Kernels by properly colored paths in arc-colored digraphs *

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September 24, 2018

Abstract

A *kernel by properly colored paths* of an arc-colored digraph D is a set S of vertices of D such that (i) no two vertices of S are connected by a properly colored directed path in D, and (ii) every vertex outside S can reach S by a properly colored directed path in D. In this paper, we conjecture that every arc-colored digraph with all cycles properly colored has such a kernel and verify the conjecture for unicyclic digraphs, semi-complete digraphs and bipartite tournaments, respectively. Moreover, weaker conditions for the latter two classes of digraphs are given.

Keywords: kernel; kernel by monochromatic (properly colored, rainbow) paths

1 Introduction

All graphs (digraphs) considered in this paper are finite and simple, i.e., without loops or multiple edges (arcs). For terminology and notation not defined here, we refer the reader to Bang-Jensen and Gutin [1].

A path (cycle) in a digraph always means a *directed* path (cycle) and a *k-cycle* C_k means a cycle of length k, where $k \geq 2$ is an integer. For a digraph D, define its *kernel* to be a set S of vertices of D such that (i) no two vertices of S are connected by an arc in D, and (ii) every vertex outside S can reach S by an arc in D. This notion was originally introduced by von Neumann and Morgenster [21] in 1944. Since it has many applications in both cooperative games and logic (see [2, 3]), its existence has been the focus of extensive study, both from the algorithmic perspective and the sufficient condition

*The first author is supported by NSFC (Nos. 11601430 and 11671320), NPU (No. 2016KY0101) and China Postdoctoral Science Foundation (No. 2016M590969); the second author is supported by JSPS KAKENHI (No. 15K04979); and the third author is supported by NSFC (Nos. 11571135 and 11671320). Part of this work was done while the first and the third author were visiting Yokohama City University and the hospitality was appreciated.

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perspective. Among them, the following results are of special importance. For more results on kernels, we refer the reader to the survey paper [4] by Boros and Gurvich.

Theorem 1 (Chvátal [6]). *It is NP-complete to recognize whether a digraph has a kernel or not.*

Theorem 2 (Richardson [18], von Neumann and Morgenster [21]). *Let D be a digraph. Then the following statements hold:*
(i) if D has no cycle, then D has a unique kernel;
(ii) if D has no odd cycle, then D has at least one kernel;
(iii) if D has no even cycle, then D has at most one kernel.

An arc $uv \in A(D)$ is called *symmetrical* if $vu \in A(D)$. For a cycle $(u_0, u_1, \ldots, u_{k-1}, u_0)$, we call two arcs u_iu_{i+2} and $u_{i+1}u_{i+3}$ *crossing consecutive*, where addition is modulo k. The following theorem has been proved.

Theorem 3 (Duchet [8], Duchet and Meyniel [9], Galeana-Sánchez and Neumann-Lara[12]). *A digraph D has a kernel if one of the following conditions holds:*
(i) each cycle has a symmetrical arc;
(ii) each odd cycle has two crossing consecutive arcs;
(iii) each odd cycle has two chords whose heads are adjacent vertices.

It is worth noting that if we replace Condition (ii) in the definition of kernels by every vertex outside S can reach S by an arc or a path of length 2, then such a vertex subset, named *quasi-kernel*, always exists. This was proved by Chvátal and Lovász [7] in 1974. Jacob and Meyniel [17] furthermore showed in 1996 that every digraph has either a kernel or three quasi-kernels. For more results on quasi-kernels, see [5, 13, 16].

Let D be a digraph and m a positive integer. Call D an *m-colored digraph* if its arcs are colored with at most m colors. Denote by $c(uv)$ the color assigned to the arc uv. A subdigraph H of an arc-colored digraph D is called *monochromatic* if all arcs of H receive the same color, and is called *rainbow* if any two arcs of H receive two distinct colors. Define a *kernel by monochromatic paths* (or an *MP-kernel* for short) of an arc-colored digraph D to be a set S of vertices of D such that (i) no two vertices of S are connected by a monochromatic path in D, and (ii) each vertex outside S can reach S by a monochromatic path in D.

The concept of MP-kernels in an arc-colored digraph was introduced by Sands, Sauer and Woodrow [19] in 1982. They showed that every 2-colored digraph has an MP-kernel. In particular, as a corollary, they showed that every 2-colored tournament has a one-vertex MP-kernel. Here note that each MP-kernel of an arc-colored tournament consists of one vertex. They also proposed the problem that whether a 3-colored tournament with no rainbow triangles has a one-vertex MP-kernel. This problem still remains open and has attracted many authors to investigate sufficient conditions for the existence of MP-kernels in arc-colored tournaments. Shen [20] showed in 1988 that for $m \geq 3$ every m-colored tournament with no rainbow triangles and no rainbow transitive triangles has a one-vertex MP-kernel, and also showed that the condition "with no rainbow triangles and no rainbow transitive triangles" cannot be improved for $m \geq 5$. In 2004, Galeana-Sánchez and Rojas-Monroythe [14] showed, by constructing a family of counterexamples, that the condition of Shen cannot be improved for $m = 4$, either. Galeana-Sánchez [10] showed in 1996 that every arc-colored tournament such that the arcs, with at most one exception, of each cycle of length at most four are assigned the same color has a one-vertex MP-kernel. Besides, Galeana-Sánchez and Rojas-Monroythe [15] showed in 2004 that every arc-colored bipartite tournament with all 4-cycles monochromatic has an MP-kernel. For more results on MP-kernels, we refer to the survey paper [11] by Galeana-Sánchez.

A subdigraph H of an arc-colored digraph D is called *properly colored* if any two consecutive arcs of H receive distinct colors. Define a *kernel by properly colored paths* (or a *PCP-kernel* for short) of an arc-colored digraph D to be a set S of vertices of D such that (i) no two vertices of S are connected by a properly colored path in D, and (ii) each vertex outside S can reach S by a properly colored path in D.

By the definitions of kernels, MP-kernels and PCP-kernels, one can see in some sense that both MP-kernels and PCP-kernels generalize the concept of kernels in digraphs.

Observation 1. *Let $D = (V(D), A(D))$ be a digraph. Then the following three statements are equivalent.*
(i) D has a kernel;
(ii) $|A(D)|$-colored D has an MP-kernel;
(iii) 1-colored D has a PCP-kernel.

In this paper we concentrate on providing some sufficient conditions for the existence PCP-kernels in arc-colored digraphs. For convenience, we write "PC path" for "properly colored path" in the following. Define the *closure* $\mathscr{C}(D)$ of an arc-colored digraph D to be a digraph with vertex set $V(\mathscr{C}(D)) = V(D)$ and arc set $A(\mathscr{C}(D)) = \{uv : \text{there is a PC } (u,v)\text{-path in } D\}$. It is not difficult to see that the following simple (but useful) result holds.

Observation 2. *An arc-colored digraph D has a PCP-kernel if and only if $\mathscr{C}(D)$ has a kernel.*

2 Main results

We first consider the computational complexity of finding a PCP-kernel in an arc-colored digraph.

Proposition 1. *It is NP-hard to recognize whether an arc-colored digraph has a PCP-kernel or not.*

Proof. Let D be a digraph and V^* a set of vertices with $V^* \cap V(D) = \emptyset$. Let D' be the digraph with $V(D') = V(D) \cup V^*$ and $A(D') = A(D) \cup \{uv : \ u \in V^*, v \in V(D)\}$, i.e., adding a set V^* of new vertices to D together with all possible arcs from V^* to

$V(D)$. We can always choose a V^* with $|\{uv: u \in V^*, v \in V(D)\}| \geq m$. Color D' by using m colors in such a way that the subdigraph D is monochromatic and the arc set $\{uv: u \in V^*, v \in V(D)\}$ is m-colored. It is not difficult to see that the m-colored D' has a PCP-kernel if and only if D has a kernel. By Theorem 1 the computational complexity of the latter problem is NP-complete. The desired result then follows directly. $\square$

Now we present the following result.

