On the spanning connectivity of tournaments

Bo zhang,   Weihua Yang,*   Shurong Zhang

Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

Abstract.   Let D be a digraph. A k-container of D between u and v, $C(u,v)$, is a set of k internally disjoint paths between u and v. A k-container $C(u,v)$ of D is a strong (resp. weak) k^*-container if there is a set of k internally disjoint paths with the same direction (resp. with different directions allowed) between u and v and it contains all vertices of D. A digraph D is k^*-strongly (resp. k^*-weakly) connected if there exists a strong (resp. weak) k^*-container between any two distinct vertices. We define the strong (resp. weak) spanning connectivity of a digraph D, $\kappa_s^*(D)$ (resp. $\kappa_w^*(D)$), to be the largest integer k such that D is ω^*-strongly (resp. ω^*-weakly) connected for all $1 \leq \omega \leq k$ if D is a 1^*-strongly (resp. 1^*-weakly) connected. In this paper, we show that a tournament with n vertices and irregularity $i(T) \leq k$, if $n \geq 6t + 5k$ ($t \geq 2$), then $\kappa_s^*(T) \geq t$ and $\kappa_w^*(T) \geq t + 1$ if $n \geq 6t + 5k - 3$ ($t \geq 2$).

Keywords: Hamiltonian path; Connectivity; Spanning connectivity; Bypass; Tournament

1   Introduction

For terminology not explicitly introduced here, we refer to [3, 11]. A digraph D consists of a set $V(D)$ of vertices and a set $A(D)$ of order pairs xy of distinct vertices called arcs, if xy is an arc of D, we say that x dominates y. A tournament T is a digraph such that each pair of vertices is joined by precisely one arc. A Hamiltonian path (resp. cycle) of a tournament is a path (resp. cycle) including all vertices of T. Bang-Jensen, Gutin and Huang [4] obtained some

---

*Corresponding author. E-mail: ywh222@163.com, yangweihua@tyut.edu.cn.

characterization for the existence of a Hamiltonian (x, y)-path in an extended tournament. If $P = x_1 x_2 ... x_m$ is a path in a tournament T and x is a vertex not in the path such that x is dominated by some x_i and dominates some x_j with $i < j$, then T contains an (x_1, x_m)-path $P' = x_1 x_2 ... x_k x x_{k+1} ... x_m$ where $i \leq k \leq j - 1$. We say that P' is an *augmentation* of P. By Menger's theorem [10], a tournament T is k-strongly connected (or called k-strong) if and only if for each ordered pair x, y of vertices, T contains k internally disjoint paths from x to y. When T is not strong, and let T_1, T_2, ..., T_k be the components. Without loss of generality, we may assume that whenever $i < j$, each vertex of T_i dominates each vertex of T_j. We refer to T_i as the i'th component of T, and to T_1 and T_k as the initial and terminal components respectively. When T is strong, T has a Hamiltonian cycle. If $uv \in A(T)$ and P is a (u, v)-path of length k, then P is called a *k-bypass* of uv. Alspach et al. in [2] investigates bypasses in asymmetric digraphs. Zhang and Wu [12] investigates conditions for arc-3-cyclicity in tournaments. Guo et al. in [5] investigates bypasses in tournaments.

We denote the in-degree and out-degree of vertex x by $d^-(x)$ and $d^+(x)$ respectively. The *irregularity $i(T)$* of a tournament T is the maximum $|d^+(x) - d^-(x)|$ over all vertices x of T. Clearly, a tournament is regular if $i(T) = 0$. If $i(T) \neq 0$, the T contains a vertex of in-degree at least $\frac{1}{2}(n - 1)$ and a vertex of out-degree at least $\frac{1}{2}(n - 1)$. Every vertex of T has out-degree at least $\frac{1}{2}(n - 1 - i(T))$.

In an undirected graph G, a k-container of G between u and v, $C(u, v)$, is a set of k internally disjoint paths between u and v. The concept of container is proposed by Hsu in [6] to evaluate the performance of communication of an interconnection networks. A k-container $C(u, v)$ of G is a k^*-container if it contains all vertices of G. A graph G is k^*-connected if there exists a k^*-container between any two distinct vertices of G. The study of k^*-connected graph is motivated by the globally 3*-connected graphs proposed by Albert et al. in [1]. Lin et al. in [9] proved that the pancake graph P_n is w^*-connected for any w with $1 \leq w \leq n - 1$ if and only if $n \neq 3$. Later, they apply the concept to discuss the spanning connectivity of graphs in [7] and discuss the spanning fan-connectivity of graphs in [8].

