphs, called weakly quasi-transitive digraphs, which contains quasi-transitive digraphs, symmetric digraphs, and extended semicomplete digraphs, but is incomparable to the class of locally semicomplete digraphs. We show that weakly quasi-transitive digraphs can be recursively constructed by simple substitutions from transitive oriented graphs, semicomplete digraphs, and symmetric digraphs. This recursive construction of weakly quasi-transitive digraphs, similar to the one for quasi-transitive digraphs, demonstrates the naturalness of the new digraph class. As a by-product, we prove that the forbidden subdigraphs for semicomplete chordal digraphs are the same for weakly quasi-transitive chordal digraphs. The forbidden subdigraph characterization of weakly quasi-transitive chordal digraphs generalizes not only the recent results on quasi-transitive chordal digraphs and extended semicomplete chordal digraphs but also the classical results on chordal graphs.

1 Introduction

We consider digraphs which do not contain loops or multiple arcs but may contain digons (i.e., pairs of arcs joing vertices in opposite directions). If an arc is contained in a digon then it is called a *symmetric arc*. A digraph which does not contain a symmetric arc is called an *oriented graph*. A digraph which contains only symmetric arcs is called a *symmetric digraph*. Graphs may be viewed as symmetric digraphs.

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Two vertices in a digraph D are *adjacent* and refered to as *neighbours* of each other if there is at least one arc between them. We say that u is an *in-neighbour* of v or v an *out-neighbour* of u if uv is an arc in D (symmetric or not). The set of all in-neighbours of a vertex v is denoted by $N^-(v)$ and the set of all out-neighbours of v is denoted by $N^+(v)$. We use $S(D)$ to denote the spanning subdigraph of D whose arc set consists of all symmetric arcs in D.

A vertex v in a digraph D is *di-simplicial* if for every $u \in N^-(v)$ and $w \in N^+(v)$ with $u \neq w$, uw is an arc of D. A digraph D is *chordal* if every induced subdigraph of D contains a di-simplicial vertex. It follows that every chordal digraph D has a vertex ordering $v_1, v_2, \ldots, v_n$ such that v_i is a di-simplicial vertex in the subdigraph of D induced by $v_i, v_{i+1}, \ldots, v_n$ for each $i \geq 1$. Such an ordering is called a *perfect elimination ordering* of D.

Perfect elimination orderings of digraphs arise in the study of sparse linear systems by Gaussian elimination, cf. [9]. When a digraph is symmetric, di-simplicial vertices coincide with simplicial vertices of its underlying graph. Thus, a symmetric digraph is chordal if and only if its underlying graph is chordal. It is well-known that chordal graphs are precisely the graphs which do not contain an induced cycle of length ≥ 4, cf. [8].

Little is known about the forbidden structure of chordal digraphs. In particular, there is no known characterization of chordal digraphs by forbidden subdigraphs. Recently, Meister and Telle [11] found a forbidden subdigraph characterization for semicomplete chordal digraphs. A digraph D is *semicomplete* if between any two vertices there is at least one arc. The following theorem is proved in [11].

Theorem 1.1. *[11] A semicomplete digraph D is chordal if and only if $S(D)$ is chordal and D does not contain any of the digraphs in Figure 1 as an induced subdigraph.* □

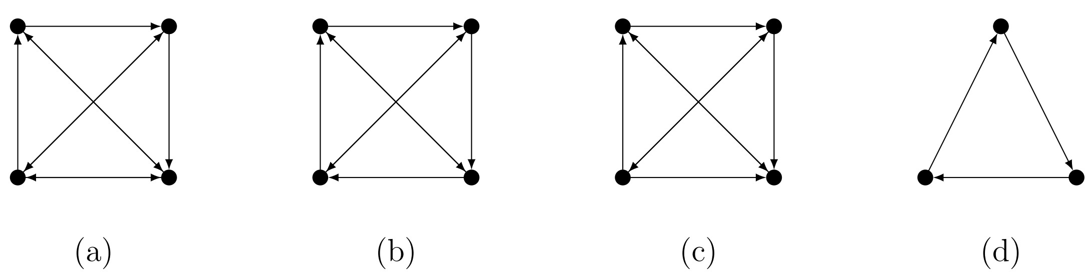

Figure 1: Semicomplete digraphs which are not chordal

A digraph D is called *locally semicomplete* if for every vertex v, $N^-(v)$ and $N^+(v)$ each induces a semicomplete subdigraph in D. Locally semicomplete digraphs are a popular generalization of semicomplete digraphs and have been extensively studied, cf. [1, 2, 3, 10]. Many properties for semicomplete digraphs hold for locally semicomplete digraphs, cf. [1]. However, there are locally semicomplete digraphs which are neither semicomplete nor chordal. Any directed cycle consisting of non-symmetric arcs is locally semicomplete but not chordal, and is not semicomplete if it has has four or more vertices.

We will prove that directed cycles with four or more vertices consisting of non-symmetric arcs are the only minimal locally semicomplete digraphs which are not chordal and which are not semicomplete.

Quasi-transitive digraphs are another well-studied class of digraphs generalizing semicomplete digraphs, cf. [4, 5, 6, 7]. A digraph $D = (V, A)$ is called *quasi-transitive* if for any three vertices u, v, w, $uv \in A$ and $vw \in A$ imply $uw \in A$ or $wu \in A$ (or both), cf. [4]. The class of quasi-transitive digraphs contains all *transitive oriented graphs*. These are the oriented graphs which satisfy the property that for any three vertices u, v, w, $uv \in A$ and $vw \in A$ imply $uw \in A$. Equivalently, they are the oriented graphs in which every vertex is a di-simplicial vertex. Quasi-transitive chordal digraphs are studied recently in [12], where it is proved that they have the same forbidden subdigraphs as for semicomplete chordal digraphs as stated in Theorem 1.1.

We introduce a new class of digraphs as a common generalization of several classes of digraphs including quasi-transitive digraphs and symmetric digraphs. Since graphs can be viewed as symmetric digraphs, the new class of digraphs contains all graphs.

Let v be a vertex and u, w be neighbours of v in a digraph D. Then u, w are called *synchronous* neighbours of v if u, w are both in $N^-(v) \setminus N^+(v)$, or in $N^+(v) \setminus N^-(v)$, or in $N^-(v) \cap N^+(v)$; otherwise they are called *asynchronous* neighbours of v. We call a digraph D *weakly quasi-transitive* if for each vertex v of D, any two asynchronous neighbours of v are adjacent.