Proposition 2. *An arc-colored digraph D has a PCP-kernel if one of the following conditions holds:*
(i) *D has no cycle;*
(ii) *the coloring of D is proper (consecutive arcs receive distinct colors);*
(iii) *D is properly-connected (each vertex can reach all other vertices by a PC path).*

Proof. Note that $\mathscr{C}(D)$ has a cycle if and only if D has a cycle. The statement (i) therefore follows directly from Theorem 2 (i) and Observation 2. Assume that the coloring of D is proper. If D is strongly connected, then each vertex forms a PCP-kernel. If D is not strongly connected, then the set of sinks is a PCP-kernel. If D is properly-connected, then by the definition of PCP-kernels each vertex forms a PCP-kernel. $\square$

By Proposition 2 (i), every arc-colored digraph containing no cycle has a PCP-kernel. It is natural to ask what is the analogous answer for a digraph D containing cycles. For the simplest case, i.e., D is a cycle, we get the following result.

Theorem 4. *An arc-colored cycle has a PCP-kernel if and only if it is not a monochromatic odd cycle.*

Call a digraph *unicyclic* if it contains exactly one cycle. Note that every cycle is unicyclic. For general arc-colored unicyclic digraphs, furthermore, for general digraphs containing cycles, a number of examples (see for example the arc-colored digraphs in Figures 1 and 3) show that additional conditions are needed to guarantee the existence of PCP-kernels. But what kind of conditions do we need? By Proposition 2 (ii) and (iii), if the coloring is proper or "close" to proper (roughly speaking), then it has a PCP-kernel. By Proposition 2 (i), the existence of cycles influences the existence of PCP-kernels. This yields a natural question to ask whether the condition "all cycles are properly colored" suffices or not. Based on this consideration, we propose the following conjecture.

Conjecture 1. *Every arc-colored digraph with all cycles properly colored has a PCP-kernel.*

Remark 1. If Conjecture 1 is true, then it is best possible in view of the two arc-colored digraphs in Figure 1, in which solid arcs, dotted arcs and dashed arcs represent arcs colored by three distinct colors respectively. It is not difficult to check that neither of them has a PCP-kernel. For any even integer $n \geq 6$ (resp. odd integer $n \geq 7$), the sharpness of Conjecture 1 can be shown by replacing the path (v_6, v_1, v_2) (resp. (u_9, u_1, u_2)) of the left digraph (resp. the right digraph) by a monochromatic path of length $n-4$ (resp. length $n-7$) using the color assigned to the previous short path. One can check that neither of the two new constructed digraphs has a PCP-kernel.

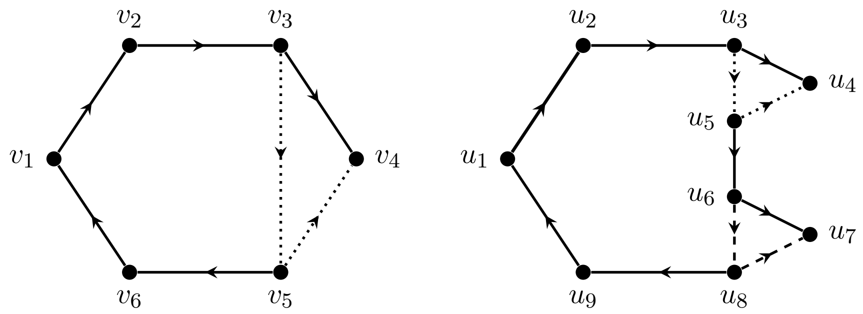

Figure 1: Two arc-colored digraphs with no PCP-kernels.

A digraph D is *semi-complete* if for every two vertices there exists at least one arc between them. A *tournament* (*bipartite tournament*) is an orientation of a complete graph (complete bipartite graph). Note that each tournament is semi-complete. Theorem 4 shows that Conjecture 1 holds for cycles. We will also show that Conjecture 1 holds for general unicyclic digraphs, semi-complete digraphs and bipartite tournaments. In fact, for the latter two classes of digraphs, weaker conditions have been obtained, respectively.

Theorem 5. *Every arc-colored unicyclic digraph with the unique cycle properly colored has a PCP-kernel.*

Remark 2. We see from the two unicyclic arc-colored digraphs in Figure 1 that the condition "the unique cycle is properly colored" cannot be dropped in Theorem 5.

Note that every two vertices in a semi-complete digraph are adjacent and thus every PCP-kernel in such a digraph consists of one vertex. We obtain the following result whose proof idea is similar to that in [20].

Theorem 6. *Every arc-colored semi-complete digraph with no monochromatic triangles has a vertex v such that all other vertices can reach v by a PC path of length at most 3.*

Corollary 1. *Every arc-colored semi-complete digraph with no monochromatic triangles has a PCP-kernel.*

Remark 3. The condition "with no monochromatic triangles" in Theorem 6 and Corollary 1 cannot be dropped. Recall that every tournament is semi-complete and one can verify that the 2-colored tournament shown in Figure 2 has no PCP-kernels and no vertex defined in Theorem 6, in which solid arcs and dotted arcs represent arcs colored by two distinct colors, respectively. Larger m-colored tournaments containing no PCP-kernel for general m can be constructed by adding new vertices together with new colors to the new added arcs such that T^* has no outneighbors in the set of new added vertices.

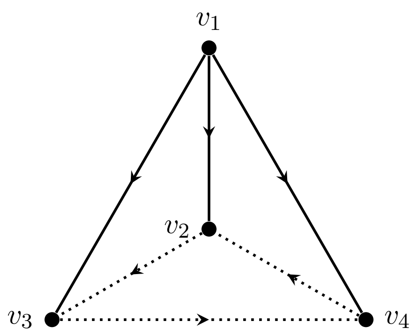

Figure 2: A 2-colored tournament with no PCP-kernels.

Theorem 7. *Every arc-colored bipartite tournament $D = (X, Y; A)$ with (i) all 4-cycles and 6-cycles properly colored, or (ii) $\min\{|X|, |Y|\} \leq 2$, has a PCP-kernel.*

Remark 4. The conditions in Theorem 7 cannot be dropped in view of the 3-colored bipartite tournament $D_6 = (X, Y; A)$ shown in Figure 3, in which solid arcs, dotted arcs and dashed arcs represent arcs colored by three distinct colors, respectively. One can see that $\min\{|X|, |Y|\} = 3$ and D_6 contains neither PC 4-cycles nor PC 6-cycles. One can also see that the closure $\mathscr{C}(D_6)$ of D_6 is semi-complete, in which the new added arcs are represented by thick dashed lines. Note that a semi-complete digraph has a kernel if and only if it has a vertex v such that all other vertices can reach v by an arc. One can see that $\mathscr{C}(D_6)$ does not contain such a vertex, so by Observation 2 we get that D_6 has no PCP-kernels. Furthermore, we can construct infinite family of bipartite tournaments which can show that the conditions in Theorem 7 cannot be dropped. Let D_{n-6} be an arbitrary m-colored bipartite tournament with $n > 6$. Define D to be the union of D_6 and D_{n-6} as follows: take all possible arcs between D_{n-6} and D_6 going from D_{n-6} to D_6 and denote this set of arcs by A^*, let $V(D) = V(D_6) \cup V(D_{n-6})$ and $A(D) = A(D_6) \cup A(D_{n-6}) \cup A^*$, let the colors on D_{n-6} and D_6 remain the same and let the coloring of A^* be arbitrary. Then D has no PCP-kernel since the proposition that D has a PCP-kernel implies that D_6 has a PCP-kernel.

