Rainbow independent sets in graphs with maximum degree two *

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Abstract

Given a graph G, let $f_G(n,m)$ be the minimal number k such that every k independent n-sets in G have a rainbow m-set. Let $\mathcal{D}(2)$ be the family of all graphs with maximum degree at most two. Aharoni et al. (2019) conjectured that (i) $f_G(n, n-1) = n-1$ for all graphs $G \in \mathcal{D}(2)$ and (ii) $f_{C_t}(n,n) = n$ for $t \geq 2n+1$. Lv and Lu (2020) showed that the conjecture (ii) holds when t = 2n + 1. In this article, we show that the conjecture (ii) holds for $t \geq \frac{1}{3}n^2 + \frac{44}{9}n$. Let C_t be a cycle of length t with vertices being arranged in a clockwise order. An ordered set $I = (a_1, a_2, \ldots, a_n)$ on C_t is called a 2-jump independent n-set of C_t if $a_{i+1} - a_i = 2 \pmod{t}$ for $any 1 \leq i \leq n-1$. We also show that a collection of 2-jump independent n-sets. F of C_t with $|\mathcal{F}| = n$ admits a rainbow independent n-set, i.e. (ii) holds if we restrict \mathcal{F} on the family of 2-jump independent n-sets. Moreover, we prove that if the conjecture (ii) holds, then (i) holds for all graphs $G \in \mathcal{D}(2)$ with $c_e(G) \leq 4$, where $c_e(G)$ is the number of components of G isomorphic to cycles of even lengths.

1 Introduction

Let $\mathcal{F} = (F_1, F_2, ..., F_n)$ be a collection of sets (not necessarily distinct), a *(partial)* rainbow set of \mathcal{F} is a set $R = \{x_{i_1}, x_{i_2}, ..., x_{i_m}\}$ such that $x_{i_j} \in F_{i_j}$ for $1 \leq i_1 < i_2 < ... < i_k \leq m \leq n$. Given a graph G and a collection \mathcal{F} of independent sets of G, a rainbow set R of \mathcal{F} is called a rainbow independent set of (\mathcal{F}, G) if R is also an independent set of G. An *m*-set is a set of size m. Let \mathcal{I} be a family of sets. Write $\mathcal{F} \subseteq \mathcal{I}$ for a collection of

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sets (not necessarily distinct) \mathcal{F} with each member of \mathcal{F} belonging to \mathcal{I} . Given a graph G and an integer n, write $\mathcal{I}(G)$ (resp. $\mathcal{I}_n(G)$ or $\mathcal{I}_{n^+}(G)$) for the family of independent sets (resp. independent sets such that each of them has uniform size n or of size at least n) of G. Given a graph G and integers m, n with $m \leq n$, define

 $f_G(n,m) = \min\{|\mathcal{F}| : \mathcal{F} \sqsubseteq \mathcal{I}_n(G) \text{ and } (\mathcal{F},G) \text{ has a rainbow indepent } m\text{-set}\}.$

Given a family \mathcal{G} of graphs, let

$$f_{\mathcal{G}}(n,m) = \sup\{f_{\mathcal{G}}(n,m) : \mathcal{G} \in \mathcal{G}\}.$$

Clearly, a rainbow matching in a graph is a rainbow independent set in its line graph. Partially motivated by the study of the rainbow matching problem in graphs (there are fruitful results about the problem, one can refer [1, 2, 4, 5, 6, 7, 8] for more details), Aharoni, Briggs, Kim and Kim studied the rainbow independent set problem in certain classes of graphs and proposed several conjectures in [3].

Let $m \leq n$. Clearly,

$$f_G(n,m) \geq m. \tag{1}$$

Aharoni, et al. [3] conjectured that the lower bound in (1) is tight for $f_G(n, n - 1)$ with $\Delta(G) \leq 2$ and $f_{C_t}(n, n)$, where C_t is a cycle with t vertices. Write $\mathcal{D}(2)$ for the family of graphs with maximum degree two.

Conjecture 1.1 (Conjecture 2.14 in [3]). $f_{\mathcal{D}(2)}(n, n-1) = n-1$.

Conjecture 1.2 (Conjecture 2.9 in [3]). If $t \ge 2n + 1$, then $f_{C_t}(n, n) = n$.

The following proposition can be easily checked (also has been observed in [2, 3, 8]).

Proposition 1. $f_{C_{2n}}(n,n) = 2n - 1.$

When t = 2n + 1, Conjecture 1.2 was confirmed by Lv and Lu [10].

Theorem 1.3 (Theorem 1 in [10]). $f_{C_{2n+1}}(n, n) = n$.

In this article, we first show that Conjecture 1.1 is true when G is 2-regular and $|V(G)| \in \{2n-1, 2n\}$ and then we show that Conjecture 1.2 seems stronger than Conjecture 1.1, i.e. Conjecture 1.2 implies Conjecture 1.1 for graphs $G \in \mathcal{D}(2)$ with $c_e(G) \leq 4$, where $c_e(G)$ is the number of components of G isomorphic to cycles of even lengths. So we concentrate our attention on Conjecture 1.2 and prove that this conjecture holds when t is large. The main results of the article are listed below.

Theorem 1.4. If G is 2-regular with $2n - 1 \le |V(G)| \le 2n$ then $f_G(n, n - 1) = n - 1$.

Theorem 1.5. If $f_{C_{\ell}}(n,n) = n$ for $\ell \geq 2n+1$, then $f_G(n,n-1) = n-1$ for all graphs $G \in \mathcal{D}(2)$ with $c_e(G) \leq 4$, provided that $\mathcal{I}_n(G) \neq \emptyset$.

Theorem 1.6. $f_{C_t}(n,n) = n \text{ for } t > \frac{1}{3}n^2 + \frac{44}{9}n.$

Given integers a, b with $a \leq b$, let $[a, b] = \{a, a+1, \ldots, b-1, b\}$, and write [b] for [1, b] for simplicity. Let C_t be a cycle with vertex set [a, a+t-1] and edge set $\{a(a+1), \ldots, (a+t-2)(a+t-1), (a+t-1)a\}$. For $t \geq 2n+1$ and $2 \leq k \leq t-2$, an ordered set $I = (a_1, a_2, \ldots, a_n)$ is called a k-jump independent set of C_t if $a_{i+1} - a_i = k \pmod{t}$ for any $1 \leq i \leq n-1$. We call a_1 the start of I, denoted by s(I), and a_n the end of I. Let $\mathcal{I}_n^k(G)$ be the family of all k-jump independent n-sets in G. We prove that Conjecture 1.2 is true if we restrict $\mathcal{F} \sqsubseteq \mathcal{I}_n^2(C_t)$.

Theorem 1.7. Given integers t, n with $t \ge 2n + 1$, if $\mathcal{F} \subseteq \mathcal{I}_n^2(C_t)$ with $|\mathcal{F}| = n$, then \mathcal{F} has a rainbow independent n-set.

Remark: Any independent *n*-set of C_{2n+1} must be a 2-jump set. So the theorem can be viewed as a generalization of Theorem 1.3 in some sense.

We give a bit more definitions and notation. For a collection \mathcal{F} of sets and a given set A, denote $\mathcal{F} - A = \{I - A : I \in \mathcal{F}\}$ and $\mathcal{F} \cap A = \{I \cap A : I \in \mathcal{F}\}$. For a set B of numbers and a given number i, let $B + i = \{b + i : b \in B\}$. For a graph G and a collection \mathcal{F} of independent sets in G, let $v \in V(G)$, define $C_{\mathcal{F}}(v) = \{I : v \in I \in \mathcal{F}\}$ to be the *list* of v and $c_{\mathcal{F}}(v) = |C_{\mathcal{F}}(v)|$ the *list number* of v. For a rainbow independent set R of \mathcal{F} and $v \in R$, let $C_R(v)$ be the color (i.e. the independent set in the list $C_{\mathcal{F}}(v)$) assigned to v and let $C_R = \bigcup_{v \in R} \{C_R(v)\}$.

