Ulam, the Man and the Mathematician

Paul Erdös
HUNGARIAN ACADEMY OF SCIENCES

First of all a few words of introduction. Ulam was a friend and collaborator of mine for about 50 years. We had innumerable mathematical and political discussions and have several joint papers. While discussing mostly our joint work I will neglect his work in physics, biology, computers and computer science.

Ulam wrote a very successful autobiography [7] and I will mention only a few incidents which as far as I remember are not mentioned in his autobiography and which I hope are accurate.

I first met Ulam in Cambridge, England in 1935 and then again in 1938–39 in Cambridge, Massachusetts at Harvard University where he was a member of the Society of Fellows. But our real mathematical contact started when I visited him twice at the University of Wisconsin between 1941 and 1943 and where we obtained our first joint results. Then I met him in Santa Fe and Los Angeles in 1946. He had a serious illness there (almost his only illness, he enjoyed very good health up to his fatal heart attack), probably encephalitis. He completely recovered and while he was recuperating I visited him on an island south of Los Angeles (all this is described in his autobiography). I visited him several times in Los Alamos, the last time was in 1952.

In 1963 there was a meeting on number theory in Boulder; we met there and also visited Aspen together. I was at his house when he got a call from the White House asking his advice about the test ban—Ulam was strongly in favor of it. Then in 1968 and 1970 I was Visiting Professor at the University of Colorado and there we wrote our first joint papers on additive number theory and set theory. In 1970 my mother, then 90 years old, was also with me, and Françoise (Mrs. Ulam) wrote a little article about her. Later in the 70's we often were together at the University of Florida. I had planned to continue our work when I learned that he suddenly died of a coronary attack in May 1984.

Ulam was clearly both a prodigy and a "dotigy". The word *dotigy* cannot be found in any dictionary and is due to Ulam himself. I gave a talk on child prodigies and Ulam remarked that we were both "dotigies", i.e., we really should be in our dotage but can still "prove and conjecture". Perhaps it is a sad commentary on human fate that the best wish we can make for a baby is "May you be a prodigy and then later a dotigy".

Ulam was certainly a prodigy, for he proved before he was 20 that in every infinite set there is a 2-valued measure where the whole set has measure 1, points have measure 0, and the measure is finitely additive. Tarski discovered this independently a few months later. Recently I found out that F. Riesz antici-

Journal of Graph Theory, Vol. 9 (1985) 445–449
 CCC 0364-9024/85/040445-05$04.00

pated them both by 20 years. He proved this in Rome at the International Congress in 1908.

Perhaps one of the most important discoveries of Ulam was the construction in a set S of power $\aleph_1$ of the so called Ulam matrix $\{A_n^{(\alpha)}\}$, $1 \le \alpha < \omega_1$, $1 \le n < \omega$. This matrix has $\aleph_0$ rows and $\aleph_1$ columns, the $A_n^{(\alpha)}$ are all subsets of a fixed set S of power $\aleph_1$, two sets in the same row are pairwise disjoint and the union of the sets in the same column contains all but denumerably many elements of S. It can be constructed by a simple transfinite induction and from it Ulam easily deduced his famous theorem that if $|S| = m$ where $m < m_0$ and m_0 is the first inaccessible cardinal, then one can not define a countably additive measure among the subsets of S so that points have measure 0, the whole set has measure 1 and every subset should be measurable. The problem for inaccessible cardinals remained open. This paper had an immense influence and later led to the development of the theory of large cardinals which in my opinion is one of the most important developments in modern mathematics and I am pleased to remember that one of the second starting points of the development in this theory was in papers with Tarski [4, 5] which continued and completed some earlier work of Tarski. Perhaps the reader will forgive me for a few words of personal reminisences. I mistakenly believed that the first inaccessible cardinal was perhaps measurable. In 1957 Hajnal and I proved a theorem from which it was trivial to deduce that the first and, in fact, many other inaccessible cardinals do not have a countably additive measure. Hajnal only noticed this after Hanf and Tarski as well as Kiesler and Tarski obtained their results. I am afraid that the fault here was mine. As Hajnal put it "I was a young man then. How could I have doubted and contradicted the "pgom" (poor great old man)?" I was old even at those distant times. In fact as Hajnal put it, one possible consequence of this oversight was that the Hanf-Kiesler-Tarski proof gave much more insight than ours and soon led to an explosive development of the theory of large cardinals. Perhaps if we had published first, this development would have been slowed down.