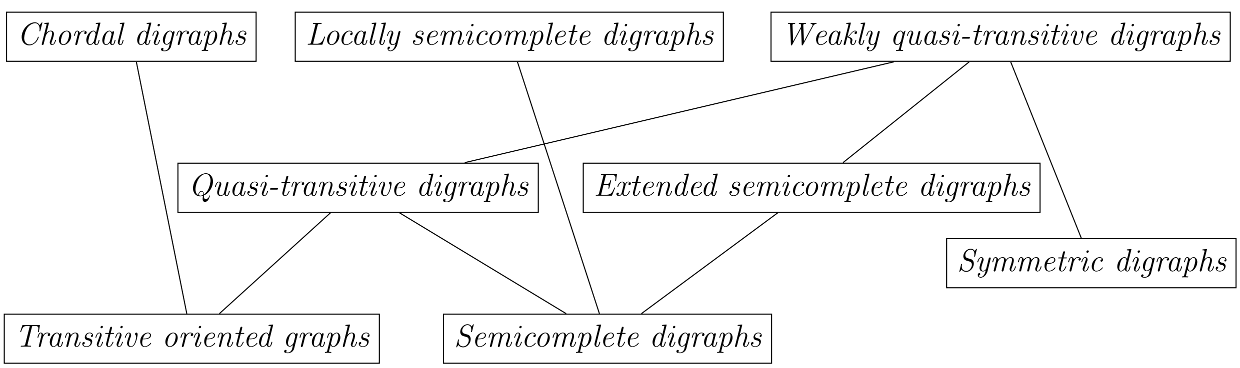

Figure 2: A containment hierarchy

The class of weakly quasi-transitive digraphs contains all quasi-transitive digraphs (and hence contains all semicomplete digraphs as well as all transitive oriented digraphs). Indeed, suppose D is not weakly quasi-transitive. Then some vertex v has two non-adjacent asynchronous neighbours u, w. Since u, w are asynchronous neighbours of v, one of of u, w is in $N^-(v)$ and the other is in $N^+(v)$. Hence D is not a quasi-transitive digraph. Clearly every symmetric digraph is weakly quasi-transitive. Symmetric digraphs have the property that the neighbours of each vertex are synchronous and any digraph having this property is weakly quasi-transitive. If a digraph D is weakly quasi-transitive then any digraph obtained from D by substituting an independent set for each vertex of D is also weakly quasi-transitive. *Extended semicomplete digraphs* are the digraphs obtained this way from semicomplete digraphs so they are all weakly quasi-transitive. Therefore the class of weakly quasi-transitive digraphs simultaneously contains quasi-transitive digraphs, symmetric digraphs, and extended semicomplete digraphs. Figure 2 depicts a containment hierarchy of the digraph classes relevant to this paper.

Let D be a digraph with vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be vertex-disjoint digraphs. A *substitution* of the digraphs H_i for the vertices v_i in D for each i is a new digraph D^* obtained from $H_1, H_2, \ldots, H_n$ by adding all possible arcs xy where $x \in V(H_i)$ and $y \in V(H_j)$ for each arc v_iv_j in D. We use $D[H_1, H_2, \ldots, H_n]$ to denote the new digraph D^* and also say that it is obtained from D by *substituting* H_i for v_i for each i. We call D *strong* if for any two vertices x, y there is a directed (x, y)-path and a directed (y, x)-path; otherwise we call D *non-strong*.

Theorem 1.2. *[4] Let D be a quasi-transitive digraph. Then the following statements hold:*

1. *If D is non-strong, then $D = T[H_1, H_2, \ldots, H_n]$ where T is a transitive oriented graph and each H_i is a strong quasi-transitive digraph.*
2. *If D is strong, then $D = S[H_1, H_2, \ldots, H_n]$ where S is a strong semicomplete digraph and each H_i is either a single-vertex digraph or a non-strong quasi-transitive digraph.*

□

Thus every quasi-transitive digraph can be obtained from transitive oriented graphs and semicomplete digraphs recursively by substitutions. Weakly quasi-transitive digraphs admit a similar construction. We will show (see Theorem 3.2) that weakly quasi-transitive digraphs can be constructed recursively from transitive oriented graphs, symmetric digraphs, and semicomplete digraphs by substitutions. As a by-product of this recursive construction, we prove that the forbidden subdigraphs for weakly quasi-transitive chordal digraphs are exactly those for semicomplete chordal digraphs. The forbidden subdigraph characterization of weakly quasi-transitive chordal digraphs generalizes not only the results of [12] on quasi-transitive chordal digraphs and extended semicomplete chordal digraphs but also the classical results on chordal graphs.

2 Locally semicomplete chordal digraphs

Let D be a digraph and $C : v_1v_2 \ldots v_kv_1$ be a directed cycle in D. If there is no arc between v_i and v_j for all i, j with $|i - j| \notin \{1, k - 1\}$, then the cycle C is called *induced* in D.

Lemma 2.1. *If D is a chordal digraph, then D does not contain an induced directed cycle consisting of non-symmetric arcs and $S(D)$ does not contain an induced directed cycle of length ≥ 4.*

Proof: Suppose that C is either an induced directed cycle in D consisting of non-symmetric arcs or an induced directed cycle of length ≥ 4 in $S(D)$. Then the subdigraph of D induced by the vertices of C has no di-simplicial vertex and hence is not a chordal digraph. Therefore D is not a chordal digraph. □

When $S(D)$ contains no induced directed cycle of length ≥ 4, $S(D)$ is a chordal digraph and hence has di-simplicial vertices. The di-simplicial vertices of $S(D)$ necessarily contain the di-simplicial vertices of D, as observed in [11].

Lemma 2.2. *[11] Every di-simplicial vertex of a digraph D is a di-simplicial vertex of $S(D)$.* □

Suppose that v is a di-simplicial vertex of $S(D)$ but not a di-simplicial vertex of D. Since v is not a di-simplicial vertex of D, there exist $u \in N^-(v)$ and $w \in N^+(v)$ such that uw is not an arc of D and we shall call such an ordered triple (u, v, w) of vertices a *violating triple* for v. We remark that a violating triple (u, v, w) exists only if v is a di-simplicial vertex of $S(D)$ and it certifies that v is not a di-simplicial vertex of D. We call v *type 1* if for every violating triple (u, v, w), both uv, vw are non-symmetric and *type 2* otherwise. The following lemma allows us to streamline the selection of violating triples.

