Inference on Weibull Parameters Under a Balanced Two Sample Type-II Progressive Censoring Scheme

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Abstract

The progressive censoring scheme has received considerable amount of attention in the last fifteen years. During the last few years joint progressive censoring scheme has gained some popularity. Recently, the authors Mondal and Kundu ("A new two sample Type-II progressive censoring scheme", arXiv:1609.05805) introduced a balanced two sample Type-II progressive censoring scheme and provided the exact inference when the two populations are exponentially distributed. In this article we consider the case when the two populations follow Weibull distributions with the common shape parameter and different scale parameters. We obtain the maximum likelihood estimators of the unknown parameters. It is observed that the maximum likelihood estimators cannot be obtained in explicit forms, hence, we propose approximate maximum likelihood estimators, which can be obtained in explicit forms. We construct the asymptotic and bootstrap confidence intervals of the population parameters. Further we derive an exact joint confidence region of the unknown parameters. We propose an objective function based on the expected volume of this confidence set and using that we obtain the optimum progressive censoring scheme. Extensive simulations have been performed to see the performances of the proposed method, and one real data set has been analyzed for illustrative purposes.

Key Words and Phrases: Type-II censoring; progressive censoring; joint progressive censoring; maximum likelihood estimator; approximate maximum likelihood estimator; joint confidence region; optimum censoring scheme.

AMS Subject Classifications: 62N01, 62N02, 62F10.

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1 Introduction

In any life testing experiment censoring in inevitable. Different censoring schemes have been introduced in the literature to optimize time, cost and efficiency. Among different censoring schemes, Type-I and Type-II are the two most popular censoring schemes. But none of these censoring schemes allows removal of units during life testing experiment. Progressive censoring scheme incorporates this flexibility in a life testing experiment. Progressive Type-II censoring scheme allows removal of experimental units during the experiment as well as ensures a certain number of failure to be observed during the experiment to make it efficient. Extensive work had been done on the different aspects of the progressive censoring since the introduction of the book by Balakrishnan and Aggarwala [2]. A comprehensive collection of different work related to progressive censoring scheme can be found in a recent book by Balakrishnan and Cramer [3].

But all these development are mainly based on a single population. Recently two sample joint censoring schemes are becoming popular for a life testing experiment mainly to optimize time and cost. In a Type-II joint censoring scheme two samples are put on a life testing experiment simultaneously and the experiment is continued until a certain number of failures are observed. Balakrishnan and Rasouli [7] first considered the likelihood inference for two exponential populations under joint a Type-II censoring scheme. Ashour and Eraki [1] extended the results for multiple populations and when the lifetime of different populations follow Weibull distributions.

Recently, Rasouli and Balakrishnan [20] introduced a joint progressive Type-II censoring (JPC) scheme and provided the exact likelihood inference for two exponential populations under this censoring scheme. Parsi and Ganjali [16] extended the results of Rasouli and Balakrishnan [20] for two Weibull populations. Doostparast and Ahmadi et al. [12] provided the Bayesian inference of the unknown parameters based on the data obtained from a JPC scheme under LINEX loss function. Balakrishnan and Su et al. [8] extended the JPC model to general K populations and studied exact likelihood inference of the unknown parameters for exponential distributions.

Mondal and Kundu [13] recently introduced a balanced joint progressive Type-II censoring

(BJPC) scheme and it is observed that it has certain advantages over the JPC scheme originally introduced by Rasouli and Balakrishnan [20]. The scheme proposed by Mondal and Kundu [13] has a close connection with the self relocating design proposed by Srivastava [21]. Mondal and Kundu [13] provided the exact likelihood inference for the two exponential populations under a BJPC scheme. The main aim of this paper is to study likelihood inference of two Weibull populations under this new scheme. We provide the maximum likelihood estimators (MLEs) of the unknown parameters, and it is observed that the MLEs of the unknown parameters cannot be obtained in explicit form. Due to this reason we propose to use approximate maximum likelihood estimators (AMLEs) of the unknown parameters, which can be obtained in explicit forms. We propose to use the asymptotic distribution of the MLEs and bootstrap method to construct confidence intervals (CI) of the unknown parameters. We have also provided an exact joint confidence region of the parameter set. Further, we propose an objective function based on the expected volume of this confidence set and this has been used to find the optimum censoring scheme (OCS). Extensive simulations have been performed to see the effectiveness of the different methods, and one real data set has been analyzed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2 we briefly describe the model and provide necessary assumptions. The MLEs and AMLEs are derived in Section 3. In Section 4 we provide the joint confidence region of the unknown parameters. Next we propose the objective function in Section 5. In Section 6 we provide the simulation results and the analysis of a real data set. Finally we conclude the paper in Section 7.

2 Model Description and Model Assumption

The balanced joint Type-II progressive censoring scheme proposed by Mondal and Kundu [13] can be briefly described as follows. Suppose there are two lines of similar products and it is important to study the relative merits of these two products. A sample of size m is drawn from one product line (say A) and another sample of size m is drawn from the other product line (say B). Let k be the total number of failures to be observed from the life testing experiment and $R_1,\dots,R_{k-1}$

are pre-specified non-negative integers satisfying $\sum_{i=1}^{k-1}(R_i+1) < m$. Under the BJPC scheme, two sets of samples from these two products are simultaneously put on a test. Suppose the first failure is coming from the product line A and the first failure time is denoted by W_1, then at W_1, R_1 units are removed randomly from the remaining $m-1$ surviving units of the product line A as well as R_1+1 units are chosen randomly from the remaining m surviving units of product line B and they are removed from the experiment. Next, if the second failure is coming from the product line B at time point W_2, R_2+1 units are withdrawn from the remaining $m-R_1-1$ units from the product line A and R_2 units withdrawn from the remaining $m-R_1-2$ units from the product line B randomly at W_2. The test is continued until k failures are observed with removal of all the remaining surviving units from both the product lines at the $k-$th failure. In this life testing experiment a new set of random variable $Z_1,\ldots,Z_k$ is introduced where $Z_i=1$ or 0 if ith failure comes from the product line A or B respectively. Under the BJPC, the data consists of $(\mathbf{W},\mathbf{Z})$ where $\mathbf{W}=(W_1,\ldots W_k)$ and $\mathbf{Z}=(Z_1,\ldots,Z_k)$. A schematic diagram of the BJPC is provided in Figure 1 and Figure 2.

A random variable X is said to follow Weibull distribution with the shape parameter $\alpha>0$ and the scale parameter $\lambda>0$ if it has the following probability density function (PDF)

$$f(x;\alpha,\lambda)=\begin{cases}\alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}} & \text{if} \quad x>0\\ 0 & \text{if} \quad x\leq 0,\end{cases} \tag{1}$$

and it will be denoted by $\mathrm{WE}(\alpha,\lambda)$. We assume the lifetimes of m units of product line A, say $X_1,\ldots,X_m$, are independent identically distributed (i.i.d) random variables from $\mathrm{WE}(\alpha,\lambda_1)$ and the lifetimes of m units of product line B, say $Y_1,\ldots,Y_m$ are i.i.d random variables from $\mathrm{WE}(\alpha,\lambda_2)$.

