# Partitioning a graph into highly connected subgraphs

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**Abstract**

Given $k \geq 1$, a *k-proper partition* of a graph $G$ is a partition $\mathcal{P}$ of $V(G)$ such that each part $P$ of $\mathcal{P}$ induces a $k$-connected subgraph of $G$. We prove that if $G$ is a graph of order $n$ such that $\delta(G) \geq \sqrt{n}$, then $G$ has a 2-proper partition with at most $n/\delta(G)$ parts. The bounds on the number of parts and the minimum degree are both best possible. We then prove that if $G$ is a graph of order $n$ with minimum degree

$$\delta(G) \geq \sqrt{c(k-1)n},$$

where $c = \frac{2123}{180}$, then $G$ has a $k$-proper partition into at most $\frac{cn}{\delta(G)}$ parts. This improves a result of Ferrara, Magnant and Wenger [Conditions for Families of Disjoint $k$-connected Subgraphs in a Graph, *Discrete Math.* **313** (2013), 760–764], and both the degree condition and the number of parts is best possible up to the constant $c$.

# 1 Introduction

A graph $G$ is *k-connected* if the removal of any collection of fewer than $k$ vertices from $G$ results in a connected graph with at least two vertices. In this paper, we

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are interested in determining minimum degree conditions that ensure that the vertex set of a graph can be partitioned into sets that each induce a $k$-connected subgraph. In a similar vein, Thomassen [17] showed that for every $s$ and $t$, there exists a function $f(s,t)$ such that if $G$ is an $f(s,t)$-connected graph, then $V(G)$ can be decomposed into sets $S$ and $T$ such that $S$ induces an $s$-connected subgraph and $T$ induces a $t$-connected subgraph. In the same paper, Thomassen conjectured that $f(s,t) = s+t+1$, which would be best possible, and Hajnal [10] subsequently showed that $f(s,t) \leq 4s + 4t - 13$.

From a vulnerability perspective, highly connected graphs represent robust networks that are resistant to multiple node failures. When a graph is not highly connected, it is useful to partition the vertices of the graph so that every part induces a highly connected subgraph. For example, Hartuv and Shamir [11] designed a clustering algorithm where the vertices of a graph $G$ are partitioned into highly connected induced subgraphs. It is important in such applications that each part is highly connected, but also that there are not too many parts.

Given a simple graph $G$ and an integer $k \geq 1$, we say a partition $\mathcal{P}$ of $V(G)$ is *$k$-proper* if for every part $P \in \mathcal{P}$, the induced subgraph $G[P]$ is $k$-connected. Ferrara, Magnant, and Wenger [5] gave a minimum-degree condition on $G$ that guarantees a $k$-proper partition.

**Theorem 1** (Ferrara, Magnant, Wenger [5])**.** *Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$. If $\delta(G) \geq 2k\sqrt{n}$, then $G$ has a $k$-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq 2kn/\delta(G)$.*

In addition, they present a graph $G$ with $\delta(G) = (1 + o(1))\sqrt{(k-1)n}$ that contains no $k$-proper partition. This example, which we make more precise below, leads us to make the following conjecture.

**Conjecture 2.** *Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$. If $\delta(G) \geq \sqrt{(k-1)n}$, then $G$ has a $k$-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n-k+1}{\delta-k+2}$.*

To see that the degree condition in Conjecture 2, if true, is approximately best possible, let $n, \ell$ and $p$ be integers such that $\ell = \sqrt{(k-1)(n-1)}$ and $p = \frac{\ell}{(k-1)} = \frac{n-1}{\ell}$. Starting from $H = pK_\ell$, so that $|H| = n-1$, construct the graph $G$ by adding a new vertex $v$ that is adjacent to exactly $k-1$ vertices in each component of $H$. Then $\delta(G) = \ell - 1$, but there is no $k$-connected subgraph of $G$ that contains $v$.

To see that the number of components in Conjecture 2 is best possible, let $r$ and $s$ be integers such that $r = \sqrt{(k-1)n} - k + 2$ and $s = \frac{n-k+1}{r}$. Consider then $G = sK_r \vee K_{k-1}$, which has minimum degree $r + k - 2 = \sqrt{(k-1)n}$, while every $k$-proper partition has at least $s = \frac{n-k+1}{\delta-k+2}$ parts.

As an interesting comparison, Nikiforov and Shelp [13] give an approximate version of Conjecture 2 with a slightly weaker degree condition. Specifically, they prove that if $\delta(G) \geq \sqrt{2(k-1)n}$, then there exists a partition of $V(G)$ such that $n - o(n)$ vertices are contained in parts that induce $k$-connected subgraphs.