Lemma 2.3. *Let D be a locally semicomplete digraph such that $S(D)$ is chordal and D does not contain an induced directed cycle consisting of non-symmetric arcs or any digraph in Figure 1 as an induced subdigraph. Suppose that (u, v, w) is a violating triple. Then the following statements hold:*

1. *If uv is a non-symmetric arc then there exists a di-simplicial vertex u' of $S(D)$ (possibly $u' = u$) such that (u', v, w) is a violating triple and $u'v$ is a non-symmetric arc.*
2. *If vw is a non-symmetric arc then there exists a di-simplicial vertex w' of $S(D)$ (possibly $w' = w$) such that (u, v, w') is a violating triple and vw' is a non-symmetric arc.*

Proof: The two statements can be obtained from each other by reversing the arcs of D. Thus we only prove the first one. Assume uv is non-symmetric. Consider $S(D) - (N^-[w] \cap N^+[w])$ where $N^-[w] = N^-(w) \cup \{w\}$ and $N^+[w] = N^+(w) \cup \{w\}$. Since uw is not an arc of D, u is not a vertex in $N^-[w] \cap N^+[w]$ and hence is a vertex of $S(D) - (N^-[w] \cap N^+[w])$. Since $S(D)$ is chordal, $S(D) - (N^-[w] \cap N^+[w])$ is also chordal. Thus each component of $S(D) - (N^-[w] \cap N^+[w])$ contains a di-simplicial vertex. Let u' be a di-simplicial vertex of the component of $S(D) - (N^-[w] \cap N^+[w])$ that contains u. Let $u_1u_2 \ldots u_k$ where $u_1 = u$ and $u_k = u'$ be a directed path in $S(D) - (N^-[w] \cap N^+[w])$. We prove by induction on k that (u_k, v, w) is a violating triple and u_kv is a non-symmetric arc of D.

This is true when $k = 1$. So assume $k > 1$, (u_{k-1}, v, w) is a violating triple and $u_{k-1}v$ is a non-symmetric arc of D. Suppose that vw is a symmetric arc. Then there is an arc between u_{k-1} and w as they are both in-neighbours of v. Since $u_{k-1}w$ is not an arc of D, wu_{k-1} is a non-symmetric arc. Thus both w and u_k are in-neighbours of u_{k-1} so there is an arc between them. Since $u_k \notin N^-[w] \cap N^+[w]$, w and u_k are joined by a non-symmetric arc. Since u_k and v are out-neighbours of u_{k-1}, they are adjacent. If u_k and v are joined by symmetric arcs, then u_k and w are in $N^-(v) \cap N^+(v)$. Since v is a di-simplicial vertex, u_k and w are joined by symmetric arcs, which contradicts the fact $u_k \notin (N^-[w] \cap N^+[w])$. If vu_k or u_kw is an arc of D, then the subdigraph of D indcued by v, w, u_{k-1}, u_k is Figure 1(a), (b) or (c), contradicting to our assumption. Hence vu_k is not an arc (i.e., u_kv is a non-symmetric arc) and u_kw is not an arc of D, that is, (u_k, v, w) is a violating triple. On the other hand, suppose that vw is a non-symmetric arc. Since $u_{k-1}w$ is not an arc and D does not contain an induced directed cycle consisting of non-symmetric arcs, there is no arc between u_{k-1} and w. This implies there is no arc between u_k and w as otherwise u_{k-1}, w are non-adjacent vertices in $N^-(u_k)$ or in $N^+(u_k)$, which contradicts that D is locally semicomplete. The vertices u_k, v are adjacent because they are out-neighbours of u_{k-1}. Since w is an out-neighbour of v and there is no arc between u_k and w, u_k cannot be an out-neighbour of v. Therefore u_kv is a non-symmetric arc of D, which means (u_k, v, w) is a violating triple. □

We call a violating triple (u, v, w) *canonical* if u is a di-simplicial vertex of $S(D)$ whenever uv is a non-symmetric arc and w is a di-simplicial vertex of $S(D)$ whenever vw is a non-symmetric arc. Lemma 2.3 ensures that if there is a violating triple for v then there exists a canonical violating triple for v. In particular, if v is a type 1 vertex then there is a violating triple (u, v, w) for v such that u, w are both di-simplicial vertices of $S(D)$.

Theorem 2.4. *A locally semicomplete digraph D is chordal if and only if $S(D)$ is chordal and D does not contain as an induced subdigraph a directed cycle consisting of non-symmetric arcs or a digraph in Figure 1.*

Proof: The necessity follows from Theorems 1.1 and Lemma 2.1. For the other direction assume that $S(D)$ is chordal and D contains neither a directed cycle consisting of non-symmetric arcs nor a digraph in Figure 1 as an induced subdigraph. To prove D is chordal it suffices to show that D has a di-simplicial vertex. Since $S(D)$ is chordal, $S(D)$ has di-simplicial vertices. If some di-simplicial vertex of $S(D)$ is a di-simplicial vertex of D then we are done. Hence we also assume that none of the di-simplcial vertices of $S(D)$ is a di-simplicial vertex of D.

First suppose that $S(D)$ has di-simplicial vertices of type 1. Let v be such a vertex. Then there is a violating triple for v and thus by Lemma 2.3 there is a canonical violating triple (u, v, w) for v. Note that u, w are both di-simplicial vertices of $S(D)$ and uw is not an arc. Since D contains no directed cycles consisting of non-symmetric arcs, wu is not an arc and so u, w are not adjacent. We claim that the triple (u, v, w) can be chosen so that u is type 1. We prove this by contradiction. So assume u is type 2. Then there is a canonical violating triple (u_1, u, w_1) for u such that exactly one of u_1u, uw_1 is a non-symmetric arc. Suppose first that u_1u is non-symmetric and uw_1 is symmetric. Since u_1, w_1 are both in-neighbours of u and D is locally semicomplete, they are adjacent. But u_1w_1 is not an arc so w_1u_1 is a non-symmetric arc. There is no arc between w_1 and w as otherwise w, u are in-neighbours or out-neighbours of w_1, which contradicts the fact that they are not adjacent. Since w is an out-neighbour of v but not adjacent to w_1, w_1