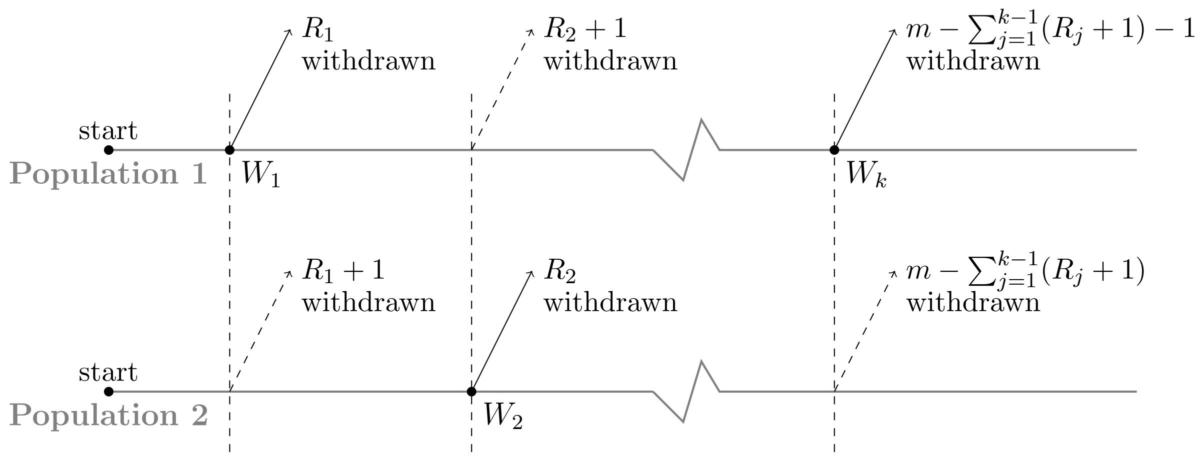

Figure 1: kth failure comes from Population 1

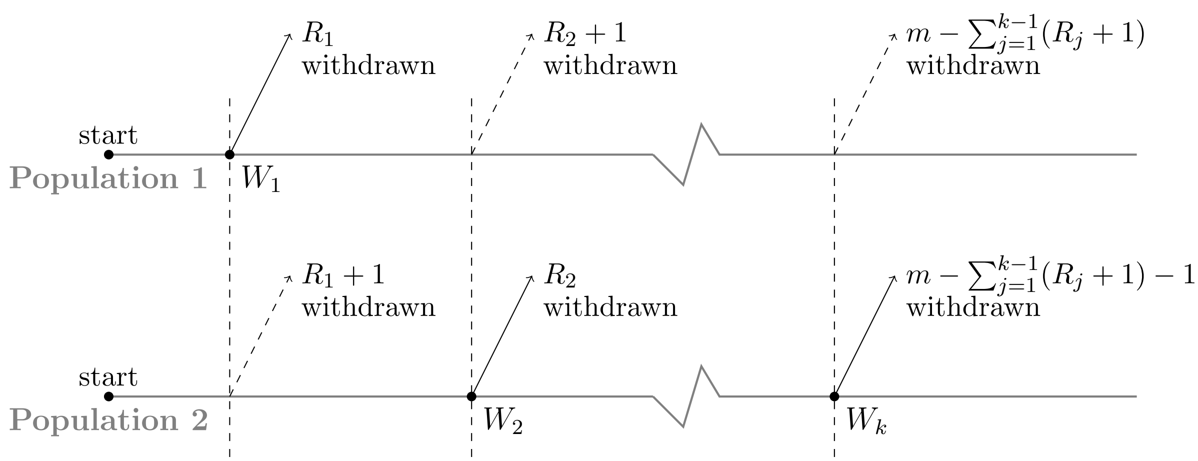

Figure 2: kth failure comes from Population 2

3 Point Estimations

3.1 Maximum Likelihood Estimators (MLEs)

The likelihood function of the unknown parameters $(\alpha, \lambda_1, \lambda_2)$ based on the observed data $(\mathbf{W}, \mathbf{Z})$, is given by

$$L(\alpha, \lambda_1, \lambda_2|\mathbf{w}, \mathbf{z}) = C\alpha^k {\lambda_1}^{k_1} {\lambda_2}^{k_2} \prod_{i=1}^{k} {w_i}^{\alpha-1} e^{-(\lambda_1+\lambda_2)A(\alpha)} \tag{2}$$

where

$$\begin{aligned} A(\alpha) &= \sum_{i=1}^{k} c_i w_i^{\alpha}, \quad c_i = R_i + 1; i = 1, \ldots, k-1, \\ c_k &= m - \sum_{i=1}^{k-1}(R_i + 1), \quad k_1 = \sum_{i=1}^{k} z_i, \quad k_2 = \sum_{i=1}^{k}(1 - z_i) = k - k_1; \\ C &= \prod_{i=1}^{k}(m - \sum_{j=1}^{i-1}(R_j + 1)). \end{aligned}$$

The log-likelihood function without the normalizing constant is given by

$$l(\alpha, \lambda_1, \lambda_2|\mathbf{w}, \mathbf{z}) = k\ln(\alpha) + k_1\ln(\lambda_1) + k_2\ln(\lambda_2) - (\lambda_1 + \lambda_2)A(\alpha) + (\alpha - 1)\sum_{i=1}^{k}\ln(w_i). \tag{3}$$

Hence, the normal equations can be obtained by taking partial derivatives of the log-likelihood function (3) and equating them to zero as given below

$$\frac{\partial l}{\partial \lambda_1} = \frac{k_1}{\lambda_1} - \sum_{i=1}^{k} c_i {w_i}^{\alpha} = 0, \tag{4}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{k_2}{\lambda_2} - \sum_{i=1}^{k} c_i {w_i}^{\alpha} = 0, \tag{5}$$

$$\frac{\partial l}{\partial \alpha} = \frac{k}{\alpha} - (\lambda_1 + \lambda_2)\sum_{i=1}^{k} c_i \ln(w_i){w_i}^{\alpha} + \sum_{i=1}^{k}\ln w_i = 0. \tag{6}$$

For a given α, when $k_1 > 0$ and $k_2 > 0$ the MLEs of λ_1 and λ_2 can be obtained from (4) and (5) as follows:

$$\widehat{\lambda}_1(\alpha) = \frac{k_1}{A(\alpha)} \quad \text{and} \quad \widehat{\lambda}_2(\alpha) = \frac{k_2}{A(\alpha)}.$$

When α is also unknown, it is possible to obtain the MLE of α from (6) by substituting λ_1 and λ_2 with $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$, respectively. Alternatively, the MLE of α can be obtained by maximizing the profile log-likelihood function of α, $l(\alpha, \widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha)) = P(\alpha)$ (say), where

$$P(\alpha) = k\ln(\alpha) - k\ln A(\alpha) + (\alpha - 1)\sum_{i=1}^{k}\ln(w_i). \tag{7}$$

We need the following result for further development.

LEMMA 1: The function $P(\alpha)$ as defined in (7) attains a unique maximum at some $\alpha^* \in (0, \infty)$ where α^* is the unique solution of

$$\frac{1}{\alpha} - H(\alpha) + \frac{1}{k}\sum_{i=1}^{k}\ln(w_i) = 0, \tag{8}$$

where $H(\alpha) = \dfrac{A'(\alpha)}{A(\alpha)} = \dfrac{\sum_{i=1}^{k} c_i \ln(w_i) w_i^{\alpha}}{\sum_{i=1}^{k} c_i w_i^{\alpha}}$.