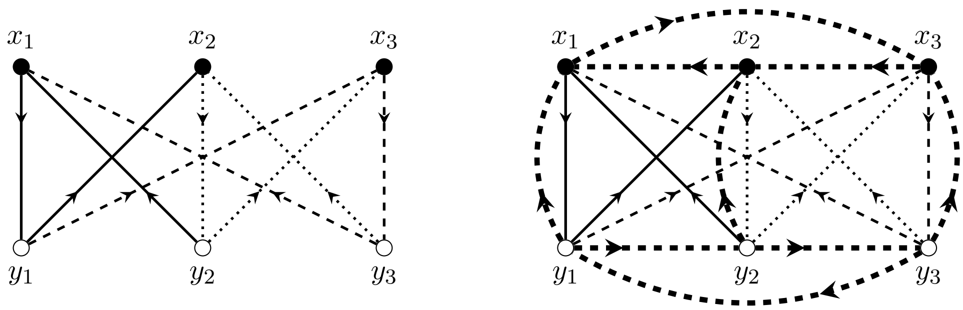

Figure 3: A 3-colored bipartite tournament D_6 and its closure $\mathscr{C}(D_6)$.

In the rest of the paper, we always use $H_1 - H_2$ to denote $H_1 - V(H_2)$ for two digraphs H_1 and H_2; if H_2 consists of a single vertex v, then we denote $H_1 - \{v\}$ by $H_1 - v$. For two vertices u and v, if uv is an arc then we say u *dominates* v and sometimes write $u \to v$.

3 Proofs of Theorem 4 and Theorem 5

Proof of Theorem 4. The necessity of Theorem 4 follows from the fact that each odd cycle has no kernel. For the sufficiency, it is equivalent to show that (i) every arc-colored odd cycle with at least two colors has a PCP-kernel and (ii) every arc-colored even cycle has a PCP-kernel. We prove the result by constructing such a kernel S.

Let $C = (v_0, \ldots, v_{n-1}, v_0)$ be an arc-colored cycle and assume w.l.o.g. that the vertices are located in a clockwise direction. If C is an monochromatic even cycle, then we can let $S = \{v_0, v_2, \ldots, v_{n-2}\}$. Now assume that C is an arc-colored cycle with at least two colors. If the coloring is proper, then clearly each vertex forms a PCP-kernel. Now assume that the coloring is not proper and assume w.l.o.g. that $P_1 = (v_{n_1}, v_{n_1+1}, \ldots, v_{n'_1} = v_{n-1})$ is a monochromatic path of maximum length (which is at least two). Put $v_{n'_1-2}, v_{n'_1-4}, \ldots, v_{n'_1-2t_1}$ into S, where t_1 is the largest integer such that $n'_1 - 2t_1 \geq n_1$. Here, note that since C is neither a monochromatic odd cycle nor a PC cycle, we have $n_1 \in \{0, 1, \ldots, n-3\}$ and $n'_1 - 2t_1 = n_1$ or $n_1 + 1$. Afterwards, we consider, in a counter-clockwise direction, the first appeared maximal monochromatic path of length at least two in $C - P_1$, say $P_2 = (v_{n_2}, v_{n_2+1}, \ldots, v_{n'_2})$. Now put $v_{n'_2-2}, v_{n'_2-4}, \ldots, v_{n'_2-2t_2}$ into S, where t_2 is the largest integer such that $n'_2 - 2t_2 \geq n_2$. Continue this procedure until there is no monochromatic path of length at least two and let $P_r = (v_{n_r}, v_{n_r+1}, \ldots, v_{n'_r})$ be the last appeared maximal monochromatic path of length at least two. It follows that

$$S = \bigcup_{i=1}^{r} \{v_{n'_i-2}, v_{n'_i-4}, \ldots, v_{n'_i-2t_i}\}, \quad \overline{S} = V(C) \backslash S = \overline{S}' \cup \overline{S}'',$$

where

$$\overline{S}' = \bigcup_{i=1}^{r} \{v_{n'_i-3}, v_{n'_i-5}, \ldots, v_{n'_i-2t_i+1}\},$$

$$\overline{S}'' = \bigcup_{i=1}^{r} \{v_{n'_i-2t_i-1}, v_{n'_i-2t_i-2}, \ldots, v_{n'_{i+1}}, v_{n'_{i+1}-1}\},$$

$n_{r+1} = n_1$, $n'_{r+1} = n'_1 = n - 1$ and addition is modulo n. It is not difficult to check that no two vertices of S are connected by a PC path in C. For each $1 \leq i \leq r$, one can also verify that each vertex in $\{v_{n'_i-3}, v_{n'_i-5}, \ldots, v_{n'_i-2t_i+1}\}$ can reach some vertex in $\{v_{n'_i-2}, v_{n'_i-4}, \ldots, v_{n'_i-2t_i+2}\}$ by a PC path of length one, and each vertex in $\{v_{n'_i-2t_i-1}, v_{n'_i-2t_i-2}, \ldots, v_{n'_{i+1}}, v_{n'_{i+1}-1}\}$ can reach $v_{n'_i-2t_i}$ by a PC path; in other words, every vertex outside S can reach S by a PC path in C. Therefore, the set S is a PCP-kernel of C. $\square$

Proof of Theorem 5. Let D be an arc-colored unicyclic digraph with a PC cycle C. Note that the cycle C must be an induced cycle since otherwise two cycles will appear.

Note also that each vertex of C forms a PCP-kernel of C. If D is strongly connected, then D is a cycle and the desired result follows directly. Now assume that D is not strongly connected. Then there exist strongly connected components $D_1, \ldots, D_k$, $k \geq 2$, of D such that there is no arc from D_i to D_j for any $i > j$. Let D_i be the component containing the cycle C. One can see that $D_i = C$. One can also see that each $D_j \neq D_i$ is a single vertex, since otherwise another cycle will appear. We distinguish two cases and show the result by constructing a PCP-kernel S.

If $i = k$, then let v be an arbitrary vertex of $D_k = C$ and we put v into S. Let $j_1 \in \{1, \ldots, k-1\}$ be the largest integer such that there is no PC (D_{j_1}, v)-path. Put D_{j_1} into S. Let $j_2 \in \{1, \ldots, j_1 - 1\}$ be the largest integer such that there is no PC $(D_{j_2}, \{v, D_{j_1}\})$-path. Put D_{j_2} into S. Continue this procedure until all the remaining vertices in $V(C)\backslash S$ can reach S by a PC path. Let D_{j_r} be the last vertex putting into S. The terminal vertex set $S = \{v, D_{j_1}, \ldots, D_{j_r}\}$ is clearly a PCP-kernel.

If $i \neq k$, then D contains at least one sink and we put all sinks, say $v_1, \ldots, v_p$, into S. By similar procedure above we can put, step by step, the vertices $D_{j_1}, \ldots, D_{j_t}$ with $j_t > i$ into S. Let $U \subseteq V(D_i)$ be the set of vertices which cannot reach the current $S = \{v_1, \ldots, v_p, D_{j_1}, \ldots, D_{j_t}\}$ by a PC path. If $U \neq \emptyset$, then put an arbitrary vertex of U (instead of all vertices of U) into S and continue the procedure. If $U = \emptyset$, then $j_{t+1} < i$ and we can use the same procedure above to get a PCP-kernel S. $\square$

4 Proof of Theorem 6

For convenience, in this proof, call a vertex v *good* if all other vertices can reach v by a PC path of length at most 3. One can see that it suffices to consider the tournament case. Let T be an m-colored tournament, where m is a positive integer. For $m = 1$, note that each monochromatic tournament with no monochromatic triangles is transitive, then the unique sink is a good vertex. So we may assume that $m \geq 2$ and T is an arc-colored tournament with at least two colors. We prove the result by induction on $|V(T)|$.

