

A characterization of Q -polynomial distance-regular graphs using the intersection numbers

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Abstract

We consider a primitive distance-regular graph Γ with diameter at least 3. We use the intersection numbers of Γ to find a positive semidefinite matrix G with integer entries. We show that G has determinant zero if and only if Γ is Q -polynomial.

1 Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 3$. In the literature there are a number of characterizations for the Q -polynomial condition on Γ . There is the balanced set characterization [9, Theorem 1.1], [10, Theorem 3.3]. There is a characterization involving the dual distance matrices [10, Theorem 3.3]. There is a characterization involving the intersection numbers a_i [8, Theorem 3.8]; cf. [3, Theorem 5.1]. There is a characterization involving a tail in a representation diagram [5, Theorem 1.1]. There is a characterization involving a pair of primitive idempotents [6, Theorem 1.1]; cf. [7, Theorem 1.1].

In this paper we obtain the following characterization of the Q -polynomial property. Assume Γ is primitive. We use the intersection numbers of Γ to find a positive semidefinite matrix G with integer entries. We show that G has determinant zero if and only if Γ is Q -polynomial. Our main result is Theorem 18.

2 Preliminaries

Let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of the matrices whose rows and columns are indexed by X and whose entries are in \mathbb{C} . For $B \in \text{Mat}_X(\mathbb{C})$ let \overline{B} and B^t denote the complex conjugate and the transpose of B , respectively. Let $V = \mathbb{C}^X$ denote the vector space over \mathbb{C} consisting of column vectors with coordinates indexed by X and entries in \mathbb{C} . Observe that $\text{Mat}_X(\mathbb{C})$ acts

on V by left multiplication. We endow V with the Hermitean inner product (\cdot, \cdot) such that $(u, v) = u^t \bar{v}$ for all $u, v \in V$. The inner product (\cdot, \cdot) is positive definite. For $B \in \text{Mat}_X(\mathbb{C})$ we obtain $(u, Bv) = (\bar{B}^t u, v)$ for all $u, v \in V$. We endow $\text{Mat}_X(\mathbb{C})$ with the Hermitean inner product $\langle \cdot, \cdot \rangle$ such that $\langle R, S \rangle = \text{tr}(R^t \bar{S})$ for all $R, S \in \text{Mat}_X(\mathbb{C})$. The inner product $\langle \cdot, \cdot \rangle$ is positive definite.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R . Let ∂ denote the shortest path-length distance function for Γ . Define the diameter $D := \max\{\partial(x, y) | x, y \in X\}$. For a vertex $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, we call the graph Γ *regular with valency k* whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq D$) and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y . The integers p_{ij}^h are called the *intersection numbers* of Γ . From now on we assume Γ is distance-regular with diameter $D \geq 3$. We abbreviate $c_i := p_{1, i-1}^i$ ($1 \leq i \leq D$), $a_i := p_{1i}^i$ ($0 \leq i \leq D$), $b_i := p_{1, i+1}^i$ ($0 \leq i \leq D-1$), $k_i := p_{ii}^0$ ($0 \leq i \leq D$), and $c_0 = 0, b_D = 0$. Observe that Γ is regular with valency $k = b_0$ and $c_i + a_i + b_i = k$ ($0 \leq i \leq D$). By [2, p. 127] the following holds for $0 \leq h, i, j \leq D$: (i) $p_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two; and (ii) $p_{ij}^h \neq 0$ if one of h, i, j equals the sum of the other two. For $0 \leq i \leq D$, let A_i denote the matrix in $\text{Mat}_X(\mathbb{C})$ with (x, y) -entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad x, y \in X.$$

We call A_i the *i -th distance matrix* of Γ . We call $A = A_1$ the *adjacency matrix* of Γ . Observe that A_i is real and symmetric for $0 \leq i \leq D$. Note that $A_0 = I$, where I is the identity matrix. Observe that $\sum_{i=0}^D A_i = J$, where J is the all-ones matrix in $\text{Mat}_X(\mathbb{C})$. Observe that for $0 \leq i, j \leq D$, $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$. For integers h, i, j ($0 \leq h, i, j \leq D$) we have

$$p_{0j}^h = \delta_{hj} \tag{1}$$

$$p_{ij}^0 = \delta_{ij} k_i \tag{2}$$

$$p_{ij}^h = p_{ji}^h \tag{3}$$

$$k_h p_{ij}^h = k_i p_{hj}^i = k_j p_{ih}^j. \tag{4}$$

For $0 \leq i, j \leq D$ we have $A(A_i A_j) = (A A_i) A_j$. This gives a recursion

$$c_{i+1} p_{i+1, j}^h + a_i p_{ij}^h + b_{i-1} p_{i-1, j}^h = c_h p_{ij}^{h-1} + a_h p_{ij}^h + b_h p_{ij}^{h+1} \tag{5}$$

that can be used to compute the intersection numbers.

Let M denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by A . By [2, p. 127] the matrices A_0, A_1, \dots, A_D form a basis for M . We call M the *Bose-Mesner algebra* of Γ . By [2, p. 45], M has a basis E_0, E_1, \dots, E_D such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $E_i^t = E_i$ ($0 \leq i \leq D$); (iv) $\overline{E_i} = E_i$ ($0 \leq i \leq D$); (v) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$). The matrices E_0, E_1, \dots, E_D are called the *primitive idempotents* of Γ , and E_0 is called the *trivial* idempotent. For $0 \leq i \leq D$ let m_i denote the rank of E_i . Let λ denote an indeterminate. Define polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{C}[\lambda]$ by $v_0 = 1$, $v_1 = \lambda$, and

$$\lambda v_i = c_{i+1} v_{i+1} + a_i v_i + b_{i-1} v_{i-1} \quad (1 \leq i \leq D),$$

where $c_{D+1} := 1$. By [2, p. 128] and [11, Lemma 3.8], the following hold: (i) $\deg v_i = i$ ($0 \leq i \leq D+1$); (ii) the coefficient of λ^i in v_i is $(c_1 c_2 \dots c_i)^{-1}$ ($0 \leq i \leq D+1$); (iii) $v_i(A) = A_i$ ($0 \leq i \leq D$); (iv) $v_{D+1}(A) = 0$; (v) the distinct eigenvalues of Γ are precisely the zeros of v_{D+1} ; call these $\theta_0, \theta_1, \dots, \theta_D$. Define a matrix $B \in \text{Mat}_{D+1}(\mathbb{C})$ as follows:

$$B = \begin{bmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_{D-1} \\ \mathbf{0} & & & c_D & a_D \end{bmatrix}.$$