The rest of the paper is arranged as follows. We give a greedy algorithm to find a rainbow independent set for a given graph G and $\mathcal{F} \sqsubseteq \mathcal{I}(G)$ and some preliminaries in Section 2. We prove Theorems 1.4 and 1.5 in Section 3. In Sections 4 and 5, we give the proofs of Theorem 1.6 and 1.7. We also give some discussions in the last section.

2 A greedy algorithm and some preliminaries

First, we give a greedy algorithm (GRIS) to find a rainbow independent set in a given graph G and $\mathcal{F} \sqsubseteq \mathcal{I}(G)$.

It is easy to check that the output $R = \text{GRIS}(G, \mathcal{F})$ is a rainbow independent set of (\mathcal{F}, G) . The following is a simple fact when we apply GRIS to a path.

Lemma 2.1. Let P_t be a path with vertex set [t] and edge set $\{12, 23, \ldots, (t-1)t\}$ and $\mathcal{F} \sqsubseteq \mathcal{I}_{(n-1)^+}(P_t)$ with $|\mathcal{F}| \ge n-1$. Suppose $t \ge 2n-1$. Let $R = GRIS(P_t, \mathcal{F})$ and $C = C(P_t, \mathcal{F})$. If |R| < n, then |R| = n-1 and, for any $I \in \mathcal{F} \setminus C$, we have |I| = n-1 and $|I \cap \{a, a+1\}| = 1$ for any $a \in R$.

Algorithm 2.1: GREEDY-RAINBOW-INDEPENDENT-SET (GRIS (G, \mathcal{F}))

Input: A graph G with ordered vertex set $A = \{a_1, a_2, \ldots, a_t\}$, and a collection of independent set $\mathcal{F} = (I_1, \ldots, I_k) \sqsubseteq \mathcal{I}(G)$

Output: A rainbow independent set R of \mathcal{F}

- 1 Set $R = C = \emptyset$ and j = 0.
- 2 Reset j := j + 1. If $C_{\mathcal{F}}(a_j) \setminus C = \emptyset$ or $C_{\mathcal{F}}(a_j) \setminus C \neq \emptyset$ but $R \cup \{a_j\}$ is not an independent set, go to step 3; otherwise, reset $R := R \cup \{a_j\}$ and choose $I_i \in C_{\mathcal{F}}(a_j) \setminus C$ with the minimal index i and reset $C := C \cup \{I_i\}$.
- **3** If j < t, return to step 2; else if j = t, then stop and output $R = \text{GRIS}(G, \mathcal{F})$ and $C = C(G, \mathcal{F})$. We call $C(G, \mathcal{F})$ a greedy color set.

Proof. Suppose |R| = |C| = k < n. Pick $I \in \mathcal{F} \setminus C$. Then, for any $i \in I$, $R \cup \{i\}$ can not be a larger independent set of P_t . Thus $i \in R$ or i is a neighbor of some vertex in R, i.e. $i \in (R-1) \cup R \cup (R+1)$. If $i \notin R \cup (R+1)$, then $i \in R-1$, i.e., $i+1 \in R$. But at step i of the algorithm GRIS, i will be added to R since $i - 1 \notin R$. This is a contradiction to $i+1 \in R$. Hence $i \in R \cup (R+1)$. Since I is an independent set, $|I \cap \{a, a+1\}| \leq 1$ for any $a \in R$. So $n-1 \leq |I| \leq |R| = k < n$. Thus we have |I| = |R| = k = n-1 and $|I \cap \{a, a+1\}| = 1$ for any $a \in R$.

In Lemma 2.1, if we choose $\mathcal{F} \in \mathcal{I}_n(P_t)$ with $|\mathcal{F}| = n$ then |R| must be n. So we have the following corollary.

Corollary 2.2. Suppose $t \ge 2n - 1$. Then $f_{P_t}(n, n) = n$.

Remark: Note that Aharoni et al. have proved that for the family of chordal graphs \mathcal{T} and $m \leq n$, $f_{\mathcal{T}}(n,m) = m$ (Theorem 3.20 in [3]). So, in fact, we have the following result.

Corollary 2.3. Suppose $t \ge 2n - 1$ and F_t is a forest of order t. Then $f_{F_t}(n,m) = m$ for $m \le n$.

Corollary 2.4. Let C_t be a cycle with vertex set [t] and edge set $\{12, 23, \ldots, (t-1)t, t1\}$. Suppose $t \ge 2n$. Then we have

(A) $f_{C_t}(n, n-1) = n - 1.$

(B) Let m be the maximum integer such that $f_{C_t}(n,m) = n$. Then $m \ge n-1$. Suppose $\mathcal{F} \sqsubseteq \mathcal{I}_n(C_t)$ with $|\mathcal{F}| = n$. If m = n-1 then the following hold.

(B1) $c_{\mathcal{F}}(i) > 0$ for any $i \in [t]$.

(B2) If $C_{\mathcal{F}}(i) = I$ for some $i \in [t]$, then $I \in C_{\mathcal{F}}(i \pm 2)$.

(B3) If $c_{\mathcal{F}}(i) = 1$ for some $i \in [t]$ and $t \ge 2n + 1$, then $c_{\mathcal{F}}(i \pm 1) > 1$.

Proof. Suppose $\mathcal{F} \sqsubseteq \mathcal{I}_n(C_t)$ with $|\mathcal{F}| = k$. Let $P_{t-1} = C_t - \{t\}$ and $\mathcal{F}' = \mathcal{F} - \{t\}$. Then $\mathcal{F}' \sqsubseteq \mathcal{I}_{(n-1)^+}(P_{t-1})$. Note that $t-1 \ge 2n-1$. We can apply the algorithm GRIS to P_{t-1} and \mathcal{F}' . Let $R = \text{GRIS}(P_{t-1}, \mathcal{F}')$. Then $|R| \le |\mathcal{F}'| = k$.

(A) If k = n - 1 then $|R| \le k < n$. By Lemma 2.1, |R| = n - 1. Clearly, R is a rainbow independent set of (\mathcal{F}, C_t) too. This implies that $f_{C_t}(n, n - 1) = n - 1$.

(B) Suppose k = n. Then $|R| \le k = n$. If |R| < n, by Lemma 2.1, we have |R| = n - 1. So we have $m \ge |R| \ge n - 1$. Now we assume m = n - 1.

(B1) If there is a vertex $i \in [t]$ with $c_{\mathcal{F}}(i) = 0$, without loss of generality, assume i = t, then $\mathcal{F}' = \mathcal{F} - \{t\} = \mathcal{F} \sqsubseteq \mathcal{I}_n(P_{t-1})$. Note that $t-1 \ge 2n-1$. By Corollary 2.2, we can find a rainbow independent *n*-set R' of (\mathcal{F}', P_{t-1}) , which is also a rainbow independent *n*-set of (\mathcal{F}, C_t) , a contradiction to the maximality of m.