PROOF: See in the Appendix. ∎

Once the unique MLE of α, say $\widehat{\alpha}_{MLE}$, is obtained as a solution of (8), then the MLEs of λ_1 and λ_2 also can be obtained uniquely as $\widehat{\lambda}_{1_{MLE}} = \widehat{\lambda}_1(\widehat{\alpha}_{MLE})$ and $\widehat{\lambda}_{2_{MLE}} = \widehat{\lambda}_2(\widehat{\alpha}_{MLE})$, respectively, provided $k_1 > 0$ and $k_2 > 0$.

3.2 Approximate Maximum Likelihood Estimators

Since the MLEs cannot be obtained in explicit forms, we propose to use approximate MLEs (AMLEs) of the unknown parameters which can be obtained in explicit forms. They are obtained by expanding the normal equations using Taylor series expansion of first order. It can be easily seen that for $i = 1, 2, \ldots, k$, the distribution of $(\lambda_1 + \lambda_2)W_i{}^{\alpha}$ is independent of the parameters α, λ_1, λ_2 (see Lemma 2 in Section 4). Let us define the following random variables:

$$U_i = \ln\left((\lambda_1 + \lambda_2)W_i{}^{\alpha}\right) \quad \text{and} \quad V_i = \ln W_i; \quad i = 1, 2, \ldots, k.$$

Therefore, if $\theta = \ln\,(\lambda_1 + \lambda_2)$, then $U_i = \alpha V_i + \theta$, and the distribution of U_i is free from α, λ_1, λ_2, for $i = 1, 2, \ldots k$.

Now from (4) and (5) using u_i and v_i as defined above, we obtain

$$\sum_{i=1}^{k} c_i e^{u_i} = k, \tag{9}$$

and from (6) we have

$$\sum_{i=1}^{k} c_i v_i e^{u_i} = \frac{k}{\alpha} + \sum_{i=1}^{k} v_i. \tag{10}$$

Using Taylor series expansion of order 1 of e^{u_i}, we obtain the AMLEs of the unknown parameters as follows. The AMLE of α, say $\widehat{\alpha}_{AMLE}$, is the positive root of

$$\alpha^2 \left(E_1 - \frac{D_1 E_2}{D_2} \right) + \alpha \left(E_3 + \frac{D_3 E_2}{D_2} \right) = k, \tag{11}$$

and the AMLEs of λ_1, λ_2 are given by

$$\widehat{\lambda}_{1,AMLE} = \frac{k_1}{A(\widehat{\alpha}_{AMLE})}, \quad \widehat{\lambda}_{2,AMLE} = \frac{k_2}{A(\hat{\alpha}_{AMLE})},$$

respectively. Here,

$$E_1 = \sum_{i=1}^{k} c_i A_i {v_i}^2, \quad E_2 = \sum_{i=1}^{k} c_i A_i v_i, \quad E_3 = \sum_{i=1}^{k} (c_i B_i - 1) v_i,$$

$$D_1 = \sum_{i=1}^{k} c_i A_i v_i, \quad D_2 = \sum_{i=1}^{k} c_i A_i, \quad D_3 = k - \sum_{i=1}^{k} c_i B_i,$$

and for $\xi_i = E(U_i)$,

$$A_i = e^{\xi_i} \quad B_i = e^{\xi_i}(1 - \xi_i); \quad i = 1, 2, \ldots, k.$$

4 Exact Confidence Set

In this section we provide a methodology to construct an exact $100(1-\gamma)\%$ confidence set of α, λ_1 and λ_2. We need the following results for further development.

Lemma 2: Suppose $G_1, G_2, \ldots, G_k$ are independent exponential random variables and

$$E(G_j) = \frac{1}{(\lambda_1 + \lambda_2)(m - \sum_{l=1}^{j-1}(R_l + 1))}; \quad j = 1, 2, \ldots, k.$$

Then $W_i^{\alpha} \stackrel{d}{=} \sum_{j=1}^{i} G_j$ for $i = 1, 2, \ldots, k$, where W_i's are same as defined before, and '$\stackrel{d}{=}$' means equal in distribution.

Proof: See in the Appendix. ∎

Let us use the following transformation:

$$\begin{aligned} S_1 &= m(\lambda_1+\lambda_2){W_1}^{\alpha} \\ S_2 &= (m-(R_1+1))(\lambda_1+\lambda_2)({W_2}^{\alpha}-{W_1}^{\alpha}) \\ &\vdots \\ S_k &= (m-\sum_{i=1}^{k-1}(R_i+1))(\lambda_1+\lambda_2)({W_k}^{\alpha}-{W_{k-1}}^{\alpha}). \end{aligned}$$

From Lemma 2, it is evident that $S_1, S_2, \ldots, S_k$ are independent identically distributed (i.i.d.) exponential random variables with mean one. Let us define

$$U = 2\sum_{i=2}^{k} S_i, \quad V = 2S_1, \quad T_1 = \frac{U}{(k-1)V}, \quad T_2 = U + V.$$

Observe that U and V are independent, $U \sim \chi^2_{2k-2}$, $V \sim \chi^2_2$, $T_1 \sim F_{2k-2,2}$ and $T_2 \sim \chi^2_{(2k)}$. Using Basu's theorem it follows that T_1 and T_2 are independently distributed. Note that

$$T_1 = \frac{U}{(k-1)V} = \frac{\sum_{i=1}^{k} S_i}{S_1(k-1)} - \frac{1}{k-1} = \frac{\sum_{i=1}^{k} c_i W_i^{\alpha}}{(k-1)mW_1^{\alpha}} - \frac{1}{k-1}. \tag{12}$$

$$T_2 = = 2(\lambda_1+\lambda_2)\sum_{i=1}^{k} c_i W_i^{\alpha}. \tag{13}$$

From (12) it is clear that T_1 is a function α, and from now on we denote it by $T_1(\alpha)$. We have the following result.

Lemma 3: Let $0 < w_1 < w_2 < \ldots < w_k$, and

$$t_1(\alpha) = \frac{\sum_{i=1}^{k} c_i w_i^{\alpha}}{(k-1)mw_1^{\alpha}} - \frac{1}{k-1},$$

then $t_1(\alpha)$ is a strictly increasing function in α and $\lim_{\alpha\to 0} t_1(\alpha) = 0$, $\lim_{\alpha\to\infty} t_1(\alpha) = \infty$. Hence, the equation $t_1(\alpha) = t$ has a unique solution for $\alpha > 0$ and for all $t > 0$.

Proof: See Lemma 1 in Wu and Shuo-Jye [23]. ∎

We introduce the following notations. Let $\varphi(\cdot) = t_1^{-1}(\cdot)$ and note that $\varphi(\cdot)$ is an increasing function. $F_{\gamma,\delta_1,\delta_2}$ denotes the upper γ-th quantile of F distribution with degrees of freedom δ_1, δ_2 and $\chi^2_{\gamma,\delta}$ denotes the upper γ-th quantile of χ^2 distribution with degrees of freedom δ.