cannot be an out-neighbour of v. But w_1 and v are adjacent as they are out-neighbours of u so w_1v is a non-symmetric arc. Since u is an out-neighbour of u_1 but not adjacent to w, w cannot be an out-neighbour of u_1. Similarly, w_1 is an in-neighbour of u_1 but not adjacent to w, w cannot be an in-neighbour of u_1. Hence w is not adjacent to u_1. There must be an arc between v and u_1 as they are out-neighbours of w_1. But u_1 cannot be an out-neighbour of v because it is not adjacent to w which is an out-neighbour of v. Hence u_1v is a non-symmetric arc and (u_1, v, w) is a violating triple. Since (u_1, u, w_1) is a canonical violating triple and u_1u is non-symmetric, u_1 is a di-simplicial vertex of $S(D)$ and hence (u_1, v, w) is a canonical violating triple. A similar proof shows that if u_1u is symmetric and uw_1 is non-symmetric then (w_1, v, w) is a canonical violating triple. Therefore we have proved that in the case when u is not type 1 there exists a vertex x (which is u_1 or w_1) such that (x, v, w) is a canonical violating triple and the arc between x and u is non-symmetric. If x is type 1 then it is a desired vertex. Otherwise x is type 2. Repeating the same argument as above with x replacing u we find the next vertex x' which is either a desired vertex or a type 2 vertex such that (x', v, w) is a canonical violating triple. Continuing this way in a finite number of steps we either find a desired vertex u (i.e., u is type 1 and (u, v, w) is a canonical violating triple) or a 'circuit' $x_1, x_2, \ldots, x_k$, along with vertices $y_1, y_2, \ldots, y_k$, such that for each $i = 1, 2, \ldots, k$,

- x_i is di-simplicial vertex of $S(D)$ of type 2,
- (x_i, v, w) and (y_i, v, w) are canonical violating triples, and
- there is a non-symmetric arc between x_i and x_{i+1} and either (x_{i+1}, x_i, y_i) or (y_i, x_i, x_{i+1}) is a canonical violating triple (subscritps are modulo k).

Assume the latter occurs and the circuit has the minimum length. Note that the vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$ are in-neighbours of v so they are pairwise adjacent. Since D does not contain any digraph in Figure 1 as an induced subdigraph, the circuit is not a directed cycle (consisting of non-symmetric arcs). Hence we may assume without loss of generality that x_1x_2, x_1x_k are non-symmetric arcs. Then (y_1, x_1, x_2) and (x_1, x_k, y_k) are canonical violating triples. If x_2y_k is a non-symmetric arc then x_1, x_2, y_k induce Figure 1(d), a contradiction to assumption. If x_2y_k is symmetric then x_1, x_2, y_1, y_k induce Figure 1(a), (b) or (c), also a contradiction. So y_kx_2 is a non-symmetric arc. Since x_k is a di-simplicial vertex of $S(D)$ and y_k is a adjacent to x_k but not to x_2 in $S(D)$, x_2 is not adjacent to x_k in $S(D)$, that is, the arc between x_2 and x_k is non-symmetric. If x_2x_k is an arc then $x_2, \ldots, x_k$ would be a shorter circuit, a contradiction to our choice of circuit. So x_kx_2 is a non-symmetric arc. There is an arc between y_1 and x_k as they are out-neighbours of x_1. If y_1x_k is non-symmetric, then y_1, x_k, x_2 induce Figure 1(d) and if x_ky_1 is non-symmetric, then x_1, x_k, y_1, y_k induce Figure 1(a), (b) or (c), a contradiction to assumption. Hence y_1x_k is a symmetric arc and $x_2, \ldots, x_k$ is a shorter circuit, which is also a contradiction. Therefore for every type 1 vertex v there exists a canonical violating triple (u, v, w) such that u is a type 1 vertex. This implies that there exists a directed cycle on type 1 vertices consisting of non-symmetric arcs. Assume that $v_1, v_2, \ldots, v_t$ is the shortest such cycle. Since D does not contain an induced directed cycle consisting of non-symmetric arcs, $t > 3$ and there is a symmetric arc joining a pair of non-consecutive vertices of the cycle. Without loss of generality assume v_1v_s is a symmetric arc of the shortest distance along the cycle, that is, v_i, v_j are not adjacent for all $1 \leq i < j-1 \leq s-1$ except $i = 1$ and $j = s$. Since v_2 and v_s are out-neighbour of v_1, they are adjacent. This implies that $s = 3$ and so v_1v_3 is a symmetric arc. Hence (v_2, v_3, v_1) is a violating triple in which v_3v_1 is a symmetric arc, which contradicts the assumption that v_3 is a type 1 vertex. Therefore $S(D)$ has no type 1 di-simplicial vertex, that is, every di-simplicial vertex of $S(D)$ is type 2.

Let v be a di-simplicial vertex of $S(D)$. Since v is type 2, there is a canonical violating triple (u, v, w) such that exactly one of uv, vw is a non-symmetric arc. If uv is non-symmetric then u is a di-simplicial vertex of $S(D)$. If vw is non-symmetric then w is a di-simplicial. This implies that for each di-simplicial vertex of $S(D)$ there is a di-simplicial vertex z of $S(D)$ such that z, v are part of a canonical violating triple for v and the arc between v and z is non-symmetric. It follows that there exists a 'circuit' $z_1, z_2, \ldots, z_r$, along with vertices $w_1, w_2, \ldots, w_r$, such that for each $i = 1, 2, \ldots, r$,

- z_i is a di-simplicial vertex of $S(D)$ of type 2,
- either (z_{i+1}, z_i, w_i) or (w_i, z_i, z_{i+1}) is a canonical violating triple,
- the arc between z_i and z_{i+1} is non-symmetric and the arcs between w_i and z_i are symmetric (subscripts are modulo r).

We again assume that the circuit is chosen to have the minimum length. Suppose $r = 2$. If the non-symmetric arc between z_1 and z_2 is z_1z_2, then (w_1, z_1, z_2) and (z_1, z_2, w_2) are the canonical violating triples where w_1z_1 and z_2w_2 are symmetric arcs. Neither w_1z_2 nor z_1w_2 is an arc. Since w_1 and z_2 are out-neighbours of z_1, they are adjacent so z_2w_1 is a non-symmetric arc. Similarly, w_2z_1 is a non-symmetric arc. There is an arc between w_1 and w_2 as they are in-neighbours of z_1. Depending the arcs between w_1 and w_2, the subdigraph induced by z_1, z_2, w_1, w_2 is Figure 1(a), (b) or (c), which contradicts the assumption. The same conclusion holds if the non-symmetric arc between z_1 and z_2 is z_2z_1. So $r \geq 3$.