(B2) Without loss of generality, assume $C_{\mathcal{F}}(t) = \{I\}$. By the symmetry of C_t , it is sufficient to prove $2 \in I$. Recall that $\mathcal{F}' = \mathcal{F} - \{t\} \sqsubseteq \mathcal{I}_{(n-1)^+}(P_{t-1})$. Let $C = C(P_{t-1}, \mathcal{F}')$. Note that $m = |R| = |\operatorname{GRIS}(P_{t-1}, \mathcal{F}')| = |C| = n - 1$ and $|\mathcal{F}'| = n$. We have $|\mathcal{F}' \setminus C| = 1$. Let $\mathcal{F}' \setminus C = \{I'\}$. By Lemma 2.1, we have |I'| = n - 1 and $|I' \cap \{a, a + 1\}| = 1$ for every $a \in R$. Since I is the only independent set containing t in \mathcal{F} , we have $I' = I \setminus \{t\}$. By (B1), $C_{\mathcal{F}'}(1) = C_{\mathcal{F}}(1) \neq \emptyset$. So by the processing of the algorithm GRIS, 1 is the first vertex added to R. Hence $|(I \setminus \{t\}) \cap \{1, 2\}| = 1$. Since $1t \in E(C_t)$ and $t \in I$, we have $1 \notin I$. This implies that $2 \in I$.

(B3) If not, there are two consecutive vertices $i, i+1 \in [t]$ with $|C_{\mathcal{F}}(i)| = |C_{\mathcal{F}}(i+1)| = 1$. Without loss of generality, assume i = t - 3 and $C_{\mathcal{F}}(t - 3) = \{I_1\}$ and $C_{\mathcal{F}}(t - 2) = \{I_2\}$. Let $P_{t-6} = C_t - [t-5,t]$ and $\mathcal{F}' = (\mathcal{F} \setminus \{I_1, I_2\}) - [t-5,t]$. Since I_1 is the only member of \mathcal{F} containing t-3 and I_2 is the only one of \mathcal{F} containing t-2, we have $I \cap \{t-3, t-2\} = \emptyset$ for any $I \in \mathcal{F} \setminus \{I_1, I_2\}$. So $|I \cap [t-5,t]| \leq 2$ for any $I \in \mathcal{F} \setminus \{I_1, I_2\}$. Thus $\mathcal{F}' \sqsubseteq \mathcal{I}_{(n-2)^+}(P_{t-6})$. Note that $t-6 \geq 2n+1-6 = 2(n-2)-1$. By Corollary 2.2, we have $f_{P_{t-6}}(n-2, n-2) = n-2$. So we have a rainbow independent (n-2)-set R of (P_{t-6}, \mathcal{F}') . This is also a rainbow independent (n-2)-set of $(\mathcal{F} \setminus \{I_1, I_2\}, C_t)$. Recall that $C_{\mathcal{F}}(t-3) = \{I_1\}$ and $C_{\mathcal{F}}(t-2) = \{I_2\}$. By (B2), $I_1 \in C_{\mathcal{F}}(t-1)$ and $I_2 \in C_{\mathcal{F}}(t-4)$. Therefore, by adding t-4 and t-1 to R, we get a rainbow independent n-set $R \cup \{t-1, t-4\}$ of (\mathcal{F}, C_t) , a contradiction to m = n - 1.

3 Proofs of Theorem 1.4 and 1.5

An odd (resp. even) cycle is a cycle of odd length (resp. even length).

Proof of Theorem 1.4. We prove by induction on the number of components c(G) of G. If c(G) = 1, then G is a cycle. By Corollaries 2.4 (A), we have $f_G(n, n - 1) = n - 1$. Now suppose c(G) > 1 and the result holds for all 2-regular graphs H with c(H) < c(G). Let $\mathcal{F} = (I_1, I_2, \ldots, I_{n-1}) \subseteq \mathcal{I}_n(G)$.

If G has a component isomorphic to a cycle C_{2m+1} for some $m \in [n-1]$, then $|I_i \cap V(C_{2m+1})| = m$ for every $i \in [n-1]$ because $2n-1 \leq |V(G)| \leq 2n$. Let $\mathcal{F}_1 = (I_1 \cap V(C_{2m+1}), \ldots, I_m \cap V(C_{2m+1}))$. Then $\mathcal{F}_1 \sqsubseteq \mathcal{I}_m(C_{2m+1})$. By Theorem 1.3, we can find a rainbow independent *m*-set R_1 of $(\mathcal{F}_1, C_{2m+1})$. Let $H = G - V(C_{2m+1})$ and $\mathcal{F}_2 = (I_{m+1} \cap V(H), \ldots, I_{n-1} \cap V(H))$. Then H is 2-regular with |V(H)| = 2(n-m) - 1 and $\mathcal{F}_2 \sqsubseteq \mathcal{I}_{n-m}(H)$. By the induction hypothesis, there is a rainbow independent (n-m-1)-set R_2 of (\mathcal{F}_2, H) . So $R_1 \cup R_2$ is a rainbow independent (n-1)-set of (\mathcal{F}, G) , we are done.

Now suppose that every component of G is an even cycle. Let $G = C_1 \cup C_2 \cup \ldots \cup C_k$ with $|V(C_i)| = 2n_i$ for $i \in [k]$. Then $|I_i \cap V(C_j)| = n_j$ for every $i \in [n-1]$ and $j \in [k]$. Let $H = C_2 \cup \ldots \cup C_k$. Then $\mathcal{F}' = (I_{n_1+1} \cap V(H), \ldots, I_{n-1} \cap V(H)) \sqsubseteq \mathcal{I}_{n-n_1}(H)$. By the induction hypothesis, there is a rainbow independent $(n - n_1 - 1)$ -set R' of (\mathcal{F}', H) . So $|R' \cap V(C_j)| = n_j$ for all $j \in [2, k]$ but exactly one exception, without loss of generality, say C_2 , with $|R' \cap V(C_2)| = n_2 - 1$. Let $J = \bigcup_{v \in R' \cap V(C_2)} C_{R'}(v)$. Then $|J| = n_2 - 1$. Let $\mathcal{F}_1 = (I_j : j \in [n_1] \text{ or } I_j \in J)$ and assume $n_1 \leq n_2$. Then $|\mathcal{F}_1| = n_1 + n_2 - 1 \geq 2n_1 - 1$ and $\mathcal{F}_1 \cap V(C_1) \sqsubseteq \mathcal{I}_{n_1}(C_1)$. So we have a rainbow independent n_1 -set R_1 of $(\mathcal{F}_1 \cap V(C_1), C_1)$ by Proposition 1. Let $\mathcal{F}_2 = \mathcal{F}_1 \setminus C_{R_1}$. Then $|\mathcal{F}_2| = n_2 - 1$ and $\mathcal{F}_2 \cap V(C_2) \sqsubseteq \mathcal{I}_{n_2}(C_2)$. By Corollary 2.4 (A), there is a rainbow independent (n_2-1) -set of $(\mathcal{F}_2 \cap V(C_2), C_2)$. Therefore, $R' \setminus (R' \cap V(C_2)) \cup R_1 \cup R_2$ is rainbow independent (n-1)-set of (\mathcal{F}, G) . This completes the proof.

Let V be a finite set and $\mathcal{I} \sqsubseteq 2^V$. For a subset $S \subset V$, define

$$h(\mathcal{I}, S) = \max\{m : |\{I \in \mathcal{I} : |I \cap S| \ge m\}| \ge m\}.$$

Lemma 3.1. Let W be a finite set and $V \subseteq W$. Suppose $\mathcal{I} \sqsubseteq 2^W$ is a collection of n^+ -sets with $|\mathcal{I}| = n$ and $h(\mathcal{I}, V) = n$. Let (V_1, V_2) be a partition of V and let $h(\mathcal{I}, V_i) = m_i$ for i = 1, 2. Then we have $m_1 + m_2 \ge n$. Furthermore,

(i) for each $\ell \leq m_1 + m_2 - n$, there exists a partition $(\mathcal{I}_1, \mathcal{I}_2)$ of \mathcal{I} such that $h(\mathcal{I}_1, V_1) = m_1 - \ell$ and $h(\mathcal{I}_2, V_2) = n - m_1 + \ell$;

(ii) if $m_1 + m_2 - n = 0$, then $|I \cap V_{3-i}| \le m_{3-i}$ for every $I \in \mathcal{I}_i$ and i = 1, 2;

(iii) if $m_1 + m_2 - n > 0$ and $\ell \leq m_1 + m_2 - n - 1$, then we can choose \mathcal{I}_2 with $|I \cap V_2| \geq n - m_1 + \ell + 1$ for every $I \in \mathcal{I}_2$.