Theorem 4.1 *(i) A* $100(1-\gamma)\%$ *confidence interval of* α *is given by*

$$\big[\varphi(F_{1-\gamma/2,2k-2,2}),\quad \varphi(F_{\gamma/2,2k-2,2})\big] = B(\gamma) \qquad (say).$$

(ii) For a given α*, a* $100(1-\gamma)\%$ *confidence set of* (λ_1, λ_2) *is given by*

$$\left\{(\lambda_1,\lambda_2);\lambda_1\geq 0,\lambda_2\geq 0,\frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha} < \lambda_1+\lambda_2 < \frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha}\right\} = C(\gamma;\alpha) \qquad (say).$$

Proof:

(i) Since $T_1(\alpha) \sim F_{2k-2,2}$, we have $P(F_{1-\gamma/2,2k-2,2} < T_1(\alpha) < F_{\gamma/2,2k-2,2}) = 1-\gamma$. As $t_1(\alpha)$ is an increasing function of α and $\varphi(t)$ is the unique solution of $t_1(\alpha) = t$ we also have

$$P(\varphi(F_{1-\gamma/2,2k-2,2}) < \alpha < \varphi(F_{\gamma/2,2k-2,2})) = 1-\gamma.$$

(ii) Since $T_2 \sim \chi^2_{2k}$, using (13), we obtain

$$P(\chi^2_{1-\gamma/2,2k} < 2(\lambda_1+\lambda_2)\sum_{i=1}^k c_i W_i^\alpha < \chi^2_{\gamma/2,2k}) = 1-\gamma.$$

Hence,

$$P\left((\lambda_1,\lambda_2);\lambda_1\geq 0,\lambda_2\geq 0,\frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha} < \lambda_1+\lambda_2 < \frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha}\right) = 1-\gamma.$$

∎

Note that $C(\gamma;\alpha)$ is a trapezoid enclosed by four straight lines

$$i)\ \lambda_1 = 0, \quad ii)\ \lambda_2 = 0, \quad iii)\ \lambda_1+\lambda_2 = \frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha} \quad iv)\ \lambda_1+\lambda_2 = \frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^k c_i w_i^\alpha}.$$

Corollary 4.2 *A* $100(1-\gamma)\%$ *joint confidence region of* α*,* λ_1*,* λ_2 *is given by*

$$D(\gamma) = \Big\{(\alpha,\lambda_1,\lambda_2);\alpha\in B(\gamma_1),\ (\lambda_1,\lambda_2)\in C(\gamma_2;\alpha)\Big\},$$

here γ_1 *and* γ_2 *are such that* $1-\gamma = (1-\gamma_1)(1-\gamma_2)$.

5 Optimum Censoring Scheme

Finding an optimum censoring scheme is an important problem in any life testing experiment. In this section we propose a new objective function and based on which we provide an algorithm to find the optimum censoring scheme.

In a progressive censoring scheme for fixed sample size (m) and for fixed effective sample size (k), the *efficiency* of the estimators depends on the censoring scheme $\{R_1, \ldots, R_{k-1}\}$. In practical situation out of all possible set of censoring schemes it is important to find out the optimal censoring scheme (OCS) i.e. the censoring scheme which provides maximum *information* about the unknown parameters. In this case, for fixed m and k, the possible set of censoring schemes consists of R_i's, $i = 1, \ldots, k-1$ such that $\sum_{i=1}^{k-1}(R_i + 1) < m$.

In case of Weibull and other lifetime distributions most of the available criteria to find the optimum censoring scheme, are based on the expected Fisher information matrix, i.e. the asymptotic variance covariance matrix of the MLEs, see for example Ng et al. [14], Pradhan and Kundu [17], Pradhan and Kundu [18], Balakrishnan and Cramer [3] and the references cited therein. In this paper we propose a new objective function based on the volume of the exact confidence set of the unknown parameters, which is more reasonable than the asymptotic variance covariance matrix.

In this case first we will show that it is possible to determine the volume of the exact joint confidence set of α, λ_1 and λ_2. From the Theorem 4.1 the area $Area(C(\gamma_2;\alpha)$ of the trapezoid $C(\gamma_2;\alpha)$ is

$$Area(C(\gamma_2;\alpha) = \frac{(\chi^2_{\gamma_2/2,2k})^2 - (\chi^2_{1-\gamma_2/2,2k})^2}{8A(\alpha)^2}.$$

The volume $V(D(\gamma))$ of the confidence region $D(\gamma)$ as in Corollary 4.2, becomes

$$V(D(\gamma)) = \frac{1}{8}((\chi^2_{\gamma_2/2,2k})^2 - (\chi^2_{1-\gamma_2/2,2k})^2) \int_{B(\gamma_1)} \frac{1}{A(\alpha)^2}\, d\alpha. \qquad (14)$$

Based on (14), we propose the objective function as $E_{data}(V(D(\gamma)))$. Therefore, for fixed m and k if $\mathcal{R}_1 = (R_{1,1}, R_{2,1}, \ldots, R_{k-1,1})$ and $\mathcal{R}_2 = (R_{1,2}, R_{2,2}, \ldots, R_{k-1,2})$ are two censoring plans then $\mathcal{R}_1$ is better than $\mathcal{R}_2$ if $\mathcal{R}_1$ provides smaller $E_{data}(V(D(\gamma)))$ than $\mathcal{R}_2$. The following algorithm can be used to compute $E_{data}(V(D(\gamma)))$, for fixed m, k and $R_1, \ldots, R_{k-1}$.

Algorithm:

- Step1: Given m, k and $R_1, \ldots, R_{k-1}$ generate the data $(\mathbf{W}, \mathbf{Z})$ under BJPC from two Weibull populations, $WE(\alpha, \lambda_1)$ and $WE(\alpha, \lambda_2)$.
- Step2: Compute $V(D(\gamma))$ based on the data, this can be done by various numerical method like trapezoidal rule.
- Step3: Repeat Step 1 to 2 say B times, and take their average which approximates $E_{data}(V(D(\gamma)))$.

6 Simulation Study And Data Analysis

6.1 Simulation Study

In this section we compare the performance of the MLEs and AMLEs based on an extensive simulation experiment. We have taken the sample size $m = 25$ and different effective sample size namely, $k = 15, 20$. For different censoring schemes and for different parameter values we compute the average estimate (AE) and mean square error (MSE) of the MLEs and AMLEs based on 10,000 replications. We have also computed the 90% asymptotic and percentile bootstrap confidence intervals, and we have reported the average length (AL) and the coverage percentages (CP) in each case. Bootstrap confidence intervals are obtained based on 1000 bootstrap samples. The results are reported in Tables 1 to 6. We have used the following notations denoting different censoring scheme. For example, the progressive censoring scheme $R_1 = 2, R_2 = 0, R_3 = 0, R_4 = 0$, has been denoted by $R = (2, 0_{(3)})$.

Some of the points are quite clear from this simulation experiment. It is observed that for fixed m (sample size) as k (effective sample size) increases the biases and MSEs decrease for both MLEs and AMLEs as expected. The performances of the MLEs and AMLEs are very close to each other in all cases considered both in terms of biases and MSEs. Hence we recommend to use AMLEs in this case as they have explicit forms. Now comparing the bootstrap and asymptotic confidence intervals in terms of the average lengths and coverage percentages, it is observed the performance of the bootstrap confidence intervals are not very satisfactory. Most of the times it cannot maintain the nominal level of the coverage percentages. Where as even for small sample sizes the performances of the asymptotic confidence intervals are quite satisfactory. In most of the cases considered the coverage percentages are very close to the nominal level. Hence, we recommend to use asymptotic confidence interval in this case.