Suppose that $z_1z_2 \ldots z_rz_1$ is a directed cycle. Since D does not contain an induced directed cycle consisting of non-symmetric arcs, $r > 3$ and there is a symmetric arc between a pair of non-consecutive vertices of the cycle. Without loss of generality assume z_1z_s is a symmetric arc of the shortest distance along the cycle, that is, z_i, z_j are not adjacent for all $1 \leq i < j - 1 \leq s - 1$ except $i = 1$ and $j = s$. Since z_2 and z_s are out-neighbour of z_1, they are adjacent. This implies that $s = 3$ and so z_1z_3 is a symmetric arc. Since z_3 and z_r are in-neighbours of z_1, they are adjacent. Since z_3 and w_r are out-neighbours of z_1, they are adjacent. The arcs between z_3 and w_r cannot be symmetric as otherwise z_1 and w_r are both neighbours of z_3 in $S(D)$ but z_1w_r is a non-symmetric arc, which contradicts the fact that z_3 is a di-simplicial vertex of $S(D)$. So z_3 and w_r are joined by a non-symmetric arc. If w_rz_3 is a non-symmetric arc, then the subdigraph induced by z_1, z_3, z_r, w_r is Figure 1(a), (b) or (c), a contradiction. Hence z_3w_r is a non-symmetric arc. The arc between z_3 and z_r must be non-symmetric as otherwise z_3 and w_r are non-adjacent neighbours of z_r in $S(D)$, which contradicts the fact that z_r is a di-simplicial vertex of $S(D)$. If z_3z_r is a non-symmetric arc then z_1, z_3, z_r, w_r induce Figure 1(c), a contradiction. On the other hand, if z_rz_3 is a non-symmetric arc, then $z_3, \ldots, z_r$ would be a directed cycle of length shorter than r consisting of non-symmetric arcs, which contradicts the choice of circuit. Therefore $z_1z_2 \ldots z_rz_1$ is not a directed cycle. Hence we may assume without loss of generality that z_1z_2 and z_1z_r are non-symmetric arcs.

Since z_1z_2 and z_1z_r are non-symmetric arcs, (w_1, z_1, z_2) and (z_1, z_r, w_r) are canonical violating triples. Since z_2 and z_r are out-neighbours of z_1, they are adjacent. So z_2 is an in-neighbour or an out-neighbour of z_r. Combining this with the fact that w_r is both an in-neighbour and an out-neighbour of z_r we see that z_2 and w_r are adjacent. If z_2 and w_r are joined by symmetric arcs then z_1, z_2, w_1, w_r induced Figure 1(a), (b) or (c), a contradiction. So z_2 and w_r are joined by a non-symmetric arc. If z_2w_r is a non-symmetric arc, then z_1, z_2, w_r induce Figure 1(d), a contradiction. Hence w_rz_2 is a non-symmetric arc. This means that w_r is not adjacent to z_2 in $S(D)$. However, w_r is adjacent to z_r in $S(D)$ and z_r is a di-simplicial vertex of $S(D)$. It follows that z_2 and z_r are joined by a non-symmetric arc. If z_2z_r is a non-symmetric arc, then $z_2, \ldots, z_r$ would be a shorter circuit, a contradiction to our choice. So z_rz_2 is a non-symmetric arc. Since w_1 and z_r are out-neighbours of z_1, they are adjacent. If w_1 and z_r are joined by symmetric arcs, then again $z_2, \ldots, z_r$ would be a shorter circuit, a contradiction. So w_1 and z_r are joined by a non-symmetric arc. It cannot be w_1z_r as otherwise w_1, z_r, z_2 induce Figure 1(d), a contradiction. Hence z_rw_1 is a non-symmetric arc. The subdigraph induced by z_1, z_r, w_1, w_r is Figure 1(a), (b) or (c), a contradiction. Therefore, D has a di-simplicial vertex. This completes the proof. □

3 Weakly quasi-transitive digraphs

According to Theorem 1.2, transitive oriented graphs and semicomplete digraphs are basic building blocks for quasi-transitive digraphs. Using these blocks one can form a class $\mathcal{Q}$ of digraphs as follows:

1. Each transitive oriented graph is in $\mathcal{Q}$.
2. Each semicomplete digraph is in $\mathcal{Q}$.
3. If $D, H_1, H_2, \ldots, H_n \in \mathcal{Q}$, then $D[H_1, H_2, \ldots, H_n] \in \mathcal{Q}$, provided that H_i is a single-vertex digraph when the vertex v_i for which H_i is substituted is incident with a symmetric arc for each i.

Transitive oriented graphs and semicomplete digraphs are quasi-transitive. Moreover, the substitution operation for defining $\mathcal{Q}$ maintain the property of being quasi-transitive. Hence the digraphs in $\mathcal{Q}$ are all quasi-transitive. Theorem 1.2 ensures that every quasi-transitive digraph can be obtained from transitive oriented graphs and semicomplete digraphs by substitutions. Therefore we have the following:

Corollary 3.1. *The class $\mathcal{Q}$ consists of quasi-transitive digraphs.* □

Interestingly, weakly quasi-transitive digraphs can also be constructed in a similar way from transitive oriented graphs, semicomplete digraphs and symmetric digraphs.

Let $\mathcal{W}$ be the class of digraphs defined as follows:

1. each transitive oriented graph is in $\mathcal{W}$;
2. each semicomplete digraph is in $\mathcal{W}$;
3. each symmetric digraph is in $\mathcal{W}$;
4. if D is in $\mathcal{W}$ then any digraph obtained from D by substituting digraphs of $\mathcal{W}$ for the vertices of D is in $\mathcal{W}$.

A *module* in a digraph D is an induced subgraph H of D such that for any vertex x not in H, either x is adjacent to no vertex in H or the vertices in H are synchronous neighbours of x. A module is called *trivial* if it has only one vertex or is the entire digraph D and *non-trivial* otherwise. An *oriented path* in D is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that v_i and v_{i+1} are joined by a non-symmetric arc for each $i = 1, 2, \ldots, k-1$.