Proof. Assume $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ with $|I_1 \cap V_1| \ge |I_2 \cap V_1| \ge \ldots \ge |I_n \cap V_1|$. By the definition of $m_1, |I_i \cap V_1| \ge m_1$ for each $1 \le i \le m_1$ and $|I_j \cap V_1| \le m_1$ for each $m_1 + 1 \le j \le n$. So we have $|I_j \cap V_2| \ge n - m_1$ for each $m_1 + 1 \le j \le n$. By the definition of m_2 , we have $m_2 \ge n - m_1$, i.e. $m_1 + m_2 \ge n$.

(i) Let $\ell \leq m_1 + m_2 - n$ and $\mathcal{I}'_1 = \{I_i : |I_i \cap V_1| \geq m_1 - \ell\}$ and $\mathcal{I}'_2 = \{I_i : |I_i \cap V_2| \geq n - m_1 + \ell\}$. By the definition of m_1 and m_2 , we have $|\mathcal{I}'_1| \geq m_1$ and $|\mathcal{I}'_2| \geq m_2$. Since every

 $|I_i| \ge n$, we have either $|I_i \cap V_1| \ge m_1 - \ell$ or $|I_i \cap V_2| > n - m_1 + \ell$. Hence $\mathcal{I}'_1 \cup \mathcal{I}'_2 = \mathcal{I}$ and so $|\mathcal{I}'_1 \setminus \mathcal{I}'_2| \le n - m_2 \le m_1 - \ell$. Therefore, we can choose a subset \mathcal{I}_1 of \mathcal{I} with $\mathcal{I}'_1 \setminus \mathcal{I}'_2 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}'_1$ and $|\mathcal{I}_1| = m_1 - \ell$. Let $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$. Clearly, $h(\mathcal{I}_1, V_1) = m_1 - \ell$ and $h(\mathcal{I}_2, V_2) = n - m_1 + \ell$.

(ii) Clearly, if $\ell = m_1 + m_2 - n = 0$ then $|I \cap V_{3-i}| \le n - m_i = m_{3-i}$ for any $I \in \mathcal{I}_i$ and i = 1, 2.

(iii) If $m_1 + m_2 - n > 0$ and $\ell \le m_1 + m_2 - n - 1$, then we can reset $\mathcal{I}'_2 = \{I_i : |I_i \cap V_2| \ge n - m_1 + \ell + 1\}$. Note that $n - m_1 + \ell + 1 \le m_2$. All discussions in (i) keep true. Thus, we have the desired partition $(\mathcal{I}_1, \mathcal{I}_2)$ of \mathcal{I} with $|I \cap V_2| \ge n - m_1 + \ell + 1$ for every $I \in \mathcal{I}_2$. \Box

Now we give the proof of Theorem 1.5.

Proof of Theorem 1.5: Suppose to the contrary that there is a graph $G \in \mathcal{D}(2)$ with $c_e(G) \leq 4$ and an $\mathcal{F} = (I_1, I_2, \ldots, I_{n-1}) \sqsubseteq \mathcal{I}_n(G)$ such that G contains no rainbow independent (n-1)-set of (\mathcal{F}, G) . Since $G \in \mathcal{D}(2)$, each component of G is a path or a cycle. Assume that G is a minimum counterexample. We claim that G contains no component isomorphic to a path or an odd cycle.

Claim 1. G contains no component H with $f_H(h,h) = h$, where $h = h(\mathcal{F}, V(H))$. In particular, G contains no component isomorphic to a path or an odd cycle.

Proof. Suppose to the contrary that there is a component H of G with $f_H(h,h) = h$, where $h = f(\mathcal{F}, V(H))$. Let G' = G - V(H). Without loss of generality, assume $|I_1 \cap V(H)| \ge |I_2 \cap V(H)| \ge \ldots \ge |I_{n-1} \cap V(H)|$. So $|I_i \cap V(H)| \ge h$ for every $1 \le i \le h$ and $|I_j \cap V(G')| \le h$ for every $h+1 \le j \le n-1$ by the definition of h. Let $\mathcal{F}_1 = (I_1, \ldots, I_h)$ and $\mathcal{F}_2 = (I_{h+1}, \ldots, I_{n-1})$. Then $\mathcal{F}_1 \sqsubseteq \mathcal{I}_{h+}(H)$ and $\mathcal{F}_2 \sqsubseteq \mathcal{I}_{(n-h)+}(G')$. Since $f_H(h,h) = h$, we can find a rainbow independent h-set R_1 of (\mathcal{F}_1, H) . Since G is a minimum counterexample, we can find a rainbow independent (n - h - 1)-set R_2 of (\mathcal{F}_2, G') . Therefore, $R_1 \cup R_2$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction.

If *H* is isomorphic to a path or an odd cycle. Then $|V(H)| \ge 2h+1$ if *H* is an odd cycle and 2h otherwise. By the assumption $f_{C_s}(h,h) = h$ when $s \ge 2h+1$ and Corollary 2.3, we always have $f_H(h,h) = h$. We are done.

By Claim 1, we may assume that G consists of k even cycles, namely $G = \bigcup_{i=1}^{k} C_i$, where $|V(C_i)| = 2n_i, 1 \le i \le k$. By Corollary 2.4 (A), $k \ge 2$. Let $V_i = V(C_i)$. By Claim 1, $h(\mathcal{F}, V_i) = n_i$ for $i \in [k]$. Let $\mathcal{F}_i = \{I_j \in \mathcal{F} : |I_j \cap V_i| = n_i\}, i \in [k]$. Then $|\mathcal{F}_i| \ge n_i$ for $i \in [k]$. Denote $t = \sum_{i=1}^{k} n_i - n$.

Claim 2. $t \ge k - 1$.

Proof. Suppose to the contrary that $t \leq k-2$. Note that $n = |I| = \sum_{i=1}^{k} |I \cap V_i|$ for every $I \in \mathcal{F}$. So there are at least two cycles $C_i, i \in [k]$ with $|I \cap V_i| \geq n_i$, i.e., every I lies in at least two of $\{\mathcal{F}_i, i \in [k]\}$ for every $I \in \mathcal{F}$. Therefore,

$$\sum_{i=1}^{k} |\mathcal{F}_i| \ge 2|\mathcal{F}| = 2(n-1) = 2\sum_{i=1}^{k} n_i - 2t - 2 > \sum_{i=1}^{k} (2n_i - 2).$$

Thus there exists at least one $i \in [k]$, say i = 1, such that $|\mathcal{F}_1| \ge 2n_1 - 1$. By Proposition 1, we can find a rainbow independent n_1 -set R_1 of (\mathcal{F}_1, C_1) . By the minimality of G, we can find a rainbow independent $(n - n_1 - 1)$ -set R_2 of $(\mathcal{F} \setminus C_{R_1}, G - V_1)$. Hence $R_1 \cup R_2$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction.