Since each arc-colored transitive triangle and each non-monochromatic triangle has a good vertex, the result holds for $|V(T)| = 3$. Now assume that T is a minimum counterexample with $|V(T)| = k \geq 4$. It follows that each m-colored tournament with no monochromatic triangles and with order less than k has a good vertex. So for each vertex v of T the subtournament $T - v$ has a good vertex. Denote by v^* the good vertex of $T - v$ corresponding to the given coloring of T. Then $v^* \to v$, since otherwise v^* is a good vertex of T. For two distinct vertices u and v, we claim that $u^* \neq v^*$. If not, then by the definition of u^* there exist a PC (v, u^*)-path in $T - u$ and a PC (u, u^*)-path in $T - v$. It follows immediately that there exist a PC (v, u^*)-path and a PC (u, u^*)-path in T. Thus, u^* is a good vertex of T, a contradiction.

Now consider the subdigraph H induced on the arc set $\{v^*v : v \in V(T)\}$. Since each vertex of H has both indegree and outdegree one, then H consists of vertex-disjoint cycles. If H has at least two cycles, then by induction hypothesis the induced subtournament on each cycle has a good vertex, which is obviously a good vertex of T, a contradiction. So H consists of one cycle.

Let $H = (v_0, v_1, \ldots, v_{n-1}, v_0)$. By the choices of the arcs, there exists no PC (v_i, v_{i-1})-path of length at most 3 in T, addition is modulo n in this proof. Consider the three vertices v_0, v_1, v_2, if $v_2 \to v_0$, then since there exists no monochromatic triangle we have either (v_2, v_0, v_1) is PC (v_2, v_1)-path of length 2 or (v_1, v_2, v_0) is a PC (v_1, v_0)-path of length 2, a contradiction. So $v_0 \to v_2$. In fact, one can see from the simple proof that $v_i \to v_{i+2}$ for any $v_i \in V(T)$.

Let s be the minimum integer such that $v_s \to v_0$ and $v_0 \to v_i$ for any $i \leq s-1$. Such an integer s exists by the fact that $v_{n-1} \to v_0$. We may assume that $v_i \to v_j$ for any $1 \leq i < j \leq s$. Since there exists no PC (v_s, v_{s-1})-path of length 2, we have $c(v_s v_0) = c(v_0 v_{s-1})$, say $c(v_s v_0) = c(v_0 v_{s-1}) = 1$. By assumption, there exists no monochromatic triangle, we have $c(v_{s-1} v_s) \neq c(v_s v_0)$. Since there exists no PC (v_{s-1}, v_{s-2})-path of length 3, we have $c(v_0 v_{s-2}) = c(v_s v_0) = 1$. Since (v_0, v_{s-2}, v_s, v_0) is not a monochromatic triangle, we have $c(v_{s-2} v_s) \neq 1$. Similarly, we can show that $c(v_0 v_i) = 1$ and $c(v_i v_s) \neq 1$ for any $1 \leq i \leq s-1$. This implies that $c(v_1 v_s) \neq c(v_s v_0) = 1$ and a PC (v_1, v_0)-path (v_1, v_s, v_0) of length 2 appears, a contradiction.

5 Proof of Theorem 7

Proof of Theorem 7 (i). For the 1-colored case, by Observation 1, it suffices to consider the existence of a kernel. We claim that either X or Y is a kernel. If X is not a kernel, then there exists $y \in Y$ such that each vertex of X is an inneighbor of y, implying that Y is a kernel. So every 1-colored bipartite tournament has a PCP-kernel (not necessary to satisfy the required condition). In the following we assume $m \geq 2$ and consider PCP-kernels in m-colored bipartite tournaments with at least two colors.

We write $u \sim v$ if $u \to v$ or $v \to u$. It is not difficult to verify, see also in [15], that the following lemma holds. We need to keep in mind of this lemma in the forthcoming proof.

Lemma 1 (Galeana-Sánchez and Rojas-Monroy [15]). *Let D be an arc-colored bipartite tournament. Then*
(i) for each directed walk $(u_0, u_1, \ldots, u_k)$ in D we have $u_i \sim u_j$ iff $j - i \equiv 1$ (mod 2);
(ii) every closed directed walk of length at most 6 is a cycle in D.

For two vertices u and v in D, denote by $dist(u, v)$ the distance from u to v. The following lemma will play a key role in the proof.

Lemma 2. *If there exists a PC (u, v)-path but exists no PC (v, u)-path in D, then $dist(u, v) \leq 2$.*

Proof. Let $P = (u_0, u_1, \ldots, u_k)$ be a shortest PC (u, v)-path, where $u = u_0$ and $v = u_k$. The result holds clearly for $k \leq 2$. Now let $k \geq 3$ and assume the opposite that each (u, v)-path has length at least 3.

Claim 1. *There exists no arc from* $\{u_0, u_1, \dots, u_{k-3}\}$ *to* u_k.

Proof. The statement holds directly for $k \le 4$. Assume that $k \ge 5$. Let $i^* = \min\{i : u_i \to u_k, 0 \le i \le k-3\}$. Then $i^* \in \{u_2, u_3, \dots, u_{k-3}\}$ and $u_k \to u_{i^*-2}$. Now $(u_k, u_{i^*-2}, u_{i^*-1}, u_{i^*}, u_k)$ is a 4-cycle and by assumption it is properly colored. So $c(u_{i^*-1}u_{i^*}) \ne c(u_{i^*}u_k)$ and $u_0 P u_{i^*} u_k$ is a PC (u, v)-path of length less than k, a contradiction. □

Claim 2. *There exists* $i \in \{0, 1, \dots, k-3\}$ *such that* $u_i \to u_{i+3}$.

Proof. Assume the opposite that $u_{i+3} \to u_i$ for each $i \in \{0, 1, \dots, k-3\}$. If k is odd, then $u_0 \sim u_k$ and either there exists a (u, v)-path of length 1 or there exists a PC (v, u)-path of length 1, a contradiction. So k is even. Recall that $k \ge 3$. If $k = 4$, then $(u_0, u_1, u_2, u_3, u_0)$ and $(u_1, u_2, u_3, u_4, u_1)$ are PC 4-cycles. So $(u_4, u_1, u_2, u_3, u_0)$ is a PC (v, u)-path, a contradiction. If $k = 6$, then $u_5 \to u_0$ since otherwise (u_0, u_5, u_6) is a (u, v)-path of length 2. Now $(u_3, u_4, u_5, u_6, u_3)$ is a PC 4-cycle and $u_0 P u_5 u_0$ is a PC 6-cycle. Thus, $u_6 u_3 P u_5 u_0$ is a PC (v, u)-path, a contradiction. If $k = 8$, then $u_7 \to u_0$ and $u_8 \to u_1$ since otherwise either (u_0, u_7, u_8) or (u_0, u_1, u_8) is a (u, v)-path of length 2. Besides, we have $u_5 \to u_0$ since otherwise $(u_0, u_5, u_6, u_7, u_0)$ and $(u_5, u_6, u_7, u_8, u_5)$ are PC 4-cycles and $(u_8, u_5, u_6, u_7, u_0)$ is a PC (v, u)-path. We also can show that $u_8 \to u_3$. If not, then $(u_3, u_8, u_1, u_2, u_3)$ is a PC 4-cycle and $(u_0, u_1, u_2, u_3, u_8)$ is a PC (u, v)-path of length less than k, a contradiction. Then there exist two PC 6-cycles $(u_0, u_1, u_2, u_3, u_4, u_5, u_0)$ and $(u_3, u_4, u_5, u_6, u_7, u_8, u_3)$. It follows that $u_8 u_3 P u_5 u_0$ is a PC (v, u)-path, a contradiction. So from now on assume that $k \ge 10$.

