Effective Behavior of Dielectric Elastomer Composites

Thesis by

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Abstract

The class of electroactive polymers has been developed to a point where real life applications as “artificial muscles” are conceivable. These actuator materials provide attractive advantages: they are soft, lightweight, can undergo large deformation, possess fast response time and are resilient. However, widespread application has been hindered by their limitations: the need for a large electric field, relatively small forces and energy density. However, recent experimental work shows great promise that this limitation can be overcome by making composites of two materials with high contrast in their dielectric modulus. In this thesis, a theoretical framework is derived to describe the electrostatic effect of the dielectric elastomers. Numerical experiments are conducted to explain the reason for the promising experimental results and to explore better microstructures of the composites to enhance the favorable properties.

The starting point of this thesis is a general variational principle, which characterizes the behavior of solids under combined mechanical and electrical loads. Based on this variational principle, we assume the electric field is small as of order $\varepsilon^{\frac{1}{2}}$, assume further the deformation is caused by the electrostatic effects; the deformation field is then of order ε. Using the tool of Γ-convergence, we derive a small-strain model in which the electric field and the deformation field are decoupled which results in a huge simplification of the problem.

Based on this small-strain model, employing the powerful tool of two-scale convergence, we derive the effective properties for dielectric composites conducting small strains. A formula of the effective electromechanical coupling coefficients is given in terms of the unit cell solutions.

Armed with these theoretical results, we carry out numerical experiments about

the effective properties of different kind of composites. A very careful analysis of the numerical results provides a deep understanding of the mechanism of the enhancement in strain by making composites of different microstructures.

Contents

List of Figures

Chapter 1

Introduction

Electroactive polymers (EAP) are polymers that can change their shape in response to electrical stimulation. These lightweight and flexible actuators can be used in a wide variety of applications such as robotic manipulators and vehicles, active damping and conformal control surfaces. Moreover, these actuators can be miniaturized and incorporated into MEMS (Micro-Electro-Mechanical Systems). In comparison with other types of active materials such as EAC (electroactive ceramics) and SMA (shape memory alloys), EAPs can undergo large strains [24, 39, 33, 41], their response time is shorter [24], their density is lower and their resilience is greater [5]. At the present, their main limitations are low actuation force and low mechanical energy density.

Broadly speaking, there are three classes of electroactive polymers: dielectric elastomers, ionic polymers and ferric/liquid crystal elastomers. The first class is the most developed and closest to application [34, 25]. Roughly speaking, they actuate by squeezing a piece of elastomer between electrodes. The second class, the ionic polymers including gels and conductive polymers, actuates by the differential deformation induced by the electric-field-induced diffusion of ions. They tend to operate under small fields, but are slow and require a controlled environment. Ferroelectric as well as liquid crystal elastomers are new and promising [6, 38], and undergoing rapid development.

The promising first class is limited by the large electric fields ($\backsim$100 MV/m) they require for meaningful actuation. The reason for this is poor electromechanical coupling due to the fact that the typical polymers have a limited ratio of dielectric to elastic modulus (flexible polymers have low dielectric modulus while high dielectric modulus polymers are stiff) [22]. Recent experimental works suggest that this limitation can be overcome by making electroactive polymer composites (EAPC) by combining an elastomer with a high dielectric or even conductive material [43, 22, 23]. Remarkably these works show that the effective electromechanical coupling is significantly larger than that can be expected naively from the ratio of effective dielectric to effective elastic moduli. The reason for this high enhancement was pointed out by Li, Huang, and Zhang [26, 27]. They pointed out that the electromechanical coupling is nonlinear, and hence the effective behavior of the composite depends on the mean square of the electric field rather than the square-mean. It follows that the contrast between component properties promotes field fluctuation, and this in turn results in the enhancement of the effective electromechanical coupling.

Within the class of electronic polymers two types of coupling between the electrical and the mechanical fields are broadly distinguished, piezoelectric and electrostrictive. The piezoelectric effect is a linear (and more generally odd) coupling between the mechanical stress/strain and the electric field/displacement current. Piezoelectric systems are reasonably well understood. For heterogeneous systems (composites), the theoretical implications of the linear coupling have been examined extensively, especially for small strains (see, e.g., [7]).

Electrostriction is a nonlinear effect where the strains depend quadratically (and more generally in an even manner) on the applied electric field. It can arise due to inherent material properties, or due to electrostatic effects through a Maxwell stress. The dielectric elastomers discussed earlier are electrostrictive effect as a result of electrostatic effects. From a theoretical point of view, the behavior of the composites of electrostrictive materials is less known. This provides one of the motivations for this thesis. Our aim is to develop a general theory for defining, calculating and understanding the effective properties of electrostrictive composites under small-strain assumption. This theory builds on the insights offered by Li et al. [26, 27] and gives an engineer a tool to develop composites with high coupling.

Another motivation of this thesis is a rigorous derivation of a small-strain approx-

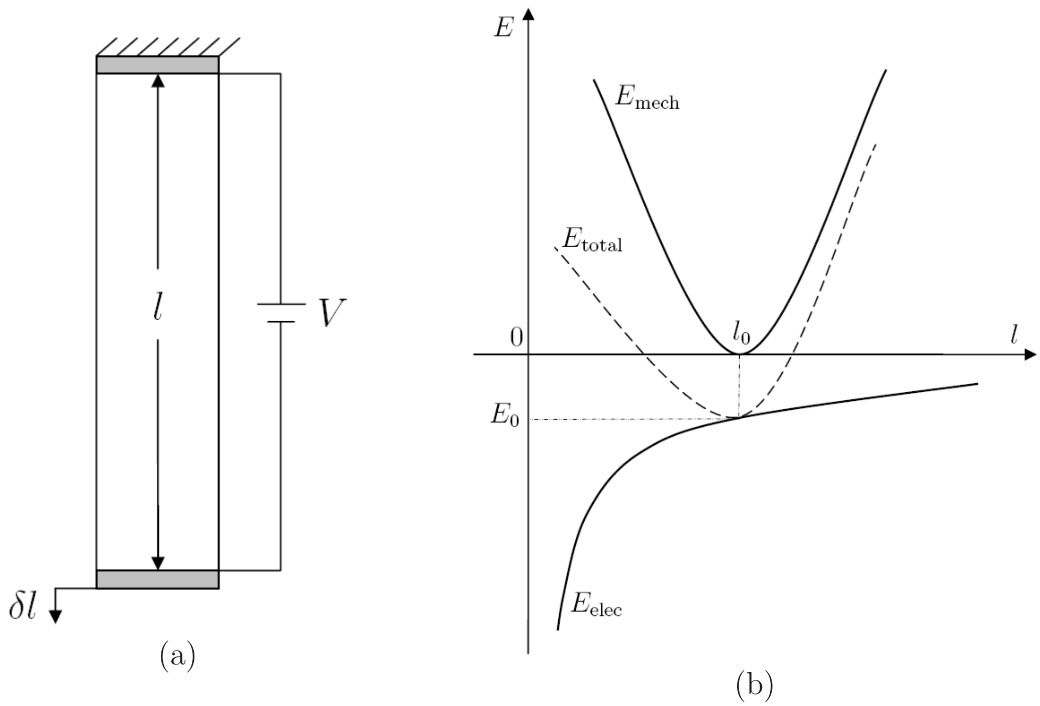


Figure 1.1: One dimensional scale analysis.

imation for electrostrictive materials. The difficulties were pointed out by Toupin [37] and are introduced in Chapter 2 as a formal computation.

The basic idea is apparent in a one-dimensional model calculation. Consider a piece of one dimensional electroactive polymer as shown in Figure 1.1(a). The natural length of the elastomer is l_0. The current length under voltage V is l. Denote the strain to be $e = \frac{\delta l}{l_0}$, where $\delta l = l - l_0$. Assume the polarization caused by the electric field is p. Then the energy of this one dimensional system is

$$E_{total} = \int_0^{l_0} W\left(\frac{l}{l_0}\right) dx + \int_0^{l} \left(\frac{\alpha}{2}|p|^2 - p \cdot \frac{V}{l}\right) dy.$$

The solutions of this system, the polarization and the deformation, are the minimizers of the above function. If we minimize E with respect to p, we get $p = \frac{V}{2\alpha}$, so that

$$E_{total} = W\left(\frac{l}{l_0}\right) l_0 - \frac{V^2}{2\alpha l}.$$

The two terms are shown in Figure 1.1(b) as E_{mech} and E_{elec}, and the sum as E_{total}.

Now assume that both V and e are small and expand the energy around $V = 0$ and $e = 0$. Assume further that $W(1) = 0, W'(1) = 0$. A simple calculation leads to

$$E \simeq \frac{l_0}{2} W''(1)\, e^2 - \frac{V^2}{2\alpha l_0} + \frac{V^2}{2\alpha l_0}\, e + \cdots$$

The first term on the right hand side is purely mechanical, the second term is purely electrostatic and the third term is the electromechanical coupling. Note that if both the electric field and the strain are small, the electromechanical coupling is smaller than the electrostatic energy, irrespective of the relative magnitude of V and e. Further, assume that the deformation is of order ε. If the electric field has the same order, the coupling term can be neglected. Thus there is no electromechanical coupling with this scaling.

If, however, the electric field is of order $\varepsilon^{\frac{1}{2}}$, things are quite different. The leading order energy is the electrostatic energy $E_0 = -\frac{V^2}{2\alpha l_0}$, which is of order ε. E_0 depends only on the electric field and does not involve the deformation of the material. The second order energy is $E - E_0 = \frac{l_0}{2} W''(1)\, e^2 + \frac{V^2}{2\alpha l_0}\, e$. It is the sum of the mechanical and the coupling terms. In fact, we may view this as the mechanical energy with the forcing provided by the electric field and it can be looked as a correction term to the leading order energy. This two-order expansion is also clear from Figure 1.1(b): The ground state is E_0 given by the electrostatics, and the correction is mechanics driven by the electrostatics. This suggests that we first compute the electrostatic field in the reference configuration and then use it as the forcing term in the mechanical problem. The second motivation is to generalize this idea to multidimensions.

This thesis is divided into seven chapters. Chapter 2 is the mathematical description of the physical problem together with formal computations. We consider a piece of electroactive polymer attached with two thin layers of conductive electrodes. The starting point is a variational principle which characterizes the behavior of the electroactive polymer under electrical loads. This goes back to Toupin [37], but follows the formulation in Shu and Bhattacharya [35], Xiao and Bhattacharya [40]. We define the relevant function spaces and give an existence result for the solution of the Maxwell equation. Next, the Euler-Lagrange equations for this variational principle are derived and analyzed under the small-strain assumption. If we consider the electrostatic or electrostrictive coupling, the formal Taylor expansion of the Euler-Lagrange equations suggests that the electromechanical coupling is present only when the deformation has the same order as the square of the electric field. Specifically, if the strain is of order ε, then the electromechanical coupling is present only if the electric field is of order $\varepsilon^{\frac{1}{2}}$. On the other hand, if we consider the piezoelectric coupling, formal calculation reveals that the electromechanical coupling occurs when the deformation field has the same order with the electric field.

By the formal calculation, the small-strain model for dielectric elastomers consists of two equations. One of them is an elliptic equation in the reference configuration for the electric field. The second equation is one of linear elasticity with the Maxwell stress that is known from the solution of the electric field in the first equation acting as the force term. This decoupling between the electric field and the strain field provides a significant simplification of the original problem that allows us to carry out a detailed analysis of the properties of dielectric EAPs.

In Chapter 3, we provide the rigorous proof for the above formal calculation of dielectric elastomers. The main tool we use in this chapter is Γ-convergence [18, 8, 29]. We rescale the electric field with $\varepsilon^{\frac{1}{2}}$ and the energy with ε. In the first part of this chapter, the energy functional is shown to Γ-converge to a limit functional which does not depend on the deformation. The equation for the electric field is derived after that as the Euler-Lagrange equation of the limit functional. The main difficultly in this part of the proof is that the strain field converges only in an integral norm, not in $\mathcal{L}^\infty$ norm. Thus the function space on the reference region is not isomorphic to the function space on the deformed region. We have to deal with this issue with care. In the second part of this chapter, the order ε correction term of the first order energy functional is derived. The Euler-Lagrange equation of this correction term gives the strain equation. In this part, to make the Maxwell stress belong to the dual space of the strain field, we assume the electric field to be quite regular. In fact, we need the electric field to be $\mathcal{L}^4$ bounded in some compact set containing the reference region.

This provides some restriction to the geometries and microstructures that one can consider in Chapter 4.

Chapter 4 is devoted to developing a homogenization theory for the heterogeneous dielectric elastomers based on the small-strain model. Using the tool of two-scale convergence, we derive a formula for the effective electromechanical coupling coefficients. From this formula, the effective electromechanical coupling is composed of two parts. One is the average Maxwell stress, and the second comes from the fluctuation of the Maxwell stress. The main difficulty of this chapter is to find out the two-scale limit of the Maxwell stress. We need the local strong convergence of the electric field and it is proven by combining the two-scale convergence with the local estimate of the oscillation terms.

In Chapter 5, we derive a simple formula to compute the effective electromechanical coupling for dielectric laminates. Numerical results are provided to show the effect of some parameters such as the volume fraction and the lamination angle on the effective electrocoupling of the laminates. An interesting phenomenon is observed which distinguishes the nonlinear electrostriction effect from the linear piezoelectric effect. Unlike the linear coupling of piezoelectric composites, the effective electromechanical coupling of the dielectric laminates can exceed the electromechanical coupling of each individual component material. More surprisingly, the example suggests that an infinite strain can be obtained by sequential laminates. The numerical experiments are consistent with the experimental results given by Huang and Zhang [22]. The careful analysis of the numerical results reveals that the key to the enhancement of the longitudinal strain is the high ratio of the dielectric modulus of the two constituent materials. This provides a fluctuation of the dielectric modulus for the heterogeneous dielectric material, which causes the fluctuation of the electric field and thus an oscillation of the Maxwell stress. The Maxwell stress in the compliant phase of the heterogeneous elastomer is very large, and a large shear strain is generated because of this.

Chapter 6 is devoted to the numerical experiments on particulate composites in order to highlight fundamental parameters that influence the overall response of the electroactive polymer composites. With the guidelines gained from the analytical study, unit cell solutions are constructed in a finite element code. We computed an ellipsoidal stiff phase inside a square compliant dielectric material. Numerical results show that for fixed volume fraction, the larger the ratio of the long axis length to the short axis length for the ellipsoid, the larger the effective longitudinal strain. This suggests that fiber like inclusion is a favorable inclusion to enhance the longitudinal strain. Another factor that affects the longitudinal strain is the distance between the inclusions. In order to make use of the squeezing effect caused by the fluctuation of the electric field, it is very crucial to take a right distance between the inclusions.

In Appendix A, we give the rigorous derivation of the small-strain model for piezoelectric materials.

This thesis is by no means the end of the story. There are still numerous interesting and open problems about this topic. In Chapter 7, we discuss some of the possible directions.

Chapter 2

Variational Principle of Electroactive Polymers

2.1 Kinematics and Electrostatics

Consider a piece of electroactive polymer occupying a domain $\Omega \subset \mathbb{R}^N$ as shown in Figure 2.1. Assume that this reference region Ω has a relatively good regularity: it satisfies the strong local Lipschitz condition, and there exists a constant r_0, $2 < r_0 < \infty$, such that $\Omega \in \mathfrak{B}^{r_0}$ (see Definition 3.2). We attach two thin layers of conductors C_0, C_1 to the material and apply an external electric field. The interaction between the electroactive polymer and the applied electric field causes a deformation $y : \Omega \to \mathbb{R}^N$ that brings the material to another shape $y(\Omega)$. We assume that the deformation is invertible and $J = \det F > 0$ almost everywhere in Ω, where $F(x) = \nabla_x y(x)$ is the deformation gradient.

Denote by $p : y(\Omega) \to \mathbb{R}^N$ the polarization of the electroactive polymer per unit deformed volume and by $p_0 : \Omega \to \mathbb{R}^N$ the polarization per unit reference volume. The relation between $p(y)$ and $p_0(x)$ is then

$$p_0(x) = \det\big(\nabla_x y(x)\big)\, p\big(y(x)\big) = J(x)\, p\big(y(x)\big). \tag{2.1}$$

The polarization of the material together with the conductors generate an electric field in the entire space. The electrostatic potential φ_e at any point in $\mathbb{R}^N$ is obtained

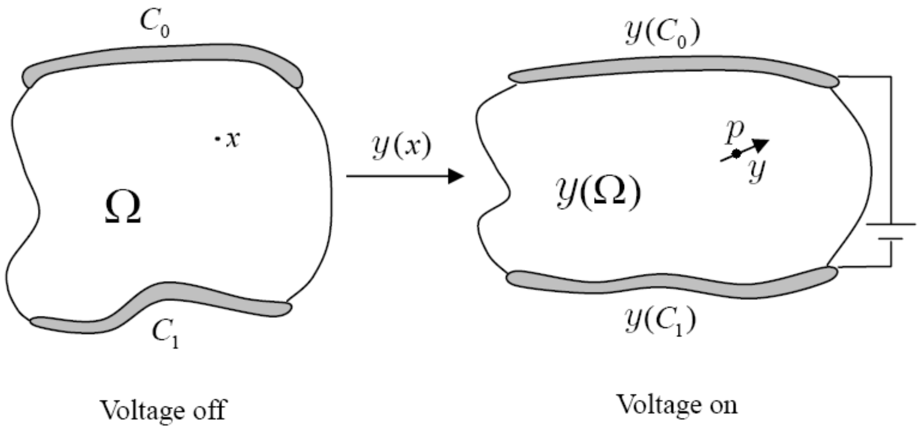


Figure 2.1: Principle of operation of electroactive polymers.

by solving the Maxwell equation

$$\begin{cases} \nabla_y \cdot \big[-\varepsilon_0 \nabla_y \varphi_e + p\,\chi\big(y(\Omega)\big)\big] = 0 & \text{in} \quad \mathbb{R}^N \backslash y(C), \\ \nabla_y \varphi_e = 0 & \text{in} \quad y(C_0), \\ \nabla_y \varphi_e = 0 & \text{in} \quad y(C_1), \end{cases} \tag{2.2}$$

subject to boundary conditions $\varphi_e = g_0$ on $\partial y(C_0)$, $\varphi_e = g_1$ on $\partial y(C_1)$, where g_0 and g_1 are two given constants. Above, we have denoted $C = C_0 \cup C_1$, and $y(C) = y(C_0) \cup y(C_1)$.