Let $n_k = \min\{n_1, \ldots, n_k\}$. By Corollary 2.4 (A), there is a rainbow independent (n_k-1) -set R_k of (\mathcal{F}_k, C_k) . Let $\mathcal{A} = \mathcal{F} \setminus C_{R_k}$. Then

$$|\mathcal{A}| = n - 1 - (n_k - 1) = \sum_{i=1}^{k-1} n_i - t$$

and

$$h(\mathcal{A}, G - V_k) = \sum_{i=1}^{k-1} n_i - t$$

because $|I \cap V(G - V_k)| \ge n - n_k = \sum_{i=1}^{k-1} n_i - t$ for every $I \in \mathcal{A}$. If k = 2, then $G - V_k = C_1$ and so $h(\mathcal{A}, C_1) = n_1 - t < n_1$ because $t \ge k - 1 = 1$. By the assumption $f_{C_s}(n, n) = n$ for $s \ge 2n + 1$, we have a rainbow independent $(n_1 - t)$ -set R_1 of (\mathcal{A}, C_1) . Hence $R_1 \cup R_2$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction.

Now assume that $k \geq 3$. Let $\hat{V}_i = V(G - V_k - V_i)$ and $h(\mathcal{A}, V_i) = m_i$ for $i \in [k-1]$. Then $\hat{V}_i \neq \emptyset$. Applying Lemma 3.1 to $V_i \cup \hat{V}_i$, we have $m_i + h(\mathcal{A}, \hat{V}_i) \geq |\mathcal{A}|$.

Claim 3. *For* $i \in [k-1]$ *,*

$$m_i + h(\mathcal{A}, \hat{V}_i) = \begin{cases} |\mathcal{A}| + 1 & \text{if } m_i = n_i \\ |\mathcal{A}| & \text{otherwise.} \end{cases}$$

Proof. We only prove the case i = 1 and the other cases can be shown similarly. If $m_1 = n_1$ and $m_1 + h(\mathcal{A}, \hat{V_1}) = |\mathcal{A}|$, then by Lemma 3.1 (i) and (ii), \mathcal{A} can be partitioned into $\mathcal{A}_1 \cup \mathcal{A}_2$ such that $h(\mathcal{A}_1, V_1) = n_1$ and $h(\mathcal{A}_1, \hat{V_1}) = |\mathcal{A}| - n_1$, furthermore, for any $I \in \mathcal{A}_1$, we have $|I \cap \hat{V_1}| \leq |\mathcal{A}| - n_1$. Thus, for every $I \in \mathcal{A}_1$, $|I \cap V_k| = |I \cap (V_1 \cup V_k)| - |I \cap V_1| \geq$ $n - (|\mathcal{A}| - n_1) - n_1 = n_k$, i.e. $\mathcal{A}_1 \subset \mathcal{F}_k$. Hence $|\mathcal{F}_k| \geq 2n_k - 1$. By Proposition 1, we can find a rainbow independent n_k -set R'_k of (\mathcal{F}_k, C_k) . Let $\mathcal{F}' = \mathcal{F} \setminus C_{R'_k}$. Then $|\mathcal{F}'| = n - 1 - n_k$ and $|I \cap V(G - V_k)| \geq n - n_k$ for every $I \in \mathcal{F}'$. By the minimality of G, there is a rainbow independent $(n-1-n_k)$ -set R' of $(\mathcal{F}', G-V_k)$. Therefore, $R'_k \cup R'$ is a rainbow independent (n-1)-set of (\mathcal{F}, G) , a contradiction.

Now assume $m_1 + h(\mathcal{A}, \hat{V_1}) \geq |\mathcal{A}| + 2$ if $m_1 = n_1$, or $m_1 + h(\mathcal{A}, \hat{V_1}) \geq |\mathcal{A}| + 1$ if $m_1 < n_1$. By applying Lemma 3.1 (i) and (iii) to $V_1 \cup \hat{V_1}$, we can partition \mathcal{A} into $\mathcal{A}_1 \cup \mathcal{A}_2$ such that (1) $h(\mathcal{A}_1, V_1) = n_1 - 1$, $h(\mathcal{A}_2, \hat{V_1}) = |\mathcal{A}| + 1 - n_1$ and for any $I \in \mathcal{A}_2$, $|I \cap \hat{V_1}| \geq |\mathcal{A}| + 2 - n_1$ if $m_1 = n_1$ (choose $\ell = 1$) (which implies that $|\mathcal{A}_1| = n_1 - 1$), or $h(\mathcal{A}_1, V_1) = m_1$, $h(\mathcal{A}_2, \hat{V_1}) =$ $|\mathcal{A}| - m_1$ and for any $I \in \mathcal{A}_2$, $|I \cap \hat{V_1}| \geq |\mathcal{A}| + 1 - m_1$ if $m_1 < n_1$ (choose $\ell = 0$) (which implies that $|\mathcal{A}_1| = m_1$). By the assumption $f_{C_s}(m, m) = m$ for $s \geq 2m + 1$, we have a rainbow independent $|\mathcal{A}_1|$ -set R_1 of (\mathcal{A}_1, C_1) . On the other hand, by the minimality of G, we have a rainbow independent $|\mathcal{A}_2|$ -set R_2 of $(\mathcal{A}_2, G - V_k - V_1)$. Hence $R_1 \cup R_2 \cup R_k$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction too.

If k = 3, then $m_i + h(\mathcal{A}, \hat{V}_i) = m_i + h(\mathcal{A}, V_{3-i}) = m_i + m_{3-i}$ for i = 1, 2. So we have either $m_i = n_i$ for i = 1, 2 or $m_i < n_i$ for i = 1, 2 by Claim 3. For the former case, we have $n_1+n_2 = m_1+m_2 = |\mathcal{A}|+1 = n-n_3+1$. So $t = n_1+n_2+n_3-n = 1$, which is a contradiction to $t \ge k - 1$. For the latter case, we have $m_1 + m_2 = |\mathcal{A}| = n - n_3$. Applying Lemma 3.1 (i) and (ii) to $V_1 \cup V_2$, we can partition \mathcal{A} into $\mathcal{A}_1 \cup \mathcal{A}_2$ such that $h(\mathcal{A}_1, V_1) = m_1 < n_1$ and $h(\mathcal{A}_2, V_2) = |\mathcal{A}| - m_1 = m_2 < n_2$. By the assumption $f_{C_s}(m, m) = m$ for $s \ge 2m + 1$, we obtain a rainbow independent m_i -set R_i of (\mathcal{A}_i, C_i) for i = 1, 2. Thus $R_1 \cup R_2 \cup R_3$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction.

Now, we assume k = 4. Hence $t \ge k - 1 = 3$. We distinguish two cases according to the relations of m_i and n_i for $i \in [3]$.