Theorem 3.2. *The class $\mathcal{W}$ consists of weakly quasi-transitive digraphs.*

Proof: Transitive oriented graphs and semicomplete digraphs are quasi-transitive, so they are weakly quasi-transitive. Symmetric digraphs are also weakly quasi-transitive because any vertex in a symmetric digraph has only synchronous neighbours. To prove the rest of digraphs in $\mathcal{W}$ are all weakly quasi-transitive, let $D^* = D[H_1, H_2, \ldots, H_n]$ where $D, H_1, H_2, \ldots, H_n$ are weakly quasi-transitive. Consider three vertices u, v, w where u, w are asynchronous neighbours of v. Assume $u \in V(H_i)$, $v \in V(H_j)$ and $w \in V(H_k)$. If $i = j = k$ then u, w are adjacent as H_i is weakly quasi-transitive. Suppose $i = j \neq k$. Since v and w are adjacent, each vertex of H_i is adjacent to all vertices of H_k and in particular, u is adjacent to w. Similarly, if $i \neq j = k$, then u and w are adjacent. Suppose that $i \neq j \neq k$. Then $i \neq k$ because u and w are asynchronous neighbours of v. Since D is weakly quasi-transitive, the two vertices of D corresponding to H_i and H_k are adjacent so u and w are adjacent. Hence all digraphs in $\mathcal{W}$ are weakly quasi-transitive.

We prove by induction on number of vertices that every weakly quasi-transitive digraph is in $\mathcal{W}$. Let D be a weakly quasi-transitive with n vertices. Assume that every weakly quasi-transitive digraph with fewer than n vertices is in $\mathcal{W}$. If D is quasi-transitive or symmetric then it is in $\mathcal{W}$. So assume that D is neither quasi-transitive nor symmetric. Since D is not quasi-transitive, there exist vertices u, v, w with $u \in N^-(v)$ and $w \in N^+(v)$ such that u and w are not adjacent in D. Thus u and w are non-adjacent neighbours of v. Since D is weakly quasi-transitive, any two asynchronous neighbours of v are adjacent. Hence u and w are synchronous neighbours of v, which implies u and w are both in $N^+(v) \cap N^-(v)$.

Suppose H is a non-trivial module in D. Let D' be the digraph obtained from D by deleting all vertices of H except one. Then $D = D'[H_1, H_2, \ldots, H_k]$ where $H_1 = H$ and each H_i with $i \geq 2$ is a single-vertex digraph. The digraphs $D', H_1, \ldots, H_k$ each has fewer than n vertices and is weakly quasi-transitive and hence they are in $\mathcal{W}$. This means that D is obtained from digraphs in $\mathcal{W}$ by substitution and by definition D is in $\mathcal{W}$. Thus, it suffices to show that there is a non-trivial module in D.

Let R be the subdigraph of D induced by $N^+(v) \cap N^-(v)$. Then u and w are a pair of non-adjacent vertices in R. Let M_1 be the subdigraph of R induced by the vertices which are connected to u by paths in $\overline{U(R)}$. Clearly, M_1 contains u and w but not v. Suppose x is a vertex in $N^+[v] \cup N^-[v]$ but not in M_1. We claim that x is completely adjacent to M_1. Indeed, if $x \in N^+[v] \cap N^-[v]$, then the definition of M_1 implies that x is completely adjacent to M_1. On the other hand, if $x \in N^+(v) \oplus N^-(v)$, then x and any vertex of M_1 are asynchronous neighbours of v so x is also completely adjacent to M_1. By definition any two vertices of M_1 are connected by a path in $\overline{U(M_1)}$. In such a path any two consecutive vertices are not adjacent in D and hence are synchronous neighbours of x. It follows that the vertice of M_1 are synchronous neighbours of x. Suppose $x \notin N^+[v] \cup N^-[v]$. If x is adjacent to some vertex y in M_1, then x and v are non-adjacent neighbours of y and hence they must be synchronous neighbours of y. The fact that v is joined to y by symmetric arcs implies x is joined to y by symmetric arcs. Thus if x is completely adjacent to M_1 then the vertices of M_1 are synchronous neighbours of x. It follows that M_1 is a module if for each $x \notin N^+[v] \cup N^-[v]$, either x is adjacent to no vertex in M_1 or completely adjacent to M_1. We may assume M_1 is not a module as otherwise we are done. This means that there exist vertices x, y, y' with $x \notin N^+[v] \cup N^-[v]$ and $y, y' \in M_1$ such that x is adjacent to y but not to y'. These three vertices x, y, y' along with M_1 will be refered to in the rest of proof.

Suppose $N^+(v) \oplus N^-(v) \neq \emptyset$. Any vertex in $N^+(v) \oplus N^-(v)$ is a neighbour of v asynchronous to those of v in $N^+(v) \cap N^-(v)$. Hence every vertex in $N^+(v) \oplus N^-(v)$ is completely adjacent to $N^+(v) \cap N^-(v)$ and in particular to M_1. Suppose that the arcs between $N^+(v) \oplus N^-(v)$ and M_1 are all symmetric. Let M_2 be the subdigraph of D induced by vertices which are connected to v by oriented paths. Clearly, M_2 contains v and all vertices in $N^+(v) \oplus N^-(v)$. We show that x is not a vertex in M_2. Suppose not; there is an oriented path connecting x and v. Then there must exists an oriented path connecting x and a vertex in $N^+(v) \oplus N^-(v)$. Let $a_1 \sim a_2 \sim \cdots \sim a_s$ be such a path where $a_1 = x$ and $a_s \in N^+(v) \oplus N^-(v)$. Note that a_s is joined to each vertex of M_1 by symmetric arcs and a_1 $(= x)$ is not adjacent to y' (in M_1). Let j be the largest subscript such that a_j is not adjacent to some vertex y'' of M_1. Then $j < k$ and $a_j \notin N^+[v] \cup N^-[v]$. Since a_j and a_{j+1} are joined by a non-symmetric arc, $a_{j+1} \notin N^+[v] \cap N^-[v]$. Either $a_{j+1} \in N^+(v) \oplus N^-(v)$ or $a_{j+1} \notin N^+[v] \cup N^-[v]$. In either case a_{j+1} is joined to each vertex of M_1 by symmetric arcs. Thus a_j and y'' are non-adjacent asynchronous neighbours of a_{j+1}, contradicting the assumption that D is weakly quasi-transitive. So x is not a vertex of M_2. We show that M_2 is a module. Let z be a vertex not in M_2. By definition z cannot be joined to any vertex of M_2 by a non-symmetric arc. Suppose z is joined to some vertex h of M_2 by symmetric arcs. Since h can reach every other vertex of M_2 by an oriented path, following such a path we see that z is joined to every vertex in the path by symmetric arcs. Hence the vertices of M_2 are synchronous neighbours of z. Therefore M_2 is a non-trivial module in D.