Since g_0 and g_1 are constants, (2.2) is equivalent to

$$\begin{cases} \nabla_y \cdot \big[-\varepsilon_0 \nabla_y \varphi_e + p\,\chi\big(y(\Omega)\big)\big] = 0 & \text{in} \quad \mathbb{R}^N \backslash y(C), \\ \varphi_e = g_0 & \text{in} \quad y(C_0), \\ \varphi_e = g_1 & \text{in} \quad y(C_1). \end{cases} \tag{2.3}$$

Again, because g_0 and g_1 are constants, there exist $\Omega_y^0 \supset y(C_0)$ and $\Omega_y^1 \supset y(C_1)$ such that $\overline{\Omega_y^0} \cap \overline{\Omega_y^1} = \varnothing$. Construct $g(y)$ such that $g(y) \in \mathcal{H}_0^1(\mathbb{R}^N)$ with compact support and satisfying

$$g(y) = \begin{cases} g_0 & \text{in} \quad \Omega_y^0, \\ g_1 & \text{in} \quad \Omega_y^1. \end{cases}$$

Then if we define $\varphi(y) = \varphi_e(y) - g(y)$, $\varphi(y)$ satisfies

$$\begin{cases} \nabla_y \cdot \big[-\varepsilon_0 \nabla_y \varphi + p\,\chi\big(y(\Omega)\big) - \varepsilon_0 \nabla_y g\,\big] = 0 & \text{in} \quad \mathbb{R}^N \backslash y(C), \\ \varphi = 0 & \text{on} \quad \partial y(C_0), \\ \varphi = 0 & \text{on} \quad \partial y(C_1). \end{cases} \tag{2.4}$$

To define the function space for $\varphi(y)$, denote first

$$\mathfrak{D}(\mathcal{O}) = \big\{\phi \,\big|\, \phi \in C_0^\infty(\mathcal{O})\big\}$$

for any open set $\mathcal{O} \subset \mathbb{R}^N$. Space $\mathfrak{D}(\mathcal{O})$ is linear. Now, equip $\mathfrak{D}$ with norm $\|\phi\|^2 = (\nabla\phi, \nabla\phi)_{\mathcal{L}^2(\mathbb{R}^N)}$ and let $\mathcal{D}(\mathcal{O})$ be the completion of $\mathfrak{D}(\mathcal{O})$ under this norm. Then $\mathcal{D}(\mathcal{O})$ is a Hilbert space. For problem (2.4), let us consider $\mathcal{D}\big(\mathbb{R}^N \backslash y(C)\big)$. We say that $\varphi \in \mathcal{D}\big(\mathbb{R}^N \backslash y(C)\big)$ is the weak solution of equation (2.4) if

$$\varepsilon_0 \int_{\mathbb{R}^N} \nabla_y \varphi \cdot \nabla_y \psi \, dy + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_y \psi \cdot \nabla_y g \, dy = \int_{y(\Omega)} p \cdot \nabla_y \psi \, dy, \tag{2.5}$$

$\forall \psi \in \mathcal{D}\big(\mathbb{R}^N \backslash y(C)\big)$.

In this space, the bilinear form

$$\mathcal{L}(\varphi, \psi) := \varepsilon_0 \int_{\mathbb{R}^N} \nabla_y \varphi \cdot \nabla_y \psi \, dy$$

is a coercive bounded bilinear operator. The integration is on the entire space, because we extend any function in $\mathcal{D}\big(\mathbb{R}^N \backslash y(C)\big)$ to the entire space by zero. $\nabla_y g \in \mathcal{L}^2(\mathbb{R}^N) \subset \mathcal{D}^{-1}\big(\mathbb{R}^N \backslash y(C)\big)$ and $p \in \mathcal{L}^2\big(y(\Omega)\big) \subset \mathcal{D}^{-1}\big(\mathbb{R}^N \backslash y(C)\big)$. Therefore, the Lax-Milgram theorem applies, i.e., there exists a unique solution for the electric potential equation (2.4) and the solution satisfies

$$\big\|\nabla_y \varphi\big\|_{\mathcal{L}^2(\mathbb{R}^N)} \leqslant \frac{1}{\varepsilon_0} \left(\big\|p\big\|_{\mathcal{L}^2(y(\Omega))} + \big\|\varepsilon_0 \nabla_y g\big\|_{\mathcal{L}^2(\mathbb{R}^N)} \right). \tag{2.6}$$

Remark If $\partial y(C)$ is regular, for example, if it is Lipschitz continuous, then space

$\mathcal{D}\big(\mathbb{R}^N\backslash y(C)\big)$ is equivalent to the space

$$\mathcal{D}_1\big(\mathbb{R}^N\backslash y(C)\big) := \big\{\psi \mid \psi\in\mathcal{L}^2_{\rm loc}(\mathbb{R}^N),\ \nabla\psi\in\mathcal{L}^2(\mathbb{R}^N),\ \gamma(\psi)=0\ \text{ on }\ \partial y(C)\big\}.$$

$\mathcal{D}_1\big(\mathbb{R}^N\backslash y(C)\big)$ is a complete Hilbert space under norm $\|\psi\|^2 = (\nabla\psi,\nabla\psi)$. In fact, for any sequence ψ^i with $\nabla\psi^i$ a Cauchy sequence in $\mathcal{L}^2(\mathbb{R}^N)$, there exists $w = (w_1,\ldots,w_N)$, $w_i\in\mathcal{L}^2(\mathbb{R}^N)$ such that $\nabla\psi^i\to w$, since $\mathcal{L}^2(\mathbb{R}^N)$ is a complete Hilbert space. Now we need to find a function $\psi_0\in\mathcal{L}^2_{\rm loc}(\mathbb{R}^N)$ such that $\nabla\psi_0 = w$. To do this, we use the following lemma (see, for example, [13]).

Lemma 2.1 *Suppose $\mathcal{O}$ is a connected open set with $\partial\mathcal{O}$ Lipschitz continuous. Assume that $\partial\mathcal{O}=\Gamma_1\cup\Gamma_2$, where Γ_1 and Γ_2 are closed sets with $\Gamma_1\cap\Gamma_2=\varnothing$. Γ_1 has positive measure. Then there exists a constant $c(\mathcal{O})$ such that*

$$\big\|u\big\|_{\mathcal{L}^2(\mathcal{O})}\leqslant c(\mathcal{O})\,\big\|\nabla u\big\|_{\mathcal{L}^2(\mathcal{O})},$$

$\forall u\in\mathcal{H}^1(\mathcal{O})$ with $\gamma(u)=0$ on Γ_1.

Armed with this lemma, we resume the above discussion. Consider a compact set $D\supset y(C)$. Using Lemma 2.1, ψ^i form a Cauchy sequence with respect to $\mathcal{H}^1(D)$ norm on D. So there exists a $\psi_0(D)\in\mathcal{H}^1(D)$ such that $\nabla\psi_0=w$ and $\psi^i\to\psi_0$ in $\mathcal{H}^1(D)$. By the continuity of trace with respect to $\mathcal{H}^1$ norm, $\gamma(\psi_0)=0$ on $\partial y(C)$. Enlarging D, ψ_0 is then defined on the entire space $\mathbb{R}^N$ and it is in $\mathcal{L}^2_{\rm loc}(\mathbb{R}^N)$. Therefore, $\mathcal{D}_1$ is complete and thus $\mathcal{D}_1\big(\mathbb{R}^N\backslash y(C)\big)=\mathcal{D}\big(\mathbb{R}^N\backslash y(C)\big)$ if $\partial y(C)$ is Liptschtz continuous.

$\square$

2.2 Variational Principle

The total energy of the system described above is

$$\begin{aligned}\mathcal{F} &= \int_\Omega W(x,F,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y \varphi_e\right|^2 dy \\ &\quad - \int_{\partial y(C)} g(y)\big(-\varepsilon_0 \nabla_y \varphi_e + p\,\chi(y(\Omega))\big)\cdot n_C\,dS_y + \int_{y(C)} W_c\,dy. \end{aligned} \tag{2.7}$$

Here, W is the stored energy per unit reference volume in the electroactive polymer. It is a function of the deformation gradient F and the polarization p_0. W_c is the elastic energy density of the conductor layers. n_C is the outward pointing normal vector to $\partial y(C)$.

Recalling that $\varphi_e = \varphi + g(y)$, (2.7) can be rewritten as

$$\begin{aligned}\mathcal{F} &= \int_\Omega W\left(x,F,p_0\right)\,dx + \int_{y(C)} W_c\,dy \\ &\quad + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi\right|^2 dy + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y g\right|^2 dy + \varepsilon_0\int_{\mathbb{R}^N}\nabla_y\varphi\cdot\nabla_y g\,dy \\ &\quad - \int_{\partial y(C)} g(y)\big(-\varepsilon_0\nabla_y\varphi - \varepsilon_0\nabla_y g + p\,\chi(y(\Omega))\big)\cdot n_C\,dS_y. \end{aligned} \tag{2.8}$$

Multiplying equation (2.4) by g, we get

$$\begin{aligned} &\varepsilon_0\int_{\mathbb{R}^N}\nabla_y g\cdot\nabla_y\varphi\,dy \\ &= -\varepsilon_0\int_{\mathbb{R}^N}|\nabla_y g|^2\,dy + \int_{y(\Omega)}\nabla_y g\cdot p\,dy \\ &\quad + \int_{\partial y(C)} g(y)\big(-\varepsilon_0\nabla_y\varphi + p\,\chi(y(\Omega))\big)\cdot n_C\,dS_y. \end{aligned} \tag{2.9}$$

The last term follows because $g \equiv g_0$ on Ω_y^0, $g \equiv g_1$ on Ω_y^1.

Plugging (2.9) into (2.8), the energy becomes,

$$\begin{aligned}\mathcal{F} &= \int_\Omega W\left(x,F,p_0\right)dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi\right|^2 dy \\ &\quad + \int_{y(\Omega)}\nabla_y g\cdot p\,dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}|\nabla_y g|^2 dy + \int_{y(C)} W_c\,dy. \qquad (2.10)\end{aligned}$$

Remark. To see the physical meaning of the energy more clearly, we split the electric field into two parts. One is φ_p, the solution of

$$\begin{cases}\nabla_y\cdot\left[-\varepsilon_0\nabla_y\varphi_p + p\,\chi\big(y(\Omega)\big)\right] = 0 & \text{in} \quad \mathbb{R}^N\backslash y(C),\\ \varphi_p = 0 & \text{on} \quad \partial y(C).\end{cases} \qquad (2.11)$$

Another is φ_{ext}, the solution of

$$\begin{cases}\nabla_y\cdot\left[-\varepsilon_0\nabla_y\varphi_{\text{ext}}\right] = 0 & \text{in} \quad \mathbb{R}^N\backslash y(C),\\ \varphi_{\text{ext}} = g_0 & \text{on} \quad \partial y(C_0),\\ \varphi_{\text{ext}} = g_1 & \text{on} \quad \partial y(C_1).\end{cases} \qquad (2.12)$$

Clearly, φ_p is the electric field induced by the polarization of the electroactive polymer, φ_{ext} is the external electric field and $\varphi_e = \varphi_p + \varphi_{\text{ext}}$.

Plug $\varphi_e = \varphi_p + \varphi_{\text{ext}}$ into $\mathcal{F}$:

$$\begin{aligned}\mathcal{F} &= \int_\Omega W\left(x,F,p_0\right)dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi_{\text{ext}}\right|^2 dy + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi_p\right|^2 dy \\ &\quad + \varepsilon_0\int_{\mathbb{R}^N}\nabla_y\varphi_{\text{ext}}\cdot\nabla_y\varphi_p\,dy - \int_{\partial y(C)} g(y)\big(-\varepsilon_0\nabla_y\varphi_p + p\,\chi(y(\Omega))\big)\cdot n_C dS_y \\ &\quad - \int_{\partial y(C)} g(y)(-\varepsilon_0\nabla_y\varphi_{\text{ext}})\cdot n_C\,dS_y + \int_{y(C)} W_c\,dy. \qquad (2.13)\end{aligned}$$

On the right hand side, we have

$$\begin{aligned}&\varepsilon_0\int_{\mathbb{R}^N}\nabla_y\varphi_{\text{ext}}\cdot\nabla\varphi_p\,dy \\ &= \int_{y(\Omega)}\nabla_y\varphi_{\text{ext}}\cdot p\,dy + \int_{\partial y(C)} g(y)\big(-\varepsilon_0\nabla_y\varphi_p + p\,\chi(y(\Omega))\big)\cdot n_C\,dS_y \qquad (2.14)\end{aligned}$$

by multiplying equation (2.11) with φ_{ext}. We also have

$$\varepsilon_0 \int_{\mathbb{R}^N} \left|\nabla_y \varphi_{\text{ext}}\right|^2 dy = \int_{\partial y(C)} g(y)\left(-\varepsilon_0 \nabla_y \varphi_{\text{ext}}\right) \cdot n_C \, dS_y \tag{2.15}$$

by multiplying equation (2.12) with φ_{ext}. Substituting (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} \mathcal{F} &= \int_\Omega W\left(x, F, p_0\right) dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi_p\right|^2 dy \\ &\quad + \int_{y(\Omega)} \nabla_y \varphi_{\text{ext}} \cdot p \, dy - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi_{\text{ext}}\right|^2 dy + \int_{y(C)} W_c \, dy. \end{aligned} \tag{2.16}$$

The physical meaning of the energy functional (2.16) is now evident. The first term is the stored energy. The second term is the electric potential energy induced by the polarization. The third term is the interaction between the polarization and the external electric field. The fourth term is the external electric field energy and the last term is the stored elastic energy in the conductive layers. The last two terms are constants if the external field is fixed, or in other words, the electrode layers are detached from the material.

□

We seek to simplify this problem by neglecting the thickness of the two thin layers of electrodes, i.e., to replace C and $y(C)$ with manifolds of dimension $N-1$. To do this, we need to address two issues. First we neglect the elastic energy of the electrodes, the last term in (2.10). This is reasonable. Second, we need to ensure that the electrostatic terms remain meaningful. This requires some care and follows from the work of Bucur and Buttazzo [10, 11, 9].

Definition 2.1 *Let $\mathcal{O}$ be an open bounded set in $\mathbb{R}^N$, and let Ω_n and Ω be open subsets of $\mathcal{O}$. We say Ω_n γ-converges to Ω if for every $f \in \mathcal{H}^{-1}(\mathcal{O})$, $u_{\Omega_n,f} \to u_{\Omega,f}$*

strongly in $\mathcal{H}_0^1(\mathcal{O})$*, where* $u_{\Omega,f}$ *is the solution for the problem*

$$\begin{cases} -\Delta u_{\Omega,f} = f, \\ u_{\Omega,f} \in \mathcal{H}_0^1(\Omega). \end{cases} \tag{2.17}$$

Definition 2.2 *The Hausdorff distance between two open set* Ω_1 *and* Ω_2 *is defined as*

$$d_{H^c}(\Omega_1, \Omega_2) = d\,(\Omega_1^c, \Omega_2^c).$$

Lemma 2.2 ([10]) $\Omega_n \rightarrow \Omega$ *in* H^c*-convergence is equivalent to* $\Omega_n \rightarrow \Omega$ *in* γ*-convergence if* Ω_n *is in any of the following list of domain classes:*

- *The class of* $\mathcal{A}_{\text{unif cone}}$ *of domains satisfying a uniform exterior cone property, i.e., for every point* x_0 *on the boundary of every* $\Omega \in \mathcal{A}_{\text{unif cone}}$*, there is a closed cone, with uniform height and opening, and with vertex in* x_0*, lying in the complement of* Ω*.*

- *The class* $\mathcal{A}_{\text{unif flat cone}}$ *of domains satisfying a uniform flat cone condition, i.e., as above, but with the weaker requirement that the cone maybe flat of dimension* $N-1$*.*

- $\mathcal{A}_{\text{cap density}}$*, satisfying a uniform capacity density condition, i.e, there exist* $c, r > 0$ *such that for every* $\Omega \in \mathcal{A}_{\text{cap density}}$*, and for every* $x \in \partial\Omega$*, we have*

$$\frac{\text{Cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\text{Cap}(B_{x,t}, B_{x,2t})} \geqslant c \qquad \forall t \in (0, r).$$

- *The class* $\mathcal{A}_{\text{unif Wiener}}$ *satisfying a uniform Wiener condition, i.e., for every point* $x \in \partial\Omega$

$$\int_r^R \frac{\text{Cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\text{Cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geqslant G(r, R, x) \qquad \forall\, 0 < r < R < 1,$$

where $G : (0,1) \times (0,1) \times \mathcal{O} \longrightarrow R_+$ *is fixed, such that for every* $R \in (0,1)$

$$\lim_{r \to 0} G(r, R, x) = +\infty \qquad \text{locally uniformly on } x.$$

- *For $N = 2$, the class of all open subsets Ω of $\mathcal{O}$ for which the number of connected components of $\bar{\mathcal{O}}\backslash\Omega$ is uniformly bounded.*

The following inclusions can be established [10]:

$$\mathcal{A}_{\text{convex}} \subseteq \mathcal{A}_{\text{unif cone}} \subseteq \mathcal{A}_{\text{unif flat cone}} \subseteq \mathcal{A}_{\text{cap density}} \subseteq \mathcal{A}_{\text{unif Wiener}}.$$

A uniform Wiener conditions is thus the weakest reasonable constraint to obtain a continuity result in the Hausdorff complementary topology; it is based on a local equicontinuity property of the solutions on the moving domain.

The above conclusion is also true for our equations in infinite domain. Indeed, we have the following result.

Proposition 2.3 *Let C^η be a sequence of domains (electrode layers) with thickness $\eta \to 0$. Assume $\mathbb{R}^N\backslash C^\eta$ is in one of the classes in Lemma 2.2. For fixed $y(\Omega)$, denote by $\varphi^\eta(y)$ the solution of*

$$\begin{cases} \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi^\eta + p\,\chi\big(y(\Omega)\big) - \varepsilon_0 \nabla_y g\,\right] = 0 \ \text{ in } \ \mathbb{R}^N\backslash y(C^\eta), \\ \varphi^\eta \in \mathcal{D}\big(\mathbb{R}^N\backslash y(C^\eta)\big), \end{cases} \tag{2.18}$$

φ the solution of

$$\begin{cases} \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi + p\,\chi\big(y(\Omega)\big) - \varepsilon_0 \nabla_y g\,\right] = 0 \ \text{ in } \ \mathbb{R}^N\backslash y(\Gamma), \\ \varphi \in \mathcal{D}\big(\mathbb{R}^N\backslash y(\Gamma)\big), \end{cases} \tag{2.19}$$

where $y(\Gamma) = \partial\overline{y(C^\eta)} \cap \partial\overline{y(\Omega)}$. Assume $y(C^\eta) \to y(\Gamma)$ as $\eta \to 0$ in d_{H^c} sense, then

$$\left\|\nabla_y \varphi^\eta - \nabla_y \varphi\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \to 0 \quad \text{as} \ \ \eta \to 0.$$

In addition, let $\mathcal{F}^\eta$ be the corresponding energy (2.10) for φ^η and define

$$\mathcal{F} \ := \ \int_\Omega W\left(x, F, p_0\right) dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi\right|^2 dy + \int_{y(\Omega)} \nabla_y g \cdot p \, dy - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y g\right|^2 dy,$$

we also have $\lim_{\eta\to 0}\mathcal{F}^\eta = \mathcal{F}$.