We have further studied the relation between $E(V(D(\gamma)))$ and the expected time of test (ETOT) for different censoring schemes and for different parameter values. $E(V(D(\gamma))$ is computed as described in Section 5 based on $B = 50,000$ samples with $\gamma = 0.1$ and $\gamma_1 = \gamma_2$. The ETOT i.e. $E(W_k)$ is computed by Monte-Carlo simulation based on 10,000 samples. The results are reported in Tables 7 to 9 for different parameter values and for different censoring schemes. We have also provided a scatter plot of ETOT vs. $E(V(D(\gamma)))$ for different censoring schemes in Figure 4. It is evident that as the ETOT increases, $E(V(D(\gamma)))$ decreases as expected.

Table 1: $m = 25, n = 25, \alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE		AMLE	
		AE	MSE	AE	MSE
k=15,R=$(7,0_{(13)})$	α	0.550	0.017	0.534	0.015
	λ_1	0.576	0.109	0.568	0.101
	λ_2	1.149	0.279	1.132	0.253
k=15,R=$(0_{(6)},7,0_{(7)})$	α	0.552	0.019	0.547	0.018
	λ_1	0.601	0.143	0.595	0.137
	λ_2	1.198	0.371	1.187	0.352
k=15,R=$(0_{(13)},7)$	α	0.564	0.024	0.559	0.023
	λ_1	0.628	0.184	0.622	0.176
	λ_2	1.248	0.514	1.236	0.491
k=20,R=$(3,0_{(18)})$	α	0.537	0.012	0.529	0.011
	λ_1	0.547	0.056	0.544	0.054
	λ_2	1.079	0.123	1.074	0.118
k=20,R=$(0_{(9)},3,0_{(9)})$	α	0.539	0.013	0.534	0.012
	λ_1	0.548	0.062	0.546	0.061
	λ_2	1.097	0.147	1.093	0.143
k=20,R=$(0_{(18)},3)$	α	0.538	0.012	0.529	0.011
	λ_1	0.542	0.055	0.539	0.054
	λ_2	1.083	0.130	1.078	0.125

Table 2: $m = 25, n = 25, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE		AMLE	
		AE	MSE	AE	MSE
k=15,R=(7,$0_{(13)}$)	α	1.096	.071	1.064	0.063
	λ_1	0.575	0.103	0.566	0.095
	λ_2	1.154	0.292	1.136	0.264
k=15,R=($0_{(6)}$,7,$0_{(7)}$)	α	1.107	0.078	1.096	0.074
	λ_1	0.602	0.155	0.597	0.148
	λ_2	1.204	0.446	1.193	0.426
k=15,R=($0_{(13)}$,7)	α	1.126	0.101	1.116	0.097
	λ_1	0.620	0.210	0.614	0.201
	λ_2	1.244	0.660	1.232	0.625
k=20,R=(3,$0_{(18)}$)	α	1.082	0.057	1.073	0.055
	λ_1	0.557	0.066	0.554	0.064
	λ_2	1.109	0.162	1.105	0.158
k=20,R=($0_{(9)}$,3,$0_{(9)}$)	α	1.080	0.052	1.071	0.050
	λ_1	0.550	0.060	0.548	0.059
	λ_2	1.093	0.139	1.089	0.135
k=20,R=($0_{(18)}$,3)	α	1.085	0.058	1.076	0.056
	λ_1	0.555	0.066	0.553	0.065
	λ_2	1.113	0.172	1.109	0.167

Table 3: $m = 25, n = 25, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE		AMLE	
		AE	MSE	AE	MSE
k=15,R=(7,$0_{(13)}$)	α	2.209	0.294	2.147	0.259
	λ_1	0.578	0.110	0.569	0.101
	λ_2	1.150	0.289	1.132	0.258
k=15,R=($0_{(6)}$,7,$0_{(7)}$)	α	2.220	0.319	2.197	0.304
	λ_1	0.597	0.132	0.592	0.126
	λ_2	1.192	0.388	1.181	0.365
k=15,R=($0_{(13)}$,7)	α	2.261	0.414	2.240	0.397
	λ_1	0.630	0.193	0.624	0.184
	λ_2	1.253	0.531	1.241	0.504
k=20,R=(3,$0_{(18)}$)	α	2.148	0.191	2.113	0.178
	λ_1	0.545	0.055	0.542	0.054
	λ_2	1.087	0.131	1.081	0.125
k=20,R=($0_{(9)}$,3,$0_{(9)}$)	α	2.158	0.207	2.140	0.199
	λ_1	0.548	0.060	0.546	0.059
	λ_2	1.098	0.141	1.094	0.137
k=20,R=($0_{(18)}$,3)	α	2.164	0.227	2.145	0.218
	λ_1	0.552	0.063	0.549	0.062
	λ_2	1.108	0.155	1.103	0.151

Table 4: AL and CP of CI's, $m = 25, n = 25, \alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI AL	Bootstrap 90% CI CP	Asymptotic 90%CI AL	Asymptotic 90%CI CP
k=15,R=$(7,0_{(13)})$	α	0.435	83.1%	0.378	90.1%
	λ_1	1.141	87.8%	0.882	90.1%
	λ_2	1.812	84.5%	1.296	92.1%
k=15,R=$(0_{(6)},7,0_{(7)})$	α	0.457	78.8%	0.378	89.8%
	λ_1	1.374	86.8%	0.937	90.6%
	λ_2	2.245	83.8%	1.430	92.9%
k=15,R=$(0_{(13)},7)$	α	0.519	79.2%	0.431	89.7%
	λ_1	1.809	84.1%	1.049	91.1%
	λ_2	3.084	82.2%	1.667	93.8%
k=20,R=$(3,0_{(18)})$	α	0.365	82.9%	0.323	89.6%
	λ_1	0.811	88.6%	0.700	88.3%
	λ_2	1.241	86.4%	1.018	90.5%
k=20,R=$(0_{(9)},3,0_{(9)})$	α	0.366	83.2%	0.323	89.3%
	λ_1	0.836	89.6%	0.711	88.8%
	λ_2	1.285	87.5%	1.044	90.5%
k=20,R=$(0_{(18)},3)$	α	0.392	84.7%	0.343	90.3%
	λ_1	0.852	88.7%	0.724	89.3%
	λ_2	1.355	85.7%	1.065	90.8%