We call B the *intersection matrix* of Γ . Note that A has the same minimal polynomial as B . Moreover the minimal polynomial of B is the characteristic polynomial of B . For an eigenvalue θ of B we have $vB = \theta v$ where v is a row vector $v = (v_0(\theta), v_1(\theta), \dots, v_D(\theta))$. Define polynomials $\{u_i\}_{i=0}^D$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda/k$, and

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \quad (1 \leq i \leq D-1).$$

Observe that $u_i = v_i/k_i$ ($0 \leq i \leq D$). For an eigenvalue θ of B we have $Bu = \theta u$ where u is a column vector $u = (u_0(\theta), u_1(\theta), \dots, u_D(\theta))^t$. By [2, p. 131, 132],

$$A_j = \sum_{i=0}^D v_j(\theta_i) E_i \quad (0 \leq j \leq D), \quad (6)$$

$$E_j = |X|^{-1} m_j \sum_{i=0}^D u_i(\theta_j) A_i \quad (0 \leq j \leq D). \quad (7)$$

Since $E_i E_j = \delta_{ij} E_i$ and by (6), (7) we have $A_j E_i = E_i A_j$ ($0 \leq i, j \leq D$).

For $1 \leq i \leq D$ let Γ_i denote the graph with vertex set X where vertices are adjacent in Γ_i whenever they are at distance i in Γ . We observe that $\Gamma_1 = \Gamma$. The graph Γ is said to be *primitive* whenever Γ_i is connected for $1 \leq i \leq D$.

Lemma 1. (See [2, Proposition 4.4.7].) *Assume Γ is primitive. Then $u_i(\theta_j) \neq 1$ for $1 \leq i, j \leq D$.*

We now define a matrix $S \in \text{Mat}_{D+1}(\mathbb{C})$.

Definition 2. Let $S \in \text{Mat}_{D+1}(\mathbb{C})$ denote the transition matrix from the basis $\{A_i\}_{i=0}^D$ of M to the basis $\{E_i\}_{i=0}^D$ of M . Thus

$$\begin{aligned} E_j &= \sum_{i=0}^D S_{ij} A_i & (0 \leq j \leq D), \\ A_j &= \sum_{i=0}^D (S^{-1})_{ij} E_i & (0 \leq j \leq D). \end{aligned}$$

Lemma 3. *The entries of S and S^{-1} are given below. For $0 \leq i, j \leq D$,*

$$S_{ij} = |X|^{-1} m_j u_i(\theta_j), \quad (S^{-1})_{ij} = v_j(\theta_i).$$

Proof. Immediate from Definition 2 and (6), (7). □

We recall the Q -polynomial property. Let \circ denote the entry-wise multiplication in $\text{Mat}_X(\mathbb{C})$. Note that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$. So M is closed under \circ . By [11, p. 377], there exist scalars $q_{ij}^h \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D). \quad (8)$$

We call the q_{ij}^h the *Krein parameters* of Γ . By [2, p. 48, 49], these parameters are real and nonnegative for $0 \leq h, i, j \leq D$. The graph Γ is said to be *Q -polynomial* with respect to the ordering E_0, E_1, \dots, E_D whenever the following hold for $0 \leq h, i, j \leq D$: (i) $q_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two; and (ii) $q_{ij}^h \neq 0$ if one of h, i, j equals the sum of the other two. Let E denote a primitive idempotent of Γ . We say that Γ is *Q -polynomial with respect to E* whenever there exists a Q -polynomial ordering E_0, E_1, \dots, E_D of the primitive idempotents such that $E = E_1$.

We recall the dual Bose-Mesner algebra of Γ . Fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with (y, y) -entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad y \in X.$$

We call E_i^* the *i -th dual idempotent* of Γ with respect to x . Observe that (i) $\sum_{i=0}^D E_i^* = I$; (ii) $E_i^{*t} = E_i^*$ ($0 \leq i \leq D$); (iii) $\overline{E_i^*} = E_i^*$ ($0 \leq i \leq D$); (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \leq i, j \leq D$). By construction $E_0^*, E_1^*, \dots, E_D^*$ are linearly independent. Let $M^* = M^*(x)$

denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ with basis $E_0^*, E_1^*, \dots, E_D^*$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x .

We now recall the dual distance matrices of Γ . For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with (y, y) -entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad y \in X. \quad (9)$$

We call A_i^* the *dual distance matrix* of Γ with respect to x and E_i . By [11, p. 379], the matrices $A_0^*, A_1^*, \dots, A_D^*$ form a basis for M^* . Observe that (i) $A_0^* = I$; (ii) $\sum_{i=0}^D A_i^* = |X|E_0^*$; (iii) $A_i^{*t} = A_i^*$ ($0 \leq i \leq D$); (iv) $\overline{A_i^*} = A_i^*$ ($0 \leq i \leq D$); (v) $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ ($0 \leq i, j \leq D$). From (6), (7) we have

$$A_j^* = m_j \sum_{i=0}^D u_i(\theta_j) E_i^* \quad (0 \leq j \leq D), \quad (10)$$

$$E_j^* = |X|^{-1} \sum_{i=0}^D v_j(\theta_i) A_i^* \quad (0 \leq j \leq D). \quad (11)$$