Proof Repeating the same arguments that lead to (2.6), we can establish the existence of the solution for (2.19) in space $\mathcal{D}\big(\mathbb{R}^N\backslash y(\Gamma)\big)$ with

$$\big\|\nabla\varphi\big\|_{\mathcal{L}^2(\mathbb{R}^N)} \leqslant \frac{1}{\varepsilon_0}\big\|p\,\chi\big(y(\Omega)\big) - \varepsilon_0\nabla_y g\big\|_{\mathcal{L}^2(\mathbb{R}^N)}. \tag{2.20}$$

The proof of Proposition 2.3 is then exactly the same as in Bucur and Zolesio [11]. □

Proposition 2.3 shows that as long as we have some regularity for the electrode layers–specifically, the complement set of the electrodes is in any of the above $\mathcal{A}$ classes–then we may replace it with a boundary of dimension $N-1$.

In summary, we have the following variational principle for electroactive polymers. Consider an electroactive polymer occupying the reference region $\Omega \subset \mathbb{R}^N$ with two thin conductive layers Γ_0 and Γ_1 attached to it. Let $y:\Omega\to\mathbb{R}^N$ be the deformation of the material. Assume $g(y)\in\mathcal{H}^1(\mathbb{R}^N)$ with compact support satisfying $g(y)\equiv g_0$ in some small neighborhood of $y(\Gamma_0)$ and $g(y)\equiv g_1$ in some neighborhood of $y(\Gamma_1)$, where g_0 and g_1 are the boundary conditions for the electric field potential. Then the energy of the described system is

$$\mathcal{F} := \int_\Omega W\,(x,F,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\varphi\big|^2 dy + \int_{y(\Omega)}\nabla_y g\cdot p\,dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y g\big|^2 dy, \tag{2.21}$$

where φ is the solution to

$$\begin{cases} \nabla_y\cdot\big[-\varepsilon_0\nabla_y\varphi + p\,\chi\big(y(\Omega)\big) - \varepsilon_0\nabla_y g\,\big] = 0 \ \text{ in } \ \mathbb{R}^N\backslash y(\Gamma), \\ \varphi\in\mathcal{D}\big(\mathbb{R}^N\backslash y(\Gamma)\big). \end{cases} \tag{2.22}$$

We seek to find the deformation $y(x)$ and the polarization p that minimize the energy $\mathcal{F}$ in (2.21).

2.3 Euler-Lagrange Equations

To derive the Euler-Lagrange equations for the above variational principle, let us first assume $p_0(x)$ and $y(x)$ are the minimizers of (2.21) subject to (2.22). Fix $y(x)$ and consider a perturbation to the polarization $p_0(x)$. Let $p_1(x) = p_0(x) + tq_0(x)$, $q_0(x) \in \mathcal{L}^2(\Omega)$ and t is a small parameter. Denote $\varphi_1(y)$ to be the solution of

$$\begin{cases} \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi_1 + (p+tq)\chi\big(y(\Omega)\big) - \varepsilon_0 \nabla_y g\right] = 0 & \text{in } \mathbb{R}^N \backslash y(\Gamma), \\ \varphi_1 \in \mathcal{D}\big(\mathbb{R}^N \backslash y(\Gamma)\big). \end{cases} \tag{2.23}$$

By the linearity of the equation, $\varphi_1(y) = \varphi(y) + t\phi(y)$, where $\phi(y)$ is the solution of

$$\begin{cases} \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \phi + q\chi\big(y(\Omega)\big)\right] = 0 & \text{in } \mathbb{R}^N \backslash y(\Gamma), \\ \phi \in \mathcal{D}\big(\mathbb{R}^N \backslash y(\Gamma)\big). \end{cases} \tag{2.24}$$

Define

$$\begin{aligned} h(t) &= \int_\Omega W(x, F, p_0 + tq_0)\, dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi_1\right|^2 dy \\ &\quad + \int_{y(\Omega)} \nabla_y g \cdot (p + tq)\, dy - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y g\right|^2 dy \\ &= \int_\Omega W(x, F, p_0 + tq_0)\, dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi + t\nabla_y \phi\right|^2 dy \\ &\quad + \int_{y(\Omega)} \nabla_y g \cdot (p + tq)\, dy - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y g\right|^2 dy. \end{aligned}$$

Since $h(t)$ obtains its minimum at $t = 0$, we have

$$\begin{aligned} 0 = h'(t)|_{t=0} &= \int_\Omega \frac{\partial W(x, F, p_0)}{\partial p_0} \cdot q_0\, dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} 2\nabla_y \varphi \cdot \nabla_y \phi\, dy + \int_{y(\Omega)} \nabla_y g \cdot q\, dy \\ &= \int_\Omega \frac{\partial W(x, F, p_0)}{\partial p_0} \cdot q_0\, dx + \int_{y(\Omega)} \nabla_y \varphi \cdot q\, dy + \int_{y(\Omega)} \nabla_y g \cdot q\, dy \\ &= \int_\Omega \left[\frac{\partial W(x, F, p_0)}{\partial p_0} + F^{-T}\left(\nabla_x \varphi + \nabla_x g\right)\right] \cdot q_0\, dx. \end{aligned}$$

Thus we get an equation for p_0 as

$$\frac{\partial W(x,F,p_0)}{\partial p_0} + F^{-T}(\nabla_x \varphi + \nabla_x g) = 0. \tag{2.25}$$

Now, fix p_0 and let $y_1(x) = y(x) + tz(x)$ be a perturbation of $y(x)$, then $F_1 = \nabla_x y_1(x) = F + tG$. Let $\psi(y)$ be the solution to equation (2.22) with deformation $y_1(x)$,

$$\begin{cases} \nabla_{y_1} \cdot \left[-\varepsilon_0 \nabla_{y_1} \psi + p\chi\big(y_1(\Omega)\big) - \varepsilon_0 \nabla_{y_1} g\right] = 0 \qquad \text{in } \mathbb{R}^N \backslash y_1(\Gamma), \\ \psi \in \mathcal{D}\big(\mathbb{R}^N \backslash y_1(\Gamma)\big). \end{cases} \tag{2.26}$$

If we denote $\psi(x) = \psi\big(y(x)\big)$, then the equation for $\psi(x)$ in the reference coordinate is

$$\begin{cases} \nabla_x \cdot \left[-\varepsilon_0 J_1 F_1^{-1} F_1^{-T}\big(\nabla_x \psi(x) + \nabla_x g(x)\big) + F_1^{-1} p_0 \chi(\Omega)\right] = 0 \qquad \text{in } \mathbb{R}^N \backslash \Gamma, \\ \psi(x) \in \mathcal{D}\big(\mathbb{R}^N \backslash \Gamma\big). \end{cases} \tag{2.27}$$

Similarly, if we denote $\varphi(x) = \varphi\big(y(x)\big)$, the equation for $\varphi(x)$ in the reference coordinate is

$$\begin{cases} \nabla_x \cdot \left[-\varepsilon_0 J F^{-1} F^{-T}\big(\nabla_x \varphi(x) + \nabla_x g(x)\big) + F^{-1} p_0 \chi(\Omega)\right] = 0 \qquad \text{in } \mathbb{R}^N \backslash \Gamma, \\ \varphi(x) \in \mathcal{D}\big(\mathbb{R}^N \backslash \Gamma\big). \end{cases} \tag{2.28}$$

Assume t is small and consider the Taylor expansion of F_1^{-1}, J_1 etc.:

$$\begin{aligned}
J_1 = \det(F_1) &= \det(F)\big(1 + t\,\mathrm{tr}(F^{-1}G)\big) + o(t) = J + tJ\,\mathrm{tr}(F^{-1}G) + o(t);\\
F_1^{-1} &= F^{-1} - tF^{-1}GF^{-1} + o(t);\\
F_1^{-T} &= F^{-T} - tF^{-T}G^TF^{-T} + o(t);\\
J_1F_1^{-1}F_1^{-T} &= JF^{-1}F^{-T} + t\big(\mathrm{tr}(F^{-1}G)F^{-1}F^{-T} - JF^{-1}GF^{-1}F^{-T}\\
&\quad -JF^{-1}F^{-T}G^TF^{-T}\big) + o(t)\\
&= A + tB + o(t),
\end{aligned}$$

where

$$\begin{aligned}
A &= JF^{-1}F^{-T},\\
B &= \mathrm{tr}(F^{-1}G)F^{-1}F^{-T} - JF^{-1}GF^{-1}F^{-T} - JF^{-1}F^{-T}G^TF^{-T}.
\end{aligned}$$

Thus, equation (2.27) can be written as

$$\nabla_x \cdot \big[-\varepsilon_0\big(A + tB + o(t)\big)(\nabla_x\psi + \nabla_x g) + \big(F^{-1} - tF^{-1}GF^{-1} + o(t)\big)p_0\big] = 0. \quad (2.29)$$

To leading order, we obtain

$$\nabla_x \cdot \big[-\varepsilon_0 A(\nabla_x\psi + \nabla_x g) + F^{-1}p_0\big] = 0.$$

This is exactly (2.28), the solution is $\varphi(x)$.

The second order equation is

$$\nabla_x \cdot (-\varepsilon_0 A\nabla_x\phi) = \nabla_x \cdot \big[\varepsilon_0 B(\nabla_x\varphi + \nabla_x g) + F^{-1}GF^{-1}p_0\big]. \quad (2.30)$$

The solution for (2.27) is then $\psi(x) = \varphi(x) + t\phi(x) + o(t)$.

Now, define

$$
\begin{aligned}
r(t) &= \int_\Omega W(x,F_1,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_{y_1}\psi\right|^2 dy_1 \\
&\quad + \int_{y_1(\Omega)} \nabla_{y_1} g\cdot p\,dy_1 - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_{y_1} g\right|^2 dy_1 \\
&= \int_\Omega W(x,F+tG,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} J_1F_1^{-1}F_1^{-T}\nabla_x\psi\cdot\nabla_x\psi\,dx \\
&\quad + \int_\Omega F_1^{-T}\nabla_x g\cdot p_0\,dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} J_1F_1^{-1}F_1^{-T}\nabla_x g\cdot\nabla_x g\,dx \\
&= \int_\Omega W(x,F+tG,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}(A+tB)(\nabla_x\varphi+t\nabla_x\phi)\cdot(\nabla_x\varphi+t\nabla_x\phi)\,dx \\
&\quad + \int_\Omega (F^{-T}-tF^{-T}G^TF^{-T})\nabla_x g\cdot p_0\,dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}(A+tB)\nabla_x g\cdot\nabla_x g\,dx + o(t) \\
&= \int_\Omega W(x,F+tG,p_0)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} A(\nabla_x\varphi\cdot\nabla_x\varphi+\nabla_x g\cdot\nabla_x g)\,dx \\
&\quad + \int_\Omega F^{-T}\nabla_x g\cdot p_0\,dx - t\int_\Omega F^{-T}G^TF^{-T}\nabla_x g\cdot p_0\,dx \\
&\quad + t\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} B\nabla_x\varphi\cdot\nabla_x\varphi + 2A\nabla_x\varphi\cdot\nabla_x\phi - B\nabla_x g\cdot\nabla_x g\,dx\right] + o(t).
\end{aligned}
$$

Since $r(t)$ is minimized at $t=0$, we have

$$
\begin{aligned}
0 = r'(0) &= \int_\Omega \frac{\partial W(x,F,p_0)}{\partial F} : G\,dx - \int_\Omega F^{-T}G^T F^{-T}\nabla_x g\cdot p_0\,dx \\
&+\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} B\nabla_x\varphi\cdot\nabla_x\varphi + 2A\nabla_x\varphi\cdot\nabla_x\phi - B\nabla_x g\cdot\nabla_x g\,dx \\
&= \int_\Omega \frac{\partial W(x,F,p_0)}{\partial F} : G\,dx - \int_\Omega F^{-T}G^T F^{-T}\nabla_x g\cdot p_0\,dx \\
&+\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} B\nabla_x\varphi\cdot\nabla_x\varphi - B\nabla_x g\cdot\nabla_x g\,dx \\
&-\int_{\mathbb{R}^N}\varepsilon_0 B(\nabla_x\varphi+\nabla_x g)\cdot\nabla_x\varphi\,dx - \int_\Omega F^{-1}GF^{-1}p_0\cdot\nabla_x\varphi\,dx \\
&= \int_\Omega \frac{\partial W(x,F,p_0)}{\partial F} : G\,dx - \int_\Omega F^{-T}G^T F^{-T}(\nabla_x\varphi+\nabla_x g)\cdot p_0\,dx \\
&-\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} B(\nabla_x\varphi+\nabla_x g)\cdot(\nabla_x\varphi+\nabla_x g)\,dx \\
&= \int_\Omega \Bigg[\frac{\partial W(x,F,p_0)}{\partial F} + \varepsilon_0 J\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)\otimes\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)F^{-T} \\
&-\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)\otimes p_0F^{-T} - \frac{\varepsilon_0}{2}\left|F^{-T}(\nabla_x\varphi+\nabla_x g)\right|^2 F^{-T}\Bigg] : G\,dx,
\end{aligned}
$$

for any $z(x)\in\mathcal{H}^1(\Omega)$. Above, we use (2.30) in the second equality. Therefore, we get the equation for the deformation:

$$
\begin{aligned}
&\nabla_x\cdot\Bigg[\frac{\partial W(x,F,p_0)}{\partial F} + \varepsilon_0 J\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)\otimes\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)F^{-T} \\
&-\big(F^{-T}(\nabla_x\varphi+\nabla_x g)\big)\otimes p_0F^{-T} - \frac{\varepsilon_0}{2}\left|F^{-T}(\nabla_x\varphi+\nabla_x g)\right|^2 F^{-T}\Bigg] = 0. \qquad (2.31)
\end{aligned}
$$

Collecting all the equations and remembering that $\varphi_e = \varphi + g$, we get

$$
\begin{cases}
\dfrac{\partial W(x,F,p_0)}{\partial p_0} + F^{-T}\nabla_x\varphi_e = 0, \\
\nabla_x\cdot\left[-\varepsilon_0 JF^{-1}F^{-T}\nabla_x\varphi_e(x) + F^{-1}p_0\chi(\Omega)\right] = 0, \\
\nabla_x\cdot\left[\dfrac{\partial W}{\partial F} + \left((F^{-T}\nabla_x\varphi_e)\otimes(\varepsilon_0 JF^{-T}\nabla_x\varphi_e - p_0) - \dfrac{\varepsilon_0}{2}\left|F^{-T}\nabla_x\varphi_e\right|^2\right)F^{-T}\right] = 0.
\end{cases}
\qquad (2.32)
$$

These are the Euler-Lagrange equations.

2.4 Formal Derivation of Small-strain Models

Starting from the above Euler-Lagrange equations, under the small deformation assumption, we can get small-strain models for dielectric elastomers and piezoelectric elastomers by taking different scalings.

2.4.1 Small-strain Model for Dielectric Elastomers

Assume the deformation is of order ε. Let $y(x) = x + \varepsilon u(x)$. Then $F(x) = I + \varepsilon \nabla u$,

$$\begin{aligned} J &= 1 + \varepsilon \operatorname{tr}(\nabla u) + o(\varepsilon), \\ F^{-1} &= I - \varepsilon \nabla u + o(\varepsilon), \\ F^{-T} &= I - \varepsilon \nabla u^T + o(\varepsilon). \end{aligned}$$

Assume that the electric field is of order ε^δ, i.e., $\varphi_e = \varepsilon^\delta \tilde{\varphi}_e$. Then, from the first equation of (2.32),

$$\left.\frac{\partial^2 W}{\partial p_0^2}\right|_{\substack{F=I\\p_0=0}} p_0 + \left.\frac{\partial^2 W}{\partial p_0 \partial F}\right|_{\substack{F=I\\p_0=0}} \varepsilon \nabla u + \varepsilon^\delta \nabla_x \tilde{\varphi}_e + o(\varepsilon^\delta) = 0.$$

Assume

$$\left.\frac{\partial^2 W}{\partial p_0 \partial F}\right|_{\substack{F=I\\p_0=0}} = 0, \qquad \left.\frac{\partial^2 W}{\partial p_0^2}\right|_{\substack{F=I\\p_0=0}} = H^{-1},$$

then p_0 should have the same order with φ_e. Let $p_0 = \varepsilon^\delta \tilde{p}_0$, then we get an equation of order ε^δ for $\tilde{p}_0$,

$$H^{-1}\tilde{p}_0 + \nabla_x \tilde{\varphi}_e = 0.$$

The leading order in the Maxwell equation is also of order ε^δ and we get

$$\nabla_x \cdot \big(-\varepsilon_0 \nabla_x \tilde{\varphi}_e + \tilde{p}_0 \chi(\Omega)\big) = 0.$$

In the third equation of (2.32), the first term is

$$\frac{\partial W}{\partial F} = \frac{\partial^2 W}{\partial F^2}\bigg|_{\substack{F=I\\ p_0=0}} \varepsilon \nabla u + \frac{\partial^2 W}{\partial F \partial p_0}\bigg|_{\substack{F=I\\ p_0=0}} \varepsilon^{\delta} \tilde{p}_0 + \frac{\partial^3 W}{\partial F \partial p_0^2}\bigg|_{\substack{F=I\\ p_0=0}} \varepsilon^{2\delta} \tilde{p}_0 \tilde{p}_0 + o(\varepsilon) + o(\varepsilon^{2\delta}),$$

where we assume

$$\frac{\partial^2 W}{\partial F \partial p_0}\bigg|_{\substack{F=I\\ p_0=0}} = 0.$$

The second term is

$$\begin{aligned}
&\left((F^{-T}\nabla_x \varphi_e) \otimes (\varepsilon_0 J F^{-T}\nabla_x \varphi_e - p_0) - \frac{\varepsilon_0}{2}\left|F^{-T}\nabla_x \varphi_e\right|^2\right) F^{-T} \\
= \quad & \varepsilon^{2\delta}\left[(\varepsilon_0 \nabla_x \tilde{\varphi}_e \otimes \nabla_x \tilde{\varphi}_e) - \nabla_x \tilde{\varphi}_e \otimes \tilde{p}_0 - \frac{\varepsilon_0}{2}\left|\nabla_x \tilde{\varphi}_e\right|^2\right] + o(\varepsilon^{2\delta}).
\end{aligned}$$

Now, if we consider an electric field induced strain, the mechanical stress should balance with the Maxwell stress, thus $\delta = \frac{1}{2}$. If we assume

$$\frac{\partial^2 W}{\partial F^2}\bigg|_{\substack{F=I\\ p_0=0}} = C, \qquad \frac{\partial^3 W}{\partial F \partial p_0^2}\bigg|_{\substack{F=I\\ p_0=0}} = A,$$

then from the third equation in (2.32), we get an order ε equation

$$\nabla_x \cdot \left[C\nabla_x u + A\tilde{p}_0\tilde{p}_0 + \varepsilon_0 \nabla_x \tilde{\varphi}_e \otimes \nabla_x \tilde{\varphi}_e - \nabla_x \tilde{\varphi}_e \otimes \tilde{p}_0 - \frac{\varepsilon_0}{2}\left|\nabla_x \tilde{\varphi}_e\right|^2 I\right] = 0.$$

For simplicity, we still denote $\tilde{p}_0$, $\tilde{\varphi}_e$ as p_0 and φ_e. The small-strain model for the dielectric elastomer is then

$$\begin{cases}
H^{-1}p_0 + \nabla_x \varphi_e = 0, \\
\nabla_x \cdot \left[-\varepsilon_0 \nabla_x \varphi_e(x) + p_0 \chi(\Omega)\right] = 0, \\
\nabla_x \cdot \left[C\nabla_x u + Ap_0p_0 + \varepsilon_0 \nabla_x \varphi_e \otimes \nabla_x \varphi_e - \nabla_x \varphi_e \otimes p_0 - \frac{\varepsilon_0}{2}\left|\nabla_x \varphi_e\right|^2 I\right] = 0.
\end{cases} \tag{2.33}$$

From the first equation in (2.33), $p_0 = -H\nabla_x\varphi_e$. Plugging this into the other equations, we get

$$\begin{cases} -\nabla_x \cdot \Big[\big(\varepsilon_0 I + H\chi(\Omega)\big)\nabla_x\varphi_e(x)\Big] = 0, \\ \nabla_x \cdot \Big[C\nabla_x u + A(H\nabla_x\varphi_e)(H\nabla_x\varphi_e) + \varepsilon_0\nabla_x\varphi_e \otimes \nabla_x\varphi_e \\ \qquad +\nabla_x\varphi_e \otimes (H\nabla_x\varphi_e) - \dfrac{\varepsilon_0}{2}\big|\nabla_x\varphi_e\big|^2 I\Big] = 0. \end{cases} \tag{2.34}$$

The first equation in (2.34) is an order $\varepsilon^{\frac{1}{2}}$ equation which decides the electric field. The second equation in (2.34) is an order ε equation. Since the electric field is known from the first equation, this equation determines the deformation field. This small-strain model effectively decouples the electric field with the deformation field and thus provides a huge simplification for the problem.