We claim first that $u_k \to u_{k-5}$. If not, then $(u_k, u_{k-3}, u_{k-6}, u_{k-5}, u_k)$ is a PC 4-cycle and thus $u_0 P u_{k-5} u_k$ is a PC (u, v)-path of length less than k, a contradiction. We also claim that $u_{k-3} \to u_{k-8}$. If not, then since $(u_{k-9}, u_{k-8}, u_{k-3}, u_{k-6}, u_{k-9})$ and $(u_{k-8}, u_{k-3}, u_{k-2}, u_{k-1}, u_k, u_{k-5}, u_{k-8})$ are PC cycles we have $c(u_{k-9}u_{k-8}) \ne c(u_{k-8}u_{k-3})$ and $c(u_{k-8}u_{k-3}) \ne c(u_{k-3}u_{k-2})$. It follows that $u_0 P u_{k-8} u_{k-3} P u_k$ is a PC (u, v)-path of length less than k, a contradiction.

Recall that $u_{i+3} \to u_i$ for each $i \in \{0, 1, \dots, k-3\}$ and all 4-cycles and 6-cycles are properly colored. Thus,

$$u_k u_{k-5} P u_{k-3} u_{k-8} u_{k-7} u_{k-10} \cdots u_{k-2i} u_{k-2i-1} u_{k-2i+2} \cdots u_0$$

is a PC (v, u)-path, contradicting the assumption in Lemma 2. □

Let i be the minimum integer in $\{0, 1, \dots, k-3\}$ such that $u_i \to u_{i+3}$ and let $j^* = \max\{j : u_i \to u_j, i+3 \le j \le k\}$. By Claim 1, we have $j^* \ne k$. If $j^* = k-1$, then $i \ne 0$; otherwise, (u_0, u_{k-1}, u_k) is a (u_0, u_k)-path of length 2. By Claim 1, we also have $u_k \to u_{i-1}$. Since $(u_{i-1}, u_i, u_{k-1}, u_k, u_{i-1})$ is a PC 4-cycle, we get that $u_0 P u_i u_{k-1} u_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. So we have $j^* \le k-2$.

By the choice of j^*, we have $u_{j^*+2} \to u_i$ and $(u_i, u_{j^*}, u_{j^*+1}, u_{j^*+2}, u_i)$ is a PC 4-cycle. Hence $c(u_i u_{j^*}) \ne c(u_{j^*} u_{j^*+1})$. If $i = 0$, then $u_0 u_{j^*} P u_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. So $i \ge 1$.

By the minimality of i we have $u_{i+2} \to u_{i-1}$. Since $u_i \sim u_{j^*}$, we have $u_{i+2} \sim u_{j^*+2}$. If $u_{j^*+2} \to u_{i+2}$, then $(u_{j^*+2}, u_{i+2}, u_{i-1}, u_i, u_{j^*}, u_{j^*+1}, u_{j^*+2})$ is a PC 6-cycle and $u_0Pu_iu_{j^*}Pu_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. So $u_{i+2} \to u_{j^*+2}$.

If $j^* \leq k-4$, then by the choice of j^* we have $u_{j^*+4} \to u_i$. Now $(u_i, u_{i+1}, u_{i+2}, u_{j^*+2}, u_{j^*+3}, u_{j^*+4}, u_i)$ is a PC 6-cycle and $u_0Pu_{i+2}u_{j^*+2}Pu_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. So $j^* \in \{k-2, k-3\}$. If $j^* = k-2$, then $(u_i, u_{i+1}, u_{i+2}, u_{j^*+2}, u_i)$ is a PC 4-cycle and $u_0Pu_{i+2}u_{j^*+2}$ is a PC (u_0, u_k)-path of length less than k, a contradiction. So $j^* = k-3$ and $u_{j^*+3} = u_k$.

Now we claim that $u_{i+1} \to u_k$. If not, then $(u_{i+1}, u_{i+2}, u_{j^*+2}, u_k, u_{i+1})$ is a PC 4-cycle and $u_0Pu_{i+2}u_{j^*+2}u_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. We may also claim that $u_{i-1} \to u_k$. If not, then $(u_{i-1}, u_i, u_{i+1}, u_k, u_{i-1})$ is a PC 4-cycle and $u_0Pu_{i+1}u_k$ is a PC (u_0, u_k)-path of length less than k, a contradiction. Similarly, we can show that $u_{i-s} \to u_k$ for any odd s with $1 \leq s \leq i$. Clearly, there will be a (u, v)-path of length at most 2.

The proof of Lemma 2 is complete. $\square$

In view of Theorem 3 (i), it suffices to show that every cycle of $\mathscr{C}(D)$ has a symmetrical arc. Assume the opposite that there exists a cycle C in $\mathscr{C}(D)$ containing no symmetrical arc and denote it by

$$C = (u_0, u_1, \ldots, u_l, u_0).$$

We will get a contradiction by showing that C has a symmetrical arc. Here we distinguish two cases.

Case 1. $l = 2$.

Since a bipartite tournament contains no odd cycle, there exists an arc of C in $A(\mathscr{C}(D)) \backslash A(D)$, say u_0u_1. By Lemma 2, there exists a (u_0, u_1)-path of length 2 in D, say (u_0, x_0, u_1).

If $u_1u_2, u_2u_0 \in A(D)$, then $(u_0, x_0, u_1, u_2, u_0)$ is a (properly colored) 4-cycle in D and $u_1u_0 \in \mathscr{C}(D)$. This implies that C has a symmetrical arc u_0u_1, a contradiction.

If $|\{u_1u_2, u_2u_0\} \cap A(D)| = 1$, then by Lemma 2 and Lemma 1 (ii) there will be a 5-cycle which contradicts the well-known fact that a bipartite tournament contains no odd cycle.

Now let $u_1u_2, u_2u_0 \notin A(D)$. Then by Lemma 2, there exist a (u_1, u_2)-path of length 2 and a (u_2, u_0)-path of length 2 in D, say (u_1, x_1, u_2) and (u_2, x_2, u_0). By Lemma 1 (ii) and our assumption we get that $(u_0, x_0, u_1, x_1, u_2, x_2, u_0)$ is a PC 6-cycle. This implies that each arc in C is a symmetrical arc, a contradiction.

Case 2. $l \geq 3$.

In view of Lemma 2, there exists a (u_i, u_{i+1})-path of length at most 2 for each $0 \leq i \leq l$ in D, where $u_{l+1} = u_0$. Let P_i be the shortest (u_i, u_{i+1})-path in D and let $C^* = \cup_{i=0}^{l} P_i$. Then C^* is a closed directed walk in D. For convenience, denote this closed walk by

$$C^* = (x_0, x_1, \ldots, x_s, x_0),$$

where $x_0 = u_0$ and $s \geq l$.

If $x_3x_0 \in A(D)$, then $(x_0, x_1, x_2, x_3, x_0)$ is a PC 4-cycle and $x_1x_0, x_2x_0 \in A(\mathscr{C}(D))$. Note that either $x_0x_1 \in A(C)$ or $x_0x_2 \in A(C)$. This implies that C has a symmetrical arc, a contradiction. Similarly, if $x_0x_{s-2} \in A(D)$, then we can show that either x_sx_0 or $x_{s-1}x_0$ is a symmetrical arc of C, a contradiction.

