H-Decomposition of r-graphs when H is an r-graph with exactly k independent edges *

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Abstract

Let $\phi_H^r(n)$ be the smallest integer such that, for all r-graphs G on n vertices, the edge set E(G) can be partitioned into at most $\phi_H^r(n)$ parts, of which every part either is a single edge or forms an r-graph isomorphic to H. The function $\phi_H^2(n)$ has been well studied in literature, but for the case $r \geq 3$, the problem that determining the value of $\phi_H^r(n)$ is widely open. Sousa (2010) gave an asymptotic value of $\phi_H^r(n)$ when H is an r-graph with exactly 2 edges, and determined the exact value of $\phi_H^r(n)$ in some special cases. In this paper, we first give the exact value of $\phi_H^r(n)$ when H is an r-graph with exactly 2 edges, which improves Sousa's result. Second we determine the exact value of $\phi_H^r(n)$ when H is an r-graph consisting of exactly k independent edges.

^{*}The work was supported by NNSF of China (No. 11671376) and NSF of Anhui Province (No. 1708085MA18).

1 Introduction

Given two r-graphs G and H, an H-decomposition of G is a partition of the edge set of G such that each part is either a single edge or forms an r-graph isomorphic to H. The minimum number of parts in an H-decomposition of G is denoted by $\phi_H^r(G)$. The H-decomposition number $\phi_H^r(n)$ is defined as

$$\phi_H^r(n) = \max\{\phi_H^r(G) : G \text{ is an } r\text{-graph with } |V(G)| = n\}.$$

An r-graph G with $\phi_H^r(G) = \phi_H^r(n)$ is called an extremal graph of H.

For the case r=2, we omit the index 2 for short in the paper, for example we write graph for 2-graph, and $\phi_H(n)$ for $\phi_H^2(n)$. The function $\phi_H(n)$ has been well studied in literature by many researchers. The first exact value of $\phi_H(n)$ when $H=K_3$ was given by Erdős, Goodman and Pósa [3] in 1966, where K_k is the complete graph on k vertices. Ten years later, Bollobás [2] generalized the result to $H=K_k$, $k\geq 3$. Much more exact values of $\phi_H(n)$ can be found in the survey by Sousa [11] in 2015. Recently, Hou, Qiu and Liu determined the exact values of $\phi_H(n)$ when H is a graph consisting of k complete graphs of order at least 3 which intersect in exactly one common vertex [7] and when H is a graph consisting of k cycles of odd lengths which intersect in exactly one common vertex [8]. An asymptotic value of the function $\phi_H(n)$ was given by Pikhurko and Sousa [9], and lately the value was improved by Allen, Böttcher, and Person [1].

For the case $r \geq 3$, the study of the function $\phi_H^r(n)$ is widely open. Sousa [10] gave an asymptotic value of $\phi_H^r(n)$ when H is an r-graph consisting of 2 edges, and determined the exact value of $\phi_H^r(n)$ in the special cases where the two edges of H intersect exactly 1, 2 and r-1 vertices. In this paper, we first generalize Sousa's result in [10], that is we obtain the exact value of $\phi_H^r(n)$ when H is an r-graph consisting of exactly 2 edges. Second we focus on the case that H is the r-graph consisting of exactly k independent edges, and we determine the exact value of $\phi_H^r(n)$ in this case.

Given positive integer n, r and k with $n \geq r \geq 2$, let K_n^r be a complete r-graph on n vertices and let $K_n^r - \ell e$ be a graph obtained from K_n^r by deleting ℓ edges from K_n^r , where ℓ is an integer so that $0 \leq \ell \leq k-1$ and $e(K_n^r - \ell e) \equiv k-1 \pmod k$. Note that ℓ is determined uniquely by n, r and k. Let $\mathcal{K}_{n,k}^r$ be the family of r-graphs $K_n^r - \ell e$. The followings are our main results.

Theorem 1. Given integers r, k satisfying $0 \le k \le r - 1$. Let H be an r-graph consisting of exactly 2 edges which intersect k vertices. If $n \ge 2r - k$, then $\phi_H^r(n) =$

 $\lceil \frac{1}{2} \binom{n}{r} \rceil$. Moreover, graphs $G \in \mathcal{K}_{n,2}^r$ and $G = K_n^r$ if $\binom{n}{r} \equiv 0 \pmod{2}$ are extremal graphs of H.

Theorem 1 improves Sousa's result in [10].

Theorem 2. Given integers $k \ge 1$, $r \ge 2$ and $n_0 = kr(k+r-2) + 2r - 1$, let H be an r-graph on n vertices consisting of exactly k independent edges. If $n \ge n_0$ then

$$\phi_H^r(n) = \begin{cases} \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 1, & \text{if } \binom{n}{r} \equiv k - 1 \pmod{k}; \\ \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 2, & \text{otherwise.} \end{cases}$$

Furthermore, G is an extremal graph of H if and only if $G \in \mathcal{K}_{n,k}^r$ or $G = K_n^r$ if $\binom{n}{r} \equiv k-2 \pmod{k}$.