2.4.2 Small-strain Model for Piezoelectric Elastomers

Now take another scale. Assume the deformation field is of order ε and the electric field is of order ε too. Let $\varphi_e = \varepsilon\tilde{\varphi}_e$, then from the first equation of (2.32),

$$\left.\frac{\partial^2 W}{\partial p_0^2}\right|_{\substack{F=I\\ p_0=0}} p_0 + \left.\frac{\partial^2 W}{\partial p_0 \partial F}\right|_{\substack{F=I\\ p_0=0}} \varepsilon\nabla u + \varepsilon\nabla_x\tilde{\varphi}_e + o(\varepsilon) = 0.$$

Denote

$$\left.\frac{\partial^2 W}{\partial p_0 \partial F}\right|_{\substack{F=I\\ p_0=0}} = A, \qquad \left.\frac{\partial^2 W}{\partial p_0^2}\right|_{\substack{F=I\\ p_0=0}} = H^{-1},$$

then p_0 should have the same order with φ_e. Let $p_0 = \varepsilon\tilde{p}_0$, we get an equation of order ε as

$$H^{-1}\tilde{p}_0 + A\nabla u + \nabla_x\tilde{\varphi}_e = 0.$$

From the second equation of (2.32), the leading order equation for the Maxwell equation is also of order ε

$$\nabla_x \cdot \big(-\varepsilon_0\nabla_x\tilde{\varphi}_e + \tilde{p}_0\chi(\Omega)\big) = 0.$$

In the third equation of (2.32),

$$\frac{\partial W}{\partial F} = \frac{\partial^2 W}{\partial F^2}\bigg|_{\substack{F=I\\p_0=0}} \varepsilon \nabla u + \frac{\partial^2 W}{\partial F \partial p_0}\bigg|_{\substack{F=I\\p_0=0}} \varepsilon \tilde{p}_0 + o(\varepsilon),$$

in which

$$\frac{\partial^2 W}{\partial F \partial p_0}\bigg|_{\substack{F=I\\p_0=0}} = A.$$

On the other hand,

$$\left((F^{-T}\nabla_x \varphi_e) \otimes (\varepsilon_0 J F^{-T} \nabla_x \varphi_e - p_0) - \frac{\varepsilon_0}{2}\left|F^{-T}\nabla_x \varphi_e\right|^2\right) F^{-T} \sim \varepsilon^2.$$

Thus we get another equation of order ε out of the third equation of (2.32),

$$\nabla \cdot [C\nabla u + A\tilde{p}_0] = 0.$$

Putting them together, and still denoting $\tilde{p}_0$, $\tilde{\varphi}_e$ as p_0, φ_e, we get

$$\begin{cases} H^{-1}p_0 + A\nabla u + \nabla_x \varphi_e = 0, \\ \nabla_x \cdot \left[-\varepsilon_0 \nabla_x \varphi_e(x) + p_0 \chi(\Omega)\right] = 0, \\ \nabla_x \cdot \left[C\nabla_x u + A p_0\right] = 0. \end{cases} \tag{2.35}$$

From (2.35), we can see that the deformation field couples linearly with the polarization and the electric field. This is a linear model for piezoelectric material.

However, the above derivation is just a formal computation. In Chapter 3, we are going to prove the small-strain model for the dielectric elastomers using the tool of Γ-convergence. The rigorous derivation of the small-strain model for piezoelectric materials will be given in Appendix A.

Chapter 3

Small-strain Model for Dielectric Elastomers

This chapter is devoted to making the formal calculation of Section 2.4.1 rigorous using Γ-convergence. Below, Section 3.1 provides a brief introduction to Γ-convergence. Section 3.2 lay out the assumptions and the main results of this chapter. In the remaining two sections, rigorous proofs are provided for these results.

3.1 An Introduction to Γ-convergence

The notion of Γ-convergence is introduced by De Giorgi [19, 17]. The importance of this notion lies in the fact that, under appropriate technical hypotheses, it implies the convergence of minimizers which in our case are the deformation field and the polarization. Below, we briefly recall definitions and some of the properties of Γ-convergence that are relevant to our development, and we refer to [4, 18, 8, 29, 20, 30] for an overview and an extensive list of references on the subject.

Let X and Y be two given metric spaces, with $X \subset Y$. Consider a functional $\mathcal{F} : X \to \bar{\mathbb{R}}$, and a one parameter family of functionals $\mathcal{F}^\varepsilon : X \to \bar{\mathbb{R}}$, with $\varepsilon \in (0, +\infty)$.

Definition 3.1 *$\mathcal{F}^\varepsilon$ $\Gamma(Y)$-converges to $\mathcal{F}$ if, for every sequence ε_j converging to zero, the following two conditions hold:*

1. *For every sequence $u^j \subset X$ such that $u^j \to u$ in Y,*

$$\liminf_{j\to+\infty} \mathcal{F}^{\varepsilon_j}(u^j) \geqslant \mathcal{F}(u); \tag{3.1}$$

2. *there exists a sequence $\bar{u}^j \subset X$ such that $\bar{u}^j \to u$ in Y and*

$$\lim_{j\to+\infty} \mathcal{F}^{\varepsilon_j}(\bar{u}^j) = \mathcal{F}(u). \tag{3.2}$$

Condition 1 is often referred as the lower bound condition; condition 2 is often called the existence of the recovery sequence. The definition above gives a notion of pointwise convergence. We say that $\mathcal{F}^\varepsilon$ $\Gamma(Y)$-converges to $\mathcal{F}$ in X or, equivalently, that $\mathcal{F}$ is the $\Gamma(Y)$-limit of $\mathcal{F}^\varepsilon$ in X if $\mathcal{F}^\varepsilon$ $\Gamma(Y)$-converges to $\mathcal{F}$ at every $u \in X$. A key property of Γ-convergence, which motivated its introduction, is given by the following result.

Proposition 3.1 *Assume that $\mathcal{F}^\varepsilon$ $\Gamma(Y)$-converges to $\mathcal{F}$ in X. Let u^ε be a sequence such that*

$$\mathcal{F}^\varepsilon(u^\varepsilon) \leqslant \inf\{\mathcal{F}^\varepsilon(v) \mid v \in X\} + \varepsilon.$$

Assume further that u^ε is compact in Y. Let u^{ε_j} be any subsequence, say $u^{\varepsilon_j} \to u$ in Y as $\varepsilon_j \to 0$. Then

1. *$\mathcal{F}(u) \leqslant \mathcal{F}(v)$, $\forall\, v \in X$;*

2. *$\mathcal{F}(u) = \lim_{j\to+\infty} \mathcal{F}^{\varepsilon_j}(u^{\varepsilon_j})$.*

For a proof see, for example, Attouch ([4], p. 39-41).

3.2 Assumptions and Main Results

Through out this thesis, we assume that the reference region Ω has a relatively good regularity: it satisfies the strong local Lipschitz condition, and there exists a constant r_0, $2 < r_0 < \infty$, such that $\Omega \in \mathfrak{B}^{r_0}$, where $\mathfrak{B}^{r_0}$ is defined as follows.

Definition 3.2 *We say that a bounded domain* $\mathcal{O} \in \mathbb{R}^N$ *is of class* $\mathfrak{B}^{r_0}$, $2 < r_0 < \infty$, *if equation*

$$\triangle v = \operatorname{div} \vec{f} \tag{3.3}$$

has a unique solution v *in* $\mathcal{W}_0^{1,r_0}$ *for every* $\vec{f} \in \mathcal{L}^{r_0}(\mathcal{O})$ *and* $\|\nabla v\|_{\mathcal{L}^{r_0}(\mathcal{O})} \leqslant c_{r_0} \|\vec{f}\|_{\mathcal{L}^{r_0}(\mathcal{O})}$ *for some constant* c_{r_0} *independent of* $\vec{f}$.

This definition constitutes a condition of regularity on the boundary of $\mathcal{O}$. It holds for any value of r_0 if the boundary is sufficiently smooth. The proof depends on the Calderon-Zygmund inequality for singular integrals (see [12] and Theorem 15.3′ of [2]).

For dielectric elastomers, we have the following assumptions on the stored energy density $W(x, F, p_0)$:

$\mathcal{A}_1$. $W(x, F, p_0)$ is a nonnegtive function and subject to the condition of frame indifference

$$W(x, QF, Qp_0) = W(x, F, p_0), \qquad \forall Q \in \mathrm{SO}(N). \tag{3.4}$$

$\mathcal{A}_2$. $W(x, F, p_0) = 0$ iff $p_0 = 0$ and $F \in \mathrm{SO}(N)$.

$\mathcal{A}_3$. $W = +\infty$ if $J = \det(F) < \delta$.

$\mathcal{A}_4$. There exists a constant $t > 0$ such that

$$\lim_{|F| \to \infty} \frac{1}{|F|^t} \inf_{x \in \Omega} W(x, F, 0) > 0. \tag{3.5}$$

Throughout this thesis, we assume this t is big enough to satisfy all the requirements of it.

$\mathcal{A}_5$. Because of the frame indifference, $W(x, F, p_0) = V(x, F^T F - I, p_0)$. Assume for fixed p_0, $V(x, \alpha(F^T F - I), p_0)$ increases monotonically with respect to α; for fixed F, $V(x, (F^T F - I), \beta p_0)$ increases monotonically with respect to β.

$\mathcal{A}_6$. Define

$$H^{-1} = \left.\frac{\partial^2 W}{\partial^2 p_0}\right|_{p_0 = 0, F = I} \tag{3.6}$$

and assume that there exists a constant c independent of x, F and p_0, such that

$$W(x,F,p_0) - \frac{1}{2}H^{-1}p_0\cdot p_0 \geqslant c\,\mathrm{dist}\big(F,\mathrm{SO}(N)\big)^2. \tag{3.7}$$

$\mathcal{A}_7$. There exists a constant ρ_0, such that for $|p_0| \leqslant \rho_0$,

$$\left|W(x,I,p_0) - \frac{H^{-1}}{2}p_0\cdot p_0\right| \leqslant w\big(|p_0|\big)\,|p_0|^2, \tag{3.8}$$

where $w(|p_0|) \to 0$ monotonically as $|p_0| \to 0$.

$\mathcal{A}_8$. There exist constants ρ_1 and ρ_2, such that if $|p_0| \leqslant \rho_1$, $|G| \leqslant \rho_2$,

$$\begin{aligned}&\left|W(x,I+G,p_0) - \frac{H^{-1}}{2}p_0\cdot p_0 - \frac{1}{2}CGG - AGp_0p_0 - Bp_0^4\right| \\ \leqslant\;& w_1(|G|,|p_0|)\,|p_0|^4 + w_2(|G|,|p_0|)\,|G|^2 + w_3(|G|,|p_0|)\,|G|\,|p_0|^2,\end{aligned}$$

where $w_1, w_2, w_3 \to 0$ monotonically as $|p_0| \to 0$ and $|G| \to 0$.

From the scale analysis for dielectric material in Section 2.4.1, the electric field is of order $\varepsilon^{\frac{1}{2}}$. Thus the leading order of the energy is the electric energy which is of order ε. Rescale the energy with ε and define

$$\mathcal{F}^\varepsilon = \frac{1}{\varepsilon}\int_\Omega W(x,F_\varepsilon,\varepsilon^{\frac{1}{2}}p_0^\varepsilon)\,dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}|\nabla_y\varphi^\varepsilon|^2\,dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}|\nabla_y g^\varepsilon|^2\,dy + \int_{y^\varepsilon(\Omega)}\nabla_y g^\varepsilon\cdot p^\varepsilon\,dy, \tag{3.9}$$

where

$$\begin{cases}\nabla_y\cdot\big[-\varepsilon_0\nabla_y(\varphi^\varepsilon+g^\varepsilon)+p^\varepsilon\,\chi\big(y^\varepsilon(\Omega)\big)\big]=0 \quad \text{in} \quad \mathbb{R}^N\backslash y^\varepsilon(\Gamma),\\ \varphi^\varepsilon\in\mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big).\end{cases} \tag{3.10}$$

We prove in Section 3.3 that under assumption $\mathcal{A}_1$ to $\mathcal{A}_8$ above, the functional $\mathcal{F}^\varepsilon$ Γ-converges to functional

$$\mathcal{F}^0 = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega H^{-1} p_0 \cdot p_0 \, dx + \dfrac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla \varphi^0\right|^2 dx & \\ \quad + \displaystyle\int_\Omega \nabla g \cdot p_0 \, dx - \dfrac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla g\right|^2 dx & F_0 = I, \\ \infty & \text{otherwise}, \end{cases} \tag{3.11}$$

where φ^0 satisfies the equation

$$\begin{cases} \nabla_x \cdot \left[-\varepsilon_0 \nabla_x (\varphi^0 + g) + p_0 \, \chi(\Omega) \right] = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Gamma, \\ \varphi^0 \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma). \end{cases} \tag{3.12}$$

In fact, we have the following theorem.

Theorem 3.2 (Γ-convergence of the first order energy) *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_8$, then the functional $\mathcal{F}^\varepsilon$ Γ-converges to functional $\mathcal{F}^0$ under the norm $\mathcal{W}^{1,t}$ for y^ε and the $\mathcal{L}^2(\Omega)$ weak norm for p_0^ε.*

Proposition 3.3 *The minimizer $\varphi^0(x)$ for functional $\mathcal{F}^0$ exists and satisfies*

$$\begin{cases} \nabla_x \cdot \left[\big(\varepsilon_0 + H \, \chi(\Omega)\big) \nabla_x (\varphi^0 + g) \right] = 0 \quad \text{in} \, \mathbb{R}^N \backslash \Gamma, \\ \varphi^0 \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma). \end{cases} \tag{3.13}$$

Proof Since φ^0 is linear with respect to p_0, $\mathcal{F}^0$ is quadratic with respect to p_0. So the minimizer exists. To derive the Euler-Lagrange equation for $\mathcal{F}^0$, denote by p a perturbation to the minimizer p_0, $p = p_0 + tq$, $q \in \big(\mathcal{L}^2(\Omega)\big)^N$. Let ϕ_q be the solution of

$$\begin{cases} \nabla_x \cdot \big(-\varepsilon_0 \nabla_x \phi_q + q \chi(\Omega) \big) = 0 \quad \text{in} \ \mathbb{R}^N \backslash \Gamma, \\ \phi_q \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma). \end{cases}$$

From the linearity of the Maxwell equation, $\nabla \varphi = \nabla \varphi^0 + t \nabla \phi_q$.

Define

$$I(t)=\int_\Omega \frac{H^{-1}}{2}(p_0+tq)^2\,dx+\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla\varphi^0+t\nabla\phi_q\right|^2dx+\int_\Omega \nabla g\cdot(p_0+tq)\,dx-\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}|\nabla g|^2\,dx.$$

Since $I(t)$ obtains its minimum at $t=0$,

$$\begin{aligned}0=I'(t)|_{t=0} &= \int_\Omega H^{-1}p_0\cdot q\,dx+\int_\Omega \nabla g\cdot q\,dx+\varepsilon_0\int_{\mathbb{R}^N}\nabla\phi_q\cdot\nabla\varphi^0\,dx\\ &= \int_\Omega (H^{-1}p_0+\nabla g+\nabla\varphi^0)\cdot q\,dx.\end{aligned}$$

So $p_0=-H(\nabla g+\nabla\varphi^0)$. Plugging it into (3.12), we get

$$\begin{cases}\nabla_x\cdot\left[\big(\varepsilon_0+H\,\chi(\Omega)\big)\nabla_x(\varphi^0+g)\,\right]=0 \quad \text{in}\,\mathbb{R}^N\backslash\Gamma,\\ \varphi^0\in\mathcal{D}(\mathbb{R}^N\backslash\Gamma).\end{cases}\tag{3.14}$$

Alternatively we can write it as

$$\begin{cases}\nabla_x\cdot\left[\big(\varepsilon_0+H\chi(\Omega)\big)\nabla_x\varphi_e^0\right]=0 \quad \text{in}\,\mathbb{R}^N\backslash\Gamma,\\ \varphi_e^0=g_0 \quad \text{on}\,\Gamma_0,\\ \varphi_e^0=g_1 \quad \text{on}\,\Gamma_1,\\ \varphi_e^0\in\mathcal{L}^2_{\text{loc}}(\mathbb{R}^N), \quad \nabla\varphi_e^0\in\mathcal{L}^2(\mathbb{R}^N).\end{cases}\tag{3.15}$$

The Euler-Lagrange equation is

$$\int_\Omega (H^{-1}p_0+\nabla g+\nabla\varphi^0)\cdot q\,dx=0 \tag{3.16}$$

for any q in $\big(\mathcal{L}^2(\Omega)\big)^N$. $\square$

Next, let us consider the ε order correction of the first order energy. Specifically, define

$$\begin{aligned}
\mathcal{F}_r^\varepsilon &= \frac{\mathcal{F}^\varepsilon - \mathcal{F}^0}{\varepsilon} \\
&= \frac{1}{\varepsilon}\Bigg[\frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx - \int_\Omega \frac{H^{-1}}{2} p_0 \cdot p_0 \, dx \\
&\quad + \int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon \, dy - \int_\Omega \nabla_x g \cdot p_0 \, dx \\
&\quad - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} |\nabla_y g^\varepsilon|^2 \, dy + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} |\nabla_x g|^2 \, dx \\
&\quad + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} |\nabla_y \varphi^\varepsilon|^2 \, dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} |\nabla_x \varphi^0|^2 \, dx\Bigg],
\end{aligned} \tag{3.17}$$

where φ^ε satisfies (3.10) and φ^0 satisfies (3.13). Define $\mathcal{F}_r^0$ as

$$\begin{aligned}
\mathcal{F}_r^0 \;:=\; & \int_\Omega \frac{1}{2} C\nabla u \nabla u + A\nabla u p_0 p_0 + B p_0^4 \, dx + \int_\Omega \frac{H^{-1}}{2} q_0 \cdot q_0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x \phi^0\right|^2 dx \\
& + \varepsilon_0 \int_{\mathbb{R}^N} (\nabla u + \nabla u^T - \mathrm{tr}(\nabla u) I)\nabla g \cdot \nabla_x \varphi^0 \, dx - \int_\Omega \nabla u^T p_0 \cdot \nabla_x \varphi^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (\nabla u + \nabla u^T - \mathrm{tr}(\nabla u) I)\nabla_x \varphi^0 \cdot \nabla_x \varphi^0 \, dx - \int_\Omega \nabla u \nabla_x g \cdot p_0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (\nabla u + \nabla u^T - \mathrm{tr}(\nabla u) I)\nabla_x g \cdot \nabla_x g \, dx,
\end{aligned} \tag{3.18}$$

where ϕ^0 is the solution to

$$\begin{cases}
\nabla_x \cdot \left(-\varepsilon_0 \nabla_x \phi^0 + q_0\, \chi(\Omega)\right) = 0 \qquad \text{in } \mathbb{R}^N \backslash \Gamma, \\
\phi^0 \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma).
\end{cases}$$

We have the following result.