Ulam once asked (I believe during one of my visits to Madison in 1943): Let S be a set of size $\aleph_1$. Can one define $\aleph_0$ measures M_k on the subsets of S so that all the measures should be 2-valued, points having measure 0, S measure 1, all the measures should be countably additive and every set should be measurable in at least one of the measures? He also asked if this is not possible for $\aleph_0$ measures. Perhaps it is possible for $\aleph_1$ measures. Alaoglu and I indeed proved that $\aleph_0$ such measures can not be defined but we could not decide the question for $\aleph_1$ such measures and indeed it later turned out that this question is undecidable. Our proof appeared in [3].

Ulam's work with Oxtoby, Mazur and Borsuk was of great importance in mathematics, but I am less competent to speak about it. His work with Hyers on the functional equation $f(x + y) = f(x) + f(y)$ is also rather interesting as is his work with Everett. But since I write in this Journal, I should perhaps also mention his famous reconstruction conjecture (the "reconstruction disease", a terminology of Harary [6]). The first results in this area were obtained by Paul Kelly, Ulam's student, and the general problem is very far from being solved and the subject is still alive. A general metaproblem of Ulam was: If $A^2 = B^2$ in some structure does it then follow that $A = B$? Usually but not always the answer seems to be negative. These problems also led to many interesting papers.

While at Los Alamos Ulam pioneered many important developments in the use of computers for problems of pure and applied mathematics. I do not want to write about this here, not because I do not think that it is important and interesting but because I feel that others who know more about this will write about it. Here I only mention that he and his collaborators obtained many interesting, fruitful and unexpected conjectures on the iteration of functions. I can talk even less about his share in the project "Orion" concerning interplanetary travel. Dyson, I understand, was very active in this project and I hope that he and others will write more about it. I only want to mention an anecdote. Ulam was rather proud about his share in starting this project and always felt sorry that it was abandoned (as far as I know it was abandoned before the treaty which forbade atomic explosions in space. Ulam certainly never wanted to break this treaty but hoped that perhaps it could be renegotiated). Once he told me that he had found a nice slogan for this project from Goethe's Faust: "Und was vor uns ein alter Mann gedacht und was wir dann so herrlich weitgebracht ja bis an die Sterne weit" (what was conceived before us by an old man and which we pushed ahead so wonderfully, yes, up to the stars). Ulam said the old man was Einstein. I immediately corrected him, "No, the old man is you and the stars should be replaced by the planets." Ulam was always afraid of getting old and was rather proud that he could play good tennis even when he was over 70. He was really fortunate to have avoided the two greatest evils, old age and stupidity, and he died suddenly of heart failure without fear or pain while he could still prove and conjecture.

During my last visit to the University of Florida, Bednarek told a nice Ulam story. Perhaps the story is a bit embellished but once Marcel Riesz told me "If you have a good story, the actual fact whether it is true should not worry you" and in this case I am sure that the story is substantially true. A few years ago Professor Gladysz of the University of Wroclaw was visiting Gainesville. It so happened that he had never met Ulam and after Bednarek introduced them they had a long conversation in Polish. When Ulam left Gladysz asked: Was this man the son of the *FAMOUS ULAM*? Bednarek was too polite to tell him the truth but was sure that Ulam would be very pleased and told him the story. As Bednarek puts it, the next day most mathematicians knew it.

During his whole mathematical life he excelled not only in proving interesting and deep theorems but perhaps even more in inventing new and stimulating problems and conjectures. He made many beautiful conjectures in subjects he never really worked on. I'll give two or three examples on subjects with which I am familiar. Anning and I proved that if $x_1, x_2, \ldots$ is an infinite set of points in a plane (or in E_n) and if all the distances are integers then the points must lie on a line. Ulam immediately asked, "Can one have infinitely many such points not all on a line with all distances rational?" I answered "Yes, Anning and I did this, but Euler anticipated us." Ulam countered, "I do not believe that the points can be everywhere dense in the plane and all distances being rational." I expect that this conjecture is probably true, but it probably is very deep. It is likely that if an infinite set has all distances rational then this set must be very restricted, but nothing is known about this.