In this paper, we generalize the concepts above to digraphs and we consider the spanning connectivity of tournaments. Let D be a digraph. A k-container of D between u and v, $C(u, v)$, is a set of k internally disjoint paths between u and v. A k-container $C(u, v)$ of D is a strong

(resp. weak) k^*-container if there is a set of k internally disjoint paths with the same direction (resp. with different directions allowed) between u and v i.e. they have either k (u, v)-paths or k (v, u)-paths and it contains all vertices of D. A digraph D is k^*-strongly (resp. k^*-weakly) connected if there exists a strong (resp. weak) k^*-container between any two distinct vertices. We define the strong (resp. weak) spanning connectivity of a digraph D, $\kappa_s^*(D)$ (resp. $\kappa_w^*(D)$), to be the largest integer k such that D is ω^*-strongly (resp. ω^*-weakly) connected for all $1 \leq \omega \leq k$ if D is a 1^*-strongly (resp. 1^*-weakly) connected. We prove that for $k \geq 1$, a $2k$-strong tournament which has at least one path of length 2 between any vertices x and y is $(k + 2)^*$-weakly connected, and that for $k \geq 0$, a $(2k + 1)$-strong tournament is $(k + 2)^*$-weakly connected. Further, for $k \geq 2$, we prove that a $2k$-strong tournament is k^*-strongly connected, and that a $(2k + 1)$-strong tournament which has at least one 2-bypass for each arc is $(k + 1)^*$-strongly connected. Finally, we also prove that a tournament with n vertices and the irregularity $i(T) \leq k$, if $n \geq 6t + 5k$ $(t \geq 2)$, then $\kappa_s^*(T) \geq t$ and that a tournament with n vertices and the irregularity $i(T) \leq k$, if $n \geq 6t + 5k - 3$ $(t \geq 2)$, then $\kappa_w^*(T) \geq t + 1$.

2 k^*-weakly connected tournaments

Several results proved in [11] are useful in this paper, as follows.

Theorem 2.1 ([11]). *Let T be a tournament and x, y distinct vertices of T. Then T has a Hamiltonian path from x to y or from y to x unless one condition of $(i) - (iv)$ below is satisfied, in which case T has no Hamiltonian path connecting x and y.*

(i) T is not strong and either the initial or the terminal component of T (or both) contains none of x, y.

(ii) T is strong, $T - x$ is not strong, and y belongs to neither the initial nor the terminal component of $T - x$.

(iii) T is strong, $T - y$ is not strong, and x belongs to neither the initial nor the terminal component of $T - y$.

(iv) T is isomorphic to T_6^s or $\overline{T}_6^s$ and x and y are as shown in Fig.2 (or interchanged on this figure).

3

Corollary 2.2 ([11]). *For any three vertices of a strong tournament there is a Hamiltonian path connecting two of them.*

Corollary 2.3 ([11]). *A 2-strong tournament is 1^*-weakly connected unless it is isomorphic to T_6^s or $\overline{T}_6^s$.*

Thomassen also prove that every arc of a 3-strong tournament is contained in a Hamiltonian cycle i.e. for any two vertices x, y of a 3-strong tournament, there is a Hamiltonian path between x and y, and that every arc of a 4-strong tournament has a Hamiltonian bypass i.e. for any two vertices x, y of a 4-strong tournament, there is a Hamiltonian path from x to y and from y to x. According to these conclusions, the following is easy to obtain.

Corollary 2.4 ([11]). *A 3-strong tournament is 1^*-weakly connected.*

Corollary 2.5 ([11]). *A 4-strong tournament is 1^*-strongly connected and 2^*-strongly connected.*

The following is also easy to obtain.

Proposition 2.6. *A strong tournament is 2^*-weakly connected.*

Proof. Let T be a strong tournament, then T has a Hamiltonian cycle. So for any two vertices x and y of T, there is a weak 2^*-container between x and y. Therefore, T is 2^*-weakly connected. □

Theorem 2.7. *For all $k \geq 0$, a $(2k+1)$-strong tournament is $(k+2)^*$-weakly connected.*

Proof. Let T be a $(2k+1)$-strong tournament and x, y be any two vertices of T. The vertices of $T - \{x, y\}$ can be partitioned into four subsets A, B, C, D such that both x and y dominate each vertex of A and are dominated by each vertex of B, each vertex of C dominates y and is dominated by x, and each vertex of D dominates x and is dominated by y. Now, we consider the proposition for $k \geq 0$ as follows.