Theorem 3.4 (Γ-convergence of the second order energy) *Under the same conditions as in Theorem 3.2, if we assume further that there exists a constant $w > 0$ and a compact set $K \supset \Omega$ such that $\nabla\varphi^0$, the solution of equation (3.13), is in $\mathcal{L}^{4+w}(K)$, then $\mathcal{F}_r^\varepsilon$ Γ-converges to $\mathcal{F}_r^0$.*

Besides the above Γ-convergence results, we also have the following compactness results.

Proposition 3.5 *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_8$ and $u^\varepsilon = 0$ on Γ_3, a part of the boundary with positive measure. Then if $\mathcal{F}^\varepsilon$ is bounded, there exists a constant c, such that*

$$\left\|p_0^\varepsilon\right\|_{\mathcal{L}^2(\Omega)} < c, \qquad \left\|\frac{F_\varepsilon - I}{\varepsilon^{\frac{1}{t}}}\right\|_{\mathcal{L}^t(\Omega)} \leqslant c.$$

Proposition 3.6 *Under the same conditions with Proposition 3.5, assume further that there exists a constant $w > 0$ and a compact set $K \supset \Omega$ such that $\nabla\varphi^0$, the solution of equation (3.13), is in $\mathcal{L}^{4+w}(K)$. Then if $\mathcal{F}_r^\varepsilon < c$, there exists a constant $\tilde{c}$, such that*

$$\left\|\frac{p_0^\varepsilon - p_0}{\varepsilon^{\frac{1}{2}}}\right\|_{\mathcal{L}^2(\Omega)} < \tilde{c} \quad \textit{and} \quad \left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} < \tilde{c}.$$

Armed with all these results, we get the small-strain model as follows.

Theorem 3.7 *Under the same conditions as in Proposition 3.6, if y^ε, p_0^ε satisfies*

$$\mathcal{F}^\varepsilon(y^\varepsilon, p_0^\varepsilon) \le \inf_{\substack{z^\varepsilon \in \mathcal{W}^{1,t}(\Omega) \\ q_0^\varepsilon \in \mathcal{L}^2(\Omega)}} \mathcal{F}^\varepsilon(z^\varepsilon, q_0^\varepsilon) + \varepsilon,$$

$u^\varepsilon = \frac{1}{\varepsilon}(y^\varepsilon - x)$ weakly convergent to u in $\mathcal{W}^{1,2}(\Omega)$ and φ^ε weakly convergent to φ^0, then φ^0 is the solution of equation (3.13), and u is the solution of

$$\nabla_x \cdot \Big[C\nabla_x u + A(H\nabla_x\varphi_e)(H\nabla_x\varphi_e) + \nabla_x\varphi_e \otimes \big[(\varepsilon_0 I + H)\nabla_x\varphi_e\big] - \frac{\varepsilon_0}{2}\big|\nabla_x\varphi_e\big|^2 I\Big] = 0. \tag{3.19}$$

Proof According to Proposition 3.1, this theorem is the direction corollary of Proposition 3.5, 3.6 and Theorem 3.2, 3.4 provided u is the minimizer of functional $\mathcal{F}_r^0$.

Actually, Recalling $\varphi_e^0 = \varphi^0 + g$, $\mathcal{F}_r^0$ can be rewritten as

$$\begin{aligned}\mathcal{F}_r^0 &= \int_\Omega \frac{1}{2} C\nabla u \nabla u + Bp_0^4 \, dx + \int_\Omega \frac{H^{-1}}{2} q_0 \cdot q_0 \, dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_x \phi^0\right|^2 dx \\ &\quad + \int_\Omega A\nabla u \left(H\nabla_x \varphi_e^0\right)\left(H\nabla_x \varphi_e^0\right) dx + \int_\Omega H^T \nabla u^T \nabla_x \varphi_e^0 \cdot \nabla_x \varphi_e^0 \, dx \\ &\quad + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left(\nabla u + \nabla u^T - \operatorname{tr}(\nabla u) I\right) \nabla_x \varphi_e^0 \cdot \nabla_x \varphi_e^0 \, dx.\end{aligned}$$

Minimizing $\mathcal{F}_r^0$ first with respect to q_0, we get $q_0 \equiv 0$ and $\nabla_x \phi^0 \equiv 0$. Next, minimizing $\mathcal{F}_r^0$ with respect to u, the minimizer exists and satisfies equation (3.19). □

(3.13) is exactly the first equation in (2.34) recalling $\varphi_e = \varphi + g$, while (3.19) is exactly the second equation in (2.34). Thus we derive rigorously the small-strain model for dielectric elastomers. The proof of Proposition 3.5 and Theorem 3.2 is given in Section 3.3, the proof of Proposition 3.6 and Theorem 3.4 is given in Section 3.4.

3.3 The First Order Limit Energy Functional

Proposition 3.8 (Lower bound for the first order energy) *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_8$, then for any sequences $y^\varepsilon \to x$ in $\mathcal{W}^{1,t}(\Omega)$ and $p_0^\varepsilon \rightharpoonup p_0$ in $\mathcal{L}^2(\Omega)$, we have*

$$\lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon \geqslant \mathcal{F}^0.$$

Before the proof of Proposition 3.8, we give some convergence result about the deformation first.

Lemma 3.9 *Denote by K a compact set in $\mathbb{R}^N$. Let $y^\varepsilon(x)$ be a sequence which satisfies* $\operatorname{supp}(y^\varepsilon - x) \subset K$ *and $J_\varepsilon \geqslant \delta$ for all ε. Assume $y^\varepsilon(x)$ converges to x strongly in $\mathcal{W}^{1,t}$. Let s_1 be the constant satisfying $\frac{1}{s_1} + \frac{1}{r_0} = \frac{1}{2}$, s_2 be the constant satisfying*

$\frac{1}{s_2} + \frac{2}{r_0} = 1$, *then*

$$\left\|F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_1}(K)} \to 0 \quad \textit{and} \quad \left\|J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_2}(K)} \to 0,$$

if $t > t_1 = \max(4s_1, 4s_2)$.

Proof

$$\begin{aligned}
\left\|F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_1}(K)} &\leqslant \left\|F_\varepsilon^{-1}\right\|_{\mathcal{L}^{2s_1}(K)} \left\|I - F_\varepsilon\right\|_{\mathcal{L}^{2s_1}(K)} \\
&= \left\|\frac{1}{J_\varepsilon}\operatorname{cof}(F_\varepsilon)\right\|_{\mathcal{L}^{2s_1}(K)} \left\|I - F_\varepsilon\right\|_{\mathcal{L}^{2s_1}(K)} \to 0 \text{ if } t > 4s_1.
\end{aligned}$$

$$\begin{aligned}
\left\|J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_2}(K)} &\leqslant \left\|\operatorname{cof}(F_\varepsilon) F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_2}(K)} \\
&\leqslant \left\|(\operatorname{cof}(F_\varepsilon) - I) F_\varepsilon^{-T}\right\|_{\mathcal{L}^{s_2}(K)} + \left\|F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_2}(K)} \\
&\leqslant \left\|\operatorname{cof}(F_\varepsilon) - I\right\|_{\mathcal{L}^{2s_2}(K)} \left\|F_\varepsilon^{-T}\right\|_{\mathcal{L}^{2s_2}(K)} + \left\|F_\varepsilon^{-T} - I\right\|_{\mathcal{L}^{s_2}(K)} \\
&\to 0 \text{ if } t > 4s_2.
\end{aligned}$$

□

Proof of Proposition 3.8

Since Ω satisfies strong local Lipschitz condition, by Stein extension theorem (see, e.g., Stein [36] or Adams [1]), $y^\varepsilon - x$ can be extended to the entire space $\mathbb{R}^N$ such that

(i) $E\big(y^\varepsilon - x\big) = y^\varepsilon - x$ a.e. in Ω;

(ii) $\left\|E(y^\varepsilon - x)\right\|_{\mathcal{W}^{1,t}(\mathbb{R}^N)} \leqslant c\left\|y^\varepsilon - x\right\|_{\mathcal{W}^{1,t}(\Omega)}$,

where E is the extension operator. We can make $y^\varepsilon - x \equiv 0$ outside some compact set K_y.

Assume there exist two open sets $\Omega_x^0 \supset \Gamma_0$ and $\Omega_x^1 \supset \Gamma_1$, such that $\Omega_x^0 \cap \Omega_x^1 = \varnothing$. Define $g(x) \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ to be a function satisfying $g(x) \equiv g_0$ in Ω_x^0 and $g(x) \equiv g_1$ in Ω_x^1. Let $g^\varepsilon(y) = g^\varepsilon\big(y^\varepsilon(x)\big) = g(x)$, then $g^\varepsilon(y) \equiv g_0$ inside $y^\varepsilon(\Omega_x^0) \supset y^\varepsilon(\Gamma_0)$, $g^\varepsilon(y) \equiv g_1$ inside $y^\varepsilon(\Omega_x^1) \supset y^\varepsilon(\Gamma_1)$ and $\nabla_y g^\varepsilon\big(y(x)\big) = F_\varepsilon^{-T}(x)\nabla_x g(x)$. Clearly, $g^\varepsilon(y) \in \mathcal{H}_0^1(\mathbb{R}^N)$

with compact support. Denote by K a big enough compact set in $\mathbb{R}^N$, such that $K \supset \big\{K_y \cup \operatorname{supp}\big(g(x)\big) \cup \operatorname{supp}\big(g^\varepsilon(y)\big)\big\}$.

To prove the lower bound of the first order energy, we examine the energy term by term.

$$\begin{aligned}
\text{term}\, a \quad &:= \quad \frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx \\
&= \quad \frac{1}{\varepsilon}\left[\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx - \frac{1}{2}\int_\Omega H^{-1}\varepsilon p_0^\varepsilon \cdot p_0^\varepsilon dx\right] + \frac{1}{2}\int_\Omega H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon\, dx \\
&\geqslant \quad \frac{1}{2}\int_\Omega H^{-1} p_0 \cdot p_0\, dx \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.20}$$

Equation (3.20) comes from (3.7) in assumption $\mathcal{A}_6$ and the lower semicontinuity of the functional $\frac{1}{2}\int_\Omega H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon dx$.

$$\begin{aligned}
\text{term}\, b \quad &:= \quad \int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon\, dy = \int_\Omega F_\varepsilon^{-T} \nabla_x g(x) \cdot p_0^\varepsilon\, dx \\
&= \quad \int_\Omega (F_\varepsilon^{-T} - I)\nabla_x g \cdot p_0^\varepsilon\, dx + \int_\Omega \nabla_x g \cdot p_0^\varepsilon\, dx \\
&\to \quad \int_\Omega \nabla_x g \cdot p_0\, dx \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.21}$$

Equation (3.21) comes from Lemma 3.9 and the fact that $p_0^\varepsilon \rightharpoonup p_0$ in $\mathcal{L}^2(\Omega)$.

$$\begin{aligned}
\text{term}\, c \quad &:= \quad -\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y g^\varepsilon\right|^2 dy \\
&= \quad -\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} \nabla_x g \cdot \nabla_x g - \nabla_x g \cdot \nabla_x g\, dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 dx \\
&= \quad -\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left(J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} - I\right) \nabla_x g \cdot \nabla_x g\, dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 dx \\
&\to \quad -\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 dx \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

The last step is from Lemma 3.9.

Now the only thing left to prove is

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \nabla_y\varphi^\varepsilon \cdot \nabla_y\varphi^\varepsilon \, dy \geqslant \int_{\mathbb{R}^N} \nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx. \tag{3.22}$$

The difficulty here is that F_ε is not in $\mathcal{L}^\infty$, so the function space $\mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$ is not isomorphic to function space $\mathcal{D}(\mathbb{R}^N\backslash\Gamma)$. To overcome this difficulty, we regularize p_0. Specifically, we introduce $\varphi_j^0 \in \mathcal{D}(\mathbb{R}^N\backslash\Gamma)$ to be the solution of

$$\begin{cases} \nabla_x \cdot \big[-\varepsilon_0\nabla_x\varphi_j^0 - \varepsilon_0\nabla_x g + p_0^j\,\chi(\Omega)\big] = 0 \quad \text{in} \quad \mathbb{R}^N\backslash\Gamma, \\ \varphi_j^0 \in \mathcal{D}(\mathbb{R}^N\backslash\Gamma), \end{cases} \tag{3.23}$$

where $p_0^j \in \mathcal{C}^\infty(\Omega)$ and $\|p_0^j - p_0\|_{\mathcal{L}^2(\Omega)} < \frac{1}{j}$. Clearly,

$$\lim_{j\to+\infty}\big\|\nabla\varphi_j^0 - \nabla\varphi^0\big\|_{\mathcal{L}^2(\mathbb{R}^N)} = 0.$$

Since Ω is in class $\mathfrak{B}^{r_0}$, following the same idea in [31], we will prove in Lemma 3.12 that $\nabla\varphi_j^0 \in \mathcal{L}^{r_0}_{\mathrm{loc}}(\mathbb{R}^N)$. Now, denote $\varphi_j^\varepsilon\big(y^\varepsilon(x)\big) = \varphi_j^0(x)$, $\varphi_j^\varepsilon\big(y^\varepsilon(\Gamma)\big) = 0$ and for any compact set D,

$$\int_D \big|\varphi_j^\varepsilon(y)\big|^2 \, dy = \int_{(y^\varepsilon)^{-1}(D)} J_\varepsilon \,\big|\varphi_j^0(x)\big|^2 \, dx < c$$

and

$$\int_{\mathbb{R}^N} \nabla_y\varphi_j^\varepsilon \cdot \nabla_y\varphi_j^\varepsilon \, dy = \int_{\mathbb{R}^N} J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} \nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 \, dx < c_j.$$

Above, we used the fact that $J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} \in \mathcal{L}^{s_2}(K)$ and $\nabla\varphi_j^0 \in \mathcal{L}^{r_0}_{\text{loc}}(\mathbb{R}^N)$. Now,

$$\begin{aligned}
\text{term}\, d \;:=\;& \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_y\varphi^\varepsilon \cdot \nabla_y\varphi^\varepsilon \, dy \\
=\;& \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y\varphi^\varepsilon - \nabla_y\varphi_j^\varepsilon\right|^2 dy + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_y\varphi^\varepsilon \cdot \nabla_y\varphi_j^\varepsilon \, dy \\
& - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y\varphi_j^\varepsilon\right|^2 dy && (3.24)\\
\geqslant\;& -\varepsilon_0 \int_{\mathbb{R}^N} \nabla_y g^\varepsilon \cdot \nabla_y\varphi_j^\varepsilon \, dy + \int_{y^\varepsilon(\Omega)} p^\varepsilon \cdot \nabla_y\varphi_j^\varepsilon \, dy - \int_\Omega p_0^j \cdot \nabla_x\varphi_j^0 \, dx \\
& + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x g \cdot \nabla_x\varphi_j^0 \, dx + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 \, dx \\
& - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_y\varphi_j^\varepsilon \cdot \nabla_y\varphi_j^\varepsilon \, dy && (3.25)\\
=\;& \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x g \cdot \nabla_x\varphi_j^0 - J_\varepsilon F_\varepsilon^{-T}\nabla_x g \cdot F_\varepsilon^{-T}\nabla_x\varphi_j^0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 \, dx \\
& + \int_\Omega p_0^\varepsilon \cdot F_\varepsilon^{-T}\nabla_x\varphi_j^0 - p_0^j \cdot \nabla_x\varphi_j^0 \, dx && (3.26)\\
=\;& \varepsilon_0 \int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x g \cdot \nabla_x\varphi_j^0 \, dx + \int_\Omega p_0^\varepsilon \cdot F_\varepsilon^{-T}\nabla_x\varphi_j^0 - p_0 \cdot \nabla_x\varphi_j^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 dx + \int_\Omega (p_0 - p_0^j) \cdot \nabla_x\varphi_j^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x\varphi_j^0 \cdot \nabla_x\varphi_j^0 \, dx. && (3.27)
\end{aligned}$$

(3.25) comes from (3.24) by dropping the first term in (3.24) and applying (3.23) and (3.10). In (3.27), the first and the last integral go to zero for fixed j as $\varepsilon \to 0$ from lemma 3.9. As to the second integral, we have

$$\int_\Omega \left(p_0^\varepsilon \cdot F_\varepsilon^{-T}\nabla_x\varphi_j^0 - p_0 \cdot \nabla_x\varphi_j^0\right) dx = \int_\Omega (p_0^\varepsilon - p_0) \cdot \nabla_x\varphi_j^0 \, dx + \int_\Omega (F_\varepsilon^{-T} - I)\, p_0^\varepsilon \cdot \nabla\varphi_j^0 \, dx \to 0$$

for fixed j and $\varepsilon \to 0$.

