ON HYPERGRAPHS WITHOUT LOOSE CYCLES

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ABSTRACT. Recently, Mubayi and Wang showed that for $r \geqslant 4$ and $\ell \geqslant 3$, the number of n-vertex r-graphs that do not contain any loose cycle of length ℓ is at most $2^{O(n^{r-1}(\log n)^{(r-3)/(r-2)})}$. We improve this bound to $2^{O(n^{r-1}\log\log n)}$.

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§1. INTRODUCTION

Let two graphs G and H be given. The graph G is called *H-free* if it does not contain any copy of H as a subgraph. One of the central problems in graph theory is to determine the extremal and typical properties of H-free graphs on n vertices. For example, one of the first influential results of this type is the Erdős–Kleitman–Rothschild theorem [3], which, for instance, implies that the number of triangle-free graphs with vertex set $[n] = \{1, \ldots, n\}$ is $2^{n^2/4+o(n^2)}$. This has inspired a great deal of work on counting the number of H-free graphs. For an overview of this line of research, the reader is referred to, e.g., [2,9]. For a recent, exciting result in the area, see [7], which also contains a good discussion of the general area, with several pointers to the literature. These problems are closely related to the so-called *Turán problem*, which asks to determine the maximum possible number of edges in an H-free graph. More precisely, given an r-uniform hypergraph (or r-graph) H, the *Turán number* $\mathrm{ex}_r(n, H)$ is the maximum number of edges in an r-graph G on n vertices that is H-free. Let $\mathrm{Forb}_r(n, H)$ be the set of all H-free r-graphs with vertex set $[n]$. Noting that the subgraphs of an H-free r-graph G are also H-free, we trivially see that $|\mathrm{Forb}_r(n, H)| \geqslant 2^{\mathrm{ex}_r(n,H)}$, by considering an H-free r-graph G on $[n]$ with the maximum number of edges and all its subgraphs. On the other hand for, fixed r and H,

$$|\mathrm{Forb}_r(n, H)| \leqslant \sum_{1\leqslant i\leqslant \mathrm{ex}_r(n,H)} \binom{\binom{n}{r}}{i} = 2^{O(\mathrm{ex}_r(n,H)\log n)}. \tag{1}$$

Hence the above simple bounds differ by a factor of $\log n$ in the exponent, and all existing results support that this $\log n$ factor should be unnecessary, i.e., the trivial lower bound should be closer to the truth.

There are very few results in the case $r > 2$ and $\mathrm{ex}_r(n, H) = o(n^r)$. The only known case is when H consists of two edges sharing t vertices [1,4]. Very recently, Mubayi and Wang [8] studied $|\mathrm{Forb}_r(n, H)|$ when H is a loose cycle. Given $\ell \geqslant 3$, an *r-uniform loose cycle* C_ℓ^r is an $\ell(r-1)$-vertex r-graph whose vertices can be ordered cyclically in such a way that the edges are sets of

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consecutive r vertices and every two consecutive edges share exactly one vertex. When r is clear from the context, we simply write C_ℓ.

Theorem 1. [8] *For every $\ell \geqslant 3$ and $r \geqslant 4$, there exists $c = c(r, \ell)$ such that*

$$|\mathrm{Forb}_r(n, C_\ell)| < 2^{cn^{r-1}(\log n)^{(r-3)/(r-2)}} \tag{2}$$

for all n. For $\ell \geqslant 4$ even, there exists $c = c(\ell)$ such that $|\mathrm{Forb}_3(n, C_\ell)| < 2^{cn^2}$ for all n.

Since $\mathrm{ex}_r(n, C_\ell) = \Omega(n^{r-1})$ for all $r \geqslant 3$ [5, 6], Theorem 1 implies that $|\mathrm{Forb}_3(n, C_\ell)| = 2^{\Theta(n^2)}$ for even $\ell \geqslant 4$. Mubayi and Wang also conjecture that similar results should hold for $r = 3$ and all $\ell \geqslant 3$ odd and for all $r \geqslant 4$ and $\ell \geqslant 3$, i.e., $|\mathrm{Forb}_r(n, C_\ell)| = 2^{\Theta(n^{r-1})}$ for all such r and ℓ. In this note we give the following improvement of Theorem 1 for $r \geqslant 4$.

Theorem 2. *For every $\ell \geqslant 3$ and $r \geqslant 4$, we have*

$$|\mathrm{Forb}_r(n, C_\ell)| < 2^{2r^2\ell n^{r-1}\log\log n} \tag{3}$$

for all sufficiently large n.