If there exists some $i \in [3]$, say i = 3, such that $m_3 < n_3$. By Claim 3, $|\mathcal{A}| = m_3 + h(\mathcal{A}, V_1 \cup V_2)$. First, we claim that we can choose \mathcal{A} (recall that $\mathcal{A} = \mathcal{F} \setminus C_{R_4}$) with $m_3 = h(\mathcal{A}, V_3) = n_3 - 1$. To show this, we choose C_{R_4} such that $|\mathcal{F}_3 \cap \mathcal{A}|$ is as large as possible. If $|\mathcal{F}_3 \cap \mathcal{A}| \geq n_3 - 1$, then $h(\mathcal{A}, V_3) \geq n_3 - 1$, we are done. So, assume $|\mathcal{F}_3 \cap \mathcal{A}| < n_3 - 1$. By applying Lemma 3.1 (i) and (ii) on $V_3 \cup \hat{V}_3$, we can partition \mathcal{A} into $\mathcal{A}_3 \cup \mathcal{A}'_3$ such that $h(\mathcal{A}_3, V_3) = m_3$ and $h(\mathcal{A}'_3, V_1 \cup V_2) = n - n_4 - m_3$, moreover, we can choose an $I_0 \in \mathcal{A}'_3$ such that $|I_0 \cap (V_1 \cup V_2)| = n - n_4 - m_3$ and $|I_0 \cap V_3| \leq m_3$. This implies that $I_0 \in \mathcal{F}_4$ but $I_0 \notin \mathcal{F}_3$. Therefore, reset \mathcal{A} by replacing I_0 with some set of $\mathcal{F}_3 \cap C_{R_4}$, we obtain a new \mathcal{A} with larger $|\mathcal{F}_3 \cap \mathcal{A}|$, a contradiction. Now we have $m_3 = h(\mathcal{A}, V_3) = n_3 - 1$ and so $h(\mathcal{A}, V_1 \cup V_2) = |\mathcal{A}| - m_3 = n_1 + n_2 - t + 1$. Again applying Lemma 3.1 (i) and (ii) to $V_3 \cup \hat{V}_3$, we can partition \mathcal{A} into \mathcal{A}_3 and \mathcal{A}'_3 with $h(\mathcal{A}_3, V_3) = n_3 - 1$ and $h(\mathcal{A}'_3, V_1 \cup V_2) = n_1 + n_2 - t + 1$, furthermore, for any $I \in \mathcal{A}'_3$, $|I \cap V_3| \leq n_3 - 1$ (this also implies that $|\mathcal{A}_3| = n_3 - 1$ and $|\mathcal{A}'_3| = n_1 + n_2 - t + 1$). By the assumption $f_{C_s}(m, m) = m$ for $s \geq 2m + 1$, we can find a rainbow independent $(n_3 - 1)$ -sets R_3 of (\mathcal{A}_3, C_3) . We claim that there is at least one of $h(\mathcal{A}'_3, V_i)$, i = 1, 2 with $h(\mathcal{A}'_3, V_i) = n_i$. Otherwise, we have $m'_i = h(\mathcal{A}'_3, V_i) < n_i$, i = 1, 2.

Applying Lemma 3.1 (i) to $V_1 \cup V_2$, we have $m'_1 + m'_2 \ge |\mathcal{A}'_3|$ and \mathcal{A}'_3 can be partitioned into \mathcal{B}_1 and \mathcal{B}_2 with $h(\mathcal{B}_1, V_1) = m'_1$ and $h(\mathcal{B}_2, V_2) = |\mathcal{A}'_3| - m'_1 \le m'_2$ (choose $\ell = 0$). By the assumption $f_{C_s}(m, m) = m$ for $s \ge 2m + 1$, we have a rainbow independent m'_1 -set R_1 of (\mathcal{B}_1, C_1) and a rainbow independent $(|\mathcal{A}'_3| - m'_1)$ -set of (\mathcal{B}_2, C_2) . Note that

$$\sum_{i=1}^{4} |R_i| = m_1' + |\mathcal{A}_3'| - m_1' + n_3 - 1 + n_4 - 1 = n - 1$$

Hence $R_1 \cup R_2 \cup R_3 \cup R_4$ is a rainbow independent (n-1)-set of (\mathcal{F}, G) , a contradiction. By this claim, without loss of generality, assume $h(\mathcal{A}'_3, V_1) = n_1$ and so $h(\mathcal{A}'_3, V_2) = n_2 - t + 1$. Again by Lemma 3.1 (i) and (ii), \mathcal{A}'_3 can be partitioned into \mathcal{B}_1 and \mathcal{B}_2 with $h(\mathcal{B}_1, V_1) = n_1$ and $h(\mathcal{B}_2, V_2) = n_2 - t + 1$, furthermore, $|I \cap V_1| = n_1$ and $|I \cap V_2| \leq n_2 - t + 1$ for any $I \in \mathcal{B}_1$. Hence $|I \cap V_4| = n - \sum_{i=1}^3 |I \cap V_i| \geq n - n_1 - (n_2 - t + 1) - (n_3 - 1) = n_4$ for any $I \in \mathcal{B}_1$ and so $\mathcal{B}_1 \subset \mathcal{F}_1 \cap \mathcal{F}_4$. Therefore, we have $|\mathcal{F}_4| \geq n_1 + n_4 > 2n_4 - 1$. By Proposition 1, we can find a rainbow independent n_4 -set R'_4 of (\mathcal{F}_4, C_4) . Let $\mathcal{F}' = \mathcal{F} \setminus C_{R'_4}$. Then $|\mathcal{F}'| = n - 1 - n_4$ and $|I \cap (V_1 \cup V_2 \cup V_3)| = n - n_4$ for every $I \in \mathcal{F}'$. By the minimality of G, there is a rainbow independent $(n - 1 - n_4)$ -set R' of $(\mathcal{F}', C_1 \cup C_2 \cup C_3)$. Therefore, $R'_4 \cup R'$ is a rainbow independent (n - 1)-set of (\mathcal{F}, G) , a contradiction too.

Now assume $m_i = n_i$ for all $i \in [3]$. By Corollary 2.4 (A), there is a rainbow independent $(n_3 - 1)$ -set R_3 of $(\mathcal{F}_3 \cap \mathcal{A}, C_3)$. Let $\mathcal{B} = \mathcal{A} \setminus C_{R_3}$, we choose C_{R_3} minimize $\max\{n_1 - |\mathcal{B} \cap \mathcal{F}_1|, n_2 - |\mathcal{B} \cap \mathcal{F}_2|\}$. By Claim 3, $m_3 + h(\mathcal{A}, V_1 \cup V_2) = |\mathcal{A}| + 1$, i.e., $h(\mathcal{A}, V_1 \cup V_2) = |\mathcal{B}| = n_1 + n_2 - t + 1$. As discussed in the above case, there is one of $h(\mathcal{B}, V_i)$, i = 1, 2 with $h(\mathcal{B}, V_i) = n_i$. Without loss of generality, assume $h(\mathcal{B}, V_1) = n_1$ and so $h(\mathcal{B}, V_2) = n_2 - t + 1$. Hence, $n_1 - |\mathcal{B} \cap \mathcal{F}_1| \leq 0$ and $n_2 - |\mathcal{B} \cap \mathcal{F}_2| \geq t - 1 \geq 2$. By Lemma 3.1 (i) and (ii), \mathcal{B} can be partitioned into \mathcal{B}_1 and \mathcal{B}_2 with $h(\mathcal{B}_1, V_1) = n_1$ and $h(\mathcal{B}_2, V_2) = n_2 - t + 1$, furthermore, for any $I \in \mathcal{B}_1$, we have $|I \cap V_1| = n_1$ and $|I \cap V_2| \leq n_2 - t + 1$. Hence, for any $I \in \mathcal{B}_1$, we have $I \notin \mathcal{F}_2$. If there is some $I \in \mathcal{F}_3$, reset \mathcal{B} by replacing I with some set in $\mathcal{F}_2 \cap C_{R_3}$ (which is can be done since $|\mathcal{F}_2 \cap C_{R_3}| = |\mathcal{A} \cap \mathcal{F}_2| \geq n_2 - |\mathcal{B} \cap \mathcal{F}_2| \geq 2$), the resulting new set \mathcal{B} has smaller $\max\{n_1 - |\mathcal{B} \cap \mathcal{F}_1|, n_2 - |\mathcal{B} \cap \mathcal{F}_2|\}$, a contradiction. So for any $I \in \mathcal{B}_1$, we have $I \in \mathcal{F}_4$, i.e., $\mathcal{B}_1 \subset \mathcal{F}_4$. Thus $|\mathcal{F}_4| \geq n_1 + n_4 - 1 \geq 2n_4 - 1$. With the same discussion with the end of the above case, we have a contradiction to the assumption that (\mathcal{F}, G) has no rainbow independent (n - 1)-set.

4 Proof of Theorem 1.6

In this section, C_t always denotes a cycle with vertex set [t] and edge set $\{12, 23, \ldots, (t-1)t, t1\}$.