Lemma 4. *The matrix $|X|S$ is the transition matrix from the basis $\{E_i^*\}_{i=0}^D$ of M^* to the basis $\{A_i^*\}_{i=0}^D$ of M^* . Thus*

$$A_j^* = |X| \sum_{i=0}^D S_{ij} E_i^* \quad (0 \leq j \leq D),$$

$$E_j^* = |X|^{-1} \sum_{i=0}^D (S^{-1})_{ij} A_i^* \quad (0 \leq j \leq D).$$

Proof. Immediate from Lemma 3 and (10), (11). □

3 The matrices S^{alt} , $(S^{-1})^{alt}$, S'

Recall the matrix S from Definition 2. We now modify the matrices S, S^{-1} to obtain $D \times D$ matrices $S^{alt}, (S^{-1})^{alt}$ as follows:

$$(S^{alt})_{ij} = S_{ij} - S_{0j} \quad (1 \leq i, j \leq D), \quad (12)$$

$$(S^{-1})_{ij}^{alt} = (S^{-1})_{ij} \quad (1 \leq i, j \leq D). \quad (13)$$

Lemma 5. *The following (i)–(iv) hold.*

(i) S^{alt} is the transition matrix from $\{A_2 E_i^* A - A E_i^* A_2\}_{i=1}^D$ to $\{A_2 A_i^* A - A A_i^* A_2\}_{i=1}^D$.

(ii) S^{alt} is the transition matrix from $\{A_3 E_i^* - E_i^* A_3\}_{i=1}^D$ to $\{A_3 A_i^* - A_i^* A_3\}_{i=1}^D$.

(iii) S^{alt} is the transition matrix from $\{A_2 E_i^* - E_i^* A_2\}_{i=1}^D$ to $\{A_2 A_i^* - A_i^* A_2\}_{i=1}^D$.

(iv) S^{alt} is the transition matrix from $\{AE_i^* - E_i^*A\}_{i=1}^D$ to $\{AA_i^* - A_i^*A\}_{i=1}^D$.

(v) $(S^{-1})^{alt}$ and S^{alt} are inverses.

Proof. (i), (v) For $1 \leq j \leq D$ we write $A_2A_j^*A - AA_j^*A_2$ in terms of $\{A_2E_i^*A - AE_i^*A_2\}_{i=1}^D$. By Lemma 4 and (12) and since $\sum_{i=0}^D E_i^* = I$, we have

$$\begin{aligned}
A_2A_j^*A - AA_j^*A_2 &= |X| \sum_{i=0}^D (A_2E_i^*A - AE_i^*A_2)S_{ij} \\
&= |X|(A_2E_0^*A - AE_0^*A_2)S_{0j} + |X| \sum_{i=1}^D (A_2E_i^*A - AE_i^*A_2)S_{ij} \\
&= |X|(A_2(I - (E_1^* + \cdots + E_D^*))A - A(I - (E_1^* + \cdots + E_D^*))A_2)S_{0j} \\
&\quad + |X| \sum_{i=1}^D (A_2E_i^*A - AE_i^*A_2)S_{ij} \\
&= |X| \sum_{i=1}^D (A_2E_i^*A - AE_i^*A_2)(S_{ij} - S_{0j}) \\
&= |X| \sum_{i=1}^D (A_2E_i^*A - AE_i^*A_2)(S^{alt})_{ij}.
\end{aligned}$$

Next, for $1 \leq j \leq D$ we write $A_2E_j^*A - AE_j^*A_2$ in terms of $\{A_2A_i^*A - AA_i^*A_2\}_{i=1}^D$. By Lemma 4 and (13) and since $A_0^* = I$, we find

$$\begin{aligned}
A_2E_j^*A - AE_j^*A_2 &= |X|^{-1} \sum_{i=0}^D (A_2A_i^*A - AA_i^*A_2)(S^{-1})_{ij} \\
&= |X|^{-1}(A_2A_0^*A - AA_0^*A_2)(S^{-1})_{0j} \\
&\quad + |X|^{-1} \sum_{i=1}^D (A_2A_i^*A - AA_i^*A_2)(S^{-1})_{ij} \\
&= |X|^{-1} \sum_{i=1}^D (A_2A_i^*A - AA_i^*A_2)(S^{-1})_{ij} \\
&= |X|^{-1} \sum_{i=1}^D (A_2A_i^*A - AA_i^*A_2)(S^{-1})_{ij}^{alt}.
\end{aligned}$$

The result follows.

(ii) – (iv) Similar to the proof of (i). □

Define a matrix

$$S' = \begin{bmatrix} S^{alt} & & & \mathbf{0} \\ & S^{alt} & & \\ & & S^{alt} & \\ \mathbf{0} & & & S^{alt} \end{bmatrix},$$

where S^{alt} is from (12). Observe that S' is $4D \times 4D$.

Lemma 6. $\det(S') = (\det(S^{alt}))^4$. Moreover S' is invertible.

Proof. Immediate from the construction of S' . □

Corollary 7. The matrix S' is the transition matrix from

$$\{A_2 E_i^* A - A E_i^* A_2\}_{i=1}^D, \{A_3 E_i^* - E_i^* A_3\}_{i=1}^D, \{A_2 E_i^* - E_i^* A_2\}_{i=1}^D, \{A E_i^* - E_i^* A\}_{i=1}^D$$

to

$$\{A_2 A_i^* A - A A_i^* A_2\}_{i=1}^D, \{A_3 A_i^* - A_i^* A_3\}_{i=1}^D, \{A_2 A_i^* - A_i^* A_2\}_{i=1}^D, \{A A_i^* - A_i^* A\}_{i=1}^D.$$

Proof. Immediate from Lemma 5. □

4 The bilinear form \langle , \rangle

Recall the positive definite Hermitean bilinear form \langle , \rangle on $\text{Mat}_X(\mathbb{C})$.