Suppose now that the arcs between $N^+(v) \oplus N^-(v)$ and M_1 are not all symmetric. Let M_3 be a subdigraph of D induced by the vertices defined recursively as follows:

- u is a vertex in M_3;
- if h is a vertex in $N^+(v) \cap N^-(v)$ that is not adjacent to a vertex in M_3 then h is a vertex in M_3;
- if h is not in $N^+(v) \cap N^-(v)$ that is joined to a vertex in M_3 by symmetric arcs then h is a vertex in M_3.

It is easy to see that M_3 contains u, v, w, x and all vertices of M_1. Let b be a vertex in $N^+(v) \oplus N^-(v)$ which is joined to a vertex in M_1 by a non-symmetric arc. Assume that $b \in N^-(v) \setminus N^+(v)$. From the above we know that the vertices of M_1 are synchronous neighbours of b. In particular, y, y' are synchronous neighbours of b. The vertex y is joined to b by a non-symmetric arc and joined to x by symmetric arcs. Thus b and x are asynchronous neighbours of y and hence they must be adjacent. So x and v are neighbours of b. Since x and v are not adjacent, they are synchronous neighbours of b. Since $b \in N^-(v) \setminus N^+(v)$, bv is a non-symmetric arc, so bx is also a non-symmetric arc. Since bx is a non-symmetric arc and x, y' are non-adjacent neighbours of b, by' is also a non-symmetric arc. The fact that the vertices of M_1 are synchronous neighbours of b so there is a non-symmetric arc from b to every vertex in M_1. Similarly, if $b \in N^+(v) \setminus N^-(v)$ is joined to a vertex in M_1 by a non-symmetric arc then xb is a non-symmetric arc and there is a non-symmetric arc from every vertex of M_1 to b.

We claim that b is not a vertex in M_3. Suppose not; b is in M_3. By the definition of M_3 there exists a sequence of vertices $h_0, h_1, \ldots, h_t$ where $h_0 = y$ and $h_t = b$ such that for each $i > 0$, $h_i \in N^+(v) \cap N^-(v)$ implies that h_i is not adjacent to h_{i-1}, and $h_i \notin N^+(v) \cap N^-(v)$ implies h_i is joined to h_{i-1} by symmetric arcs. We choose such a vertex b so that the sequence is as short as possible. Assume $b \in N^-(v) \setminus N^+(v)$. Then b $(= h_t)$ is joined to h_{t-1} by symmetric arcs. We claim $h_{t-1} \in N^+(v) \cap N^-(v)$. Indeed, since b is joined to h_{t-1} by symmetric arcs, $h_{t-1} \in N^+(v) \cup N^-(v)$. Suppose $h_{t-1} \in N^-(v) \setminus N^+(v)$. The choice of b implies that there can only be symmetric arcs between h_{t-1} and M_1. Since h_{t-1} and x are asynchronous neighbours of b, they are adjacent. In particular, $h_{t-1}x$ is a non-symmetric arc. Thus x, y' are non-adjacent asynchronous neighbours of h_{t-1}, a contradiction. So $h_{t-1} \notin N^-(v) \setminus N^+(v)$. A similar proof shows $h_{t-1} \notin N^+(v) \setminus N^-(v)$. So $h_{t-1} \in N^+(v) \cap N^-(v)$. Since b is joined to h_{t-1} by symmetric arcs and joined to each vertex of M_1 by a non-symmetic arc, $h_{t-1} \notin M_1$ and thus $t > 2$. Hence h_{t-1} is not adjacent to h_{t-2} and is completely adjacent to M_1. If $h_{t-2} \in N^+[v] \cup N^-[v]$, then h_{t-2} must be in $N^+(v) \cup N^-(v)$ and hence adjacent to b. Thus h_{t-1}, h_{t-2} are neighbours of b. Since h_{t-1}, h_{t-2} are not adjacent, they are synchronous neighbours of b, which implies b is joined to h_{t-2} by symmetric arcs. This contradicts the choice of the sequence as $h_0, h_1, \ldots, h_{t-2}, b$ is a shorter sequence. So $h_{t-2} \notin N^+[v] \cup N^-[v]$. Let ℓ be the largest integer such that $h_{t-2}, \ldots, h_{t-\ell}$ are not in $N^+[v] \cup N^-[v]$. Then h_{t-i} is joined to h_{t-i-1} by symmetric arcs for each $i = 2, \ldots, \ell$. We must have $h_{t-\ell-1} \in N^+(v) \cap N^-(v)$. The vertex b is not adjacent to h_{t-2} as otherwise h_{t-1}, h_{t-2} are non-adjacent asynchronous neighbours of b, a contradiction. For the same reason, we see that b is not adjacent to h_{t-i} for each $i = 2, \ldots, \ell$. Since $b, h_{t-\ell-1}$ are asynchronous neighbours of v, they are adjacent. They must be joined by symmetric arcs, as otherwise $b, h_{t-\ell}$ are non-adjacent asynchronous neighbours of $h_{t-\ell-1}$, a contradiction. But this contradicts the choice of the sequence because $h_0, h_1, \ldots, h_{t-\ell-1}, b$ is a shorter sequence. Therefore b is not a vertex in M_3. So if $b \in N^-(v) \setminus N^+(v)$ is joined to a vertex in M_1 with a non-symmetric arc then $b \notin M_3$ and there is a non-symmetric arc from b to every vertex in M_1. A similar proof shows that if $b \in N^+(v) \setminus N^-(v)$ is joined to a vertex in M_1 with a non-symmetric arc then $b \notin M_3$ and there is a non-symmetric arc from each vertex of M_1 to b.