Case 1. $C \cup D = \emptyset$. T has no path of length 2 between x and y. Since T is $(2k+1)$-strong, there are at least $2k+1$ arcs from A to B. Thus, T has at least $2k+1$ internally disjoint paths

of length 3 between x and y. We choose k internally disjoint paths of length 3 among them and delate those $2k$ intermediate vertices of paths of length 3, the resulting tournament is strong. By Proposition 2.6, there is a weak 2*-container between x and y. Combining the k internally disjoint paths of length 3 between x and y, then T has a weak $(k+2)$*-container between x and y.

Case 2. $1 \leq |C \cup D| \leq k - 1$. We assume $|C \cup D| = i$. Since T is $(2k + 1)$-strong, as an argument similar to that used above in the case 2 of Proposition 2.8, there are at least $k - i$ arcs from A to B. Then T has at least $k - i$ paths of length 3 between x and y. Delate those $2(k - i)$ intermediate vertices of paths of length 3 and those i vertices of $C \cup D$, then T is $(i + 1)$-strong. So the resulting tournament is at least strong and has a weak 2*-container between x and y. Combining the $k - i$ internally disjoint paths of length 3 and the i internally disjoint paths of length 2 between x and y, then T has a weak $(k + 2)$*-container between x and y.

Case 3. $|C \cup D| \geq k$. T has at least k paths of length 2 between x and y. Delate the k intermediate vertices of paths of length 2, the resulting tournament is strong, by Proposition 2.6, there is a weak 2*-container between x and y. Combining the k internally disjoint paths of length 2 between x and y, then T has a weak $(k + 2)$*-container between x and y.

Hence, T is $(k + 2)$*-weakly connected. The proof is completed. $\square$

By an argument similar to that of Theorem 2.7, we obtain the following proposition in which the connectivity can be weaken slightly.

Proposition 2.8. *If a $2k$-strong tournament T has at least one path of length 2 between any two vertices x and y, then T is $(k + 2)$*-weakly connected.*

Proof. Let x, y be any two vertices of T. The vertices of $T - \{x, y\}$ can be partitioned into four subsets A, B, C, D such that both x and y dominate each vertex of A and are dominated by each vertex of B, each vertex of C dominates y and is dominated by x, and each vertex of D dominates x and is dominated by y. We assume that there are i paths of length 2 between x and y i.e. $|C \cup D| = i$ and $i \geq 1$.

Case 1. $|C \cup D| = 1$. Since T is $2k$-strong, T has at least $2k - 1$ paths of length 3 between x and y. We choose $k - 1$ paths of length 3 between x and y, delete those $2(k - 1)$ intermediate

vertices of paths of length 3 and the vertex in C or D. The resulting tournament is strong, by Proposition 2.6, there is a weak 2^*-container between x and y. Combining the $k-1$ internally disjoint paths of length 3 and the path of length 2 between x and y, then T has a weak $(k+2)^*$-container between x and y.

Case 2. $2 \leq |C \cup D| \leq k-1$. If $|C| = |D| = \frac{i}{2}$, since T is $2k$-strong, there are at least $2k - \frac{i}{2}$ arcs from A to B. Then T has at least $2k - \frac{i}{2}$ paths of length 3 between x and y. So we choose $k - i$ paths of length 3 between x and y. Delate those $2(k-i)$ intermediate vertices of paths of length 3 and those i intermediate vertices of paths of length 2. The resulting tournament is strong, by Proposition 2.6, there is a weak 2^*-container between x and y. Combining the $k-i$ internally disjoint paths of length 3 and the i internally disjoint paths of length 2 between x and y, then T has a weak $(k+2)^*$-container between x and y. If $|C| < |D|$ or $|D| < |C|$, we assume $min\{|C|, |D|\} = m$ and $m < \frac{i}{2}$. Since T is $2k$-strong, there are at least $2k - m - 1$ arcs from A to B. Then T has at least $2k - m - 1$ paths of length 3 between x and y. So we choose $k-i$ paths of length 3 between x and y. By an argument similar to that used above in this case, one can see that T has a weak $(k+2)^*$-container between x and y.