Lemma 8. (See [11, Lemma 3.2].) For $0 \leq h, i, j, r, s, t \leq D$,

$$(i) \langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir} \delta_{js} \delta_{ht} k_h p_{ij}^h,$$

$$(ii) \langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir} \delta_{js} \delta_{ht} m_h q_{ij}^h.$$

Corollary 9. (See [11, Lemma 3.2].) For $0 \leq h, i, j \leq D$,

$$(i) E_i^* A_j E_h^* = 0 \text{ if and only if } p_{ij}^h = 0,$$

$$(ii) E_i A_j^* E_h = 0 \text{ if and only if } q_{ij}^h = 0.$$

Lemma 10. For $0 \leq h, i, j, r, s, t \leq D$ we have

$$\langle A_i E_j^* A_h, A_r E_s^* A_t \rangle = \sum_{\ell=0}^D k_\ell p_{ir}^\ell p_{js}^\ell p_{ht}^\ell.$$

Proof. Since $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$ and $E_i E_j = \delta_{ij} E_i$ ($0 \leq h, i, j \leq D$) and by Lemma 8 and (4), we obtain

$$\begin{aligned}
\langle A_i E_j^* A_h, A_r E_s^* A_t \rangle &= \text{tr}((A_i E_j^* A_h)^t (\overline{A_r E_s^* A_t})) \\
&= \text{tr}(A_h E_j^* A_i A_r E_s^* A_t) \\
&= \sum_{\ell=0}^D p_{ir}^\ell \text{tr}(A_h E_j^* A_\ell E_s^* A_t) \\
&= \sum_{\ell=0}^D p_{ir}^\ell \text{tr}(E_j^* A_\ell E_s^* A_t A_h) \\
&= \sum_{\ell=0}^D \sum_{w=0}^D p_{ir}^\ell p_{ht}^w \text{tr}(E_j^* A_\ell E_s^* A_w) \\
&= \sum_{\ell=0}^D \sum_{w=0}^D p_{ir}^\ell p_{ht}^w \text{tr}(E_j^* E_j^* A_\ell E_s^* E_s^* A_w) \\
&= \sum_{\ell=0}^D \sum_{w=0}^D p_{ir}^\ell p_{ht}^w \text{tr}(E_j^* A_\ell E_s^* E_s^* A_w E_j^*) \\
&= \sum_{\ell=0}^D \sum_{w=0}^D p_{ir}^\ell p_{ht}^w \text{tr}((E_s^* A_\ell E_j^*)^t (\overline{E_s^* A_w E_j^*})) \\
&= \sum_{\ell=0}^D \sum_{w=0}^D p_{ir}^\ell p_{ht}^w \langle E_s^* A_\ell E_j^*, E_s^* A_w E_j^* \rangle \\
&= \sum_{\ell=0}^D \sum_{w=0}^D \delta_{\ell w} p_{ir}^\ell p_{ht}^w p_s^j k_j \\
&= \sum_{\ell=0}^D k_\ell p_{ir}^\ell p_{js}^\ell p_{ht}^\ell.
\end{aligned}$$

□

Definition 11. Let G denote the matrix of inner products for

$$A_2 E_i^* A - A E_i^* A_2, A_3 E_i^* - E_i^* A_3, A_2 E_i^* - E_i^* A_2, A E_i^* - E_i^* A,$$

where $1 \leq i \leq D$. Thus the matrix G is $4D \times 4D$.

Theorem 12. *The entries of G are as follows: For $1 \leq i, j \leq D$, where $\phi/2$ is a weighted sum involving the following terms and coefficients:*

$\langle \cdot, \cdot \rangle$	$A_2E_j^*A - AE_j^*A_2$	$A_3E_j^* - E_j^*A_3$	$A_2E_j^* - E_j^*A_2$	$AE_j^* - E_j^*A$
$A_2E_i^*A - AE_i^*A_2$	ϕ	$2k_2b_2(p_{ij}^1 - p_{ij}^2)$	$2k_2a_2(p_{ij}^1 - p_{ij}^2)$	$2k_2c_2(p_{ij}^1 - p_{ij}^2)$
$A_3E_i^* - E_i^*A_3$	$2k_2b_2(p_{ij}^1 - p_{ij}^2)$	$2k_3(\delta_{ij}k_i - p_{ij}^3)$	0	0
$A_2E_i^* - E_i^*A_2$	$2k_2a_2(p_{ij}^1 - p_{ij}^2)$	0	$2k_2(\delta_{ij}k_i - p_{ij}^2)$	0
$AE_i^* - E_i^*A$	$2k_2c_2(p_{ij}^1 - p_{ij}^2)$	0	0	$2k(\delta_{ij}k_i - p_{ij}^1)$

term	coefficient
p_{ij}^0	kk_2
p_{ij}^1	$k_2a_1a_2 - kb_1^2$
p_{ij}^2	$k_2(c_2(b_1 - 1) - a_2(a_1 + 1) + b_2(c_3 - 1))$
p_{ij}^3	$-k_3c_3^2$