The fourth integral satisfies

$$\int_\Omega (p_0 - p_0^j) \cdot \nabla_x \varphi_j^0 \, dx \geqslant -\left\|p_0 - p_0^j\right\|_{\mathcal{L}^2(\Omega)} \left\|\nabla_x \varphi_j^0\right\|_{\mathcal{L}^2(\Omega)} \geqslant -\frac{c}{j}.$$

Putting them together, we get

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \nabla_y \varphi^\varepsilon \cdot \nabla_y \varphi^\varepsilon \, dy \geqslant \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \nabla_x \varphi_j^0 \cdot \nabla_x \varphi_j^0 \, dx - \frac{c}{j}.$$

Let $j \to +\infty$, on the right hand side,

$$\lim_{j\to+\infty} \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \nabla_x \varphi_j^0 \cdot \nabla_x \varphi_j^0 \, dx \to \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \nabla_x \varphi^0 \cdot \nabla_x \varphi^0 \, dx.$$

In the end,

$$\text{term}\, d = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi^\varepsilon\right|^2 dy \geqslant \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_x \varphi^0\right|^2 dx \quad \text{as } \varepsilon \to 0.$$

Thus we proved

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon = \lim_{\varepsilon\to 0} (\text{term}\, a + \text{term}\, b + \text{term}\, c + \text{term}\, d) \geqslant \mathcal{F}^0.$$

$\square$

Proposition 3.10 (The recovery sequences for the first order energy) *Under the same assumption as in Proposition 3.8, there exist a sequence $y^\varepsilon \to x$ in $\mathcal{W}^{1,t}(\Omega)$ and a sequence $p_0^\varepsilon \rightharpoonup p_0$ in $\mathcal{L}^2(\Omega)$ such that $\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon = \mathcal{F}^0$.*

Proof Let $y^\varepsilon \equiv x$. For p_0, we construct p_0^ε as

$$p_0^\varepsilon = \begin{cases} p_0 & \text{if } |p_0| \leqslant Q_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

with $Q_\varepsilon \to +\infty$ to be decided. We have $p_0^\varepsilon \to p_0$ because

$$\int_\Omega \left|p_0^\varepsilon - p_0\right|^2 dx = \int_{|p_0|>Q_\varepsilon} \left|p_0\right|^2 dx \to 0 \quad \text{as } \varepsilon \to 0.$$

If we take $Q_\varepsilon < \rho_0 \varepsilon^{-\frac{1}{2}}$, then from assumption $\mathcal{A}_7$,

$$\frac{1}{\varepsilon}\int_\Omega W(x, I, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx \leqslant \frac{1}{2}\int_\Omega H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon\, dx + w(\varepsilon^{\frac{1}{2}} Q_\varepsilon)\, Q_\varepsilon^2.$$

Let $Q_\varepsilon = \min\{\varepsilon^{-\frac{1}{4}}, w(\varepsilon^{\frac{1}{4}})^{-\frac{1}{4}}\}$. Since $w(|p_0|) \to 0$ monotonically as $|p_0| \to 0$,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega W(x, I, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx \leqslant \frac{1}{2}\int_\Omega H^{-1} p_0 \cdot p_0\, dx.$$

The opposite inequality holds because of the lower semicontinuity of the functional $\frac{1}{2}\int_\Omega H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon dx$, thus

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega W(x, I, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx \to \frac{1}{2}\int_\Omega H^{-1} p_0 \cdot p_0\, dx.$$

The convergence of the other terms in the energy is obvious since $p_0^\varepsilon \to p_0$ and $y^\varepsilon \equiv x$

□

Proof of Theorem 3.2 By the definition of Γ-convergence, Theorem 3.2 is the direct conclusion of Proposition 3.8 and Proposition 3.10 .

□

Now, let us prove one of the compactness results.

Proof of Proposition 3.5

Assume there exist Ω_y^0 and Ω_y^1 such that $\Omega_y^0 \cap \Omega_y^1 = \varnothing$, $\Omega_y^0 \supset y^\varepsilon(\Gamma_0)$, $\Omega_y^1 \supset y^\varepsilon(\Gamma_1)$ for all ε. Define function $g(y) \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ as a fixed function in y-space, satisfying $g(y) \equiv g_0$ in Ω_y^0 and $g(y) \equiv g_1$ in Ω_y^1. Let $g^\varepsilon(y) = g(y)$ for each ε. In this case, the energy functional becomes

$$\mathcal{F}^\varepsilon = \frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y \varphi^\varepsilon\right|^2 dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y g\right|^2 dy + \int_{y^\varepsilon(\Omega)} \nabla_y g \cdot p^\varepsilon\, dy,$$

where φ^ε is the solution of

$$\begin{cases} \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi^\varepsilon - \varepsilon_0 \nabla_y g + p^\varepsilon \chi\big(y^\varepsilon(\Omega)\big)\right] = 0 \quad \text{in} \quad \mathbb{R}^N, \\ \varphi^\varepsilon \in \mathcal{D}\big(\mathbb{R}^N \backslash y^\varepsilon(\Gamma)\big). \end{cases} \tag{3.28}$$

In the energy, the term $\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}|\nabla_y g|^2 dy$ is a constant since $g(y)$ is independent of ε. Now if $\mathcal{F}^\varepsilon \leqslant c$, there exists a constant $\tilde{c}$ such that

$$\begin{aligned}
\tilde{c} &\geqslant \mathcal{F}^\varepsilon + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y g\right|^2 dy \\
&= \frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y \varphi^\varepsilon\right|^2 dy + \int_{y^\varepsilon(\Omega)} \nabla_y g \cdot p^\varepsilon\, dy \\
&\geqslant \frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx - c\left\|\nabla_y g\right\|_{\mathcal{L}^\infty(\mathbb{R}^N)}\left\|p_0^\varepsilon\right\|_{\mathcal{L}^2(\Omega)}.
\end{aligned}$$

From (3.7) in assumption $\mathcal{A}_6$,

$$W(x, F, 0) \geqslant c\,\mathrm{dist}\big(F, \mathrm{SO}(N)\big)^2. \tag{3.29}$$

In addition, from assumption $\mathcal{A}_4$, there exists a constant M and c such that if $|F| > M$,

$$W(x, F, 0) > c\,|F|^t > \tilde{c}\ \mathrm{dist}\big(F, \mathrm{SO}(N)\big)^t. \tag{3.30}$$

From (3.29) and (3.30), there exists a small enough constant $\beta_0 > 0$ such that

$$W(x, F, 0) > \beta_0\,\mathrm{dist}\big(F, \mathrm{SO}(N)\big)^t \qquad \forall F. \tag{3.31}$$

(3.31) together with assumption $\mathcal{A}_5$, we get

$$\begin{aligned}
\tilde{c} &\geqslant \frac{1}{\varepsilon}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx - c\left\|\nabla_y g\right\|_{\mathcal{L}^\infty(\mathbb{R}^N)}\left\|p_0^\varepsilon\right\|_{\mathcal{L}^2(\Omega)} \\
&\geqslant \frac{1}{\varepsilon}\left[\frac{1}{2}\int_\Omega W(x, F_\varepsilon, 0)\, dx + \frac{1}{2}\int_\Omega W(x, I, \varepsilon^{\frac{1}{2}} p_0^\varepsilon)\, dx\right] - c\left\|p_0^\varepsilon\right\|_{\mathcal{L}^2(\Omega)} \\
&\geqslant \frac{1}{\varepsilon}\left[\frac{\beta_0}{2}\int_\Omega \mathrm{dist}\big(F_\varepsilon, \mathrm{SO}(N)\big)^t\, dx + \frac{1}{4}\int_\Omega H^{-1}\varepsilon p_0^\varepsilon \cdot p_0^\varepsilon\, dx\right] - c\left\|p_0^\varepsilon\right\|_{\mathcal{L}^2(\Omega)}.
\end{aligned}$$

From rigidity theorem [16], we can find a rotation $R_0^\varepsilon \in \mathrm{SO}(N)$ such that

$$\int_\Omega \mathrm{dist}\big(F_\varepsilon, \mathrm{SO}(N)\big)^t\, dx \geqslant c\int_\Omega \mathrm{dist}(F_\varepsilon, R_0^\varepsilon)^t\, dx.$$

Since the material is pinned on Γ_3, $R_0^\varepsilon = I$ and

$$\begin{aligned}
\tilde{c} \;&\geqslant\; \frac{1}{\varepsilon}\left[\frac{\beta_0}{2}\int_\Omega \operatorname{dist}\big(F_\varepsilon, \mathrm{SO}(N)\big)^t\,dx + \frac{1}{4}\int_\Omega H^{-1}\varepsilon p_0^\varepsilon\cdot p_0^\varepsilon\,dx\right] - c\,\big\|p_0^\varepsilon\big\|_{\mathcal{L}^2(\Omega)}\\
&\geqslant\; \frac{1}{\varepsilon}\left[\frac{c\,\beta_0}{2}\int_\Omega \operatorname{dist}(F_\varepsilon, I)^t\,dx + \frac{1}{4}\int_\Omega H^{-1}\varepsilon p_0^\varepsilon\cdot p_0^\varepsilon\,dx\right] - c\,\big\|p_0^\varepsilon\big\|_{\mathcal{L}^2(\Omega)}\\
&\geqslant\; \frac{1}{\varepsilon}\cdot\frac{c\,\beta_0}{2}\big\|F_\varepsilon - I\big\|_{\mathcal{L}^t(\Omega)}^t + c\,\big\|p_0^\varepsilon\big\|_{\mathcal{L}^2(\Omega)}^2 - c\,\big\|p_0^\varepsilon\big\|_{\mathcal{L}^2(\Omega)}.
\end{aligned}$$

Therefore, there exists a constant c, such that

$$\big\|p_0^\varepsilon\big\|_{\mathcal{L}^2(\Omega)} < c, \qquad \left\|\frac{F_\varepsilon - I}{\varepsilon^{\frac{1}{t}}}\right\|_{\mathcal{L}^t(\Omega)} \leqslant c.$$

Thus, if $\mathcal{F}^\varepsilon$ is bounded, $F_\varepsilon \to I$ in $\mathcal{L}^t(\Omega)$ and p_0^ε is uniformly bounded in $\mathcal{L}^2(\Omega)$. This gives the compactness of the electric and the deformation field.

□

Now, let us prove the local regularity of φ_j^0.

Definition 3.3 *Let q be a number such that $1 < q < \infty$. Define q' by $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define q^* by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{N}$ if $q < N$ and to be any number in the range $1 < q^* < \infty$ if $q \geqslant N$.*

Consider differential equation

$$\nabla\cdot\big(a(x)\nabla v(x)\big) = \operatorname{div}\vec{f}(x) + h(x) \quad \text{in } \mathcal{O}, \tag{3.32}$$

where $\mathcal{O}$ is a bounded domain of class $\mathfrak{B}^r$ for some r, $2 < r < \infty$. Assume there exist constants M and λ_1 such that for any $\xi \in \mathbb{R}^N$, $a(x)$ satisfies the following condition

$$\begin{cases} \lambda_1|\xi|^2 \leqslant \big(a(x)\,\xi\,,\xi\,\big),\\ |a(x)\,\xi| \leqslant M\,|\xi|. \end{cases} \tag{3.33}$$

Define quantity θ by means of the equation

$$1-\theta = \min_{c\geqslant 0} \frac{(M^2+c^2)^{\frac{1}{2}}}{\lambda_1+c}. \tag{3.34}$$

Since the ratio in (3.34) is less than 1 for large values of c, we have $1-\theta<1$. Denote c_1 the value of c at which the minimum in (3.34) is attained. Setting $\lambda = \frac{\lambda_1+c_1}{M+c_1}$.

Theorem 3.11 (Meyers' theorem [31]) *Assume that equation (3.32) holds with $a(x)$ satisfying all the above conditions. Then (3.32) has a unique solution in $\mathcal{W}_0^{1,p}(\mathcal{O})$ for every vector field $\vec{f}(x)\in\mathcal{L}^p(\mathcal{O})$ and every function $h(x)\in\mathcal{L}^q(\mathcal{O})$ with $q^*\geqslant p$, provided $r'\leqslant Q'<p<Q\leqslant r$.*

Here $Q>2$ depends only on $\mathcal{O}$ and constant $\theta\lambda$ in such a way that $Q\to r$ as $\theta\lambda\to 1$ and $Q\to 2$ as $\theta\lambda\to 0$. The solution satisfies

$$\big\|\nabla v(x)\big\|_{\mathcal{L}^p(\mathcal{O})} \leqslant c\big\{\big\|\vec{f}(x)\big\|_{\mathcal{L}^p(\mathcal{O})}+\big\|h(x)\big\|_{\mathcal{L}^q(\mathcal{O})}\big\}, \tag{3.35}$$

where c is a constant depending only on $\mathcal{O}$, $\theta\lambda$, p, and q.

Lemma 3.12 (The local $\mathcal{L}^{r_0}$ regularity of φ_j^0) *Assume φ_j^0 is the solution of*

$$\begin{cases} \nabla_x\cdot\big[-\varepsilon_0\nabla_x\varphi_j^0-\varepsilon_0\nabla_x g+p_0^j\,\chi(\Omega)\big]=0 \quad \text{in} \quad \mathbb{R}^N\backslash\Gamma, \\ \varphi_j^0\in\mathcal{D}(\mathbb{R}^N\backslash\Gamma), \end{cases} \tag{3.36}$$

in which $p_0^j\in\mathcal{C}^\infty(\Omega)$, $g(x)\in\mathcal{C}_0^\infty(\mathbb{R}^N)$, $\Omega\in\mathfrak{B}^{r_0}$. Then for any given compact set $D\subset\mathbb{R}^N$, there exists a constant c such that

$$\big\|\nabla_x\varphi_j^0(x)\big\|_{\mathcal{L}^{r_0}(D)} \leqslant c\big\{\big\|\nabla_x g\big\|_{\mathcal{L}^{r_0}(D)}+\big\|p_0^j\chi(\Omega)\big\|_{\mathcal{L}^{r_0}(D)}\big\}. \tag{3.37}$$

Proof Equation (3.36) satisfies the hypothesis of Meyers' theorem. But we have an unbounded domain. Therefore, we need to prove it.

Let $\zeta(x)$ be an infinitely continuously differentiable function such that

$$\zeta(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant \frac{1}{2}, \\ 0 & \text{for} \quad |x| \geqslant \frac{3}{4}. \end{cases}$$

Assume $\Omega \subset \operatorname{supp}\big(g(x)\big) \subset D \subset B_R$. Let $\xi = \zeta\left(\frac{x}{2R}\right)$, then

$$\xi(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant R, \\ 0 & \text{for} \quad |x| \geqslant \frac{3}{2}R. \end{cases}$$

Denote by ψ_j^0 the unique solution of equation (3.36) in space $\mathcal{H}_0^1(B_{2R}\backslash\Gamma)$. Then by Meyers' theorem, $\psi_j^0 \in \mathcal{W}_0^{1,r_0}(B_{2R}\backslash\Gamma)$. Here $p = r_0$ since $\lambda\theta = 1$. Set $\phi_j^0 = \varphi_j^0 - \psi_j^0$, then for any $\psi \in \mathcal{H}_0^{1,2}(B_{2R}\backslash\Gamma)$, we have

$$\int_{B_{2R}} \varepsilon_0 \nabla_x \phi_j^0 \cdot \nabla\psi \, dx = 0. \tag{3.38}$$

Now consider $v(x) = \xi(x)\phi_j^0(x)$. $v(x) = \phi_j^0(x)$ for $|x| \leqslant R$ and $v(x) = 0$ for $|x| \geqslant \frac{3R}{2}$. For any $\psi \in \mathcal{H}_0^{1,2}(B_{2R}\backslash\Gamma)$, we then have

$$\begin{aligned} & \int_{B_{2R}} \varepsilon_0 \nabla\big(\xi(x)\phi_j^0(x)\big) \cdot \nabla\psi \, dx \\ = & \int_{B_{2R}} \varepsilon_0 \nabla\phi_j^0 \cdot \nabla\big(\xi(x)\psi(x)\big) \, dx - \int_{B_{2R}} \varepsilon_0 \psi \, \nabla\phi_j^0 \cdot \nabla\xi - \varepsilon_0 \phi_j^0 \, \nabla\xi \cdot \nabla\psi \, dx. \end{aligned}$$

The first integral on the right hand side is zero because of (3.38). Thus $v(x)$ satisfies

$$\begin{cases} \nabla\cdot\big(\varepsilon_0 \nabla v\big) = \nabla\cdot\big(\phi_j^0\, \varepsilon_0 \nabla\xi\big) + \varepsilon_0 \nabla\xi \cdot \nabla\phi_j^0 & \text{in} \quad B_{2R}\backslash\Gamma, \\ v \in \mathcal{H}_0^1(B_{2R}\backslash\Gamma). \end{cases} \tag{3.39}$$

$\nabla\phi_j^0 \in \mathcal{L}^2(B_{2R})$, hence $\phi_j^0 \in \mathcal{L}^{N^*}$ from embedding theorem. Therefore, $v \in \mathcal{W}_0^{1,p_1}(B_{2R}\backslash\Gamma)$ for all $p_1 \leqslant \min(N^*, r_0) = r_0$ from Meyers' theorem. Thus $\phi_j^0(x) \in \mathcal{W}_0^{1,r_0}(B_R\backslash\Gamma)$ and so $\varphi_j^0 \in \mathcal{W}_0^{1,r_0}(B_R\backslash\Gamma)$. In the end, we get $\nabla\varphi_j^0 \in \mathcal{L}^{r_0}(D)$. $\square$

With the same idea to prove Lemma 3.12, we can prove the following local regularity result for the minimizer of $\mathcal{F}^0$.