Now assume that $x_0x_3, x_{s-2}x_0 \in A(D)$. Let i be the minimum integer such that $x_0x_i, x_{i+2}x_0 \in A(D)$. Then $(x_0, x_i, x_{i+1}, x_{i+2}, x_0)$ is a PC 4-cycle in D and $x_{i+1}x_i, x_{i+2}x_i \in A(\mathscr{C}(D))$. If $x_i \in V(C)$, then $\{x_ix_{i+1}, x_ix_{i+2}\} \cap A(C) \neq \emptyset$ and thus either $x_{i+1}x_i$ or $x_{i+2}x_i$ is a symmetrical arc in C, a contradiction. So $x_i \notin V(C)$ and $x_{i-1}x_{i+1} \in A(C)$. By the choice of i, we have $x_0x_{i-2} \in A(D)$. Then $(u_0, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_0)$ is a PC 6-cycle and there exists a PC (x_{i+1}, x_{i-1})-path. So $x_{i-1}x_{i+1}$ is a symmetrical arc in C, a contradiction. □

Proof of Theorem 7 (ii). If $\min\{|X|, |Y|\} = 1$, then D has no cycle and the result follows from Proposition 2 (i). So we can assume w.l.o.g. that $|X| = \min\{|X|, |Y|\} = 2$. By contradiction, suppose the opposite that D has no PCP-kernel. By Proposition 2 we can assume that D has a cycle. It is not difficult to check that if $y \in Y$ is a source then D has a PCP-kernel if and only if $D - y$ has a PCP-kernel. So we assume also that D has no source in Y. Let $X = \{x_1, x_2\}$ and let

$$Y_0 = \{y \in Y : \ x_1 \to y, \ x_2 \to y\},$$

$$Y_1 = \{y \in Y \backslash Y_0 : \ \textit{there exists a PC } (y, Y_0)\text{-path in } D\},$$

$$Y_2 = Y \backslash (Y_0 \cup Y_1).$$

If $Y_2 = \emptyset$, then Y_0 is a PCP-kernel. So we assume that $Y_2 \neq \emptyset$. Two vertices v_1 and v_2 are called *contractible* if for any vertices u and w we have $v_1 \to u$ iff $v_2 \to u$, $w \to v_1$ iff $w \to v_2$, and $c(v_1u) = c(v_2u), c(wv_1) = c(wv_2)$ whenever $v_1u, v_2u, wv_1, wv_2 \in A(D)$. Recall that all digraphs we consider here are simple, that is, contain no loops. So there exists no arc between any two contractible vertices. We now show the following claim.

Lemma 3. *Let v_1, v_2 be two contractible vertices in an arc-colored digraph D'. Then D' has a PCP-kernel if and only if $D' - v_2$ has a PCP-kernel.*

Proof. For the necessity, let S be a PCP-kernel of D'. If $\{v_1, v_2\} \subseteq S$, then by the definition of contractible vertices $S \backslash v_2$ is a PCP-kernel of $D' - v_2$. If $v_2 \in S$ and $v_1 \notin S$, then $S \cup \{v_1\}$ is a PCP-kernel of $D' - v_2$. If $\{v_1, v_2\} \cap S = \emptyset$, then S is also a PCP-kernel of $D' - v_2$. For the sufficiency, let S' be a PCP-kernel of $D' - v_2$. If $v_1 \notin S'$, then S' is a PCP-kernel of D'. Now assume that $v_1 \in S'$. If there exists a PC (v_2, v_1)-path, then S' is a PCP-kernel of D'. Otherwise, there exists no PC (v_1, v_2)-path and $S' \cup \{v_2\}$ is a PCP-kernel of D'. □

Now we assume that D does not contain two contractible vertices and distinguish two cases in the following.

Case 1. $Y_0 \neq \emptyset$.

Since D has no source in Y, each vertex in $Y \backslash Y_0$ has one outneighbor and one inneighbor in $\{x_1, x_2\}$. For a vertex $x \in X$ which has at least one inneighbor in Y_2, since there exists no PC path from Y_2 to Y_0, we have $c(xy_0') = c(xy_0'')$ for any $y_0', y_0'' \in Y_0$; otherwise, for each $y \in Y_2$ with $y \to x$ there exists $y_0 \in Y_0$ with $c(yx) \neq c(xy_0)$, which yields a PC path (y, x, y_0) from Y_2 to Y_0, a contradiction. For convenience, denote by $c(xY_0)$ the common color assigned for the arcs from x to Y_0. By the definition of Y_2, the following claim holds.

Claim 1. *For two vertices $y', y'' \in Y_2$, if $\{y', y''\} \to x$ for some $x \in X$, then $c(y'x) = c(y''x) = c(xY_0)$.*

Let S' be a maximal subset of Y_2 such that no two vertices of S' are connected by a PC path. If $S' = Y_2$, then $Y_0 \cup S'$ is a PCP-kernel. Assume that $S' \neq Y_2$. Let

$$R = \{y \in Y_2 \backslash S' \colon \text{ there exists no PC } (y, S')\text{-path in } D\}.$$

If $R = \emptyset$, then $Y_0 \cup S'$ is a PCP-kernel. So assume that $R \neq \emptyset$. Let r be an arbitrary vertex in R. Then by the definitions of S' and R, there exists a PC (s', r)-path for some $s' \in S'$.

Claim 2. *Every PC (s', r)-path has length 2.*

Proof. By contradiction, assume w.l.o.g. that there exists a PC (s', r)-path of length 4, say (s', x_1, y, x_2, r), where $y \in Y \backslash Y_0$. Since $s' \notin Y_1$, we have $c(s'x_1) = c(x_1Y_0)$ and $c(yx_2) = c(x_2Y_0)$. We show that there exists a PC (z, r)-path for any $z \in Y_2 - \{s', r\}$. If $z = y$, then (z, x_2, r) is a desired path. Now let $z \neq y$. Since $z \notin Y_0$, we have either $z \to x_1$ or $z \to x_2$. If $z \to x_1$, then since $z \in Y_2$ we have $c(zx_1) = c(x_1Y_0) = c(s'x_1)$ and (z, x_1, y, x_2, r) is a desired path. If $z \to x_2$, then similarly $c(zx_2) = c(x_2Y_0) = c(yx_2)$ and (z, x_2, r) is a desired path. It follows that $Y_0 \cup \{r\}$ is a PCP-kernel, a contradiction. □

Now we can assume w.l.o.g. that (s', x_1, r) is a PC (s', r)-path. Remark that $c(s'x_1) \neq c(x_1r)$ and, by Claim 1, each vertex $y \in Y_2$ with $y \to x_1$ can reach r by a PC path (y, x_1, r). Let

$$Q = \{y \in Y_2 \backslash r : \ x_1 \to y\}.$$

If $Q = \emptyset$, then $Y_0 \cup \{r\}$ is a PCP-kernel, a contradiction. So assume that $Q \neq \emptyset$.