Proof of Theorem 1.6. Let $\mathcal{F} \sqsubseteq \mathcal{I}_n(C_t)$ with $|\mathcal{F}| = n$. We show that there is a rainbow independent *n*-set of (\mathcal{F}, C_t) if $t > \frac{1}{3}n^2 + \frac{44}{9}n$. Suppose to the contrary that (C_t, \mathcal{F}) has no rainbow independent *n*-set. Choose a member $J \in \mathcal{F}$. By Corollary 2.4 (A), $(\mathcal{F} \setminus \{J\}, C_t)$ has a rainbow independent (n-1)-set R. Let $R' = R \cap N_{C_t}[J]$. Then R' is a rainbow independent set of $(\mathcal{F} \setminus \{J\}, C_t)$ too. Choose such a R with the smallest |R'|. We claim that $|R'| \ge \lceil \frac{n}{2} \rceil$. Otherwise, suppose $|R'| < \frac{n}{2}$. Since $|N_{C_t}[i] \cap N_{C_t}[j]| \le 1$ for any two vertices $i, j \in J$ with $i \ne j$, every vertex of R' is contained in at most two members of $\{N_{C_t}[i] : i \in J\}$. So there exists a $j_0 \in J$ with $N_{C_t}[j_0] = [j_0 - 1, j_0 + 1] \cap R = \emptyset$. So we can enlarge R by adding j_0 with color J to R, a contradiction to the assumption. Now let $F = C_t - N_{C_t}[J \cup R]$. By the definition of R and $R', J \subseteq N_{C_t}[R'] \subseteq N_{C_t}[R]$. So we have

$$|N_{C_t}[J \cup R]| = |N_{C_t}[J]| + |N_{C_t}[R]| - |N_{C_t}[J] \cap N_{C_t}[R]| \le 3n + 3n - |J| = 5n.$$

Recall that $C_{R'}$ is the set of the corresponding colors assigned to vertices in R'. We claim that $C_{\mathcal{F}}(i) \cap C_{R'} = \emptyset$ for any $i \in V(F)$. If not, assume there is an $I \in C_{\mathcal{F}}(i) \cap C_{R'}$ for some $i \in V(F)$ and assume I is the color of j in R', i.e. $C_{R'}(j) = \{I\}$. Let $\tilde{R} = (R \setminus \{j\}) \cup \{i\}$. Then $\tilde{R}' = \tilde{R} \cap N_{C_t}[J] = R' \setminus \{j\}$, a contradiction to the minimality of |R'|. This claim implies that for any $I \in C_{R'} \cup \{J\}$, we have $I \subseteq N_{C_t}[J \cup R]$. Let $A = \{(i,I) : i \in I \text{ and } I \in \mathcal{F}\}$, $B = \{(i,I) : i \in V(F) \text{ and } i \in I, I \in \mathcal{F}\}$ and $C = \{(i,I) : i \notin V(F) \text{ and } i \in I, I \in \mathcal{F}\}$. So, with a double-counting argument, we have

$$|B| = \sum_{i \in V(F)} c_{\mathcal{F}}(i) = |A| - |C| \le n^2 - n|C_{R'} \cup \{J\}| \le \frac{1}{2}n^2 - n,$$
(2)

where the inequality holds since $|C_{R'}| = |R'| \ge \lceil \frac{n}{2} \rceil$. Note that $N_{C_t}[J \cup R]$ is a union of intervals of length at least 3 and $|N_{C_t}[J \cup R]| \le 5n$. So $N_{C_t}[J \cup R]$ consists of at most $\frac{5n}{3}$ intervals. This implies that $F = C_t - N_{C_t}[J \cup R]$ has at least t - 5n vertices and consists of $m \le \frac{5n}{3}$ paths, say P_1, P_2, \ldots, P_m . By Corollary 2.4 (B1), $c_{\mathcal{F}}(a) \ge 1$ for any $a \in [1, t]$. By Corollary 2.4 (B3), every path P_j contains at most $\frac{|V(P_j)|+1}{2}$ vertices a with $c_{\mathcal{F}}(a) = 1$. Hence,

$$\sum_{i \in F} c_{\mathcal{F}}(i) = \sum_{j=1}^{m} \sum_{i \in V(P_j)} c_{\mathcal{F}}(i)$$

$$\geq \sum_{j=1}^{m} \left(2|V(P_j)| - \frac{|V(P_j)| + 1}{2} \right)$$

$$= \frac{3}{2} \sum_{j=1}^{m} |V(P_j)| - \frac{m}{2}$$

$$\geq \frac{3}{2} (t - 5n) - \frac{m}{2}$$

$$\geq \frac{3}{2} t - \frac{25}{3} n.$$
(3)

From (2) and (3), we have

$$\frac{3}{2}t - \frac{25}{3}n \le \sum_{i \in F} c_{\mathcal{F}}(i) \le \frac{1}{2}n^2 - n_i$$

i.e. $t \leq \frac{1}{3}n^2 + \frac{44}{9}n$, a contradiction.

5 Proof of Theorem 1.7

Let \mathbb{Z}_t be the additive group of remainder of modulo t. For $a, b \in \mathbb{Z}_t$, a > b means $a \pmod{t} > b \pmod{t}$. For two sequences $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}_t^n, (a_1, \ldots, a_n) > (b_1, \ldots, b_n)$ if and only if there exists some $i \in [n]$ such that $a_i > b_i$ and $a_j = b_j$ for all j < i.

Throughout this section, let C_t be a cycle with vertex set \mathbb{Z}_t and edge set $\{01, 12, ..., (t-1)0\}$. If t = 2n + 1, by Theorem 1.3, the result holds. So we assume $t \ge 2n + 2$ in the following. Let $\mathcal{F} = (B_1, \ldots, B_n) \sqsubseteq \mathcal{I}_n^2(C_t)$. Choose a maximal rainbow independent set $A = \{a_1, a_2, \ldots, a_r\}$ of (C_t, \mathcal{F}) with the property that a_1, \ldots, a_r are in a clockwise order in C_t and $D_A = (a_r - a_1, a_2 - a_1, a_3 - a_2, \ldots, a_r - a_{r-1})$ is minimal. Without loss of generality, assume $C_A = \{B_1, B_2, \ldots, B_r\}$ and $a_1 = 1$. Then we have $a_1 < a_2 < \ldots < a_r$ by the assumption of A. It is sufficient to show that r = n. By Corollary 2.4 (A), we know $r \ge n-1$. Suppose r = n-1. Define $A_i = \{a_i, a_i + 1\}$ for $i = 1, 2, \ldots, n-1$. We have the following claims.

Claim 4. $B_n \subseteq A \cup (A+1) \cup \{a_1 - 1\}$. Moreover, $0 \in B_n$ and $|B_n \cap A_i| = 1$ for each $i \in [n-1]$.

Proof. If there exists one $b \in B_n$ such that $b \notin A \cup (A+1) \cup (A-1)$, then $\{b\} \cup A$ is a larger rainbow independent set than A, a contradiction. Now suppose that there exists one $b \in B_n$ and an $i \in [2, r]$ with $b = a_i - 1$ but $b \neq a_{i-1} + 1$. Then we can replace a_i by b in A and get another rainbow independent set $\{a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r\}$ of (\mathcal{F}, C_t) with $b - a_{i-1} < a_i - a_{i-1}$, a contradiction to the minimality of D_A . So $B_n \subseteq A \cup (A+1) \cup \{a_1-1\}$.

Note that $A \cup (A + 1)$ consists of exactly two independent sets of size n - 1. Since $|B_n| = n$, we have $a_1 - 1 \in B_n$, i.e. $0 \in B_n$ and $|B_n \cap \{a_i, a_i + 1\}| = 1$ for any $i \in [n - 1]$. \Box

Claim 5. If $C_A(a_i) = B_{k_i}$ and $B_n \in C_F(a_i)$ for some $i \ge 2$, then $B_{k_i} = B_n$.