Proposition 3.13 *Assume Ω is in class $\mathfrak{B}^{r_0}$, then there exists a number $r_1 \leqslant r_0$ such that the solution for equation (3.13) has higher local regularity: $\nabla\varphi^0 \in \mathcal{L}^{r_1}(\Omega)$.*

Proof Define $\zeta(x)$ and $\xi(x)$ as in Lemma 3.12. Denote by ψ^0 the unique solution of equation (3.13) in space $\mathcal{H}_0^1(B_{2R}\backslash\Gamma)$. Then by Meyers' theorem, $\psi^0 \in \mathcal{W}_0^{1,p}(B_{2R}\backslash\Gamma)$, for any $2 < p < Q \leqslant r_0$. Here, Q depends only on N and $\lambda\theta\big(\varepsilon_0 I + H(x)\chi(\Omega)\big)$. Set $\phi = \varphi^0 - \psi^0$, then for any $\psi \in \mathcal{H}_0^{1,2}(B_{2R}\backslash\Gamma)$,

$$\int_{B_{2R}} (H\chi(\Omega) + \varepsilon_0 I)\nabla_x\phi \cdot \nabla\psi \, dx = 0.$$

Now consider $v(x) = \xi(x)\phi(x)$. $v(x) = \phi(x)$ for $|x| \leqslant R$ and $v(x) = 0$ for $|x| \geqslant \frac{3R}{2}$. Then for any $\psi \in \mathcal{H}_0^{1,2}(B_{2R}\backslash\Gamma)$,

$$\begin{aligned}
&\int_{B_{2R}} \big(H\chi(\Omega) + \varepsilon_0 I\big)\,\nabla\big(\xi(x)\phi(x)\big) \cdot \nabla\psi \, dx \\
= &\int_{B_{2R}} \big(H\chi(\Omega) + \varepsilon_0 I\big)\nabla\phi \cdot \nabla\big(\xi(x)\psi(x)\big) \, dx \\
&- \int_{B_{2R}} \big(H\chi(\Omega) + \varepsilon_0 I\big)\psi\nabla\phi \cdot \nabla\xi - \big(H\chi(\Omega) + \varepsilon_0 I\big)\phi\nabla\xi \cdot \nabla\psi \, dx.
\end{aligned}$$

The first integral on the right hand side is zero, so $v(x)$ satisfies

$$\begin{cases} \nabla\cdot\big[\big(\varepsilon_0 I + H\chi(\Omega)\big)\nabla v\big] = \nabla\cdot\big[\phi\big(\varepsilon_0 I + H\chi(\Omega)\big)\nabla\xi\big] + \nabla\xi\cdot\big(H\chi(\Omega) + \varepsilon_0 I\big)\nabla\phi, \\ v \in \mathcal{H}_0^1(B_{2R}\backslash\Gamma). \end{cases} \tag{3.40}$$

$\nabla\phi \in \mathcal{L}^2(B_{2R})$, hence $\phi \in \mathcal{L}^{N^*}(B_{2R})$ by embedding theorem. Therefore, v is in $\mathcal{H}_0^{1,p_1}(B_{2R})$ for all $2 < p_1 < \min(N^*, Q)$. Thus $\phi(x) \in \mathcal{H}_0^{1,p_1}(B_R)$ and then $\varphi^0 \in \mathcal{H}_0^{1,p_1}(B_R)$. In the end, we get $\nabla\varphi^0 \in \mathcal{L}^{r_1}(\Omega)$ for any $2 < r_1 < p_1 \leqslant r_0$. $\square$

Remark Here, p_1 depends only on Q, while Q depends on the quantity $\lambda\theta$. If $\lambda\theta \to 1$, $p_1 \to r_0$; if $\lambda\theta \to \infty$, then $p_1 \to 2$.

In the second order energy part, to make $\nabla\varphi^0 \cdot \nabla\varphi^0 \in \mathcal{H}^{-1}(\Omega)$, $\nabla\varphi^0$ has to be a little bit more regular than $\mathcal{L}^4(\Omega)$. That requires strong assumptions on $\lambda\theta$. However, the above conclusion is a general result for any distribution $H(x)$. For particulate composite, we have better regularity for $H(x)$.

3.4 The Second Order Limit Energy Functional

Proposition 3.14 (Lower bound for the second order energy) *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_8$. Assume further that there exists a constant $w > 0$ and a compact set $K \supset \Omega$ such that $\nabla\varphi^0$, the solution of equation (3.13), is in $\mathcal{L}^{4+w}(K)$. Let y^ε be any sequence satisfying $y^\varepsilon \to x$ in $\mathcal{W}^{1,t}$ and $\nabla u^\varepsilon = \frac{1}{\varepsilon}\nabla(y^\varepsilon - x) \rightharpoonup \nabla u^0$ in $\mathcal{L}^2(\Omega)$. Let p_0^ε be any sequence such that $q_0^\varepsilon = \varepsilon^{-\frac{1}{2}}(p_0^\varepsilon - p_0) \rightharpoonup q_0$ in $\mathcal{L}^2(\Omega)$. Then for the functional defined in (3.17), we have*

$$\lim_{\varepsilon\to 0} \mathcal{F}_r^\varepsilon \geqslant \mathcal{F}_r^0,$$

where $\mathcal{F}_r^0$ is defined in (3.18).

Proof Again, examine the energy term by term.

$$\begin{aligned}
\text{term 1} &:= \frac{1}{\varepsilon}\left[\int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon \, dy - \int_\Omega \nabla_x g \cdot p_0 \, dx\right] \\
&= \frac{1}{\varepsilon}\left[\int_\Omega F_\varepsilon^{-T}\nabla_x g \cdot (p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon) \, dx - \int_\Omega \nabla_x g \cdot p_0 \, dx\right] \\
&= \int_\Omega \frac{F_\varepsilon^{-T} - I}{\varepsilon}\nabla_x g \cdot p_0 \, dx + \frac{1}{\varepsilon}\int_\Omega \nabla_x g \cdot \varepsilon^{\frac{1}{2}} q_0^\varepsilon \, dx + \varepsilon^{\frac{1}{2}}\int_\Omega \frac{F_\varepsilon^{-T} - I}{\varepsilon}\nabla_x g \cdot q_0^\varepsilon \, dx \\
&= \int_\Omega \frac{F_\varepsilon^{-T} - I}{\varepsilon}\nabla_x g \cdot p_0 \, dx + \varepsilon^{-\frac{1}{2}}\int_\Omega \nabla_x g \cdot q_0^\varepsilon \, dx + o(1).
\end{aligned}$$

Note that we have

$$\frac{F_\varepsilon^{-1} - I}{\varepsilon} = \frac{F_\varepsilon^{-1}(I - F_\varepsilon)}{\varepsilon} = F_\varepsilon^{-1}\frac{I - F_\varepsilon}{\varepsilon},$$

$\frac{I-F_\varepsilon}{\varepsilon} \rightharpoonup -\nabla u$, and $F_\varepsilon^{-1} \to I$, thus

$$\frac{F_\varepsilon^{-1} - I}{\varepsilon} \rightharpoonup -\nabla u.$$

Similarly,

$$\frac{F_\varepsilon^{-T} - I}{\varepsilon} \rightharpoonup -\nabla u^T \quad \text{and} \quad \frac{J_\varepsilon - 1}{\varepsilon} \rightharpoonup \operatorname{tr}(\nabla u).$$

Therefore,

$$\text{term}\,1 \to \varepsilon^{-\frac{1}{2}} \int_\Omega \nabla_x g \cdot q_0^\varepsilon \, dx + \int_\Omega (-\nabla u)^T \nabla_x g \cdot p_0 \, dx.$$

$$\begin{aligned}
\text{term}\,2 \;&:=\; \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y g^\varepsilon\right|^2 dy\right] \\
&=\; \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \frac{I - J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T}}{\varepsilon} \nabla_x g \cdot \nabla_x g \, dx \\
&\to\; \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (\nabla u + \nabla u^T - \operatorname{tr}(\nabla u) I) \nabla_x g \cdot \nabla_x g \, dx.
\end{aligned}$$

To address

$$\text{term}\,3 := \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y \varphi^\varepsilon\right|^2 dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x \varphi^0\right|^2 dx\right], \tag{3.41}$$

denote $\varphi^0_{y^\varepsilon}\big(y^\varepsilon(x)\big) = \varphi^0(x)$, then $\varphi^0_{y^\varepsilon}\big(y^\varepsilon(\Gamma)\big) = 0$,

$$\int_D \left|\varphi^0_{y^\varepsilon}(y)\right|^2 dy = \int_{(y^\varepsilon)^{-1}(D)} J_\varepsilon \left|\varphi^0(x)\right|^2 dx < c$$

for any compact set D, and

$$\int_{\mathbb{R}^N} \nabla_y \varphi^0_{y^\varepsilon} \cdot \nabla_y \varphi^0_{y^\varepsilon} \, dy = \int_{\mathbb{R}^N} J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} \nabla_x \varphi^0 \cdot \nabla_x \varphi^0 \, dx < c.$$

Above, we used the fact that $\nabla\varphi^0 \in \mathcal{L}^{4+w}$ and $F_\varepsilon \to I$ in $\mathcal{L}^t(\Omega)$ for t big enough. Thus $\varphi^0_{y^\varepsilon}(y) \in \mathcal{D}\big(\mathbb{R}^N \backslash y(\Gamma)\big)$. Therefore,

$$
\begin{aligned}
& \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_y\varphi^\varepsilon \cdot \nabla_y\varphi^\varepsilon \, dy \\
= \; & \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_y\varphi^\varepsilon \cdot \nabla_y\varphi^0_{y^\varepsilon}\, dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy \\
= \; & \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy - \varepsilon_0 \int_{\mathbb{R}^N} \nabla_y g^\varepsilon \cdot \nabla_y\varphi^0_{y^\varepsilon}\, dy + \int_{y^\varepsilon(\Omega)} p^\varepsilon \cdot \nabla_y\varphi^0_{y^\varepsilon}\, dy \\
& - \int_\Omega p_0 \cdot \nabla_x\varphi^0 \, dx + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x g \cdot \nabla_x\varphi^0 \, dx + \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx \\
& - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_y\varphi^0_{y^\varepsilon} \cdot \nabla_y\varphi^0_{y^\varepsilon}\, dy \\
= \; & \varepsilon_0 \int_{\mathbb{R}^N} \nabla_x g \cdot \nabla_x\varphi^0 - J_\varepsilon F_\varepsilon^{-T}\nabla_x g \cdot F_\varepsilon^{-T}\nabla_x\varphi^0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy \\
& + \int_\Omega p_0^\varepsilon \cdot F_\varepsilon^{-T}\nabla_x\varphi^0 - p_0 \cdot \nabla_x\varphi^0 \, dx \\
= \; & \varepsilon_0 \int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x g \cdot \nabla_x\varphi^0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T})\nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx \\
& + \int_\Omega p_0^\varepsilon \cdot F_\varepsilon^{-T}\nabla_x\varphi^0 - p_0 \cdot \nabla_x\varphi^0 \, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{term}\,3 \; = \; & \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2 \, dy\right] + \varepsilon_0 \int_{\mathbb{R}^N} \frac{I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}}{\varepsilon} \nabla g \cdot \nabla_x\varphi^0 dx \\
& + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \frac{I - J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}}{\varepsilon} \nabla_x\varphi^0 \cdot \nabla_x\varphi^0 \, dx + \int_\Omega \frac{F_\varepsilon^{-1} - I}{\varepsilon}\, p_0 \cdot \nabla_x\varphi^0 \, dx \\
& + \varepsilon^{-1/2}\int_\Omega q_0^\varepsilon \cdot \nabla_x\varphi^0 \, dx \; + \; \varepsilon^{1/2}\int_\Omega \frac{F_\varepsilon^{-1} - I}{\varepsilon}\, q_0^\varepsilon \cdot \nabla_x\varphi^0 \, dx.
\end{aligned}
\tag{3.42}
$$

Similar to the idea in Proposition 3.8, we define $\phi_j^0(x)$ as the solution of

$$\begin{cases} \nabla_x \cdot \big(-\varepsilon_0 \nabla_x \phi_j^0 + q_0^j \chi(\Omega)\big) = 0 \quad \text{ in } \mathbb{R}^N \backslash \Gamma, \\ \phi_j^0 \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma), \end{cases}$$

in which $q_0^j \in \mathcal{C}^\infty(\Omega)$ and $\|q_0^j - q_0\|_{\mathcal{L}^2(\Omega)} \leqslant \frac{1}{j}$.

Denote $\phi_j^\varepsilon\big(y^\varepsilon(x)\big) = \phi_j^0(x)$, then $\phi_j^\varepsilon(y) \in \mathcal{D}\big(\mathbb{R}^N \backslash y^\varepsilon(\Gamma)\big)$. We have

$$\begin{aligned}
& \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon}\big|^2\,dy\right] \\
= \;& \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon} - \varepsilon^{\frac{1}{2}}\nabla_y\phi_j^\varepsilon\big|^2\,dy\right] - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\phi_j^\varepsilon\big|^2\,dy \\
& + \frac{1}{\varepsilon}\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} 2(\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon})\cdot\varepsilon^{\frac{1}{2}}\nabla_y\phi_j^\varepsilon\,dy \\
\geqslant \;& \frac{\varepsilon_0}{\varepsilon}\int_{\mathbb{R}^N}\Big(\varepsilon^{\frac{1}{2}}\nabla_y\varphi^\varepsilon\cdot\nabla_y\phi_j^\varepsilon - \varepsilon^{\frac{1}{2}}\nabla_y\varphi^0_{y^\varepsilon}\cdot\nabla_y\phi_j^\varepsilon\Big)\,dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\phi_j^\varepsilon\big|^2 dy \\
= \;& \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\Big[\Big(-\varepsilon^{\frac{1}{2}}\varepsilon_0\nabla_y g^\varepsilon + \varepsilon^{\frac{1}{2}}p^\varepsilon\Big)\cdot\nabla_y\phi_j^\varepsilon - \varepsilon^{\frac{1}{2}}\varepsilon_0\nabla_y\varphi^0_{y^\varepsilon}\cdot\nabla\phi_j^\varepsilon\Big]\,dy - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_y\phi_j^\varepsilon\big|^2\,dy \\
= \;& \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\varepsilon^{\frac{1}{2}}\Big[-\varepsilon_0 J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}\nabla_x g\cdot\nabla_x\phi_j^0 + F_\varepsilon^{-T}\nabla_x\phi_j^0\cdot p_0 + \varepsilon^{\frac{1}{2}}F_\varepsilon^{-T}\nabla_x\phi_j^0\cdot q_0^\varepsilon \\
& - \varepsilon_0 J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}\nabla_x\varphi^0\cdot\nabla_x\phi_j^0\Big]\,dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}\nabla_x\phi_j^0\cdot\nabla_x\phi_j^0\,dx \\
= \;& \int_{\mathbb{R}^N} F_\varepsilon^{-T}\nabla_x\phi_j^0\cdot q_0^\varepsilon\,dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}\nabla_x\phi_j^0\cdot\nabla_x\phi_j^0\,dx \\
& + \varepsilon^{\frac{1}{2}}\int_{\mathbb{R}^N}\varepsilon^{-1}\big(-\varepsilon_0\nabla_x g\cdot\nabla_x\phi_j^0 + \nabla_x\phi_j^0\cdot p_0 - \varepsilon_0\nabla_x\phi_j^0\cdot\nabla_x\varphi^0\big)\,dx + o(1) \\
\to \;& \int_{\mathbb{R}^N}\nabla_x\phi_j^0\cdot q_0\,dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_x\phi_j^0\big|^2\,dx \quad \text{as } \varepsilon\to 0 \\
\to \;& \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\big|\nabla_x\phi^0\big|^2\,dx \quad \text{as } j\to+\infty. \qquad (3.43)
\end{aligned}$$

Now we can estimate term 3.

$$
\begin{aligned}
\text{term}\,3 \;=\;& \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi-\nabla_y\varphi^0_{y^\varepsilon}\right|^2 dy\right]+\varepsilon_0\int_{\mathbb{R}^N}\frac{I-J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}}{\varepsilon}\nabla g\cdot\nabla_x\varphi^0\,dx\\
&+\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\frac{I-J_\varepsilon F_\varepsilon^{-1}F_\varepsilon^{-T}}{\varepsilon}\nabla_x\varphi^0\cdot\nabla_x\varphi^0\,dx+\int_\Omega\frac{F_\varepsilon^{-T}-I}{\varepsilon}p_0\cdot\nabla_x\varphi^0\,dx\\
&+\varepsilon^{-1/2}\int_\Omega q_0^\varepsilon\cdot\nabla_x\varphi^0\,dx+o(1) \qquad (3.44)\\
\geqslant\;& \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_x\phi^0\right|^2dx+\varepsilon_0\int_{\mathbb{R}^N}(\nabla u+\nabla u^T-\mathrm{tr}(\nabla u)I)\nabla g\cdot\nabla_x\varphi^0\,dx\\
&+\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}(\nabla u+\nabla u^T-\mathrm{tr}(\nabla u)I)\nabla_x\varphi^0\cdot\nabla_x\varphi^0\,dx-\int_\Omega\nabla u\,p_0\cdot\nabla_x\varphi^0\,dx\\
&+\varepsilon^{-1/2}\int_\Omega q_0^\varepsilon\cdot\nabla_x\varphi^0\,dx.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\text{term}\,4 \;:=\;& \frac{1}{\varepsilon^2}\int_\Omega\left[W(x,F_\varepsilon,\varepsilon^{\frac12}p_0^\varepsilon)-\frac{\varepsilon H^{-1}}{2}p_0\cdot p_0\right]dx\\
=\;& \frac{1}{\varepsilon^2}\int_\Omega\left[W(x,F_\varepsilon,\varepsilon^{\frac12}p_0^\varepsilon)-\frac{\varepsilon H^{-1}}{2}p_0^\varepsilon\cdot p_0^\varepsilon\right]dx\\
&+\frac{1}{\varepsilon}\int_\Omega\frac{H^{-1}}{2}(p_0^\varepsilon-p_0)\cdot(p_0^\varepsilon-p_0)\,dx+\frac{1}{\varepsilon}\int_\Omega H^{-1}(p_0^\varepsilon-p_0)\cdot p_0\,dx\\
=\;& \text{termA}+\text{termB}+\text{termC}\,.
\end{aligned}
$$

On the right hand side,

$$
\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_\Omega\frac{H^{-1}}{2}(p_0^\varepsilon-p_0)\cdot(p_0^\varepsilon-p_0)\,dx=\lim_{\varepsilon\to0}\int_\Omega\frac{H^{-1}}{2}q_0^\varepsilon\cdot q_0^\varepsilon\,dx\geqslant\int_\Omega\frac{H^{-1}}{2}q_0\cdot q_0\,dx.
$$

$$
\text{termC}=\frac{1}{\varepsilon}\int_\Omega H^{-1}(p_0^\varepsilon-p_0)\cdot p_0\,dx=\varepsilon^{-\frac12}\int_\Omega H^{-1}q_0^\varepsilon\cdot p_0\,dx.
$$

TermC will be a part of the Euler-Lagrange equation (3.16).

Now look at termA,

$$
\text{termA}:=\frac{1}{\varepsilon^2}\int_\Omega\left[W(x,F_\varepsilon,\varepsilon^{\frac12}p_0^\varepsilon)-\frac{\varepsilon H^{-1}}{2}p_0^\varepsilon\cdot p_0^\varepsilon\right]dx
$$

where $p_0^\varepsilon = p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon$.

Since the deformation field is not in $\mathcal{L}^\infty$, we use truncation lemma to separate the fast growing part from the deformation field so that we can use Taylor expansion.