Claim 3. *There exists no PC (r, Q)-path.*

Proof. Assume the opposite that there exists a PC (r, Q)-path, say (r, x_2, y, x_1, q), for some $q \in Q$. Then $c(yx_1) = c(x_1Y_0)$ since otherwise (r, x_2, y, x_1, y_0) is a PC (r, y_0)-path for each $y_0 \in Y_0$, contradicting that $r \in Y_2$. Now we show that $Y_0 \cup \{q\}$ is a PCP-kernel. Since $Q \cup \{r\} \subseteq Y_2$, we have $c(qx_2) = c(rx_2) = c(x_2Y_0)$ for each $q \in Q$. So (q', x_2, y, x_1, q) is a PC (q', q) path for each $q' \in Q \backslash q$. For each $y' \in Y_2 \backslash Q$, note that $y' \to x_1$, since $y' \in Y_2$, we have $c(y'x_1) = c(x_1Y_0) = c(yx_1)$. Then (y', x_1, q) is a PC (y', q)-path. It therefore follows that $Y_0 \cup \{q\}$ is a PCP-kernel. $\square$

Claim 4. *There exists no PC path connecting two vertices of Q.*

Proof. By symmetry, assume that (q', x_2, y, x_1, q'') is a PC path for some two vertices $q', q'' \in Q$. Note that $y \neq r$, otherwise, there exists a PC (r, Q)-path (y, x_1, q''), contradicting Claim 3. Since $c(rx_2) = c(q'x_2)$, we get that (r, x_2, y, x_1, q'') a PC (r, q'')-path, a contradiction. $\square$

Let $Q' \subseteq Q$ be the set of vertices which cannot reach r by a PC path. By Claims 3 and 4, no two vertices of $Q' \cup \{r\}$ are connected by a PC path. It follows that $Y_0 \cup Q' \cup \{r\}$ is a PCP-kernel, a contradiction.

Case 2. $Y_0 = \emptyset$.

Recall that D has no source in Y. By the assumption we have that every vertex in Y has one outneighbor and one inneighbor in $\{x_1, x_2\}$. Let

$$Y' = \{y \in Y : \ x_1 \to y, \ y \to x_2\}, \quad Y'' = Y \backslash Y' = \{y \in Y : \ x_2 \to y, \ y \to x_1\},$$

$$Y^* = \{y \in Y' : \ c(x_1y) \neq c(yx_2)\}, \quad Y^{**} = \{y \in Y'' : \ c(x_2y) \neq c(yx_1)\}.$$

In the following proof we need to keep in mind that each vertex in Y' can reach x_2 by a PC path and each vertex in Y'' can reach x_1 by a PC path. If $Y^* \cup Y^{**} = \emptyset$, i.e., there exists no PC path connecting x_1 and x_2, then clearly $\{x_1, x_2\}$ is a PCP-kernel. Now let $Y^* \cup Y^{**} \neq \emptyset$ and assume w.o.l.g. that $Y^* \neq \emptyset$. If there exist $y_1', y_2' \in Y^*$ with $c(x_1y_1') \neq c(x_1y_2')$, then since each vertex in Y'' can reach x_2 by a PC path passing through either $\{x_1, y_1'\}$ or $\{x_1, y_2'\}$ we get that $\{x_2\}$ is a PCP-kernel. So we can assume that $c(x_1y') = \alpha$ for each $y' \in Y^*$. Let

$$Y'_\alpha = \{y \in Y' : \ c(x_1y) = c(yx_2) = \alpha\}, \quad Y''_\alpha = \{y \in Y'' : \ c(x_2y) = c(yx_1) = \alpha\}.$$

We now claim that $Y^{**} \neq \emptyset$. Assume the opposite that $Y^{**} = \emptyset$. If $Y''_\alpha = \emptyset$, then since each vertex in Y'' can reach x_2 by a PC path passing through x_1 and an arbitrary vertex in Y^* we get that $\{x_2\}$ is a PCP-kernel. If $Y''_\alpha \neq \emptyset$ and $Y'_\alpha = \emptyset$, then since each vertex in Y' can reach Y''_α by a PC path passing through x_2, and each vertex in $Y'' \backslash Y''_\alpha$ can reach Y''_α by a PC path passing through x_1 together with an arbitrary vertex in Y^* and x_2, we can get that Y''_α is a PCP-kernel. If $Y''_\alpha \neq \emptyset$ and $Y'_\alpha \neq \emptyset$, then by a similar analysis and the observation that no two vertices of $Y'_\alpha \cup Y''_\alpha$ are connected by a PC path we have that $Y'_\alpha \cup Y''_\alpha$ is a PCP-kernel. So $Y^{**} \neq \emptyset$.

If there exist $y_1'', y_2'' \in Y^{**}$ with $c(x_2y_1'') \neq c(x_2y_2'')$, then similar to the analysis for Y^* we have that $\{x_1\}$ is a PCP-kernel. Thus, we can assume that $c(x_2y'') = \beta$ for each $y'' \in Y^{**}$. For the sake of a better presentation, define the following vertex sets, see also in

Figure 5 in which a vertex encircled may represent a set of vertices, and solid arcs, dotted arcs, dashed arcs represent respectively the arcs colored by α, β and a color not in $\{\alpha,\beta\}$.

$$Y'_\beta = \{y \in Y' :\ c(x_1y) = c(yx_2) = \beta\},\ \ Y''_\beta = \{y \in Y'' :\ c(x_2y) = c(yx_1) = \beta\},$$

$$Y'_\gamma = \{y \in Y' :\ c(x_1y) = c(yx_2) \notin \{\alpha,\beta\}\},\ \ Y''_\gamma = \{y \in Y'' :\ c(x_2y) = c(yx_1) \notin \{\alpha,\beta\}\},$$

$$Y'_{\alpha\beta} = \{y \in Y' :\ c(x_1y) = \alpha,\ c(yx_2) = \beta\},\ \ Y'_{\alpha\gamma} = \{y \in Y' :\ c(x_1y) = \alpha,\ c(yx_2) \notin \{\alpha,\beta\}\},$$

$$Y''_{\beta\alpha} = \{y \in Y'' :\ c(x_2y) = \beta,\ c(yx_1) = \alpha\},\ \ Y'_{\beta\gamma} = \{y \in Y'' :\ c(x_2y) = \beta,\ c(yx_1) \notin \{\alpha,\beta\}\}.$$

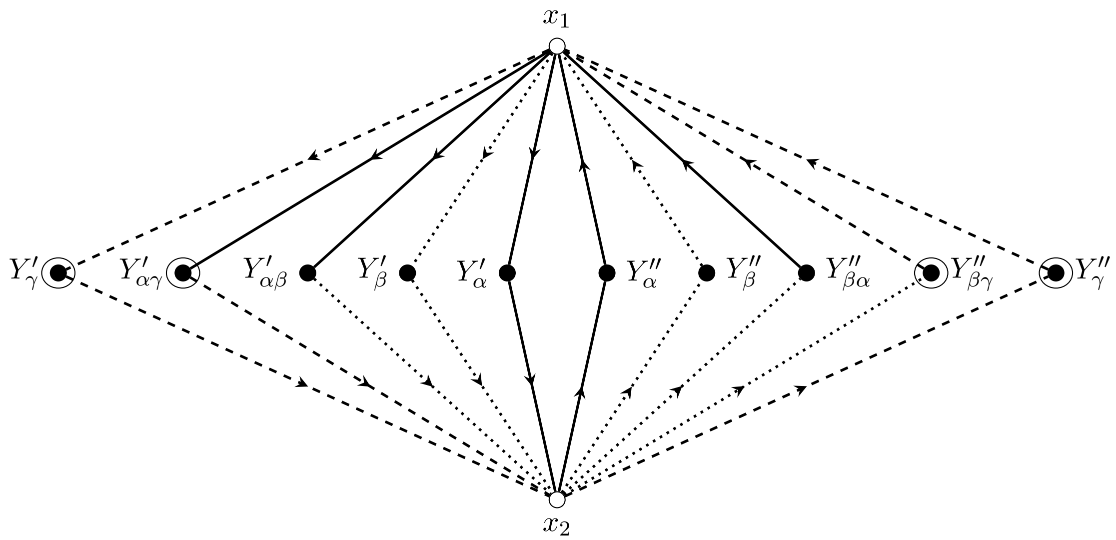

Figure 4: An arc-colored bipartite tournament with no sink and no source.