In Section 2, we verify Conjecture 2 in the case $k = 2$.

**Theorem 3.** *Let $G$ be a graph of order $n$. If $\delta(G) \geq \sqrt{n}$, then $G$ has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq (n-1)/\delta(G)$.*

Ore's Theorem [14] states that if $G$ is a graph of order $n \geq 3$ such that $\sigma_2(G) = \min\{d(u)+d(v) \mid uv \notin E(G)\} \geq n$, then $G$ is hamiltonian, and therefore has a trivial 2-proper partition. As demonstrated by Theorem 3 however, the corresponding minimum degree threshold is considerably different. Note as well that if $G$ has a 2-factor $\mathcal{F}$, then $G$ has a 2-proper partition, as each component of $\mathcal{F}$ induces a hamiltonian, and therefore 2-connected, graph. Consequently, the problem of determining if $G$ has a 2-proper partition can also be viewed as an extension of the 2-factor problem [1, 15], which is itself one of the most natural generalizations of the hamiltonian problem [6, 7, 8].

In Section 3, we improve the bound on the minimum degree to guarantee a $k$-proper partition for general $k$, as follows.

**Theorem 4.** *If $G$ is a graph of order $n$ with*

$$\delta(G) \geq \sqrt{\frac{2123}{180}(k-1)n}$$

*then $G$ has a $k$-proper partition into at most $\frac{2123n}{180\delta}$ parts.*

Conjecture 2 yields that both the degree condition and the number of parts in the partition in Theorem 4 are best possible up to the constant $\frac{2123}{180}$. Our proof of Theorem 4 has several connections to work of Mader [12] and Yuster [18], discussed in Section 3. One interesting aspect of our proof is that under the given conditions, the greedy method of building a partition by iteratively removing the largest $k$-connected subgraph will produce a $k$-proper partition.

### Definitions and Notation

All graphs considered in this paper are finite and simple, and we refer the reader to [4] for terminology and notation not defined here. Let $H$ be a subgraph of a graph $G$, and for a vertex $x \in V(H)$, let $N_H(x) = \{y \in V(H) \mid xy \in E(H)\}$.

A subgraph $B$ of a graph $G$ is a *block* if $B$ is either a bridge or a maximal 2-connected subgraph of $G$. It is well-known that any connected graph $G$ can be decomposed into blocks. A pair of blocks $B_1, B_2$ are necessarily edge-disjoint, and if two blocks intersect, then their intersection is exactly some vertex $v$ that is necessarily a cut-vertex in $G$. The *block-cut-vertex graph* of $G$ is defined to be the bipartite graph $T$ with one partite set comprised of all cut-vertices of $G$ and the other partite set comprised of all blocks of $G$. For a cut-vertex $v$ and a block $B$, $v$ and $B$ are adjacent in $T$ if and only if $v$ is a vertex of $B$ in $G$.

## 2 2-Proper Partitions

It is a well-known fact that the block-cut-vertex graph of a connected graph is a tree. This observation makes the block-cut-vertex graph, and more generally the block structure of a graph, a useful tool, specifically when studying graphs with connectivity one. By definition, each block of a graph $G$ consists of at least two vertices. A block $B$ of $G$ is *proper* if $|V(B)| \geq 3$. When studying a block decomposition of $G$, the structure of proper blocks is often of interest. In particular, at times one might hope that the proper blocks will be pairwise vertex-disjoint. In general, however, such an ideal structure is not possible. However, the general problem of determining conditions that ensure a graph has a 2-proper partition, addressed in one of many possible ways by Theorem 3, can be viewed as a vertex analogue to that of determining when a graph has vertex-disjoint proper blocks.

*Proof of Theorem 3.* We proceed by induction on $n$, with the base cases $n \leq 4$ being trivial. Thus we may assume that $n \geq 5$.

First suppose that $G$ is disconnected, and let $G_1, \cdots, G_m$ be the components of $G$. For each $1 \leq i \leq m$, since

$$\delta(G_i) \geq \delta(G) \geq \sqrt{n} > \sqrt{|V(G_i)|},$$

$G_i$ has a 2-proper partition $\mathcal{P}_i$ with at most $(|V(G_i)|-1)/\delta(G_i)$ $(\leq (|V(G_i)|-1)/\delta(G))$ parts, by induction. Therefore, $\mathcal{P} = \bigcup_{1\leq i\leq m} \mathcal{P}_i$ is a 2-proper partition of $G$ with

$$|\mathcal{P}| = \sum_{1\leq i\leq m} |\mathcal{P}_i| \leq \sum_{1\leq i\leq m} (|V(G_i)| - 1)/\delta(G) < (n-1)/\delta(G).$$

Hence we may assume that $G$ is connected. If $G$ is 2-connected, then the trivial partition $\mathcal{P} = \{V(G)\}$ is a desired 2-proper partition of $G$, so we proceed by supposing that $G$ has at least one cut-vertex.