He published a very nice and useful book on mathematical problems. A second volume was planned in collaboration with Mauldin. Now Mauldin will have to finish this work alone and I will try to help him in this work to the best of my ability. One new problem which emerged from our talks and discussions with Graham states as follows: Is it true that if $n > n_0$ and $a_1, a_2, \dots, a_n$ is a permutation of $1, 2, \dots, n$ then there is an arithmetic progression of three terms $x, x + d, x + 2d$ so that a_x, a_{x+d}, a_{x+2d} also forms an arithmetic progression?

Even though Ulam was not a number theorist, he posed several intriguing questions in number theory, many of which were given at the meeting on number theory in Boulder in 1963. He also introduced the "lucky numbers" which were discovered independently by Eri Jabotinsky in Haifa.

In the 70's and early 80's Ulam and I often were together at the University of Florida in Gainesville and we published several papers on combinatorics and set theory. Here I would only like to mention a problem of Ulam which led us to many interesting problems and results in graph theory.

Here is one of our problems about which we first of all wrote a paper [2] with five authors: The problem is as follows: Let $G(n)$ and $G'(n)$ be two graphs on n vertices. Denoted by $e(G)$ the number of edges of G. We assume $e(G) = e(G')$. By a U-*decomposition* we mean a partition of the edge set $E(G) = E_1 + \cdots + E_n$, $E(G') = E_1' + \cdots + E_n'$ such that the graphs E_i and E_i' are isomorphic for all i. Such a decomposition always exists if the graphs G and G' have the same number of edges. The function $U(G, G')$ is defined to be the minimum value of n for which such a U decomposition exists. Let

$$U(n) = \max_{G,G'}(U(G, G')) .$$

We prove

$$U(n) = \frac{2}{3}n + o(n) .$$

We published many further papers on this and related topics. There are interesting extensions to hypergraphs and the problem is still alive.

We hope for further interesting developments. Fan Chung and I recently finished a paper on this subject [1].

Now I would like to mention some of the work we did at Madison. We considered the Boolean algebra of the set of all subsets of the integers modulo a finitely additive ideal. We conjectured but did not have a completely satisfactory proof of the fact that there are $2^{2^{\aleph_0}}$ nonisomorphic Boolean algebras of such a type. This was later proved by Monk and in a more general form by Shelah.

We also considered in particular three special Boolean algebras. Let B_1 be the algebra modulo finite sets, B_2 the algebra modulo the sequences of density 0 and B_3 the algebra modulo the ideal of sequences of logarithmic density 0. We easily proved that B_1 is not isomorphic to B_2 and B_3 and thought that we proved that B_2 is not isomorphic to B_3. We could never reconstruct our proof and suspected that perhaps B_2 is not isomorphic to B_3. I offered 100 dollars for a proof or disproof. Finally Just and Krawczyk, under the assumption of the Continuum Hypothesis, and Franluaiorczy under the assumption of Martin's Axiom proved that B_2 is not isomorphic to B_3. More work on this subject was done by Monk.

Ulam was a good friend and collaborator for nearly 50 years and clearly the world of mathematics in particular and science and society in general will never be the same.

In the 1001 nights, the king was greeted by "O King may you live forever". A mathematician and scientist can be greeted by the more realistic "O Mathematician, may your theorems live forever". I wish and expect this fate for Stan.

References

[1] F. R. K. Chung and P. Erdös, On unavoidable hypergraphs (to appear in *J. Graph Theory*).

[2] F. R. K. Chung, P. Erdös, R. L. Graham, S. M. Ulam and F. F. Yao, Minimal decompositions of two graphs into pairwise isomorphic subgraphs. *Proc. Tenth Southeastern Conf. on Combinatorics, Graph Theory and Computing* (1979) 3–18.

[3] P. Erdös, Some remarks on set theory, *Proc. Amer. Math. Soc.* **1** (1950) 121–141.

[4] P. Erdös and A. Tarski, On families of mutually exclusive sets. *Annals of Math.* **44** (1943), 315–329.

[5] P. Erdös and A. Tarski, On some problems involving inaccessible cardinals. *Essays on the Foundations of Mathematics*, Hebrew University, Jerusalem (1961) 50–82.

[6] F. Harary, The Four Color Conjecture and other Graphical Diseases. *Proof Techniques in Graph Theory*, Academic Press, New York (1969).

[7] S. M. Ulam, *Adventures of a Mathematician*, Scribner, New York (1976).