Case 3. $|C \cup D| \geq k$. T has at least k paths of length 2 between x and y. Delate the k intermediate vertices of paths of length 2, the resulting tournament is strong, by Proposition 2.6, there is a weak 2^*-container between x and y. Combining the k internally disjoint paths of length 2 between x and y, then T has a weak $(k+2)^*$-container between x and y.

Hence, T is $(k+2)^*$-weakly connected. The proof is completed. $\square$

3 k^*-strongly connected tournaments

Theorem 3.1. *For all $k \geq 2$, a $2k$-strong tournament is k^*-strongly connected.*

Proof. Let T be a $2k$-strong tournament and x, y be any two vertices of T. Without loss of generality, we assume $xy \in A(T)$. The vertices of $T - \{x, y\}$ can be partitioned into four subsets A, B, C, D such that both x and y dominate each vertex of A and are dominated by each vertex of B, each vertex of C dominates y and is dominated by x, and each vertex of D dominates x and is dominated by y. Now, we consider three cases as follows.

Case 1. $C = \emptyset$. Since T is $2k$-strong, there are at least $2k - 1$ arcs from A to B. Thus, T has $2k - 1$ internally disjoint (x, y)-paths of length 3. We choose $k - 2$ internally disjoint (x, y)-paths among them and delate the $2(k - 2)$ intermediate vertices of the (x, y)-paths. The resulting tournament is 4-strong, by Corollary 2.5, there is a strong 2^*-container from x to y. Combining the $k - 2$ internally disjoint (x, y)-paths of length 3, T has a strong k^*-container from x to y.

Case 2. $1 \leq |C| \leq k - 3$. Since T is $2k$-strong, there are at least $2k - 1 - |C|$ arcs from A to B. Thus, T has $2k - 1 - |C|$ internally disjoint (x, y)-paths of length 3. We choose $k - 2 - |C|$ internally disjoint (x, y)-paths among them, then delete the $2(k - 2 - |C|)$ intermediate vertices of (x, y)-paths of length 3 and the vertices of C. The resulting tournament is $(4 + |C|)$-strong, by Corollary 2.5, there is a strong 2^*-container from x to y. Combining the $k - 2 - |C|$ internally disjoint (x, y)-paths of length 3 and the $|C|$ internally disjoint (x, y)-paths of length 2, T has a strong k^*-container from x to y.

Case 3. $|C| \geq k - 2$. T has at least $k - 2$ internally disjoint (x, y)-paths of length 2. Delate the $k - 2$ intermediate vertices of (x, y)-paths of length 2, the resulting tournament is $(k + 2)$-strong and it is at least 4-strong, by Corollary 2.5, there is a strong 2^*-container from x to y. Combining the $k - 2$ internally disjoint (x, y)-paths of length 2, T has a strong k^*-container from x to y.

Hence, T is k^*-strongly connected. The proof is completed. $\qquad\square$

By an argument similar to that of the theorem above, we obtain the following theorem in which the connectivity can be weaken slightly and we omit the detailed proof.

Theorem 3.2. *For all $k \geq 2$, let T be a $2k + 1$-strong tournament. If T has at least one 2-bypass for each arc of T, then T is $(k + 1)^*$-strongly connected.*

4 Spanning connectivity of tournaments

Lemma 4.1 ([11]). *Let T be a tournament with n vertices and the irregularity $i(T) \leq k$. Then the connectivity of T is at least $\frac{1}{3}(n - 2k)$.*

Theorem 4.2. *Let T be a tournament with n vertices and $i(T) \leq k$. If $n \geq 6t + 5k$ $(t \geq 2)$,*

then $\kappa_s^*(T) \geq t$.

Proof. By Lemma 4.1, T is $2t$-strong. As $t \geq 2$, then T is 4-strong. By Corollary 2.5, T is i^*-strongly connected for $i \in \{1, 2\}$. Let x, y be any two vertices of T. Without loss of generality, we assume x dominates y. Let A (resp. B) be the set of vertices in $T - \{x, y\}$ dominated by x (resp. dominating y). Now we consider the following three cases.

Case 1. $A \cap B = \emptyset$.