Since D contains no two contractible vertices, we have $|Y'_\alpha|, |Y'_\beta|, |Y'_{\alpha\beta}|, |Y''_\alpha|, |Y''_\beta|, |Y'_{\beta\alpha}| \leq 1$. Note that no two vertices of $Y''_\alpha \cup Y''_{\beta\alpha}$ are connected by a PC path in D and also the following holds.

$$Y^* = Y'_{\alpha\beta} \cup Y'_{\alpha\gamma} \neq \emptyset,\ Y^{**} = Y''_{\beta\alpha} \cup Y'_{\beta\gamma} \neq \emptyset,$$

$$Y' = Y'_\alpha \cup Y'_\beta \cup Y'_\gamma \cup Y'_{\alpha\beta} \cup Y'_{\alpha\gamma},\ \ Y'' = Y''_\alpha \cup Y''_\beta \cup Y''_\gamma \cup Y''_{\beta\alpha} \cup Y'_{\beta\gamma}.$$

We distinguish two subcases.

Subcase 2.1. $\alpha = \beta$.

It follows that $Y'_{\alpha\beta} = Y''_{\beta\alpha} = \emptyset$ and $Y'_{\alpha\gamma} = Y^* \neq \emptyset$. If $Y''_\alpha = \emptyset$, then each vertex in Y'' can reach x_2 by a PC path passing through x_1 and an arbitrary vertex in $Y'_{\alpha\gamma}$, which implies that $\{x_2\}$ is a PCP-kernel. If $Y''_\alpha \neq \emptyset$, then since each vertex in $Y' \backslash Y'_\alpha$ can reach Y''_α by a PC path passing through x_2, and each vertex in $Y'' \backslash Y''_\alpha$ can reach Y''_α by a PC path passing through x_1 together with an arbitrary vertex in $Y'_{\alpha\gamma}$ and x_2, together with the observation that no two vertices in $Y'_\alpha \cup Y''_\alpha$ are connected by a PC path, we can get that $Y'_\alpha \cup Y''_\alpha$ is a PCP-kernel.

Subcase 2.2. $\alpha \neq \beta$.

If $Y''_\alpha = Y''_{\beta\alpha} = \emptyset$, then since $Y'_{\alpha\beta} \cup Y'_{\alpha\gamma} \neq \emptyset$ we get that each vertex in Y'' can reach x_2 by a PC path passing through x_1 and an arbitrary vertex in $Y'_{\alpha\beta} \cup Y'_{\alpha\gamma}$. It follows that $\{x_2\}$ is a PCP-kernel.

If $Y''_\alpha \neq \emptyset$ and $Y''_{\beta\alpha} \neq \emptyset$, then each vertex in Y' can reach $Y''_\alpha \cup Y''_{\beta\alpha}$ by a PC path passing through x_2, and each vertex in $Y'' \backslash (Y''_\alpha \cup Y''_{\beta\alpha})$ can reach Y''_α by a PC path passing through x_1, an arbitrary vertex in $Y'_{\alpha\beta} \cup Y'_{\alpha\gamma}$ and x_2. Recall that no two vertices of $Y''_\alpha \cup Y''_{\beta\alpha}$ are connected by a PC path in D. So $Y''_\alpha \cup Y''_{\beta\alpha}$ is a PCP-kernel.

If $Y''_\alpha \neq \emptyset$ and $Y''_{\beta\alpha} = \emptyset$, then we can show that either Y''_α or $Y'_\alpha \cup Y''_\alpha$ is a PCP-kernel. If $Y'_\alpha = \emptyset$, then each vertex in Y' can reach Y''_α by a PC path passing through x_2, and each vertex in $Y'' \backslash Y''_\alpha$ can reach Y''_α by a PC path passing through x_1, an arbitrary vertex in $Y'_{\alpha\beta} \cup Y'_{\alpha\gamma}$ and x_2. It follows that Y''_α is a PCP-kernel. If $Y'_\alpha \neq \emptyset$, noting that no two vertices of $Y'_\alpha \cup Y''_\alpha$ are connected by a PC path, then we can similarly show that $Y'_\alpha \cup Y''_\alpha$ is a PCP-kernel.

Now assume that $Y''_\alpha = \emptyset$ and $Y''_{\beta\alpha} \neq \emptyset$. If $Y'_\beta = Y'_{\alpha\beta} = \emptyset$, then $Y'_{\alpha\gamma} = Y^* \neq \emptyset$, each vertex in $Y'' \backslash Y''_{\beta\alpha}$ can reach $Y''_{\beta\alpha}$ by a PC path passing through x_1, an arbitrary vertex in $Y'_{\alpha\gamma}$ and x_2, and clearly every vertex in Y' can reach $Y''_{\beta\alpha}$ by a PC path passing through x_2. It follows that $Y''_{\beta\alpha}$ is a PCP-kernel. If $Y'_\beta = \emptyset$ and $Y'_{\alpha\beta} \neq \emptyset$, then each vertex in $Y'' \backslash Y''_{\alpha\beta}$ can reach $Y'_{\alpha\beta}$ by a PC path passing through x_1 and each vertex in $Y' \backslash Y'_{\alpha\beta}$ can reach $Y''_{\beta\alpha}$ by a PC path passing through x_2. Recall that no two vertices of $Y'_{\alpha\beta} \cup Y''_{\beta\alpha}$ are connected by a PC path. Then $Y'_{\alpha\beta} \cup Y''_{\beta\alpha}$ is a PCP-kernel. If $Y'_\beta \neq \emptyset$ and $Y'_{\alpha\beta} \neq \emptyset$, noting that no two vertices of $Y'_\beta \cup Y'_{\alpha\beta}$ are connected by a PC path, then since each vertex in Y'' can reach $Y'_\beta \cup Y'_{\alpha\beta}$ by a PC path passing through x_1 and each vertex in $Y' \backslash (Y'_\beta \cup Y'_{\alpha\beta})$ can reach $Y'_\beta \cup Y'_{\alpha\beta}$ by a PC path passing through x_2, $Y''_{\beta\alpha}$ and x_1, we can obtain that $Y'_\beta \cup Y'_{\alpha\beta}$ is a PCP-kernel. Now let $Y'_\beta \neq \emptyset$ and $Y'_{\alpha\beta} = \emptyset$. If $Y''_\beta = \emptyset$, then since each vertex in $Y' \backslash Y'_\beta$ can reach Y'_β by a PC path passing through $x_2, Y''_{\beta\alpha}, x_1$, and each vertex in Y'' can reach Y'_β by a PC path passing through x_1, we can get that Y'_β is a PCP-kernel. If $Y''_\beta \neq \emptyset$, then by observing that no two vertices of $Y'_\beta \cup Y''_\beta$ are connected by a PC path we can similarly show that $Y'_\beta \cup Y''_\beta$ is a PCP-kernel. $\square$

6 An extension

Recall that an arc-colored digraph is rainbow if any two arcs receive two distinct colors. Another interesting topic deserving further consideration is the existence of a *kernel by rainbow paths* in an arc-colored digraph D, which is defined, similar to the definition of MP-kernels or PCP-kernels, as a set S of vertices of D such that (i) no two vertices of S are connected by a rainbow path in D, and (ii) every vertex outside S can reach S by a rainbow path in D. Similar to the proof of Proposition 1, we can get the computational complexity of finding a kernel by rainbow paths in an arc-colored digraph.

Proposition 3. *It is NP-hard to recognize whether an arc-colored digraph has a kernel by rainbow paths or not.*

Proof. Let D and D' be defined as in Proposition 1. Color D' by using m colors in such a way that the subdigraph D is monochromatic and the arc set $\{uv:\ u \in V^*, v \in V(D)\}$ is m-colored. Then one can see that the m-colored D' has a kernel by rainbow paths if and only if D has a kernel. By Theorem 1 the desired result holds. $\square$

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