Table 5: AL and CP of CI's, $m = 25, n = 25, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI AL	Bootstrap 90% CI CP	Asymptotic 90%CI AL	Asymptotic 90%CI CP
k=15,R=$(7,0_{(13)})$	α	0.866	83.6%	0.759	89.9%
	λ_1	1.089	89.7%	0.869	89.4%
	λ_2	1.745	86.2%	1.293	92.0%
k=15,R=$(0_{(6)},7,0_{(7)})$	α	0.914	78.4%	0.758	90.0%
	λ_1	1.525	86.8%	0.947	90.5%
	λ_2	2.683	82.8%	1.451	93.5%
k=15,R=$(0_{(13)},7)$	α	1.027	81.7%	0.862	90.1%
	λ_1	1.639	88.4%	1.053	91.4%
	λ_2	2.810	84.4%	1.667	93.7%
k=20,R=$(3,0_{(18)})$	α	0.726	82.3%	0.643	90.3%
	λ_1	0.808	87.2%	0.697	88.5%
	λ_2	1.222	86.1%	1.012	90.3%
k=20,R=$(0_{(9)},3,0_{(9)})$	α	0.7355	82.5%	0.648	89.9%
	λ_1	0.835	88.9%	0.712	89.2%
	λ_2	1.292	86.0%	1.039	90.7%
k=20,R=$(0_{(18)},3)$	α	0.789	81.9%	0.684	90.2%
	λ_1	0.920	86.7%	0.723	89.7%
	λ_2	1.419	84.8%	1.063	90.7%

Table 6: AL and CP of CI's,$m = 25, n = 25, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI		Asymptotic 90%CI	
		AL	CP	AL	CP
k=15,R=(7,0$_{(13)}$)	α	1.774	81.4%	1.515	90.1%
	λ_1	1.169	87.6%	0.873	89.8%
	λ_2	1.875	85.7%	1.300	91.9%
k=15,R=(0$_{(6)}$,7,0$_{(7)}$)	α	1.813	79.7%	1.521	89.1%
	λ_1	1.332	88.1%	0.948	90.0%
	λ_2	2.227	83.1%	1.440	93.0%
k=15,R=(0$_{(13)}$,7)	α	2.101	78.6%	1.722	90.0%
	λ_1	1.765	86.4%	1.074	91.2%
	λ_2	3.010	83.5%	1.708	93.7%
k=20,R=(3,0$_{(18)}$)	α	1.461	80.9%	1.294	89.8%
	λ_1	0.821	88.0%	0.699	88.9%
	λ_2	1.237	86.6%	1.014	90.0%
k=20,R=(0$_{(9)}$,3,0$_{(9)}$)	α	1.478	81.0%	1.296	89.8%
	λ_1	0.844	88.7%	0.713	89.0%
	λ_2	1.294	86.2%	1.044	90.5%
k=20,R=(0$_{(18)}$,3)	α	1.555	83.3%	1.374	89.3%
	λ_1	0.883	89.6%	0.724	89.3%
	λ_2	1.386	86.0%	1.066	91.5%

Table 7: $\alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=(5,0$_{(18)}$)	12.463	6.420
m=25,k=20,R=(0,5,0$_{(17)}$)	12.583	6.383
m=25,k=20,R=(0$_{(2)}$,5,0$_{(16)}$)	12.845	6.369
m=25,k=20,R=(0$_{(3)}$,5,0$_{(15)}$)	13.032	6.245
m=25,k=20,R=(0$_{(4)}$,5,0$_{(14)}$)	13.243	6.181
m=25,k=20,R=(0$_{(8)}$,5,0$_{(10)}$)	14.614	6.043
m=25,k=20,R=(0$_{(14)}$,5,0$_{(4)}$)	17.319	5.092
m=25,k=20,R=(0$_{(16)}$,5,0$_{(2)}$)	20.768	4.458
m=25,k=20,R=(0$_{(17)}$,5,0)	20.918	3.884
m=25,k=20,R=($_{(18)}$,5)	22.883	3.023
m=30,k=25,R=(5,0$_{(23)}$)	9.616	7.181
m=30,k=25,R=(0,5,0$_{(22)}$)	9.718	7.145
m=30,k=25,R=(0$_{(2)}$,5,0$_{(21)}$)	9.743	7.081
m=30,k=25,R=(0$_{(3)}$,5,0$_{(20)}$)	9.834	7.074
m=30,k=25,R=(0$_{(5)}$,5,0$_{(18)}$)	9.908	6.986
m=30,k=25,R=(0$_{(8)}$,5,0$_{(15)}$)	10.304	6.954
m=30,k=25,R=(0$_{(12)}$,5,0$_{(11)}$)	10.680	6.675
m=30,k=25,R=(0$_{(15)}$,5,0$_{(8)}$)	11.217	6.454
m=30,k=25,R=(0$_{(18)}$,5,0$_{(5)}$)	12.045	5.934
m=30,k=25,R=(0$_{(23)}$,5)	14.197	3.451

Table 8: $\alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=$(5,0_{(18)})$	24.360	2.393
m=25,k=20,R=$(0,5,0_{(17)})$	25.214	2.384
m=25,k=20,R=$(0_{(2)},5,0_{(16)})$	25.524	2.374
m=25,k=20,R=$(0_{(3)},5,0_{(15)})$	26.034	2.366
m=25,k=20,R=$(0_{(4)},5,0_{(14)})$	26.513	2.359
m=25,k=20,R=$(0_{(8)},5,0_{(10)})$	28.065	2.300
m=25,k=20,R=$(0_{(14)},5,0_{(4)})$	36.513	2.120
m=25,k=20,R=$(0_{(16)},5,0_{(2)})$	38.831	1.963
m=25,k=20,R=$(0_{(17)},5,0)$	40.552	1.816
m=25,k=20,R=$(0_{(18)},5)$	46.317	1.582
m=30,k=25,R=$(5,0_{(23)})$	19.331	2.549
m=30,k=25,R=$(0,5,0_{(22)})$	19.414	2.536
m=30,k=25,R=$(0_{(2)},5,0_{(21)})$	19.538	2.524
m=30,k=25,R=$(0_{(3)},5,0_{(20)})$	19.895	2.523
m=30,k=25,R=$(0_{(5)},5,0_{(18)})$	20.094	2.522
m=30,k=25,R=$(0_{(8)},5,0_{(15)})$	20.665	2.488
m=30,k=25,R=$(0_{(12)},5,0_{(11)})$	21.478	2.445
m=30,k=25,R=$(0_{(15)},5,0_{(8)})$	22.538	2.374
m=30,k=25,R=$(0_{(18)},5,0_{(5)})$	23.846	2.274
m=30,k=25,R=$(0_{(23)},5)$	28.662	1.692

Table 9: $\alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=$(5,0_{(18)})$	49.927	1.523
m=25,k=20,R=$(0,5,0_{(17)})$	50.653	1.522
m=25,k=20,R=$(0_{(2)},5,0_{(16)})$	51.265	1.515
m=25,k=20,R=$(0_{(3)},5,0_{(15)})$	51.578	1.512
m=25,k=20,R=$(0_{(4)},5,0_{(14)})$	52.719	1.508
m=25,k=20,R=$(0_{(8)},5,0_{(10)})$	57.245	1.492
m=25,k=20,R=$(0_{(14)},5,0_{(4)})$	67.359	1.424
m=25,k=20,R=$(0_{(16)},5,0_{(2)})$	77.601	1.365
m=25,k=20,R=$(0_{(17)},5,0)$	88.436	1.322
m=25,k=20,R=$(0_{(18)},5)$	89.061	1.229
m=30,k=25,R=$(5,0_{(23)})$	38.486	1.572
m=30,k=25,R=$(0,5,0_{(22)})$	38.571	1.572
m=30,k=25,R=$(0_{(2)},5,0_{(21)})$	38.781	1.569
m=30,k=25,R=$(0_{(3)},5,0_{(20)})$	39.411	1.568
m=30,k=25,R=$(0_{(5)},5,0_{(18)})$	40.220	1.561
m=30,k=25,R=$(0_{(8)},5,0_{(15)})$	41.074	1.560
m=30,k=25,R=$(0_{(12)},5,0_{(11)})$	43.549	1.538
m=30,k=25,R=$(0_{(15)},5,0_{(8)})$	45.367	1.517
m=30,k=25,R=$(0_{(18)},5,0_{(5)})$	48.084	1.486
m=30,k=25,R=$(0_{(23)},5)$	55.966	1.277