**Claim 1.** *If $G$ has a block $B$ of order at least $2\delta(G)$, then $G$ has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq (n-1)/\delta(G)$.*

*Proof.* It follows that

$$|V(G) - V(B)| \leq n - 2\delta(G) \leq n - 2\sqrt{n},$$

and

$$\delta(G - V(B)) \geq \delta(G) - 1 \geq \sqrt{n} - 1.$$

Since $\sqrt{n} - 1 = \sqrt{n - 2\sqrt{n} + 1} > \sqrt{n - 2\sqrt{n}}$,

$$\delta(G - V(B)) \geq \sqrt{n} - 1 > \sqrt{|V(G) - V(B)|}.$$

Applying the induction hypothesis, $G - V(B)$ has a 2-proper partition $\mathcal{P}$ with

$$|\mathcal{P}| \leq (n - |V(B)| - 1)/\delta(G - V(B)) \leq (n - 2\delta(G) - 1)/(\delta(G) - 1).$$

Since $(n-1)(\delta(G)-1) - (n - \delta(G) - 2)\delta(G) = \delta(G)^2 - n + \delta(G) + 1 > n - n = 0$, $(n-1)/\delta(G) \geq (n - \delta(G) - 2)/(\delta(G) - 1)$, and hence

$$|\mathcal{P} \cup \{V(B)\}| \leq \frac{n - 2\delta(G) - 1}{\delta(G) - 1} + 1 = \frac{n - \delta(G) - 2}{\delta(G) - 1} \leq \frac{n-1}{\delta(G)}.$$

Consequently $\mathcal{P} \cup \{V(B)\}$ is a 2-proper partition of $G$ with $|\mathcal{P} \cup \{V(B)\}| \leq (n-1)/\delta(G)$. □

By Claim 1, we may assume that every block of $G$ has order at most $2\delta(G)-1$. Let $\mathcal{B}$ be the set of blocks of $G$. For each $B \in \mathcal{B}$, let $X_B = \{x \in V(B) \mid x \text{ is not a cut-vertex of } G\}$. Note that $N_G(x) \subseteq V(B)$ for every $x \in X_B$. Let $X = \bigcup_{B\in\mathcal{B}} X_B$. For each vertex $x$ of $G$, let $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in V(B)\}$. In particular, for each cut-vertex $x$ of $G$ we have $|\mathcal{B}_x| \geq 2$.

**Claim 2.** *Let $x$ be a cut-vertex of $G$, and let $C$ be a component of $G - x$. Then $|V(C)| \geq \delta(G)$. In particular, every end-block of $G$ has order at least $\delta(G)+1$.*

*Proof.* Let $y \in V(C)$. Note that $d_C(y) \geq d_G(y) - 1 \geq \delta(G) - 1$. Since $N_C(y) \cup \{y\} \subseteq V(C)$, $(\delta(G)-1)+1 \leq d_C(y)+1 \leq |V(C)|$. □

**Claim 3.** *For each $x \in V(G)$, $|N_G(x) \cap X| \geq 2$. In particular, for a block $B$ of $G$, if $X_B \neq \emptyset$, then $|X_B| \geq 3$.*

*Proof.* Suppose that $|N_G(x) \cap X| \leq 1$. For each vertex $y \in N_G(x) - X$, since $y$ is a cut-vertex of $G$, there exists a component $C_y$ of $G - y$ such that $x \notin V(C_y)$. By Claim 2, $|V(C_y)| \geq \delta(G)$. Futhermore, for distinct vertices $y_1, y_2 \in N_G(x) - X$, we have $V(C_{y_1}) \cap (V(C_{y_2}) \cup N_G(x)) = \emptyset$. Hence

$$\begin{aligned} n &\geq |N_G(x) \cup \{x\}| + \sum_{y \in N_G(x) - X} |V(C_y)| \\ &\geq (\delta(G)+1) + |N_G(x) - X|\delta(G) \\ &\geq (\delta(G)+1) + (\delta(G)-1)\delta(G) \\ &= \delta(G)^2 + 1 \\ &\geq n+1, \end{aligned}$$

which is a contradiction. □

**Claim 4.** *Let $B$ be a block of $G$ with $X_B \neq \emptyset$, and let $x \in V(B)$ be a cut-vertex of $G$. Then there exists a block $C$ of $B - x$ with $X_B \subseteq V(C)$. In particular, if $B$ is an end-block of $G$, then $B - x$ is 2-connected.*