The proofs of Theorems 1 and 2 will be given in Sections 2 and 3, respectively. Before giving the proofs, we first introduce some definitions and notation. Let H be an r-graph with vertex set V(H) and E(H). For a vertex $v \in V(H)$, the degree of v, denoted by $d_H(v)$, is the number of edges of H containing v, and the minimum degree of H is denoted by $\delta(H)$. The matching number of H is the maximum number of independent edges in H. We write e(H) for the number of edges of H, that is e(H) = |E(H)|.

2 Proof of Theorem 1

We need some basic facts in algebraic graph theory. A graph G is called *vertex-transitive* if its automorphism group acts transitively on V(G). Given nonnegative integers n, r and k, let J(n, r, k) be the graph with vertex set $E(K_n^r)$, where two vertices are adjacent if and only if their intersection has size k. For $n \geq r$, the graphs J(n, r, r - 1) and J(n, r, 0) are known as the Johnson graphs and the Kneser graphs, respectively.

Fact 1 (See page 9 and page 35 in [5]). (1) J(n,r,k) has $\binom{n}{r}$ vertices, and each vertex has degree $\binom{r}{k}\binom{n-r}{r-k}$.

- (2) The graphs J(n,r,k) are vertex-transitive.
- (3) If $n \ge r \ge k$, $J(n, r, k) \cong J(n, n r, n 2r + k)$.

Lemma 3 (Theorem 3.5.1 in [5]). If G is a connected vertex-transitive graph, then G has a matching that misses at most one vertex.

Lemma 4 (Theorem 2.3 in [10]). Let H be a fixed r-graph with 2 edges and G an r-graph with n vertices. Then $\phi_H^r(G) \leq \phi_H^r(K_n^r)$.

Proof of Theorem 1: Lemma 4 implies that $\phi_H^r(n) = \phi_H^r(K_n^r)$. So, to prove the result, it is sufficient to show that $\phi_H^r(K_n^r) = \lceil \frac{1}{2} \binom{n}{r} \rceil$. Clearly, $\phi_H^r(K_n^r) \geq \lceil \frac{1}{2} \binom{n}{r} \rceil$ as e(H) = 2. To prove $\phi_H^r(K_n^r) \leq \lceil \frac{1}{2} \binom{n}{r} \rceil$, it is sufficient to find an H-decomposition of K_n^r with $\lceil \frac{1}{2} \binom{n}{r} \rceil$ parts.

By the definition of J(n, r, k), K_n^r has an H-decomposition with $\lceil \frac{1}{2} \binom{n}{r} \rceil$ parts is equivalent to the statement that J(n, r, k) has a matching missing at most one vertex. By (3) of Fact 1, we may assume $n \geq 2r$. If k = 0 and n = 2r, then J(2r, r, 0) consists of exactly $\frac{1}{2} \binom{n}{r}$ independent edges, so the statement holds.

Now we assume k>0 or n>2r. By (2) of Fact 1, J(n,r,k) is vertex-transitive. Hence, by Lemma 3, to show J(n,r,k) has a matching missing at most one vertex, it is sufficient to show that J(n,r,k) is connected. That is, we need to show that any pair of vertices e, f of J(n,r,k) are connected. Suppose $|e \cap f| = i$. We prove e and f are connected by induction on i. If i=r, then statement is trivial. If i=r-1, assume $e=\{1,2,\ldots,r-1,r\}$ and $f=\{1,2,\ldots,r-1,r+1\}$. Then ehf with $h=\{1,2,\ldots,k,r+2,\ldots,2r+1-k\}$ is a walk connecting e and f in J(n,r,k). So the result is true for the base case. Now suppose i< r-1 and the statement is true for any large i. By symmetry, one could assume that $e=\{1,2,\ldots,r-1,r\}$ and $f=\{1,2,\ldots,i,r+1,\ldots,2r-i\}$. Let $h=\{1,\ldots,i,i+1,r+1,\ldots,2r-i-1\}$. Then $|e\cap h|=i+1>i$ and $|f\cap h|=r-1>i$. By induction hypothesis, e and e0 (resp. e1 and e2) are connected in e3. By the transitivity of connectivity, e3 and e4 are connected in e4.

3 Proof of Theorem 2

We need two known theorems to prove our result. Given graphs G and H, we say G has an H-factor if G contains $\left|\frac{|V(G)|}{|V(H)|}\right|$ vertex-disjoint copies of H.

Theorem 5 (Hajnal, Szemerédi [6]). Let k be a positive integer. If G is a graph on n vertices with minimum degree

$$\delta(G) \ge (1 - \frac{1}{k})n,$$

then G has a K_k -factor.