We shall show that there are at least $t - 2$ vertices of A dominate at least $t - 2$ vertices of B. Since $|A| \geq \frac{1}{2}(n - 3 - k)$, $|B| \geq \frac{1}{2}(n - 3 - k)$, we have $|C| = |V(T) \backslash (A \cup B \cup \{x, y\})| \leq n - 2 - \frac{1}{2}(n - 3 - k) - |A|$. Without loss of generality, we may suppose each vertex $u \notin \{u_1, u_2, ..., u_{t-2}\}$ satisfying $d^-(u) \leq d^-(u_i)$ for $i \in \{1, 2, ..., t - 2\}$ in the sub-tournament $T[A]$. Recall that every tournament T' has a vertex of in-degree at least $\frac{1}{2}(|T'| - 1)$. This means that the in-degree of u_i in $T[A]$ satisfies $d_A^-(u_i) \geq \frac{1}{2}(|A| - t + 2)$ for $i \in \{1, 2, ..., t - 2\}$. Otherwise, there would be a vertex in $A \backslash \{u_1, u_2, ..., u_{t-2}\} + u_i$ such that the in-degree of it larger than u_i, contradicting the choice of u_i. So the out-degree of u_i in $T[A]$ satisfies $d_A^+(u_i) = |A| - 1 - d_A^-(u_i) \leq \frac{1}{2}|A| + \frac{1}{2}(t - 4)$. Therefore, the number of vertices in B dominated by u_i is at least $\frac{1}{2}(n - 1 - k) - d_A^+(u_i) - d_C^+(u_i) \geq t + \frac{5}{4}$, where $d_C^+(u_i)$ denotes the number of vertices in C dominated by u_i. It can be seen that every vertex u_i dominates at least $t - 2$ vertices of B. Thus, we obtain $t - 2$ internally disjoint paths xu_iv_jy of length 3, where $v_j \in B$ and $j \in \{1, 2, ..., t - 2\}$. Let $\{P_1, P_2, ..., P_m\}$ $(1 \leq m \leq t - 2)$ denote the m internally disjoint (x, y)-path of length 3. Deleting the $2m$ vertices from T, the resulting tournament is 4-strong. By Corollary 2.5, the resulting tournament is 2^*-strongly connected. As x dominates y, there is a strong 2^*-container from x to y. Let $\{P_{m+1}, P_{m+2}\}$ denote the two internally disjoint (x, y)-paths. Then $\{P_1, P_2, ..., P_{m+2}\}$ forms a strong $(m + 2)^*$-container from x to y. Hence, T has a strong i^*-container from x to y for all $i \in \{3, 4, ..., t\}$.

Case 2. $|A \cap B| \geq t - 2$.

We choose m $(1 \leq m \leq t - 2)$ vertices from $A \cap B$ and the m vertices imply a set of m internally disjoint (x, y)-paths of length 2, denoted by $\{P_1, P_2, ..., P_m\}$. Deleting the m vertices, the resulting tournament is $(2t - m)$-strong. Clearly, the resulting tournament is 4-strong. By Corollary 2.5, it is 2^*-strongly connected. As x dominates y, there is a strong 2^*-container from x to y. Let $\{P_{m+1}, P_{m+2}\}$ denote the two internally disjoint (x, y)-paths. Thus, $\{P_1, P_2, ..., P_{m+2}\}$

8

forms a strong $(m+2)^*$-container from x to y. Hence, T has a strong i^*-container from x to y for all $i \in \{3, 4, ..., t\}$.

Case 3. $1 \le |A \cap B| \le t - 3$.