Proof. If $B_{k_i} \neq B_n$, then we can reset the color of a_i by B_n and apply Claim 4 to B_{k_i} , we have $0 \in B_{k_i}$ and $|B_{k_i} \cap A_\ell| = 1$ for every $\ell \in [n-1]$. Since $B_{k_i} \neq B_n$, we have $s(B_{k_i}) \neq s(B_n)$. Without loss of generality, we assume $0 \leq s(B_{k_i}) < s(B_n)$. Let $B_n =$ $(j, j+2, \ldots, 0, 2, \ldots, 2n-t+j-2)$. Then j > 0 and 2n-t+j-2 < j (otherwise, we have $t \leq 2n-2$, a contradiction). So, by Claim 4, t-2 must be in A_{n-1} and j must be in A_s ,

where $s = n - \frac{t-j}{2}$. If $s(B_{k_i}) = 0$ then $B_{k_i} = (0, 2, ..., 2n - 2)$. So we have $2n - 2 \in A_{n-1}$, i.e. $a_{n-1} = 2n - 3$ or 2n - 2. In any case, we have $t \leq 2n + 1$, a contradiction. Now assume $s(B_{k_i}) = h \neq 0$. Then $B_{k_i} = \{h, h + 2, ..., 0, 2, ..., 2n - t + h - 2\}$. Hence h and j have the same parity. So $j - 2 \in B_{k_i}$ but $j - 2 \notin B_n$ because $j - (2n - t + j - 2) = t - 2n + 2 \geq 4$. Since $j \in B_n \cap A_s$, we have 2n - t + j - 2 must be in $B_n \cap A_{s-1}$. Since $A_s \cap A_{s-1} = \emptyset$ and 2n - t + j - 2 < j - 2 < j, there is no A_ℓ containing j - 2 for $\ell \in [n - 1]$. This is a contradiction to $|B_{k_i} \cap A_\ell| = 1$ for all $\ell \in [n - 1]$.

If $s(B_n) = 0$, i.e., $B_n = \{0, 2, ..., 2n-2\}$. Since A is an independent set with $a_1 = 1 \in A$ and $|B_n \cap \{a_i, a_i + 1\}| = 1$ for every $i \in [n-1]$, there exists an integer k with $1 < k \le n$ such that $a_i = 2i - 1$ for $1 \le i < k$ and $a_i = 2i$ for $i \ge k$ (if k < n). By Claim 5, we have $B_i = B_n$ for all $i \ge k$. Clearly, $2n \notin B_i$ for all $i \in [k, n]$. If t = 2n + 2, since $a_i = 2i - 1 \in B_i$ and B_i is 2-jump, we have $2n \notin B_i$ for all $i \in [k - 1]$. Therefore, $2n \notin B_i$ for all $i \in [n]$, which is a contradiction to Corollary 2.4 (B1). If t > 2n + 2, then $t - 3 > 2n - 1 > a_{n-1}$. Clearly, $t - 3 \notin B_n$. Let $\ell \le k - 1$ be the minimal number with $t - 3 \in B_\ell$ (If such ℓ does not exist, then $t - 3 \notin B_i$ for all $i \in [n]$, a contradiction to Corollary 2.4 (B1), too). Since $B_i \in \mathcal{I}_n^2(C_t)$ with $a_i = 2i - 1 \in B_i$ and $t - 3 \notin B_i$ for any $i < \ell$, we have $a_i + 2 = a_{i+1} \in B_i$. So we can recolor a_{i+1} with B_i for all $i \in [\ell - 1]$, remain the color of a_i unchanged for $i \in [\ell + 1, n - 1]$ and color t - 3 with B_ℓ and 0 with B_n , i.e. $(A \setminus \{1\}) \cup \{0, t - 3\}$ is a rainbow independent n-set of (\mathcal{F}, C_t) , a contradiction.

Now suppose $s(B_n) = -2(n-m)$ for some n > m > 0, i.e., $B_n = \{t - 2(n-m), ..., t - m\}$ $2, 0, 2, \ldots, 2m - 2$. If t = 2n + 2, then $B_n = \{2m + 2, \ldots, 2n, 0, 2, \ldots, 2m - 2\}$. Clearly, $2m-1 \notin A$ (otherwise, $B_n \cap \{2m-1, 2m\} = \emptyset$, a contradiction to Claim 4). With the same reason as the case $s(B_n) = 0$, we have an integer k with $1 < k \le m$ such that $a_i = 2i - 1$ for any $i \in [k-1]$ and $a_i = 2i$ for any $i \in [k, m-1]$. With the same discussion with the case $s(B_n) = 0$, we have $2m \notin B_i$ for all $i \in [m-1]$. For i = m, we have $a_m \neq 2m$ (otherwise, $B_n \cap \{a_m, a_m + 1\} = \emptyset$, a contradiction to Claim 4). So $a_i > 2m$ for any $i \in [m, n-1]$. Since B_j is 2-jump, all elements of B_j have the same parity. By Claim 5, $B_j = B_n$ if $a_j \in B_j$ is an even numbers when $j \in [m, n-1]$. So $2m \notin B_j$ for any $j \in [m, n-1]$. Therefore, we have $2m \notin B_i$ for all $i \in [n]$, a contradiction to Corollary 2.4 (B1). If t > 2n + 2, then by Claim 4, we have $a_i \in \{t - 2n + 2i, t - 2n + 2i - 1\}$ for $i \in [m, n - 1]$, and $a_i \in \{2i-1, 2i\}$ for $i \in [2, m-1]$. In particular we have $a_{n-1} \in \{-2, -3\}$. If $a_{n-1} = -2$, then $a_1 - a_{n-1} = 3$ and $a_m - a_{m-1} \ge t - 2n + 2m - 1 - 2(m-1) > 3$, we have a contradiction to the minimality of D_A by reordering A with $A = \{a_m, a_{m+1}, \dots, a_{n-1}, a_1, \dots, a_{m-1}\}$. So we have $a_{n-1} = -3$, which implies that $a_i = t - 2n + 2i - 1$ for every $i \in [m, n-1]$. In particular, $a_m = t - 2(n-m) - 1 \in B_m$. Note that $a_m - 2 = t - 2(n-m) - 3 > 2m - 1 > a_{m-1}$. By the minimality of D_A , we have $-2(n-m) - 3 \notin B_m$ (otherwise, we can reduce $a_m - a_{m-1}$ by replacing a_m with $a_m - 2$ in A). Since B_m is 2-jump, we have $s(B_m) = t - 2(n-m) - 1$ and so $-1 \in B_m$. Then $A' = (A \setminus \{a_m\}) \cup \{-1\} = \{a_{m+1}, \dots, a_{n-1}, -1, a_1, \dots, a_{m-1}\}$ is a rainbow independent set of (\mathcal{F}, C_t) with $a_{m+1} - a_{m-1} \ge t - 2n + 2m + 1 - 2(m-1) > 5 > 4 = a_1 - a_{n-1}$, a contradiction to the minimality of D_A .

This completes the proof of Theorem 1.7.

6 Concluding remarks and discussions

In this article, we show that (1) Conjecture 1.1 is true when $|V(G)| \in \{2n - 1, 2n\}$ and (2) if Conjecture 1.2 is ture then Conjecture 1.1 holds for graphs $G \in \mathcal{D}(2)$ with $c_e(G) \leq 3$. We believe that Conjecture 1.2 implies Conjecture 1.1, we leave this as an open problem. For Conjecture 1.2, it has been shown that this conjecture is true for the base case (Theorem 1.3) and for large t (Theorem 1.6). Could we show the rest case of Conjecture 1.2?

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