

Modified Erdős–Ginzburg–Ziv constants for \mathbb{Z}_2^d

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Abstract

Let G be a finite abelian group written additively, and let r be a multiple of its exponent. The modified Erdős–Ginzburg–Ziv constant $s'_r(G)$ is the smallest integer s such that every zero-sum sequence of length s over G has a zero-sum subsequence of length r . We find exact values of $s'_{2k}(\mathbb{Z}_2^d)$ for $d \leq 2k + 1$.

Keywords: Erdős–Ginzburg–Ziv constant, zero-sum sequence
2010 MSC: 05C35, 20K01

Let G be a finite abelian group written additively. We denote by $\exp(G)$ the *exponent* of G that is the least common multiple of the orders of its elements. Let r be a multiple of $\exp(G)$. The *generalized Erdős–Ginzburg–Ziv constant* $s_r(G)$ is the smallest integer s such that every sequence of length s over G has a zero-sum subsequence of length r . If $r = \exp(G)$, then $s(G) = s_{\exp(G)}(G)$ is the classical Erdős–Ginzburg–Ziv constant. The constants $s_r(G)$ have been studied extensively, see for example [4–10, 12]. The following variation of these constants was introduced in [1] and further studied in [2, 3, 11]. The *modified Erdős–Ginzburg–Ziv constant* $s'_r(G)$ is the smallest integer s such that every *zero-sum* sequence of length s over G has a zero-sum subsequence of length r .

By the definition, $s'_r(G) \leq s_r(G)$. On the other hand, if g_1, g_2, \dots, g_s is a sequence over G that does not contain a zero-sum subsequence of size r , and s is mutually prime with $\exp(G)$, then there exists $x \in G$ such that $g_1 + x, g_2 + x, \dots, g_s + x$ is a zero-sum subsequence (see [1, 11]). Thus, $s'_r(G) \geq s_r(G) - (\exp(G) - 1)$, and if $s_r(G) - 1$ is mutually prime with $\exp(G)$, then $s'_r(G) = s_r(G)$.

In this note, we consider the case $\exp(G) = 2$, so $G \cong \mathbb{Z}_2^d$. By the above-mentioned argument,

$$s'_r(\mathbb{Z}_2^d) = s_r(\mathbb{Z}_2^d) \quad \text{if } s_r(\mathbb{Z}_2^d) \text{ is even,} \quad (1)$$

and

$$s_r(\mathbb{Z}_2^d) - 1 \leq s'_r(\mathbb{Z}_2^d) \leq s_r(\mathbb{Z}_2^d). \quad (2)$$

The exact values of generalized Erdős–Ginzburg–Ziv constants $s_{2k}(\mathbb{Z}_2^d)$ have been found for $d \leq 2k + 1$:

Theorem 1 ([12]).

$$s_{2k}(\mathbb{Z}_2^d) = \begin{cases} 2k + d & \text{for } d < 2k; \\ 4k + 1 & \text{for } d = 2k; \\ 4k + 2 & \text{for } d = 2k + 1, \text{ } k \text{ is even;} \\ 4k + 5 & \text{for } d = 2k + 1, \text{ } k \text{ is odd.} \end{cases}$$

In the present note, we extend this result to the *modified* Erdős–Ginzburg–Ziv constants.

Theorem 2. *Let $d \leq 2k + 1$. Then $s'_{2k}(\mathbb{Z}_2^d) = s_{2k}(\mathbb{Z}_2^d) - 1$ in the following cases:*

- $d = 2k - 1$;
- $d = 2k - 3$, k is even;
- $d \leq 2k - 5$, d is odd.

In all other cases, $s'_{2k}(\mathbb{Z}_2^d) = s_{2k}(\mathbb{Z}_2^d)$.