*Proof.* For the moment, we show that any two vertices in $X_B$ belong to a common block of $B - x$. By way of contradiction, we suppose that there are distinct vertices $y_1, y_2 \in X_B$ such that no block of $B - x$ contains both $y_1$ and $y_2$. In particular, $y_1y_2 \notin E(G)$. Then $|N_{B-x}(y_1) \cap N_{B-x}(y_2)| \leq 1$, and hence $|N_{B-x}(y_1) \cup N_{B-x}(y_2) \cup \{y_1, y_2\}| = |N_{B-x}(y_1)| + |N_{B-x}(y_2)| - |N_{B-x}(y_1) \cap N_{B-x}(y_2)| + 2 \geq 2(\delta(G)-1)+1$. It follows that

$$|V(B) - \{x\}| \geq |N_{B-x}(y_1) \cup N_{B-x}(y_2) \cup \{y_1, y_2\}| \geq 2\delta(G) - 1,$$

and hence $|V(B)| \geq 2\delta(G)$, which contradicts the assumption that every block of $G$ has order at most $2\delta(G)-1$. Thus any two vertices in $X_B$ belong to a common block of $B - x$. This together with the definition of a block implies that a block $C$ of $B - x$ satisfies $X_B \subseteq V(C)$. □

Fix an end-block $B_0$ of $G$. Then we can regard the block-cut-vertex graph $T$ of $G$ as a rooted tree with the root $B_0$. For a block $B$ of $G$, let $G(B)$ denote the subgraph which consists of $B$ and the descendant blocks of $B$ with respect to $T$ (i.e., $G(B)$ is the graph formed by the union of all blocks of $G$ contained in the rooted subtree of $T$ with the root $B$). A 2-proper partition $\mathcal{P}$ of a subgraph of $G$ is *extendable* if $|P| \geq \delta(G)$ for every $P \in \mathcal{P}$.

**Claim 5.** *Let $B^*$ be a block of $G$ with $B^* \neq B_0$, and let $u \in V(B^*)$ be the parent of $B^*$ with respect to $T$. Then $G(B^*) - u$ has an extendable 2-proper partition. Furthermore, if $X_{B^*} \neq \emptyset$, then $G(B^*)$ has an extendable 2-proper partition.*

*Proof.* We proceed by induction on the height $h$ of the block-cut-vertex graph of $G(B^*)$ with the root $B^*$. If $h = 0$, then $G(B^*)$ $(= B^*)$ is an end-block of $G$, and hence the desired conclusion holds by Claims 2 and 4. Thus we may assume that $h \geq 2$ (i.e., $B^*$ has a child in $T$). By the assumption of induction, for $x \in V(B^*) - (X_{B^*} \cup \{u\})$ and $B \in \mathcal{B}_x - \{B^*\}$, $G(B) - x$ has an extendable 2-proper partition $\mathcal{P}_{x,B}$. For each $x \in V(B^*) - (X_{B^*} \cup \{u\})$, let $\mathcal{P}_x = \bigcup_{B \in \mathcal{B}_x - \{B^*\}} \mathcal{P}_{x,B}$ and fix a block $B_x \in \mathcal{B}_x - \{B^*\}$ so that $X_{B_x}$ is not empty, if possible.

Suppose that $X_{B^*} = \emptyset$. Fix a vertex $x \in V(B^*) - \{u\}$. Then by Claim 3, we may assume that $X_{B_x} \neq \emptyset$. By the assumption of induction, $G(B_x)$ has an extendable 2-proper partition $\mathcal{Q}_x$. This together with the assumption that $X_{B^*} = \emptyset$ implies that $\bigcup_{x \in V(B^*) - \{u\}} ((\mathcal{P}_x - \mathcal{P}_{x,B_x}) \cup \mathcal{Q}_x)$ is an extendable 2-proper partition of $G(B^*) - u$, as desired. Thus we may assume that $X_{B^*} \neq \emptyset$.