Suppose $A \cap B = S$ and $|S| = s$. Since $|A| \ge \frac{1}{2}(n - 3 - k)$, $|B| \ge \frac{1}{2}(n - 3 - k)$, we have $|C| = |V(T) \backslash (A \cup B \cup \{x, y\})| \le n - 2 - \frac{1}{2}(n - 3 - k) - |A| + s$. We assume each vertex $u \notin \{u_1, u_2, ..., u_{t-2-s}\}$ satisfying $d^-(u) \le d^-(u_i)$ for all $i \in \{1, 2, ..., t - 2 - s\}$ in the sub-tournament $T[A \backslash B]$. The in-degree of u_i in $T[A \backslash B]$ satisfies $d^-_{A \backslash B}(u_i) \ge \frac{1}{2}(|A \backslash B| - t + 2 + s)$ for each $i \in \{1, 2, ..., t - 2 - s\}$. Otherwise, there would be a vertex in $A \backslash (B \cup \{u_1, u_2, ..., u_{t-2-s}\}) + u_i$ such that the in-degree of it larger than u_i, contradicting the choice of u_i. So the out-degree of u_i in $T[A \backslash B]$ satisfies $d^+_{A \backslash B}(u_i) = |A \backslash B| - 1 - d^-_{A \backslash B}(u_i) \le \frac{1}{2}|A \backslash B| + \frac{1}{2}(t - 4 - s)$. Therefore, the number of vertices in B dominated by u_i is at least $\frac{1}{2}(n - 1 - k) - d^+_{A \backslash B}(u_i) - d^+_C(u_i) \ge t + \frac{5}{4}$, where $d^+_C(u_i)$ denote the number of vertices in C dominated by u_i. It can be seen that every vertex u_i dominates at least $(t - 2 - s)$ vertices of $B \backslash A$. Thus, we obtain $(t - 2 - s)$ internally disjoint paths $x u_i v_j y$ of length 3 ($v_j \in B, j \in \{1, 2, ..., t - 2 - s\}$) and s internally disjoint (x, y)-paths of length 2. Let $\{P_1, P_2, ..., P_s\}$ ($1 \le s \le t - 3$) denotes the s internally disjoint (x, y)-paths of length 2 and $\{Q_1, Q_2, ..., Q_m\}$ ($1 \le m \le t - 2 - s$) denotes the m internally disjoint (x, y)-paths of length 3. Deleting the $2m + s$ vertices from T, the resulting tournament is $(4 + s)$-strong. By Corollary 2.5, the resulting tournament is 2^*-strongly connected. As x dominates y, there is a strong 2^*-container from x to y. Let $\{R_1, R_2\}$ denote the two internally disjoint (x, y)-paths. Then $\{P_1, P_2, ..., P_s, Q_1, Q_2, ..., Q_m, R_1, R_2\}$ forms a strong $(s + m + 2)^*$-container from x to y. Hence, T has a strong i^*-container from x to y for all $i \in \{3, 4, ..., t\}$.

Thus, T is i^*-strongly connected for $i \in \{3, 4, ..., t\}$. As T is i^*-strongly connected for $i \in \{1, 2\}$., then $\kappa^*_s(T) \ge t$. The proof is completed. $\square$

By an argument similar to that used above, it is not difficult to obtain the following. We omit the detailed proof.

Theorem 4.3. *Let T be a tournament with n vertices and $i(T) \le k$. If $n \ge 6t + 5k - 3$ ($t \ge 2$), then $\kappa^*_w(T) \ge t + 1$.*

9

5 Acknowledgements

References

[1] M. Albert, R.E.L. Aldred, and D. Holton, On 3*-connected graphs, Australas. J. Combin. 24 (2001) 193-208.

[2] B. Alspach, K.B. Reid, and D.P. Roselle, Bypasses in asymmetric digraphs, J. Combinatorial Theory Ser. B 17 (1974) 11-18.

[3] J. Bang-Jensen, and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, London, 2000.

[4] J. Bang-Jensen, G. Gutin, and J. Huang. Weakly Hamiltonian-connected ordinary multipartite tournaments. Discrete Math, 138 (1-3): 63-74, 1995. 14th British Combinatorial Conference (Keele, 1993).

[5] Y. Guo and L. Volkmann, Bypaths in tournaments, Discrete Applied Mathematics 79 (1997) 127-135.

[6] D. Hsu, On container width and length in graphs, groups, and networks, IEICE Trans. Fund. E77-A (1994) 668-680.

[7] C.-K. Lin, H.-M. Huang, and L.-H. Hsu, On the spaning connectivity of graphs, Discrete Mathematics 307 (2007) 285-289.

[8] C.-K. Lin, J.-M. Tan, D.-F. Hsu, and L.-H. Hsu, On the spanning fan-connectivity of graphs, Discrete Appl. Math. 157 (2009) 1342-1348.

[9] C.-K. Lin, H.-M. Huang, and L.-H. Hsu, The super connectivity of the pancake graphs and star graphs, Theoret. Comput. Sci. 339 (2005) 257-271.

[10] K. Menger, Zur allgemeinen kurventheorie, Fund. Math. 10 (1927) 95-115.

[11] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory Ser. B 28 (1980) 142-163.

[12] K. Zhang and Z. Wu, A necessary and sufficient condition for completely strong path-connected tournaments, Chinese Ann. Math. Ser. A 4 (1983) 385-392.