Proof. We start with the cases where we claim $s'_{2k}(\mathbb{Z}_2^d) = s_{2k}(\mathbb{Z}_2^d)$. Among them, cases $d < 2k$ with even d , and $d = 2k + 1$ with even k follow from Theorem 1 and (1). The other three cases are $d = 2k$, $d = 2k + 1$ with odd k , and $d = 2k - 3$ with odd k . Since $s'_{2k}(\mathbb{Z}_2^d) \leq s_{2k}(\mathbb{Z}_2^d)$, it is sufficient to construct a zero-sum sequence of length $s_{2k}(\mathbb{Z}_2^d) - 1$ that does not contain a zero-sum subsequence of length $2k$. For $d = 2k$, we select a sequence of length $4k$ which consists of $2k - 1$ copies of the zero vector, the $2k$ basis vectors e_1, e_2, \dots, e_{2k} , and the vector $e_1 + e_2 + \dots + e_{2k}$. For odd k and $d = 2k + 1, 2k - 3$, we select a sequence of length $2d + 2$ which consists of $0, e_1, e_2, \dots, e_{d-1}, e_1 + e_2 + \dots + e_{d-1}, e_d, e_d + e_1, e_d + e_2, \dots, e_d + e_{d-1}, e_d + e_1 + e_2 + \dots + e_{d-1}$.

To solve the three cases where we claim $s'_{2k}(\mathbb{Z}_2^d) = s_{2k}(\mathbb{Z}_2^d) - 1$, in the light of (2), it is sufficient to prove that any zero-sum sequence of length $s_{2k}(\mathbb{Z}_2^d) - 1$ over \mathbb{Z}_2^d contains a zero-sum subsequence of length $2k$. First consider the case $d = 2k - 1$. Let $x_2, x_3, \dots, x_{4k-1} \in \mathbb{Z}_2^{2k-1}$ where $x_2 + x_3 + \dots + x_{4k-1} = 0$. Set $x_1 = x_2$. As $s_{2k}(\mathbb{Z}_2^{2k-1}) = 4k - 1$, there is $A \subset \{1, 2, \dots, 4k - 1\}$ such that $|A| = 2k$ and $\sum_{i \in A} x_i = 0$. If $1 \notin A$, then we have found a zero-sum subsequence of length $2k$ among $x_2, x_3, \dots, x_{4k-1}$. Suppose, $1 \in A$. If $2 \notin A$, then $(A \setminus \{1\}) \cup \{2\}$ points to a zero-sum subsequence of length $2k$. Suppose, $1, 2 \in A$. Set $B := (\{1, 2, \dots, 4k - 1\} \setminus A) \cup \{2\}$. Then $|B| = 2k$ and

$$\begin{aligned} \sum_{i \in B} x_i &= x_1 + x_2 + \dots + x_{4k-1} - \sum_{i \in A} x_i + x_2 \\ &= x_1 + x_2 + (x_2 + \dots + x_{4k-1}) - \sum_{i \in A} x_i = x_1 + x_2 = 0. \end{aligned}$$

Finally, let d be odd, and $d \leq 2k - 3$ if k is even, or $d \leq 2k - 5$ if k is odd. We are going to show that every zero-sum sequence of length $2k + d - 1$ over \mathbb{Z}_2^d contains a zero-sum subsequence of length $2k$. Let $x_1, x_2, \dots, x_{2k+d-1} \in \mathbb{Z}_2^d$

where $x_1 + x_2 + \dots + x_{2k+d-1} = 0$. By Theorem 1, $s_{d-1}(\mathbb{Z}_2^d) = 2d$ if $d \equiv 1 \pmod{4}$, and $s_{d-1}(\mathbb{Z}_2^d) = 2d + 3$ if $d \equiv 3 \pmod{4}$. In both cases, $s_{d-1}(\mathbb{Z}_2^d) \leq 2k + d - 1$. Thus, there is $A \subset \{1, 2, \dots, 2k + d - 1\}$ such that $|A| = d - 1$ and $\sum_{i \in A} x_i = 0$. Set $B := \{1, 2, \dots, 2k + d - 1\} \setminus A$. Then $|B| = 2k$ and $\sum_{i \in B} x_i = \sum_{i=1}^{2k+d-1} x_i - \sum_{i \in A} x_i = 0 - 0 = 0$, so B points to a zero-sum subsequence of length $2k$ within x_1, \dots, x_{2k+d-1} . \square

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