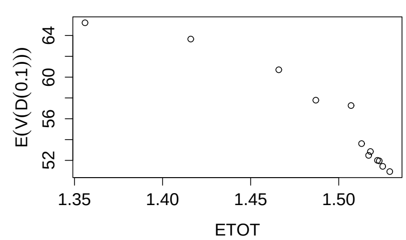

Figure 3: The ETOT and $E(D(\gamma))$ for $m = 22, k = 20, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

6.2 Data Analysis

In this section we perform the analysis of a real data set to illustrate how the propose methods can be used in practice. We have used the following data set originally obtained from Proschan [19] and here the data indicate the failure times (in hour) of air-conditioning system of two airplanes.

The data are provided below.

Plane 7914: 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Plane 7913: 1, 4, 11, 16, 18,18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 141, 163, 191, 206, 216.

From the above data sets we have generated two different jointly progressively censored samples with the censoring schemes Scheme 1: $k = 20$ and $R = (14, 0_{(8)})$ and Scheme 2: $k = 10$, $R = (2_{(7)}, 0_{(2)})$. The generated data sets are provided below.

SCHEME 1:

$$\mathbf{w} = (1, 4, 5, 13, 15, 16, 22, 36, 80, 97) \quad \mathbf{z} = (0, 0, 1, 1, 1, 0, 1, 1, 0, 0);$$

For the above data set the MLEs, AMLEs, and the two different 90% confidence intervals are provided in Tables 10 and 11. In Figure 4 we have provided the profile log-likelihood function $P(\alpha)$ of the shape parameter α and it is clear that $P(\alpha)$ attains a unique maximum. To get an idea about the joint confidence region of α, λ_1, λ_2, we have provided the confidence set of (λ_1, λ_2) for different values of α in Figure 5.

Table 10: real data analysis(scheme-1)

Parameter	MLE	AMLE
α	0.983459	0.982218
λ_1	0.017541	0.017622
λ_2	0.017541	0.017622

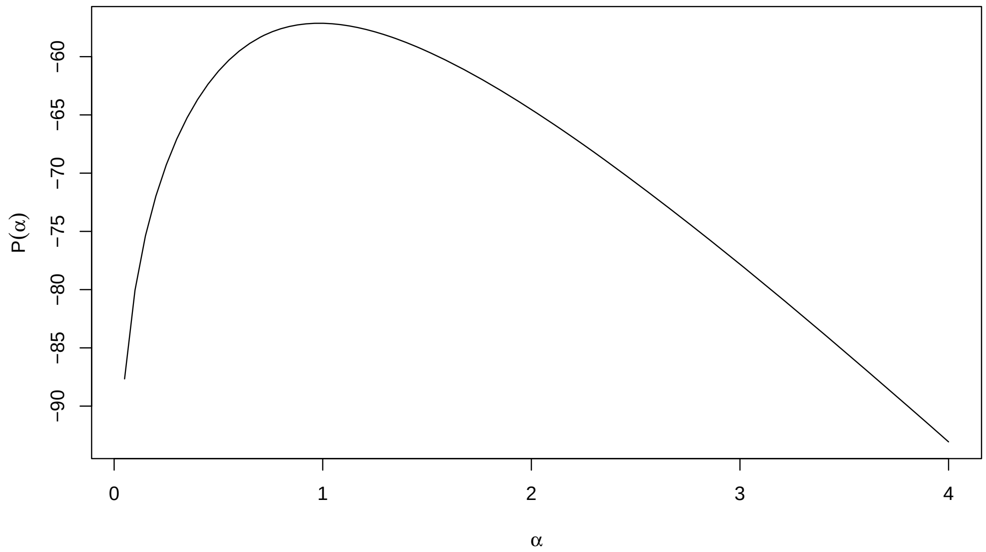

Figure 4: profile-loglikelihood function of shape parameter α for scheme-1

Table 11: real data analysis (90% CI)(scheme-1)

Parameter	90% Asymptotic CI		90% Bootstrap CI	
	LL	UL	LL	UL
α	0.6508	1.3160	0.7253	1.5900
λ_1	0	0.0426	0.001641	0.05592
λ_2	0	0.0426	0.001644	0.05623

Scheme 2:

$$\mathbf{w} = (1, 3, 4, 5, 5, 13, 14, 31, 44, 51), \mathbf{z} = (0, 1, 0, 1, 1, 1, 1, 0, 1, 0);$$

In this case the estimates and the associated confidence intervals are reported in Tables 12 and 13. The profile log-likelihood function $P(\alpha)$ has been provided in Figure 6 and it indicates that it attains a unique maximum. The confidence set of (λ_1, λ_2) for different values of α is provided in Figure 7.

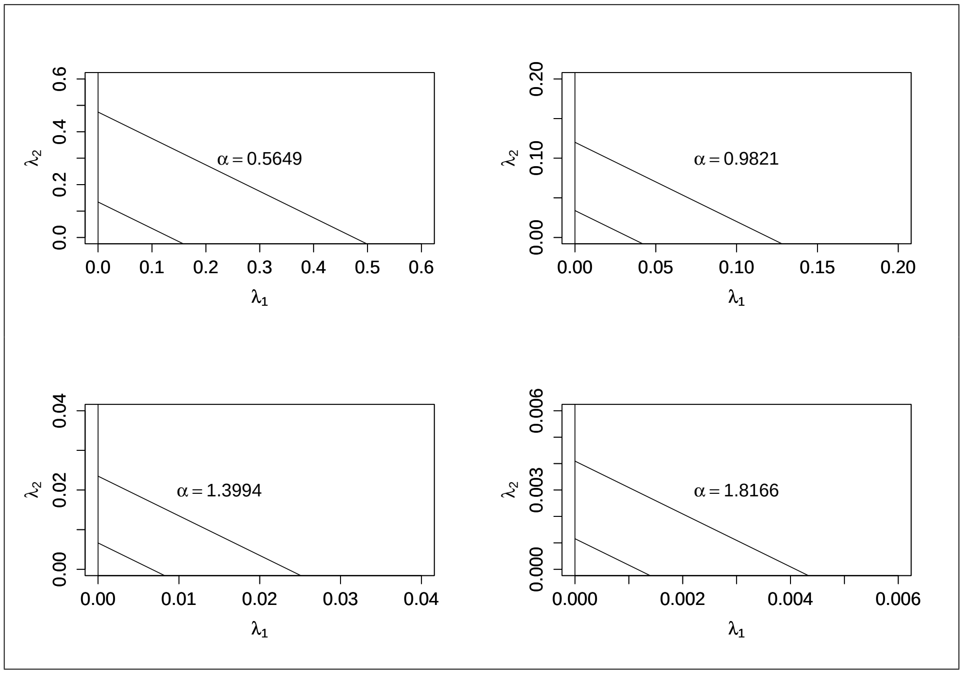

Figure 5: Confidence region of λ_1 and λ_2 for different values α for scheme-1

Table 12: real data analysis(scheme-2)