Proof. By Lemma 10 and using (1)–(5), we obtain

$$\begin{aligned}
& \langle A_2E_i^*A - AE_i^*A_2, A_2E_j^*A - AE_j^*A_2 \rangle \\
&= \langle A_2E_i^*A, A_2E_j^*A \rangle - \langle A_2E_i^*A, AE_j^*A_2 \rangle - \langle AE_i^*A_2, A_2E_j^*A \rangle + \langle AE_i^*A_2, AE_j^*A_2 \rangle \\
&= \sum_{\alpha=0}^D k_\alpha p_{22}^\alpha p_{ij}^\alpha p_{11}^\alpha - \sum_{\beta=0}^D k_\beta p_{21}^\beta p_{ij}^\beta p_{12}^\beta - \sum_{\gamma=0}^D k_\gamma p_{12}^\gamma p_{ij}^\gamma p_{21}^\gamma + \sum_{\eta=0}^D k_\eta p_{11}^\eta p_{ij}^\eta p_{22}^\eta \\
&= 2 \left(\sum_{\alpha=0}^2 k_\alpha p_{22}^\alpha p_{ij}^\alpha p_{11}^\alpha - \sum_{\beta=1}^3 k_\beta (p_{12}^\beta)^2 p_{ij}^\beta \right) \\
&= 2(k_0 p_{22}^0 p_{ij}^0 p_{11}^0 + k_1 p_{22}^1 p_{ij}^1 p_{11}^1 + k_2 p_{22}^2 p_{ij}^2 p_{11}^2 - k_1 (p_{12}^1)^2 p_{ij}^1 - k_2 (p_{12}^2)^2 p_{ij}^2 - k_3 (p_{12}^3)^2 p_{ij}^3) \\
&= 2(kk_2 p_{ij}^0 + (k_2 a_1 a_2 - kb_1^2) p_{ij}^1 + k_2(c_2(b_1 - 1) - a_2(a_1 + 1) + b_2(c_3 - 1)) p_{ij}^2 \\
&\quad - k_3 c_3^2 p_{ij}^3). \tag{14}
\end{aligned}$$

Similarly, for $1 \leq h \leq 3$,

$$\begin{aligned}
& \langle A_h E_i^* - E_i^* A_h, A_2 E_j^* A - AE_j^* A_2 \rangle \\
&= \langle A_h E_i^*, A_2 E_j^* A \rangle - \langle A_h E_i^*, AE_j^* A_2 \rangle - \langle E_i^* A_h, A_2 E_j^* A \rangle + \langle E_i^* A_h, AE_j^* A_2 \rangle \\
&= \langle A_h E_i^* A_0, A_2 E_j^* A \rangle - \langle A_h E_i^* A_0, AE_j^* A_2 \rangle - \langle A_0 E_i^* A_h, A_2 E_j^* A \rangle \\
&\quad + \langle A_0 E_i^* A_h, AE_j^* A_2 \rangle \\
&= \sum_{\alpha=0}^D k_\alpha p_{h2}^\alpha p_{ij}^\alpha p_{01}^\alpha - \sum_{\beta=0}^D k_\beta p_{h1}^\beta p_{ij}^\beta p_{02}^\beta - \sum_{\gamma=0}^D k_\gamma p_{02}^\gamma p_{ij}^\gamma p_{h1}^\gamma + \sum_{\eta=0}^D k_\eta p_{01}^\eta p_{ij}^\eta p_{h2}^\eta \\
&= 2(kp_{h2}^1 p_{ij}^1 - k_2 p_{h1}^2 p_{ij}^2) \\
&= 2(k_2 p_{1h}^2 p_{ij}^1 - k_2 p_{1h}^2 p_{ij}^2) \\
&= 2k_2 p_{1h}^2 (p_{ij}^1 - p_{ij}^2). \tag{15}
\end{aligned}$$

Similarly, for $1 \leq h, \ell \leq 3$,

$$\begin{aligned}
& \langle A_h E_i^* - E_i^* A_h, A_\ell E_j^* - E_j^* A_\ell \rangle \\
&= \langle A_h E_i^*, A_\ell E_j^* \rangle - \langle A_h E_i^*, E_j^* A_\ell \rangle - \langle E_i^* A_h, A_\ell E_j^* \rangle + \langle E_i^* A_h, E_j^* A_\ell \rangle \\
&= \langle A_h E_i^* A_0, A_\ell E_j^* A_0 \rangle - \langle A_h E_i^* A_0, A_0 E_j^* A_\ell \rangle - \langle A_0 E_i^* A_h, A_\ell E_j^* A_0 \rangle \\
&\quad + \langle A_0 E_i^* A_h, A_0 E_j^* A_\ell \rangle \\
&= \sum_{\alpha=0}^D k_\alpha p_{h\ell}^\alpha p_{ij}^\alpha p_{00}^\alpha - \sum_{\beta=0}^D k_\beta p_{h0}^\beta p_{ij}^\beta p_{0\ell}^\beta - \sum_{\gamma=0}^D k_\gamma p_{0\ell}^\gamma p_{ij}^\gamma p_{h0}^\gamma + \sum_{\eta=0}^D k_\eta p_{00}^\eta p_{ij}^\eta p_{h\ell}^\eta \\
&= 2(k_0 p_{h\ell}^0 p_{ij}^0 - \delta_{h\ell} k_h p_{ij}^h) \\
&= 2(\delta_{h\ell} \delta_{ij} k_h k_i - \delta_{h\ell} k_h p_{ij}^h) \\
&= 2\delta_{h\ell} k_h (\delta_{ij} k_i - p_{ij}^h). \tag{16}
\end{aligned}$$

The result follows. \square

In Appendix 2, we give the matrix G for $D = 3$.

Definition 13. For $1 \leq i \leq D$ let B_i denote the matrix of inner products for

$$A_2 A_i^* A - A A_i^* A_2, A_3 A_i^* - A_i^* A_3, A_2 A_i^* - A_i^* A_2, A A_i^* - A_i^* A.$$

So the matrix B_i is 4×4 .

Definition 14. Let \tilde{G} denote the matrix of inner products for

$$A_2 A_i^* A - A A_i^* A_2, A_3 A_i^* - A_i^* A_3, A_2 A_i^* - A_i^* A_2, A A_i^* - A_i^* A,$$

where $1 \leq i \leq D$. Thus the matrix \tilde{G} is $4D \times 4D$.

Lemma 15. *The matrix \tilde{G} has the form*

$$\tilde{G} = \begin{bmatrix} B_1 & & & \mathbf{0} \\ & B_2 & & \\ & & \ddots & \\ \mathbf{0} & & & B_D \end{bmatrix},$$

where B_1, B_2, \dots, B_D are from Definition 13.