In what follows, logarithms have base 2.

§2. Edge-colored r-graphs

Let $r \geqslant 2$ be an integer. An *r-uniform hypergraph G* (or *r-graph*) on a *vertex set X* is a collection of r-element subsets of X, called *hyperedges* or simply *edges*. The vertex set X of G is denoted $V(G)$. We write $e(G)$ for the number of edges in G. An *r-partite r-graph G* is an r-graph together with a vertex partition $V(G) = V_1 \cup \cdots \cup V_r$, such that every edge of G contains exactly one vertex from each V_i ($i \in [n]$). If all such edges are present in G, then we say that G is *complete*. We call an r-partite r-graph *balanced* if all parts in its vertex partition have the same size. Let $K_r(s)$ denote the complete r-partite r-graph with s vertices in each vertex class.

We now introduce some key definitions from [8], which are also essential for us. Given an $(r-1)$-graph G with $V(G) \subseteq [n]$, a *coloring function* for G is a function $\chi\colon G \to [n]$ such that $\chi(e) = z_e \in [n] \smallsetminus e$ for every $e \in G$. We call z_e the *color* of e. The pair (G, χ) is an *edge-colored* $(r-1)$-graph.

Given G, each edge-coloring χ of G gives an r-graph $G^\chi = \{e \cup \{z_e\} \colon e \in G\}$, called the *extension of G by χ*. When there is only one coloring that has been defined, we write G^* for G^χ. Clearly any subgraph $G' \subseteq G$ also admits an extension by χ, namely, $(G')^\chi = \{e \cup \{z_e\} \colon e \in G'\} \subseteq G^*$. If $G' \subseteq G$ and $\chi\restriction_{G'}$ is one-to-one and $z_e \notin V(G')$ for all $e \in G'$, then G' is called *strongly rainbow colored*. We state the following simple remark explicitly for later reference.

Remark 3. A strongly rainbow colored copy of C_ℓ^{r-1} in G' gives rise to a copy of C_ℓ^r in G^*.

The following definition is crucial.

Definition 4 ($g_r(n, \ell)$, [8])**.** For $r \geqslant 4$ and $\ell \geqslant 3$, let $g_r(n, \ell)$ be the number of edge-colored $(r-1)$-graphs G with $V(G) \subseteq [n]$ such that the extension G^* is C_ℓ^r-free.

The function $g_r(n,\ell)$ above counts the number of pairs (G,χ) with $G^\chi \in \mathrm{Forb}_r(n,C_\ell^r)$. Mubayi and Wang [8] proved that $g_r(n,\ell)$ is non-negligible in comparison with $|\mathrm{Forb}_r(n,C_\ell)|$ and were thus able to deduce Theorem 1. The following estimate on $g_r(n,\ell)$ is proved in [8].

Lemma 5 ([8], Lemma 8)**.** *For every $r \geqslant 4$ and $\ell \geqslant 3$ there is $c = c(r,\ell)$ such that for all large enough n we have $\log g_r(n,\ell) \leqslant cn^{r-1}(\log n)^{(r-3)/(r-2)}$.*

We improve Lemma 5 as follows.

Lemma 6. *For every $r \geqslant 4$ and $\ell \geqslant 3$ we have*

$$\log g_r(n,\ell) \leqslant 2rn^{r-1}\log\log n \tag{4}$$

for all large enough n.

Theorem 2 can be derived from Lemma 6 in the same way that Theorem 1 is derived from Lemma 5 in [8]. It thus remains to prove Lemma 6.

§3. Proof of Lemma 6

To bound $g_r(n,\ell)$, we should consider all possible $(r-1)$-graphs G and their 'valid' edge-colorings. Let an $(r-1)$-graph G be fixed. The authors of [8] consider decompositions of G into balanced complete $(r-1)$-partite $(r-1)$-graphs G_i, and obtain good estimates on the number of edge-colorings of each G_i. In our proof of Lemma 6, we also decompose G into balanced $(r-1)$-partite $(r-1)$-graphs G_i, but with each G_i not necessarily complete. We get our improvement because certain quantitative aspects of our decomposition are better, and similar estimates can be shown for the number of edge-colorings of each G_i.