Theorem 6 (Frankl [4]). If H is an r-graph on n vertices with matching number of size k and $n \ge (2k+1)r - k$, then

$$e(H) \le \binom{n}{r} - \binom{n-k}{r}.$$

Proof of Theorem 2: Let G be an r-graph on $n \geq n_0$ vertices with $\phi_H(G) = \phi_H(n)$. Let $p_H(G)$ denote the maximum number of pairwise edge-disjoint copies of H in G. Then we have

$$\phi_H(G) = e(G) - (k-1)p_H(G). \tag{1}$$

If we remove the edges of $p_H(G)$ pairwise edge-disjoint copies of H from G, then we obtain an H-free graph, that is a graph with matching number at most k-1. Hence by Theorem 6, we have

$$\binom{n}{r} - \binom{n-k+1}{r} \ge e(G) - kp_H(G). \tag{2}$$

On the other hand,

$$\phi_H(G) \ge \phi_H(K_n^r) \ge \frac{1}{k} \binom{n}{r}.$$
 (3)

From (1), (2) and (3), we have

$$e(G) \ge \binom{n}{r} - (k-1) \left\lceil \binom{n}{r} - \binom{n-k+1}{r} \right\rceil. \tag{4}$$

Now we define an auxiliary graph L_G as follows: let $V(L_G) = E(G)$ and two vertices $e_1, e_2 \in V(L_G)$ is adjacent if and only if $e_1 \cap e_2 = \emptyset$ in G. Hence the edge set of a copy of H in E(G) induces a clique of order k in L_G . Therefore, a collection of edge-disjoint copies of H in G corresponds to a collection of vertex-disjoint K_k in L_G .

Claim 1. L_G has a K_k -factor. In particular, $L_{K_n^r-\ell e}$ has a K_k -factor for every ℓ with $0 \le \ell \le k-1$.

Proof of the claim: By definition of L_G , we have

$$\delta(L_G) \ge \binom{n-r}{r} - \left[\binom{n}{r} - e(G)\right].$$

By Theorem 5, it suffices to show that

$$\binom{n-r}{r} - \left[\binom{n}{r} - e(G) \right] \ge (1 - \frac{1}{k})e(G),$$

that is

$$e(G) \ge k \left[\binom{n}{r} - \binom{n-r}{r} \right].$$
 (5)

To show (5), by (4) it suffices to show

$$\binom{n}{r} - (k-1) \left[\binom{n}{r} - \binom{n-k+1}{r} \right] \ge k \left[\binom{n}{r} - \binom{n-r}{r} \right],$$

that is, we need to show

$$k\binom{n-r}{r} + (k-1)\binom{n-k+1}{r} \ge (2k-2)\binom{n}{r}.$$
 (6)

By the inequality

$$\frac{\binom{n-t}{r}}{\binom{n}{r}} \ge \left(\frac{n-t-r+1}{n-r+1}\right)^r \ge 1 - \frac{rt}{n-r+1}, (r \ge t \ge 0)$$

and $n \ge n_0 = kr(k+r-2) + 2r - 1$, it can be easily check that (6) holds. This completes the proof of the claim.

Now suppose $e(G) \equiv i \pmod{k}$ and $e(G) = tk + i (0 \le i \le k - 1)$ for some $t \le \lfloor \frac{1}{k} \binom{n}{r} \rfloor$. By Claim 1, $p_H(G) = t$ and hence by (1), we have $\phi_H(G) = t + i$. In particular,

$$\phi_H^r(K_n^r - \ell e) = \begin{cases} \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 1, & \text{if } \binom{n}{r} \equiv k - 1 \pmod{k} \\ \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 2, & \text{otherwise} \end{cases}.$$

If $\binom{n}{r} \equiv k-1 \pmod k$, then $\phi_H^r(n) = \phi_H^r(G) = t+i \leq \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k-1$, and the equality holds if and only if $G = K_n^r \in \mathcal{K}_{n,k}^r$. Otherwise, $\phi_H^r(n) = \phi_H^r(G) = t+i \leq \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k-2$, the equality holds if and only if $t = \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor - 1$ and i = k-1 or $t = \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor$ and i = k-2, in the former case $G \in \mathcal{K}_{n,k}^r$ and in the latter case it happens if and only if $G = K_n^r$ and $\binom{n}{r} \equiv k-2 \pmod k$.

4 Concluding Remarks

In this paper we determine the exact value of of $\phi_H^r(n)$ when H is an r-graph consisting of exactly 2 edges or consisting of exactly k INDEPENDENT edges. We believe that Theorem 2 still holds when H consists of exactly k edges which intersect the same set of size i ($0 \le i \le r - 1$), we leave this as an open problem.

Question 7. Is the following statement true? Given integers $k \geq 1$, $r \geq 2$, let H be an r-graph consisting of exactly k edges which intersect the same set of size i $(0 \leq i \leq r - 1)$. If n is sufficiently large, then

$$\phi_H^r(n) = \begin{cases} \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 1, & if \binom{n}{r} \equiv k - 1 \pmod{k}; \\ \left\lfloor \frac{1}{k} \binom{n}{r} \right\rfloor + k - 2, & otherwise. \end{cases}$$

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