Proof. Recall that $A_0^*, A_1^*, \dots, A_D^*$ form a basis for M^* . Therefore the sum $MM^*M = \sum_{i=0}^D M A_i^* M$ is direct. The summands are mutually orthogonal by Lemma 8(ii). The result follows. \square

Lemma 16. $\det(\tilde{G}) = \prod_{i=1}^D \det(B_i)$.

Proof. Immediate from Lemma 15. \square

5 The main result

In this section we obtain our main result, which is Theorem 18.

Lemma 17. *The following (i)–(iii) hold.*

$$(i) \quad \tilde{G} = (S')^t G S'.$$

$$(ii) \quad \det(G) = (\det(S'))^{-2} \det(\tilde{G}).$$

$$(iii) \quad \det(G) = (\det(S^{alt}))^{-8} \prod_{i=1}^D \det(B_i).$$

Proof. (i) Immediate from Definition 11, Definition 14, and Corollary 7.

(ii) Follows from (i).

(iii) Follows from (ii) and Lemmas 6, 16. \square

Theorem 18. *Let Γ denote a primitive distance-regular graph with diameter $D \geq 3$. Then Γ is Q -polynomial if and only if $\det(G) = 0$.*

Proof. To prove the theorem in one direction, assume that Γ is Q -polynomial with respect to the ordering E_0, E_1, \dots, E_D . Write $A^* = A_1^*$. By [10, Theorem 3.3] and Lemma 1, we obtain $A^*A_3 - A_3A^* \in \text{Span}\{AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^*A - AA^*\}$. Thus $AA^*A_2 - A_2A^*A, A^*A_3 - A_3A^*, A^*A_2 - A_2A^*, A^*A - AA^*$ are linearly dependent. Now the matrix B_1 from Definition 13 has determinant zero. Now $\det(G) = 0$ by Lemma 17(iii).

For the other direction, assume $\det(G) = 0$. By Lemma 17(iii) and since S^{alt} is invertible, there exists an integer t ($1 \leq t \leq D$) such that $\det(B_t) = 0$. Now $AA_t^*A_2 - A_2A_t^*A, A_t^*A_3 - A_3A_t^*, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*$ are linearly dependent. We will show that $A_t^*A_3 - A_3A_t^* \in \text{Span}\{AA_t^*A_2 - A_2A_t^*A, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*\}$. To do this we show that $AA_t^*A_2 - A_2A_t^*A, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*$ are linearly independent. Suppose not. Then there exist scalars a, b, c , not all zero, such that

$$a(AA_t^*A_2 - A_2A_t^*A) + b(A_t^*A_2 - A_2A_t^*) + c(A_t^*A - AA_t^*) = 0. \quad (17)$$

Abbreviate $\theta_i^* = m_t u_i(\theta_t)$ ($0 \leq i \leq D$). So $A_t^* = \sum_{i=0}^D \theta_i^* E_i^*$. By Lemma 1,

$$\theta_i^* \neq \theta_0^* \quad (1 \leq i \leq D). \quad (18)$$

For $1 \leq h \leq 3$ pick $z \in X$ such that $\partial(x, z) = h$. Compute the (x, z) -entry in (17). For $h = 3$ we get $ac_3(\theta_1^* - \theta_2^*) = 0$. For $h = 2$ we get $aa_2(\theta_1^* - \theta_2^*) + b(\theta_0^* - \theta_2^*) = 0$. For $h = 1$ we get $ab_1(\theta_1^* - \theta_2^*) + c(\theta_0^* - \theta_1^*) = 0$. Solving these equations we obtain $a(\theta_1^* - \theta_2^*) = 0$ and $b = 0, c = 0$. Recall that a, b, c are not all zero, so $a \neq 0$ and $\theta_1^* = \theta_2^*$. Now (17) becomes $AA_t^*A_2 - A_2A_t^*A = 0$. Recall $c_2A_2 = A^2 - a_1A - kI$. We have $AA_t^*A^2 + kA_t^*A = A^2A_t^*A + kAA_t^*$. Thus $[A, AA_t^*A + kA_t^*] = 0$. For $0 \leq i, j \leq D$

such that $i \neq j$ we have $E_i A_t^* E_j (\theta_i \theta_j + k) = 0$. By Corollary 9, $E_i A_t^* E_j \neq 0$ if and only if $q_{ij}^t \neq 0$, and in this case $\theta_i \theta_j + k = 0$. Since $q_{0t}^t = 1$ and $\theta_0 = k$, we have $k\theta_t + k = 0$ and hence $\theta_t = -1$. Define a diagram with nodes $0, 1, \dots, D$. There exists an arc between nodes i, j if and only if $i \neq j$ and $q_{ij}^t \neq 0$. Observe that node 0 is connected to node t and no other nodes. By [2, Proposition 2.11.1] and Lemma 1, the diagram is connected. Thus there exists a node s with $s \neq 0$ and $s \neq t$ that is connected to node t by an arc. In other words $q_{st}^t \neq 0$. So $\theta_s \theta_t + k = 0$ and hence $\theta_s = k$, a contradiction. Therefore $AA_t^* A_2 - A_2 A_t^* A, A_t^* A_2 - A_2 A_t^*, A_t^* A - AA_t^*$ are linearly independent. So $A_t^* A_3 - A_3 A_t^* \in \text{Span}\{AA_t^* A_2 - A_2 A_t^* A, A_t^* A_2 - A_2 A_t^*, A_t^* A - AA_t^*\}$. Now by [10, Theorem 3.3] and (18), Γ is a Q -polynomial with respect to $E = E_t$. \square