**Subclaim 5.1.** *There exists a block $A$ of $B^* - u$ such that*

(i) $X_{B^*} \subseteq V(A)$,

(ii) $|V(A)| \geq \delta(G)$, *and*

(iii) *for $x \in V(B^*) - (V(A) \cup \{u\})$, there exists a block $B'_x \in \mathcal{B}_x - \{B^*\}$ with $X_{B'_x} \neq \emptyset$.*

*Proof.* By Claim 4, there exists a block $A$ of $B^* - u$ satisfying (i). We first show that $A$ satisfies (ii). Suppose that $|V(A)| \leq \delta(G) - 1$. By the definition of a block, for any $x, x' \in X_{B^*}$ with $x \neq x'$, $N_{B^*-u}(x) \cap N_{B^*-u}(x') \subseteq V(A)$, and so $|(N_{B^*-u}(x) - V(A)) \cup (N_{B^*-u}(x') - V(A))| = |N_{B^*-u}(x) - V(A)| + |N_{B^*-u}(x') - V(A)|$. For each $x \in X_{B^*}$, since $x \in V(A) - N_{B^*-u}(x)$, $|N_{B^*-u}(x) - V(A)| \geq \delta(G) - 1 - (|V(A)| - 1)$. By Claim 2, $|V(G(B_x)) - \{x\}| \geq \delta(G)$ for every $x \in V(B^*) - (X_{B^*} \cup \{u\})$. Hence by Claim 3,

$$\begin{aligned}
n &\geq \left| (V(B^*) - \{u\}) \cup \left( \bigcup_{x \in V(B^*)-(X_{B^*}\cup\{u\})} (V(G(B_x)) - \{x\}) \right) \right| \\
&= |V(B^*) - \{u\}| + \sum_{x \in V(B^*)-(X_{B^*}\cup\{u\})} |V(G(B_x)) - \{x\}| \\
&\geq |V(B^*) - \{u\}| + \delta(G) \left(|V(B^*) - \{u\}| - |X_{B^*}|\right) \\
&= (\delta(G)+1)|V(B^*) - \{u\}| - \delta(G)|X_{B^*}| \\
&\geq (\delta(G)+1) \left| V(A) \cup \left( \bigcup_{x \in X_{B^*}} (N_{B^*-u}(x) - V(A)) \right) \right| - \delta(G)|X_{B^*}| \\
&= (\delta(G)+1) \left( |V(A)| + \sum_{x \in X_{B^*}} |N_{B^*-u}(x) - V(A)| \right) - \delta(G)|X_{B^*}| \\
&\geq (\delta(G)+1) \left( |V(A)| + \sum_{x \in X_{B^*}} (\delta(G) - 1 - (|V(A)|-1)) \right) - \delta(G)|X_{B^*}| \\
&= (\delta(G)+1)(|V(A)| + |X_{B^*}|(\delta(G) - |V(A)|)) - \delta(G)|X_{B^*}| \\
&= |X_{B^*}|\delta(G)^2 - |V(A)|(\delta(G)+1)(|X_{B^*}|-1) \\
&\geq |X_{B^*}|\delta(G)^2 - (\delta(G)-1)(\delta(G)+1)(|X_{B^*}|-1) \\
&= \delta(G)^2 + |X_{B^*}| - 1 \\
&\geq n + 3 - 1,
\end{aligned}$$

which is a contradiction. Thus $|V(A)| \geq \delta(G)$.

We next check that $A$ satisfies (iii). Let $x \in V(B^*) - (V(A) \cup \{u\})$. Since $A$ is a block of $B^* - u$ and satisfies (i), $|N_G(x) \cap X_{B^*}| \leq 1$. This together with Claim 3 implies that there exists a block $B'_x \in \mathcal{B}_x - \{B^*\}$ with $X_{B'_x} \neq \emptyset$. □

Let $A$ and $B'_x \in \mathcal{B}_x - \{B^*\}$ $(x \in V(B^*) - (V(A) \cup \{u\}))$ be as in Subclaim 5.1. By the assumption of induction, for $x \in V(B^*) - (V(A) \cup \{u\})$, $G(B'_x)$ has an extendable 2-proper partition $\mathcal{Q}'_x$. Then

$$\{V(A)\} \cup \left( \bigcup_{x \in V(B^*)-(V(A)\cup\{u\})} ((\mathcal{P}_x - \mathcal{P}_{x,B'_x}) \cup \mathcal{Q}'_x) \right) \cup \left( \bigcup_{x \in V(A)-X_{B^*}} \mathcal{P}_x \right)$$

is an extendable 2-proper partition of $G(B^*) - u$.