Parameter	MLE	AMLE
α	1.1740	1.1612
λ_1	0.01367	0.01421
λ_2	0.009116	0.009479

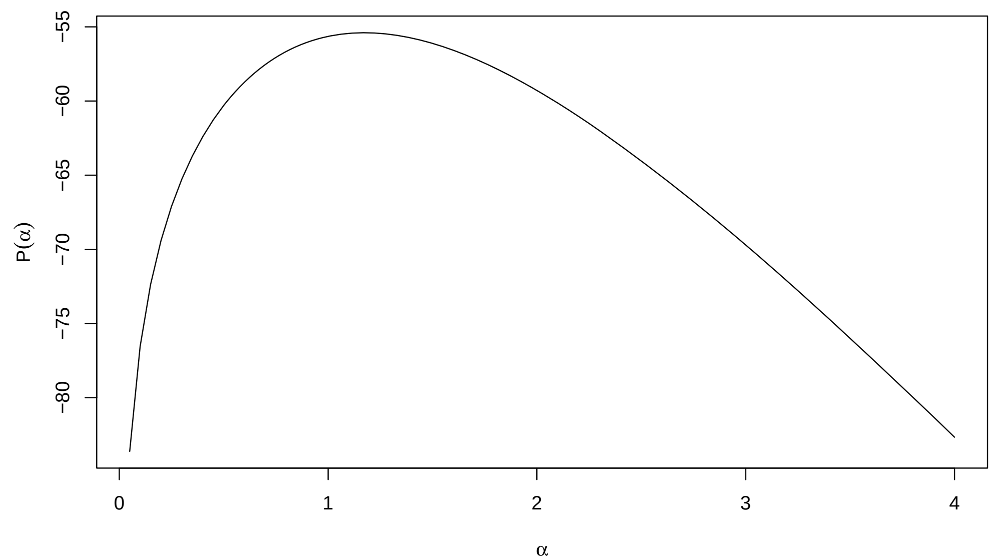

Figure 6: profile-loglikelihood function of shape parameter α for scheme-2

Table 13: real data analysis (90% CI)(scheme-2)

Parameter	90% Asymptotic CI		90% Bootstrap CI	
	LL	UL	LL	UL
α	0.7533	1.5947	0.91046	2.036402
λ_1	0	0.03351	0.001241	0.03696
λ_2	0	0.02303	0.0007122	0.02532

7 Conclusion

In this paper we analyze the new joint progressive censoring (BJPC) for two populations. It is assumed that the lifetimes of the two populations follow Weibull distribution with the same shape parameter but different scale parameters. We obtained the MLEs of the unknown parameters and since they cannot be obtained in explicit forms we have proposed to use AMLEs which can be obtained explicitly. Based on extensive simulation experiments it is observed that the performances of MLEs and AMLEs are very similar in nature. We obtained asymptotic and bootstrap confidence

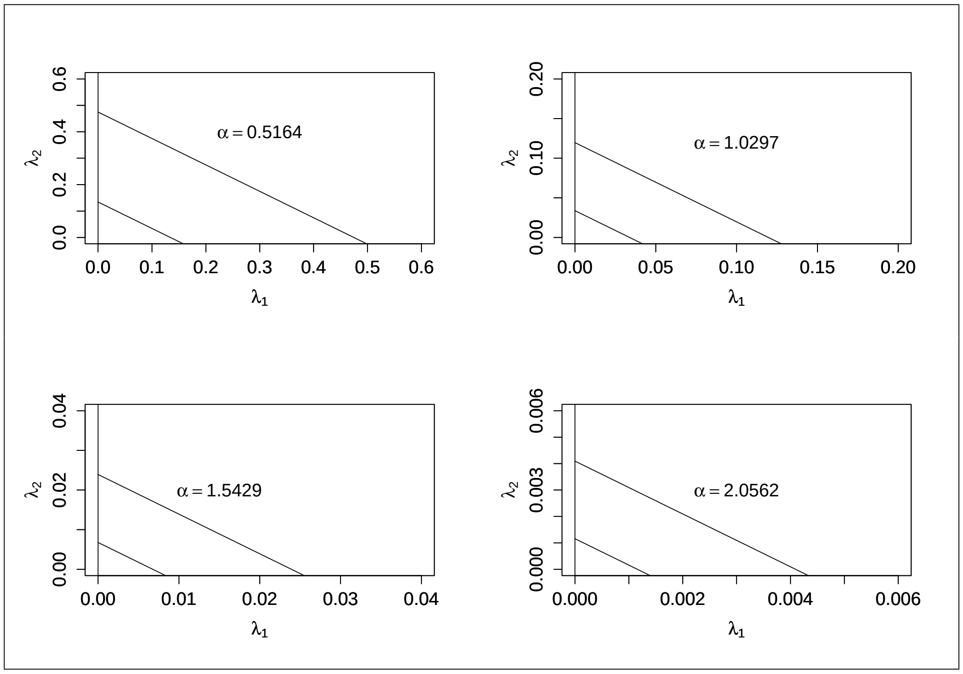

Figure 7: Confidence region of λ_1 and λ_2 for different values of α for Scheme-2

intervals and it is observed that the asymptotic confidence intervals perform quite well even for small sample sizes. We further construct an exact joint confidence set of the unknown parameters and based on the expected volume of the joint confidence set we have proposed an objective function and it has been used to obtain optimum censoring scheme. Note that all the developments in this paper are mainly based on the classical approach. It will be important to develop the necessary Bayesian inference. It may be mentioned that in this paper we have considered the sample sizes to be equal from both the populations, although most of the results can be extended even when they are not equal.

Appendix

PROOF OF LEMMA 1:

$P(\alpha) \to -\infty \quad \text{as} \quad \alpha \to 0$

$$\begin{aligned}
\lim_{\alpha\to\infty} P(\alpha) &= -k \lim \alpha \to \infty \frac{(\sum_{i=1}^{k-1}(R_i+1)\ln(w_i){w_i}^{\alpha} + (m - \sum_{i=1}^{k-1}(R_i+1))\ln(w_i){w_k}^{\alpha}))}{(\sum_{i=1}^{k-1}(R_i+1){w_i}^{\alpha} + (m - \sum_{i=1}^{k-1}(R_i+1)){w_k}^{\alpha}))} + \sum_{i=1}^{k}\ln(w_i) \\
&= -k\ln(w_k) + \sum_{i=1}^{k}\ln(w_i) \\
&= \sum_{i=1}^{k}\ln(\frac{w_i}{w_k}) < 0
\end{aligned}$$

$\Rightarrow P(\alpha) \to -\infty \quad as \quad \alpha \to \infty$ This concludes mle α^* is attained in $(0, \infty)$.

According to Balakrishnan and Kateri [6] $H(\alpha)$ is increasing function of α and $\frac{1}{\alpha}$ is decreasing in α resulting unique solution of equation of equation(5).

PROOF OF LEMMA 2: The proof can be obtained similarly as the proof of Lemma 2 of Mondal and Kundu [13] using $m = n$.

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