6 Appendix 1

Recall the distance-regular graph Γ with diameter D . Recall for $0 \leq h \leq D$

$$\begin{aligned} p_{1,h-1}^h &= c_h, & p_{1h}^h &= a_h, & p_{1,h+1}^h &= b_h, \\ p_{h,h-1}^1 &= \frac{k_h c_h}{k}, & p_{hh}^1 &= \frac{k_h a_h}{k}, & p_{h,h+1}^1 &= \frac{k_h b_h}{k}. \end{aligned}$$

We now give p_{2j}^h for $h-2 \leq j \leq h+2$.

$$\begin{aligned} p_{2,h-2}^h &= \frac{c_{h-1} c_h}{c_2}, \\ p_{2,h-1}^h &= \frac{c_h (a_{h-1} + a_h - a_1)}{c_2}, \\ p_{2h}^h &= \frac{c_h (b_{h-1} - 1) + a_h (a_h - a_1 - 1) + b_h (c_{h+1} - 1)}{c_2}, \\ p_{2,h+1}^h &= \frac{b_h (a_{h+1} + a_h - a_1)}{c_2}, \\ p_{2,h+2}^h &= \frac{b_h b_{h+1}}{c_2}. \end{aligned}$$

We now give p_{3j}^h for $h-3 \leq j \leq h+3$.

$$\begin{aligned} p_{3,h-3}^h &= \frac{c_{h-2} c_{h-1} c_h}{c_2 c_3}, \\ p_{3,h-2}^h &= \frac{(a_h - a_2) c_{h-1} c_h}{c_2 c_3} + \frac{c_{h-1} c_h (a_{h-2} + a_{h-1} - a_1)}{c_2 c_3}, \\ p_{3,h-1}^h &= \frac{c_{h-1} c_h (b_{h-2} - 1) + c_h a_{h-1} (a_{h-1} - a_1 - 1) + c_h b_{h-1} (c_h - 1)}{c_2 c_3} \\ &\quad + \frac{c_h (a_h - a_2) (a_{h-1} + a_h - a_1)}{c_2 c_3} + \frac{b_h c_h c_{h+1}}{c_2 c_3} - \frac{b_1 c_h}{c_3}, \end{aligned}$$

$$\begin{aligned}
p_{3h}^h &= \frac{c_h b_{h-1} (a_h + a_{h-1} - a_1)}{c_2 c_3} \\
&\quad + \frac{(a_h - a_2)(c_h (b_{h-1} - 1) + a_h (a_h - a_1 - 1) + b_h (c_{h+1} - 1))}{c_2 c_3} \\
&\quad + \frac{b_h c_{h+1} (a_h + a_{h+1} - a_1)}{c_2 c_3} - \frac{b_1 a_h}{c_3}, \\
p_{3,h+1}^h &= \frac{c_h b_{h-1} b_h}{c_2 c_3} + \frac{b_h (a_h - a_2)(a_{h+1} + a_h - a_1)}{c_2 c_3} \\
&\quad + \frac{b_h (c_{h+1} (b_h - 1) + a_{h+1} (a_{h+1} - a_1 - 1) + b_{h+1} (c_{h+2} - 1))}{c_2 c_3} - \frac{b_1 b_h}{c_3}, \\
p_{3,h+2}^h &= \frac{(a_h - a_2) b_h b_{h+1}}{c_2 c_3} + \frac{b_h b_{h+1} (a_{h+2} + a_{h+1} - a_1)}{c_2 c_3}, \\
p_{3,h+3}^h &= \frac{b_h b_{h+1} b_{h+2}}{c_2 c_3}.
\end{aligned}$$

7 Appendix 2

Recall the matrix G from Theorem 12. In this appendix we give G for $D = 3$.

Example 19. Assume $D = 3$. The rows and columns of G are indexed by the following matrices, in the specified order:

block 1	$A_3 E_1^* - E_1^* A_3$	$A_3 E_2^* - E_2^* A_3$	$A_3 E_3^* - E_3^* A_3$
block 2	$A_2 E_1^* - E_1^* A_2$	$A_2 E_2^* - E_2^* A_2$	$A_2 E_3^* - E_3^* A_2$
block 3	$A E_1^* - E_1^* A$	$A E_2^* - E_2^* A$	$A E_3^* - E_3^* A$
block 4	$A_2 E_1^* A - A E_1^* A_2$	$A_2 E_2^* A - A E_2^* A_2$	$A_2 E_3^* A - A E_3^* A_2$

So the matrix G is 12×12 . G has the form

$$G = \begin{bmatrix} \mathbb{X} & 0 & 0 & \mathbb{S} \\ 0 & \mathbb{Y} & 0 & \mathbb{T} \\ 0 & 0 & \mathbb{Z} & \mathbb{U} \\ \mathbb{S} & \mathbb{T} & \mathbb{U} & \mathbb{W} \end{bmatrix},$$

where each block is a 3×3 symmetric matrix as shown below.