Since $N_G(x) \cup \{x\} \subseteq V(B^*)$ for $x \in X_{B^*}$, $|V(B^*)| \geq \delta(G) + 1$, and hence $\{V(B^*)\} \cup (\bigcup_{x \in V(B^*)-(X_{B^*}\cup\{u\})} \mathcal{P}_x)$ is an extendable 2-proper partition of $G(B^*)$. □

By Claim 5, $G - V(B_0)$ has an extendable 2-proper partition $\mathcal{P}_0$. Hence $\mathcal{P} = \{V(B_0)\} \cup \mathcal{P}_0$ is a 2-proper partition of $G$. Furthermore, since $|V(B_0)| \geq \delta(G) + 1$ by Claim 2, $n = \sum_{P \in \mathcal{P}} |P| = |V(B_0)| + \sum_{P \in \mathcal{P}_0} |P| \geq (\delta(G) + 1) + (|\mathcal{P}| - 1)\delta(G) = |\mathcal{P}|\delta(G) + 1$, and hence $|\mathcal{P}| \leq (n-1)/\delta(G)$.

This completes the proof of Theorem 3. □

## 3 $k$-Proper Partitions

Let $e(k,n)$ be the maximum number of edges in a graph of order $n$ with no $k$-connected subgraph. Define $d(k)$ to be

$$\sup\left\{\frac{2e(k,n)+2}{n} : n > k\right\}$$

and

$$\gamma = \sup\{d(k)/(k-1) : k \geq 2\}.$$

Recall that the average degree of a graph $G$ of order $n$ with $e(G)$ edges is $\frac{2e(G)}{n}$. This leads to the following useful observation.

**Observation 5.** *If $G$ is a graph with average degree at least $\gamma(k-1)$, then $G$ contains a $k$-connected subgraph.*

In [12], Mader proved that $3 \leq \gamma \leq 4$ and constructed a graph of order $n$ with $\left(\frac{3}{2}k-2\right)(n-k+1)$ edges and without $k$-connected subgraphs. This led him to make the following conjecture.

**Conjecture 6.** *If $k \geq 2$, then $e(k,n) \leq \left(\frac{3}{2}k-2\right)(n-k+1)$. Consequently, $d(k) \leq 3(k-1)$ and $\gamma = 3$.*

Note that Conjecture 6 holds when $k=2$, as it is straightforward to show that $e(2,n) = n-1$. The most significant progress towards Conjecture 6 is due to Yuster [18].

**Theorem 7.** *If $n \geq \frac{9}{4}(k-1)$, then $e(k,n) \leq \frac{193}{120}(k-1)(n-k+1)$.*

Note that Theorem 7 requires $n \geq \frac{9}{4}(k-1)$, which means that we cannot immediately obtain a bound on $\gamma$. The following corollary, however, shows that we can use this result in a manner similar to Observation 5.

**Corollary 8.** *Let $G$ be a graph of order $n$ with average degree $\overline{d}$. Then $G$ contains a $\lfloor\frac{60\overline{d}}{193}\rfloor$-connected subgraph.*

*Proof.* Let $k = \lfloor\frac{60\overline{d}}{193}\rfloor$ and suppose that $G$ does not contain a $k$-connected subgraph. If $n \geq \frac{9}{4}(k-1)$, then Theorem 7 implies

$$\frac{1}{2}\overline{d}n = e(G) \leq \frac{193}{120}(k-1)(n-k+1) < \frac{193}{120}\left(\frac{60}{193}\overline{d}\right)n = \frac{1}{2}\overline{d}n.$$

Thus, assume that $n < \frac{9}{4}(k-1)$. This implies that

$$n < \frac{9}{4}(k-1) < \frac{9}{4}\frac{60}{193}\overline{d} < \frac{7}{10}\Delta(G),$$

a contradiction. □

Finally, prior to proving our main result, we require the following simple lemma, which we present without proof.

**Lemma 9.** *If $G$ is a graph of order $n \geq k+1$ such that $\delta(G) \geq \frac{n+k-2}{2}$, then $G$ is $k$-connected.*

We prove the following general result, and then show that we may adapt the proof to improve Theorem 4.