We show that M_3 is a module. Let z be a vertex that is not in M_3. For each vertex $h \in M_3$, there is a sequence of vertices $h_0, h_1, \ldots, h_t$ where $h_0 = y$ and $h_t = h$ such that for each $i > 0$, $h_i \in N^+(v) \cap N^-(v)$ implies that h_i is not adjacent to h_{i-1}, and $h_i \notin N^+(v) \cap N^-(v)$ implies h_i is joined to h_{i-1} by symmetric arcs. Suppose first that $z \in N^-(v) \setminus N^+(v)$. We know from the above that zx is a non-symmetric arc and zh is a non-symmetric arc for all $h \in M_1$. In particular, zy $(= zh_0)$ is a non-symmetric arc. Suppose $k > 0$ and zh_{k-1} is a non-symmetric arc. If $h_k \in N^+(v) \cap N^-(v)$, then h_{k-1}, h_k are non-adjacent neighbours of z so zh_k is a non-symmetric arc. If $h_k \notin N^+(v) \cap N^-(v)$, then z, h_k are asynchronous neighbours of h_{k-1} so they are adjacent. There are two cases. Either $h_k \in N^+(v) \oplus N^-(v)$ or $h_k \notin N^+(v) \cup N^-(v)$. If $h_k \notin N^+(v) \cup N^-(v)$, then clearly zh_k is a non-symmetric arc. Assume $h_k \in N^+(v) \oplus N^-(v)$. Since $h_k \notin M_3$, h_k is joined to each vertex in M_1 by symmetric arcs. In particular, h_k is joined to y' by symmetric arcs. Since y' is not adjacent to x, h_k and x cannot be adjacent as otherwise y' and x are non-adjacent asynchronous neighbours of h_k, a contradiction. Hence h_k and x are synchronous neighbours of z. Since zx is a non-symmetric arc, zh_k is a non-symmetric arc. Therefore zh is a non-symmetric arc for all $h \in M_3$. A similar proof shows that if $z \in N^+(v) \setminus N^-(v)$ then hz is a non-symmetric arc for all $h \in M_3$. Suppose next that $z \in N^+(v) \cap N^-(v)$. Since z is not in M_3, z is adjacent to every vertex in M_3. In particular, z is adjacent to x. Note that z and x are joined by symmetric arcs. Since x and y' are not adjacent, z is adjacent to y' by symmetric arcs. This implies z is also joined to y by symmetric arcs. Suppose that $k > 0$ and z is joined to h_{k-1} by symmetric arcs. If $h_k \notin N^+(v) \cup N^-(v)$ then clearly z is joined to h_k by symmetric arcs. If $h_k \in N^+(v) \cap N^-(v)$, then h_k is not adjacent to h_{k-1} and thus h_k, h_{k-1} are non-adjacent neighbours of z. Since z is joined to h_{k-1} by symmetric arcs, z is joined to h_k by symmetric arcs. If $h_k \in N^+(v) \oplus N^-(v)$, then h_k is joined to y' by symmetric arcs. Since y' and x are not adjacent, h_k and x are not adjacent. Thus h_k and x are non-adjacent neighbours of z, which implies z is joined to h_k by symmetric arcs. Suppose now that $z \notin N^+[v] \cup N^-[v]$. Since z is not in M_3, it is not adjacent to any vertex in M_1. In particular, z is not adjacent to y. Suppose that $k > 0$ and z is not adjacent to h_{k-1}. If $h_k \in N^+(v) \cap N^-(v)$, then z is not adjacent to h_k as otherwise z is joined to h_k by symmetric arcs, which implies $z \in M_3$, a contradiction to assumption. If $h_k \notin N^+(v) \cap N^-(v)$, then h_k is joined to h_{k-1} by symmetric arcs. Since z is not adjacent to h_{k-1}, z cannot be joined to h_k by a non-symmetric arc. Since $z \notin M_3$ and $h_k \in M_3$, z cannot be joined to h_k by symmetric arcs. Hence z is not adjacent to h_k.

The only case remaining is that $N^+(v) \oplus N^-(v) = \emptyset$. Since D is not a symmetric digraph, it has a non-symmetric arc. Suppose fg is a non-symmetric arc in D. Let M_4 be the subdigraph induced by the vertices which are connected to f by oriented paths. Then any two vertices in M_4 are connected by an oriented path. Since $N^+(v) \oplus N^-(v) = \emptyset$, there is no oriented path connecting f and v. So v is not a vertex in M_4. Suppose z is not in M_4 but is adjacent to a vertex h in M_4. Then z is joined to h by symmetric arcs. Each vertex of M_4 is connected to h by an oriented path. Following these oriented paths we see that z is joined to each vertex of M_4 by symmetric arcs and hence the vertices of M_4 are synchronous neighbours of z. Therefore M_4 is a non-trivial module. □

The class of weakly quasi-transitive digraphs strictly contains quasi-transitive digraphs and extended semicomplete digraphs, which in turn as classes strictly contain all semicomplete digraphs. Surprisingly, these four classes of digraphs share the same forbidden subdigraphs for being chordal.

Theorem 3.3. *A weakly quasi-transitive digraph D is chordal if and only if $S(D)$ is chordal and D does not contain any digraph in Figure 1 as an induced subdigraph.*

Proof: If D is chordal then it does not contain any digraph in Figure 1 as an induced subdigraph. Suppose D does not contain any digraph in Figure 1 as an induced subdigraph. We prove by induction on the number of vertices that D is chordal. It suffices to show that D has a di-simplicial vertex. This is true if D is a transitive oriented graph, a semicomplete digraph, or a symmetric digraph. Assume D is a weakly quasi-transitive digraph but not a transitive oriented graph, a semicomplete digraph, or a symmetric digraph. For the inductive hypothesis, assume that any induced subdigraph of D with fewer vertices than D has a di-simplicial vertex. By Theorem 3.2, $D = D'[H_1, H_2, \ldots, H_n]$ where D' and one of H_i's have at least two vertices. Then D' and each H_i is an induced subdigraph of D with fewer vertices than D and by the inductive hypothesis each of them has a di-simplicial vertex. Suppose that v is a di-simplicial vertex of D' and H_j is substituted for v. Then it is easy to verify that a di-simplicial vertex of H_j is a di-simplicial vertex of D. □

Corollary 3.4. *[12] Let D be a quasi-transitive digraph or an extended semicomplete digraph. Then D is chordal if and only if $S(D)$ is chordal and D does not contain any digraph in Figure 1 as an induced subdigraph.* □

Since graphs can be viewed as symmetric digraphs which are a subclass of the class of weakly quasi-transitive digraphs and none of the digraphs in Figure 1 is symmetric, Theorem 3.3 implies that the cycles of length ≥ 4 are precisely the forbidden induced subgraphs of chordal graphs.

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