$$\begin{aligned}
\mathbb{X} &= \begin{bmatrix} 2k_3 k & -2k_3 c_3 & -2k_3 a_3 \\ -2k_3 c_3 & 2k_3 (k_2 - p_{22}^3) & -2k_3 p_{23}^3 \\ -2k_3 a_3 & -2k_3 p_{23}^3 & 2k_3 (k_3 - p_{33}^3) \end{bmatrix}, \\
\mathbb{Y} &= \begin{bmatrix} 2k_2 (k - c_2) & -2k_2 a_2 & -2k_2 b_2 \\ -2k_2 a_2 & 2k_2 (k_2 - p_{22}^2) & -2k_2 p_{23}^2 \\ -2k_2 b_2 & -2k_2 p_{23}^2 & 2k_2 (k_3 - p_{33}^2) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned} \mathbb{Z} &= \begin{bmatrix} 2k(k - a_1) & -2kb_1 & 0 \\ -2kb_1 & 2k(k_2 - p_{22}^1) & -2kp_{23}^1 \\ 0 & -2kp_{23}^1 & 2k(k_3 - p_{33}^1) \end{bmatrix}, \\ \mathbb{S} &= \begin{bmatrix} 2k_2b_2(a_1 - c_2) & 2k_2b_2(b_1 - a_2) & -2k_2b_2^2 \\ 2k_2b_2(b_1 - a_2) & 2k_2b_2(p_{22}^1 - p_{22}^2) & 2k_2b_2(p_{23}^1 - p_{23}^2) \\ -2k_2b_2^2 & 2k_2b_2(p_{23}^1 - p_{23}^2) & 2k_2b_2(p_{33}^1 - p_{33}^2) \end{bmatrix}, \\ \mathbb{T} &= \begin{bmatrix} 2k_2a_2(a_1 - c_2) & 2k_2a_2(b_1 - a_2) & -2k_2a_2b_2 \\ 2k_2a_2(b_1 - a_2) & 2k_2a_2(p_{22}^1 - p_{22}^2) & 2k_2a_2(p_{23}^1 - p_{23}^2) \\ -2k_2a_2b_2 & 2k_2a_2(p_{23}^1 - p_{23}^2) & 2k_2a_2(p_{33}^1 - p_{33}^2) \end{bmatrix}, \\ \mathbb{U} &= \begin{bmatrix} 2k_2c_2(a_1 - c_2) & 2k_2c_2(b_1 - a_2) & -2k_2c_2b_2 \\ 2k_2c_2(b_1 - a_2) & 2k_2c_2(p_{22}^1 - p_{22}^2) & 2k_2c_2(p_{23}^1 - p_{23}^2) \\ -2k_2c_2b_2 & 2k_2c_2(p_{23}^1 - p_{23}^2) & 2k_2c_2(p_{33}^1 - p_{33}^2) \end{bmatrix}. \end{aligned}$$

The matrix \mathbb{W} is symmetric with entries

$$\begin{aligned} \mathbb{W}_{11} &= 2(k^2k_2 + (k_2a_1a_2 - kb_1^2)a_1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\ &\quad + b_2(c_3 - 1)) - k_2a_2^2)c_2), \\ \mathbb{W}_{12} &= 2((k_2a_1a_2 - kb_1^2)b_1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) \\ &\quad - k_2a_2^2)a_2 - k_3c_3^3), \\ \mathbb{W}_{13} &= 2((k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) - k_2a_2^2)b_2 - k_3c_3^2a_3), \\ \mathbb{W}_{22} &= 2(kk_2^2 + (k_2a_1a_2 - kb_1^2)p_{22}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\ &\quad + b_2(c_3 - 1)) - k_2a_2^2)p_{22}^2 - k_3c_3^2p_{22}^3), \\ \mathbb{W}_{23} &= 2((k_2a_1a_2 - kb_1^2)p_{23}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) \\ &\quad - k_2a_2^2)p_{23}^2 - k_3c_3^2p_{23}^3), \\ \mathbb{W}_{33} &= 2(kk_2k_3 + (k_2a_1a_2 - kb_1^2)p_{33}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\ &\quad + b_2(c_3 - 1)) - k_2a_2^2)p_{33}^2 - k_3c_3^2p_{33}^3). \end{aligned}$$

From Appendix 1, we find

$$\begin{aligned} p_{22}^1 &= \frac{k_2a_2}{k}, \quad p_{22}^2 = \frac{c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)}{c_2}, \quad p_{22}^3 = \frac{c_3(a_2 + a_3 - a_1)}{c_2}, \\ p_{23}^1 &= \frac{k_2b_2}{k}, \quad p_{23}^2 = \frac{b_2(a_3 + a_2 - a_1)}{c_2}, \quad p_{23}^3 = \frac{c_3(b_2 - 1) + a_3(a_3 - a_1 - 1) - b_3}{c_2}, \\ p_{33}^1 &= \frac{k_3a_3}{k}, \quad p_{33}^2 = \frac{b_2(c_3(b_2 - 1) + a_3(a_3 - a_1 - 1) - b_3)}{c_2c_3}, \\ p_{33}^3 &= \frac{c_3b_2(a_3 + a_2 - a_1)}{c_2c_3} + \frac{(a_3 - a_2)(c_3(b_2 - 1) + a_3(a_3 - a_1 - 1) - b_3)}{c_2c_3} - \frac{b_1a_3}{c_3}. \end{aligned}$$

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References

- [1] E. Bannai, T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1984.
- [2] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.
- [3] E. Hanson, *A characterization of Leonard pairs using the parameters $\{a_i\}_{i=0}^d$* , Linear Algebra Appl. 438 (2013), 2289–2305.
- [4] T. Ito, K. Tanabe, and P. Terwilliger, *Some algebra related to P - and Q -polynomial association schemes*, In Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp.167–192; arXiv:math.CO/0406556.
- [5] A. Jurišić, P. Terwilliger, and A. Žitnik, *The Q -polynomial idempotents of a distance-regular graph*, J. Combin. Theory Ser. B 100 (2010), 683–690.
- [6] H. Kurihara and H. Nozaki, *A characterization of Q -polynomial association schemes*, J. Combin. Theory Ser. A 119 (2012), 57–62.
- [7] K. Nomura and P. Terwilliger, *Tridiagonal matrices with nonnegative entries*, Linear Algebra Appl. 434 (2011), 2527–2538.
- [8] A.A. Pascasio, *A characterization of Q -polynomial distance-regular graphs*, Discrete Math. 308 (2008), 3090–3096.
- [9] P. Terwilliger, *A characterization of P - and Q -polynomial association schemes*, J. Combin. Theory Ser. A 45 (1987), 8–26.
- [10] P. Terwilliger, *A new inequality for distance-regular graphs*, Discrete Math. 137 (1995), 319–332.
- [11] P. Terwilliger, *The subconstituent algebra of an association scheme I*, J. Algebraic Combin. 1 (1992), 363–388.