**Theorem 10.** *Let $k \geq 2$ and $c \geq \frac{11}{3}$. If $G$ is a graph of order $n$ with minimum degree $\delta$ with $\delta \geq \sqrt{c\gamma(k-1)n}$, then $G$ has a $k$-proper partition into at most $\lfloor \frac{c\gamma n}{\delta} \rfloor$ parts.*

*Proof.* Since $n > \delta \geq \sqrt{c\gamma(k-1)n}$, we have $n^2 > c\gamma(k-1)n$ and hence $n > c\gamma(k-1) \geq 11(k-1)$. Therefore, by Lemma 9, it follows that

$$\delta < \frac{n+k-2}{2} < \frac{n+(k-1)}{2} \leq \frac{n+\frac{1}{11}n}{2} \leq \frac{6}{11}n.$$

Let $G_0 = G$, $\delta_0 = \delta$, and $n_0 = |V(G)|$. We will build a sequence of graphs $G_i$ of order $n_i$ and minimum degree $\delta_i$ by selecting a $k$-connected subgraph $H_i$ of largest order from $G_i$ and assigning $G_{i+1} = G_i - V(H_i)$. This process terminates when either $G_i$ is $k$-connected or $G_i$ does not contain a $k$-connected subgraph. We claim the process terminates when $G_i$ is $k$-connected and $H_i = G_i$.

By Observation 5, $G_i$ contains a $(\lfloor \frac{\delta_i}{\gamma} \rfloor + 1)$-connected subgraph $H_i$. If $\frac{\delta_i}{\gamma} \geq k-1$, then $H_i$ is $k$-connected and has order at least $\lfloor \frac{\delta_i}{\gamma} \rfloor + 1 > \frac{\delta_i}{\gamma}$. Since $H_i$ is a maximal $k$-connected subgraph in $G_i$, every vertex $v \in V(G_i) \setminus V(H_i)$ has at most $k-1$ edges to $H_i$ by a simple consequence of Menger's Theorem. Therefore, we have

$$\delta_{i+1} \geq \delta_i - (k-1)$$

and

$$n_{i+1} = n_i - |H_i| < n_i - \delta_i/\gamma.$$

This gives us the estimates on $\delta_i$ and $n_i$ of

$$\delta_i \geq \delta - i(k-1),$$

and

$$n_i \leq n - \sum_{j=0}^{i-1} \delta_j/\gamma \leq n - \frac{1}{\gamma}\sum_{j=0}^{i-1} [\delta - j(k-1)] = n - \frac{1}{\gamma}\left[i\delta - (k-1)\binom{i}{2}\right].$$

Let $t = \lceil \frac{c\gamma n}{\delta} - 4 \rceil = \frac{c\gamma n}{\delta} - (4-x)$, where $x \in [0,1)$. We claim that the process terminates with a $k$-proper partition at or before the $(t+1)^{\text{st}}$ iteration (that is, at or before the point of selecting a $k$-connected subgraph from $G_t$). First, we have

$$\delta_{t-1} \geq \delta - (t-1)(k-1) > \delta - \left(\frac{c\gamma n}{\delta} - 4\right)(k-1) = \delta - \frac{c\gamma(k-1)n}{\delta} + 4(k-1).$$

Note that $\delta^2 \geq c\gamma(k-1)n$ and hence $\delta - \frac{c\gamma(k-1)n}{\delta} \geq 0$. Therefore,

$$\delta_{t-1} > 4(k-1) \geq \gamma(k-1) \quad \text{and} \quad \delta_t \geq 3(k-1).$$

As the bound on $\delta_i$ is a decreasing function of $i$, we have $\delta_i > 4(k-1)$ for all $0 \le i \le t-1$. Thus each $G_i$ with $i < t$ contains a $k$-connected subgraph. Next, consider $n_t$.

$$
\begin{aligned}
n_t &\le n - \frac{1}{\gamma}\left[t\delta - (k-1)\binom{t}{2}\right] \\
&= n - \frac{1}{\gamma}\left[c\gamma n - (4-x)\delta - \frac{1}{2}(k-1)\left(\frac{c\gamma n}{\delta} - (4-x)\right)\left(\frac{c\gamma n}{\delta} - (5-x)\right)\right] \\
&= n - \frac{1}{\gamma}\left[c\gamma n - (4-x)\delta - \frac{c^2\gamma^2(k-1)}{2}\frac{n^2}{\delta^2} + \frac{c\gamma(9-2x)(k-1)}{2}\frac{n}{\delta} - \frac{1}{2}(k-1)(4-x)(5-x)\right] \\
&= \frac{1}{\delta^2}\left[\frac{4-x}{\gamma}\delta^3 + \frac{c^2\gamma(k-1)}{2}n^2 - (c-1)n\delta^2\right] + \frac{(4-x)(5-x)}{2\gamma}(k-1) - \frac{c(9-2x)(k-1)}{2}\frac{n}{\delta}.
\end{aligned}
$$