Definition 7 ($f_r(n,\ell,G)$)**.** Let $r \geqslant 3$ and $\ell \geqslant 3$ be given and let G be a balanced $(r-1)$-partite $(r-1)$-graph with $V(G) \subseteq [n]$. Let $f_r(n,\ell,G)$ be the number of edge-colorings $\chi\colon G \to [n]$ such that G^χ is C_ℓ^r-free.

Lemma 8. *For every $r \geqslant 4$ and $\ell \geqslant 3$ there is $c = c(r,\ell)$ such that, for any G as in Definition 7, we have*

$$f_r(n,\ell,G) \leqslant n^{cs^{r-2}}(cs^{r-2})^{e(G)}, \tag{5}$$

where $s = |V(G)|/(r-1)$.

We use the following result in the proof of Lemma 8.

Lemma 9. [8] *For every $r \geqslant 3$ and $\ell \geqslant 3$ there is $c = c(r,\ell)$ for which the following holds. Let G be an r-partite r-graph with vertex classes $V_1,\dots,V_r$ with $|V_i| = s$ for all i. If $e(G) > cs^{r-1}$, then G contains a loose cycle of length ℓ.*

Proof of Lemma 8. Let G be a balanced $(r-1)$-partite $(r-1)$-graph with each part of size s such that $V(G) \subseteq [n]$. For any edge-coloring $\chi\colon G \to [n]$, let $Z = \mathrm{im}\,\chi = \{z_e\colon e \in G\} \subseteq [n]$ be the set of all used colors. We first argue that if G^* is C_ℓ^r-free, then $|Z| < (c_9 + r)s^{r-2}$, where $c_9 = c_9(r-1,\ell)$ is the constant from Lemma 9. Indeed, if $|Z| \geqslant (c_9 + r)s^{r-2}$, then $|Z \smallsetminus V(G)| \geqslant$

$(c_9 + r)s^{r-2} - s(r-1) > c_9 s^{r-2}$. For each color $v \in Z \smallsetminus V(G)$ pick an edge in G with color v. We get a subgraph $G' \subseteq G$ that is strongly rainbow colored with $e(G') = |Z \smallsetminus V(G)| > c_9 s^{r-2}$. By Lemma 9, there is a C_ℓ^{r-1} in G' that is strongly rainbow, which contradicts the fact that G^* is C_ℓ^r-free (see Remark 3).

Let $c = c(r,\ell) = c_9 + r$. We now estimate the number $f_r(n,\ell,G)$ of valid edge-colorings χ the graph G may have as follows: choose at most cs^{r-2} colors and then color each edge of G in all possible ways. We obtain

$$f_r(n,\ell,G) \leqslant n^{cs^{r-2}} (cs^{r-2})^{e(G)}, \tag{6}$$

as required. □

Our next result gives a decomposition of any r-graph G into balanced r-partite r-graphs that are not necessarily complete.

Lemma 10. *Suppose $r \geqslant 2$ and*

$$1 \leqslant s \leqslant \left(1 - \frac{1}{r}\right) n. \tag{7}$$

Then any n-vertex r-graph G can be decomposed into t balanced r-partite r-graphs $G_i \subseteq K_r(s)$ $(1 \leqslant i \leqslant t)$, where $t \leqslant (n/s)^r \lceil c \log n \rceil$ and

$$c = c(r) = \frac{-r}{\log\left(1 - r!/r^r\right)}. \tag{8}$$

Proof. Generate independently and uniformly $\lceil c \log n \rceil$ random r-partitions $\Pi_i = (V_j^i)_{1 \leqslant j \leqslant r}$ of $[n]$ $(1 \leqslant i \leqslant \lceil c \log n \rceil)$. Thus, $v \in V_j^i$ happens with probability $1/r$ for each v, i and j independently. Note that each r-tuple is 'captured' by a given Π_i with probability $r!/r^r$. Thus the probability that a given r-tuple is *not* captured by *any* r-partition Π_i is at most

$$\left(1 - \frac{r!}{r^r}\right)^{\lceil c \log n \rceil} \leqslant 2^{-r \log n} = n^{-r}. \tag{9}$$

The union bound implies that there is a collection $\Pi_i = (V_j^i)_{1 \leqslant j \leqslant r}$ $(1 \leqslant i \leqslant \lceil c \log n \rceil)$ of r-partitions of $[n]$ capturing all r-tuples of $[n]$.