We have $\delta^2 \ge c\gamma(k-1)n$ and $(c-1)^2 \ge c^2/2$, so

$$
\frac{(c-1)}{c}((c-1)n\delta^2) \ge (c-1)^2\gamma(k-1)n^2 \ge \frac{c^2}{2}\gamma(k-1)n^2.
$$

Also, we have $n > \frac{11}{6}\delta$, and $\frac{c-1}{c} \ge \frac{8}{11}$, hence

$$
\frac{1}{c}((c-1)n\delta^2) > \frac{8}{11}\cdot\frac{11}{6}\delta^3 = \frac{4}{3}\delta^3 \ge \frac{4-x}{\gamma}\delta^3.
$$

Summing these inequalities, we get that

$$
\left[\frac{4-x}{\gamma}\delta^3 + \frac{c^2\gamma(k-1)}{2}n^2 - (c-1)n\delta^2\right] < 0
$$

and hence $n_t < \frac{(4-x)(5-x)}{2\gamma}(k-1) \le \frac{20}{2\gamma}(k-1) \le \frac{10}{3}(k-1)$. However, $\delta_t \ge 3(k-1)$, so if the process has not terminated prior to the $(t+1)^{\text{st}}$ iteration, $G_t$ is $k$-connected by Lemma 9. □

Theorem 10 immediately yields the following.

**Corollary 11.** *Suppose Conjecture 6 holds. We then see that if $G$ is a graph with minimum degree $\delta$ where $\delta \ge \sqrt{11(k-1)n}$, then $G$ has a $k$-proper partition into at most $\frac{11n}{\delta}$ parts.*

We are now ready to prove Theorem 4.

*Proof.* Observe that the proof of Theorem 10 holds at every step when substituting $\gamma = \frac{193}{60}$ by using Corollary 8 to imply that $G_i$ contains a $\lfloor\frac{60\delta_i}{193}\rfloor$-connected subgraph. Finally, note that $\left(\frac{11}{3}\right)\frac{193}{60} = \frac{2123}{180}$. □

# 4 Application: Edit Distance to the Family of $k$-connected Graphs

Define the *edit distance* between two graphs $G$ and $H$ to be the number of edges one must add or remove to obtain $H$ from $G$ (edit distance was introduced independently in [2, 3, 16]). More generally, the edit distance between a graph $G$ and a set of graphs $\mathcal{G}$ is the minimum edit distance between $G$ and some graph in $\mathcal{G}$.

Utilizing Theorem 4 and observing that $2123/180 = 11.79\overline{4} < 11.8$ we obtain the following corollary, which is a refinement of Corollary 11 in [5] for large enough $k$.

**Corollary 12.** *Let $k \geq 2$ and let $G$ be a graph of order $n$. If $\delta(G) \geq \sqrt{11.8(k-1)n}$, then the edit distance between $G$ and the family of $k$-connected graphs of order $n$ is at most $\frac{11.8kn}{\delta(G)} - k < k(4\sqrt{n} - 1)$.*

*Proof.* Let $H_1, \ldots, H_l$ be the $k$-connected subgraphs of the $k$-proper partition of $G$ guaranteed by Theorem 4; note that $l \leq \frac{11.8n}{\delta(G)}$. For each $i \in \{1, \ldots, l-1\}$, it is possible to produce a matching of size $k$ between $H_i$ and $H_{i+1}$ by adding at most $k$ edges between $H_i$ and $H_{i+1}$. Thus, adding at most $k\left(\frac{11.8n}{\delta(G)}\right)$ edges yields a $k$-connected graph. □

# 5 Conclusion

We note here that it is possible to slightly improve the degree conditions in Theorems 4 and 10 at the expense of the number of parts in the partition. In particular, a greedy approach identical to that used to prove Theorem 10 can be used to prove the following.

**Theorem 13.** *Let $k \geq 2$, $c_k \geq \frac{k-1}{k} \cdot 2\gamma$, and $p = \sqrt{\frac{c_k n}{k}}$. If $G$ is a graph of order $n$ with $\delta(G) \geq kp = \sqrt{c_k k n}$, then $G$ has a $k$-proper partition into at most $\frac{k}{k-1}p$ parts.*

This gives rise to the following, which improves on the degree condition in Theorem 4.

**Theorem 14.** *If $G$ is a graph of order $n$ with minimum degree*

$$\delta(G) \geq kp = \sqrt{\frac{193}{30}(k-1)n},$$

*then $G$ has a $k$-proper partition into at most $\frac{k}{k-1}p$ parts.*

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