Let us now fix i and consider Π_i. We now produce, from Π_i, at most $(n/s)^r$ subgraphs G' of G with $G' \subseteq K_r(s)$, with the collection of such G' capturing every r-tuple captured by Π_i. Note that doing this for every i completes the proof of our lemma. To simplify notation, let $V_j = V_j^i$ $(1 \leqslant j \leqslant r)$.

Partition each V_j into blocks W_k^j $(1 \leqslant k \leqslant \lceil |V_j|/s \rceil)$ arbitrarily, but with $|W_k^j| \leqslant s$ for all k. We now consider all the vectors

$$W(k_1, \ldots, k_r) = (W_{k_1}^1, \ldots, W_{k_r}^r), \tag{10}$$

where $1 \leqslant k_j \leqslant \lceil |V_j|/s \rceil$ for all $1 \leqslant j \leqslant r$. Clearly, each such $W(k_1, \ldots, k_r)$ induces, in a natural way, an r-partite subgraph $G(k_1, \ldots, k_r)$ of G with $G(k_1, \ldots, k_r) \subseteq K_r(s)$. Moreover, those $G(k_1, \ldots, k_r)$ capture all the r-tuples captured by Π_i. It now suffices to prove that the number of such $G(k_1, \ldots, k_r)$ is at most $(n/s)^r$. The number of choices we have for $(k_j)_{1 \leqslant j \leqslant r}$ is

$$\left\lceil \frac{|V_1|}{s} \right\rceil \cdots \left\lceil \frac{|V_r|}{s} \right\rceil < \frac{1}{s^r}(|V_1| + s) \ldots (|V_r| + s)$$
$$\leqslant \frac{1}{s^r}\Big(\frac{1}{r} \sum_{1 \leqslant j \leqslant r} (|V_j| + s)\Big)^r \leqslant \Big(\frac{1}{sr}(n + sr)\Big)^r \leqslant \Big(\frac{n}{s}\Big)^r, \tag{11}$$

where we used (7) in the last inequality above. □

Now we are ready to prove Lemma 6.

Proof of Lemma 6. Let G_0 be an $(r-1)$-graph with $V(G_0) \subseteq [n]$. In what follows, we assume n is large enough for our inequalities to hold. We first count the number of edge-colored $(r-1)$-graphs G such that its underlying $(r-1)$-graph is G_0 and G^* is C_ℓ-free. By Lemma 10, we decompose G_0 into balanced $(r-1)$-partite $(r-1)$-graphs $G_1, \ldots, G_t$, such that, for all $i \in [t]$, each vertex class of G_i contains $s = \lfloor (\log n)^2 \rfloor$ vertices and $t \leqslant 2c_{10}(n/s)^{r-1} \log n$, where $c_{10} = c_{10}(r-1)$ is as given by Lemma 10. Note that $ts^{r-2} \leqslant 3c_{10}n^{r-1}/\log n$ and, since $G_1, \ldots, G_t$ form a decomposition of G_0, we have $\sum_{i=1}^t e(G_i) = e(G_0)$. Moreover, since G^* is C_ℓ-free, each G_i^* has to be C_ℓ-free. By Lemma 8, the number of valid edge-colorings of G_0 is at most

$$\prod_{i=1}^t f_r(n, \ell, G_i) \leqslant \prod_{i=1}^t n^{c_8 s^{r-2}} (c_8 s^{r-2})^{e(G_i)} = n^{c_8 t s^{r-2}} (c_8 s^{r-2})^{e(G_0)}$$
$$\leqslant n^{3c_8 c_{10} n^{r-1}/\log n} (c_8 s^{r-2})^{n^{r-1}}. \tag{12}$$

Since there are at most $2^{n^{r-1}}$ graphs G_0 as above and $s = \lfloor (\log n)^2 \rfloor$, summing over all G_0 gives

$$\begin{aligned} \log g_r(n, \ell) &\leqslant \log \left(2^{n^{r-1}} n^{3c_8 c_{10} n^{r-1}/\log n} (c_8 s^{r-2})^{n^{r-1}}\right) \\ &\leqslant (3c_8 c_{10} + 1)n^{r-1} + (\log c_8 + (r-2)\log s)n^{r-1} \\ &\leqslant (3c_8 c_{10} + 1)n^{r-1} + (\log c_8 + 2(r-2)\log\log n)n^{r-1} \\ &\leqslant 2rn^{r-1} \log\log n, \end{aligned} \tag{13}$$

where the last inequality follows for all large enough n. □

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