Lemma 3.15 (Truncation Lemma [16]) *Let $\mathcal{O}$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a constant c depending on $\mathcal{O}$ such that for every $M > m > 1$ and every function $u : \mathcal{O} \longrightarrow \mathbb{R}^N$ with $\nabla u \in \mathcal{L}^2(\mathcal{O}; \mathbb{R}^{N^2})$, there exists a $\lambda \in [m, M]$ and a function $u^\lambda : \mathcal{O} \longrightarrow \mathbb{R}^N$ such that $|\nabla u^\lambda| \leqslant \lambda$ and*

$$\lambda^2 \big|\{u^\lambda \neq u\}\big| \leqslant \frac{c \left\|\nabla u\right\|^2_{\mathcal{L}^2(\Omega)}}{\ln(M/m)}.$$

Applying the truncation lemma to u^ε with $m = 1$, $M = M_\varepsilon$ yields truncation function v^ε, which satisfies

$$\left|\nabla v^\varepsilon\right| \leqslant \lambda_\varepsilon, \qquad 1 \leqslant \lambda_\varepsilon \leqslant M_\varepsilon,$$

and

$$\lambda_\varepsilon^2 \left|Z_u^c(\varepsilon)\right| \leqslant \frac{c}{\ln(M_\varepsilon)}, \qquad \text{where } Z_u(\varepsilon) := \left\{x | u^\varepsilon(x) = v^\varepsilon(x)\right\}.$$

For some suitable number $Q_\varepsilon \to +\infty$ which is to be decided later, define two sets $Z_p(\varepsilon)$ and $Z_q(\varepsilon)$ as

$$\begin{aligned} Z_p(\varepsilon) &= \left\{x \in \Omega : \left|p_0(x)\right| \leqslant \frac{1}{2} Q_\varepsilon\right\}, \\ Z_q(\varepsilon) &= \left\{x \in \Omega : \left|q_0^\varepsilon(x)\right| \leqslant \frac{1}{2} Q_\varepsilon\right\}. \end{aligned}$$

Denote $Z_u^c(\varepsilon)$, $Z_p^c(\varepsilon)$, $Z_q^c(\varepsilon)$ to be the complements to $Z_u(\varepsilon)$, $Z_p(\varepsilon)$, $Z_q(\varepsilon)$ in Ω respectively. For simplicity, we write them as Z_u, Z_p, Z_q, but they depend on ε.

Assume Q_ε and M_ε satisfy

$$Q_\varepsilon < \rho_1 \varepsilon^{-\frac{1}{2}} \quad \text{and} \quad M_\varepsilon < \rho_2 \varepsilon^{-1},$$

then

$$
\begin{aligned}
\left|\varepsilon\nabla u^\varepsilon\right| &= \left|\varepsilon\nabla v^\varepsilon\right| \leqslant \varepsilon M_\varepsilon < \rho_2 \quad \text{in } \ Z_u,\\
\left|\varepsilon^{\frac12} p_0^\varepsilon\right| &= \left|\varepsilon^{\frac12} p_0 + \varepsilon q_0^\varepsilon\right| \leqslant \rho_1 \qquad \text{in } \ Z_p \cap Z_q.
\end{aligned}
$$

Now we can use Taylor expansion on termA.

$$
\begin{aligned}
\text{termA} &= \frac{1}{\varepsilon^2}\int_\Omega \left[W\big(x, I+\varepsilon\nabla u^\varepsilon, \varepsilon^{\frac12}(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\big) - \frac{\varepsilon H^{-1}}{2}(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\cdot(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\right] dx\\
&= \frac{1}{\varepsilon^2}\int_{Z_u\cap Z_p\cap Z_q} \left[W - \frac{\varepsilon H^{-1}}{2}(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\cdot(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\right] dx\\
&\quad + \frac{1}{\varepsilon^2}\int_{Z_p^c\cup Z_q^c} \left[W - \frac{\varepsilon H^{-1}}{2} p_0^\varepsilon\cdot p_0^\varepsilon\right] dx + \frac{1}{\varepsilon^2}\int_{Z_u^c\cap Z_p\cap Z_q} \left[W - \frac{\varepsilon H^{-1}}{2} p_0^\varepsilon\cdot p_0^\varepsilon\right] dx\\
&\geqslant \int_{Z_u\cap Z_p\cap Z_q} \frac12 C\nabla v^\varepsilon\nabla v^\varepsilon + A\nabla v^\varepsilon(p_0+\varepsilon^{\frac12}q_0^\varepsilon)(p_0+\varepsilon^{\frac12}q_0^\varepsilon) + B(p_0+\varepsilon^{\frac12}q_0^\varepsilon)^4\, dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} w_1\big(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|\big)\,|p_0^\varepsilon|^4 + w_2\big(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|\big)\,|\nabla v^\varepsilon|^2 dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} w_3(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|)\,|\nabla v^\varepsilon||p_0^\varepsilon|^2 dx\\
&\geqslant \int_{Z_u\cap Z_p\cap Z_q} \frac12 C\nabla v^\varepsilon\nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B(p_0)^4\, dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} \left|A\nabla v^\varepsilon \varepsilon^{\frac12} q_0^\varepsilon(p_0+\varepsilon^{\frac12}q_0^\varepsilon)\right| dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} \left|B\big((\varepsilon^{\frac12}q_0^\varepsilon)^4 + 3(\varepsilon^{\frac12}q_0^\varepsilon)^3 p_0 + 6(\varepsilon^{\frac12}q_0^\varepsilon)^2 p_0^2 + 3(\varepsilon^{\frac12}q_0^\varepsilon)p_0^3\,\big)\right| dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} w_1\big(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|\big)\,|p_0^\varepsilon|^4 + w_2\big(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|\big)\,|\nabla v^\varepsilon|^2 dx\\
&\quad - \int_{Z_u\cap Z_p\cap Z_q} w_3(\varepsilon|\nabla v^\varepsilon|, \varepsilon^{\frac12}|p_0^\varepsilon|)\,|\nabla v^\varepsilon||p_0^\varepsilon|^2 dx\\
&\geqslant \int_{Z_u\cap Z_p\cap Z_q} \frac12 C\nabla v^\varepsilon\nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B(p_0)^4\, dx\\
&\quad - c\left(\, M_\varepsilon \varepsilon^{\frac12} Q_\varepsilon^2 + \varepsilon^2 Q_\varepsilon^4 + \varepsilon^{\frac32} Q_\varepsilon^4 + \varepsilon Q_\varepsilon^4 + \varepsilon^{\frac12} Q_\varepsilon^4\,\right)\\
&\quad - \big(\, w_1(\varepsilon M_\varepsilon, \varepsilon^{\frac12} Q_\varepsilon) Q_\varepsilon^4 + w_2(\varepsilon M_\varepsilon, \varepsilon^{\frac12} Q_\varepsilon) M_\varepsilon^2 + w_3(\varepsilon M_\varepsilon, \varepsilon^{\frac12} Q_\varepsilon) M_\varepsilon Q_\varepsilon^2\big).
\end{aligned}
$$

Take

$$
\begin{aligned}
M_\varepsilon &= \min\left\{\varepsilon^{-\frac{1}{4}}, w_1(\varepsilon^{\frac{3}{4}},\varepsilon^{\frac{2}{5}})^{-\frac{1}{4}}, w_2(\varepsilon^{\frac{3}{4}},\varepsilon^{\frac{2}{5}})^{-\frac{1}{4}}, w_3(\varepsilon^{\frac{3}{4}},\varepsilon^{\frac{2}{5}})^{-\frac{1}{4}}, |Z_p^c|^{-\frac{1}{4}}, |Z_q^c|^{-\frac{1}{4}}\right\},\\
Q_\varepsilon &= \min\left\{\varepsilon^{-\frac{1}{10}}, w_1(\varepsilon^{\frac{3}{4}},\varepsilon^{\frac{2}{5}})^{-\frac{1}{8}}, w_3(\varepsilon^{\frac{3}{4}},\varepsilon^{\frac{2}{5}})^{-\frac{1}{4}}\right\},
\end{aligned}
$$

then

$$
M_\varepsilon \varepsilon^{\frac{1}{2}} Q_\varepsilon^2 + \varepsilon^2 Q_\varepsilon^4 + \varepsilon^{\frac{3}{2}} Q_\varepsilon^4 + \varepsilon Q_\varepsilon^4 + \varepsilon^{\frac{1}{2}} Q_\varepsilon^4 \to 0 \qquad \text{as } \varepsilon \to 0,
$$

and

$$
w_1(\varepsilon M_\varepsilon, \varepsilon^{\frac{1}{2}} Q_\varepsilon) Q_\varepsilon^4 + w_2(\varepsilon M_\varepsilon, \varepsilon^{\frac{1}{2}} Q_\varepsilon) M_\varepsilon^2 + w_3(\varepsilon M_\varepsilon, \varepsilon^{\frac{1}{2}} Q_\varepsilon) M_\varepsilon Q_\varepsilon^2 \to 0 \qquad \text{as } \varepsilon \to 0.
$$

Thus we get

$$
\begin{aligned}
\text{termA} \;&\geqslant\; \int_\Omega \left(\frac{1}{2} C\nabla v^\varepsilon \nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B p_0^4\right) dx\\
&\quad - \int_{Z_u^c \cap Z_p \cap Z_q} \left(\frac{1}{2} C\nabla v^\varepsilon \nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B p_0^4\right) dx\\
&\quad - \int_{Z_p^c \cup Z_q^c} \left(\frac{1}{2} C\nabla v^\varepsilon \nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B p_0^4\right) dx\\
&= \int_\Omega \left(\frac{1}{2} C\nabla v^\varepsilon \nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B p_0^4\right) dx\\
&\quad - \int_{Z_p^c \cup Z_q^c \cup (Z_u^c \cap Z_p \bigcap Z_q)} \left(\frac{1}{2} C\nabla v^\varepsilon \nabla v^\varepsilon + A\nabla v^\varepsilon p_0 p_0 + B p_0^4\right) dx.
\end{aligned}
$$

The second integral in the above formula goes to zero. Actually,

$$
\int_{Z_p^c \cup Z_q^c} \frac{1}{2} C \left|\nabla v^\varepsilon\right|^2 dx \leqslant c\left(|Z_p^c| + |Z_q^c|\right) M_\varepsilon^2 \leqslant c\left(|Z_p^c| + |Z_q^c|\right)^{\frac{1}{2}} \to 0,
$$

$$
\int_{Z_u^c} \frac{1}{2} C \left|\nabla v^\varepsilon\right|^2 dx \leqslant c\,\lambda_\varepsilon^2 \left|Z_u^c\right| \leqslant \frac{c}{\ln M_\varepsilon} \to 0,
$$

$$
\int_{Z_u^c \cap Z_p \cap Z_q} B\, p_0^4\, dx \to 0 \quad \text{and} \quad \int_{Z_p^c \cup Z_q^c} B\, p_0^4\, dx \to 0,
$$

because $|Z_p^c| + |Z_q^c| \to 0$ and $|Z_u^c \cap Z_p \cap Z_q| \to 0$. Further,

$$\begin{aligned}
&\int_{Z_p^c \cup Z_q^c \cup (Z_u^c \cap Z_p \cap Z_q)} A\nabla v^\varepsilon p_0 p_0 \, dx \\
\leqslant\ & c\left(\int_{Z_p^c \cup Z_q^c \cup (Z_u^c \cap Z_p \cap Z_q)} |p_0|^4 \, dx\right)^{\frac{1}{2}} \left(\int_{Z_p^c \cup Z_q^c \cup (Z_u^c \cap Z_p \cap Z_q)} \left|\nabla v^\varepsilon\right|^2 dx\right)^{\frac{1}{2}} \\
\to\ & 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{termA} &= \frac{1}{\varepsilon^2}\left[\int_\Omega W(x, I + \varepsilon\nabla u^\varepsilon, \varepsilon^{\frac{1}{2}} p_0 + \varepsilon q_0^\varepsilon) - \frac{\varepsilon H^{-1}}{2}\left(p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon\right) \cdot \left(p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon\right) dx\right] \\
&\geqslant \int_\Omega \frac{1}{2} C\nabla u \nabla u + A\nabla u p_0 p_0 + B p_0^4 \, dx
\end{aligned}$$

because $\nabla v_\varepsilon \rightharpoonup \nabla u$.

Putting all the terms together and using the Euler-Lagrange equation (3.16), we get for any $\nabla u^\varepsilon \rightharpoonup \nabla u$ and $q_0^\varepsilon \rightharpoonup q_0$,

$$\begin{aligned}
\mathcal{F}_r^\varepsilon \geqslant \mathcal{F}_r^0 &= \int_\Omega \frac{1}{2} C\nabla u \nabla u + A\nabla u p_0 p_0 + B p_0^4 \, dx + \int_\Omega \frac{H^{-1}}{2} q_0 \cdot q_0 \, dx \\
&+ \frac{\varepsilon_0}{2} \int_{R^N} \left|\nabla_x \phi^0\right|^2 dx + \varepsilon_0 \int_{R^N} \big(\nabla u + \nabla u^T - \operatorname{tr}(\nabla u) I\big) \nabla g \cdot \nabla_x \varphi^0 \, dx \\
&+ \frac{\varepsilon_0}{2} \int_{R^N} \big(\nabla u + \nabla u^T - \operatorname{tr}(\nabla u) I\big) \nabla_x \varphi^0 \cdot \nabla_x \varphi^0 \, dx \\
&- \int_\Omega \nabla u \, p_0 \cdot \nabla_x \varphi^0 dx - \int_\Omega \nabla u^T \nabla_x g \cdot p_0 \, dx \\
&+ \frac{\varepsilon_0}{2} \int_{R^N} \big(\nabla u + \nabla u^T - \operatorname{tr}(\nabla u) I\big) \nabla_x g \cdot \nabla_x g \, dx.
\end{aligned}$$

This completes the proof of Proposition 3.14 □

Proposition 3.16 (Recovery sequences for the second order energy) *Under the same assumption in Proposition 3.14, we can find a sequence y^ε satisfying $y^\varepsilon \to x$ in $\mathcal{W}^{1,t}$, $\nabla u^\varepsilon = \frac{1}{\varepsilon}\nabla(y^\varepsilon - x) \rightharpoonup \nabla u$ in $\mathcal{L}^2(\Omega)$ and a sequence p_0^ε satisfying $q_0^\varepsilon =$*

$\varepsilon^{-\frac{1}{2}}(p_0^\varepsilon - p_0) \rightharpoonup q_0$ in $\mathcal{L}^2(\Omega)$, such that

$$\lim_{\varepsilon\to 0} \mathcal{F}_r^\varepsilon(y^\varepsilon, p_0^\varepsilon) = \mathcal{F}_r^0(u, q_0).$$

Proof If $\nabla u^\varepsilon \rightharpoonup \nabla u$ and $q_0^\varepsilon \rightharpoonup q_0$, term1 and term2 converge from the proof of Proposition 3.14. We now prove that if $q_0^\varepsilon \to q_0$, term3 also converges. From (3.42) and (3.43), to do that, we only need to prove

$$\frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N}\left|\nabla_y\varphi^\varepsilon - \nabla_y\varphi^0_{y^\varepsilon} - \varepsilon^{\frac{1}{2}}\nabla\phi_j^\varepsilon\right|^2 dy\right] \to 0 \qquad \text{as } \varepsilon\to 0,\ j\to+\infty. \tag{3.45}$$

First, let us find out the equation for $\nabla_y\varphi^0_{y^\varepsilon}$ in the y-space for each fixed ε. $\forall\psi(y)\in\mathfrak{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, we define $\psi\big(y(x)\big) = \psi_x^\varepsilon(x)$, then $\psi_x^\varepsilon(x)\in\mathcal{D}(\mathbb{R}^N\backslash\Gamma)$, since $\psi_x^\varepsilon(x)\in\mathcal{L}^2(\mathbb{R}^N)$, $\nabla_x\psi_x^\varepsilon(x) = F_\varepsilon\nabla_y\psi(y)\in\mathcal{L}^2(\mathbb{R}^N)$ and $\psi_x^\varepsilon(\Gamma) = \psi\big(y^\varepsilon(\Gamma)\big) = 0$.
Now we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{\varepsilon_0}{J_\varepsilon}F_\varepsilon F_\varepsilon^T\nabla_y\varphi^0_{y^\varepsilon}\cdot\nabla_y\psi(y)\,dy\\
=\;&\int_{\mathbb{R}^N}\varepsilon_0\nabla_x\varphi^0\cdot\nabla_x\psi_x^\varepsilon\,dx\\
=\;&\int_{\mathbb{R}^N}-\varepsilon_0\nabla_x g\cdot\nabla_x\psi_x^\varepsilon + p_0\cdot\nabla_x\psi_x^\varepsilon\,dx\\
=\;&\int_{\mathbb{R}^N}\left(-\frac{\varepsilon_0}{J_\varepsilon}F_\varepsilon F_\varepsilon^T\nabla_y g^\varepsilon\cdot\nabla_y\psi + F_\varepsilon\,p_0^{y^\varepsilon}\chi\big(y^\varepsilon(\Omega)\big)\cdot\nabla_y\psi\right)dy,
\end{aligned}$$

where $p_0^{y^\varepsilon}\big(y(x)\big) = \frac{p_0(x)}{J(x)}$.

Because $\mathfrak{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$ is dense in $\mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, $\frac{\varepsilon_0}{J_\varepsilon}F_\varepsilon F_\varepsilon^T\nabla_y\varphi^0_{y^\varepsilon}\in\mathcal{D}^{-1}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$ and $F_\varepsilon\,p_0^{y^\varepsilon}\in\mathcal{D}^{-1}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, the above equality holds for any $\psi(y)\in\mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$. Thus the equation for $\nabla_y\varphi^0_{y^\varepsilon}$ is

$$\nabla_y\cdot\left[\frac{\varepsilon_0}{J_\varepsilon}F_\varepsilon F_\varepsilon^T\nabla_y\varphi^0_{y^\varepsilon} + \frac{\varepsilon_0}{J_\varepsilon}F_\varepsilon F_\varepsilon^T\nabla_y g^\varepsilon - F_\varepsilon p_0^{y^\varepsilon}\chi\big(y^\varepsilon(\Omega)\big)\right] = 0.$$

Similarly, ϕ_j^ε satisfies

$$\nabla_y \cdot \left[\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y \phi_j^\varepsilon - F_\varepsilon q_j^\varepsilon \chi\big(y^\varepsilon(\Omega)\big) \right] = 0,$$

where $q_j^\varepsilon = \frac{q_0^j(x)}{J_\varepsilon(x)}$.

Since the equation for $\nabla_y \varphi^\varepsilon$ is

$$\nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi^\varepsilon - \varepsilon_0 \nabla_y g^\varepsilon + p_0^{y^\varepsilon} \chi\big(y^\varepsilon(\Omega)\big) + \varepsilon^{\frac{1}{2}} q^\varepsilon \chi\big(y^\varepsilon(\Omega)\big) \right] = 0,$$

we get

$$\begin{aligned}
& \nabla_y \cdot \left[\varepsilon_0 \nabla_y \varphi^\varepsilon - \varepsilon_0 \nabla_y \varphi_{y^\varepsilon}^0 - \varepsilon_0 \varepsilon^{\frac{1}{2}} \nabla_y \phi_j^\varepsilon \right] \\
= \; & \nabla_y \cdot \left[-\varepsilon_0 \nabla_y \varphi_{y^\varepsilon}^0 - \varepsilon_0 \varepsilon^{\frac{1}{2}} \nabla_y \phi_j^\varepsilon - \varepsilon_0 \nabla_y g^\varepsilon + p_0^{y^\varepsilon} \chi\big(y^\varepsilon(\Omega)\big) + \varepsilon^{\frac{1}{2}} q^\varepsilon \chi\big(y^\varepsilon(\Omega)\big) \right] \\
= \; & \nabla_y \cdot \left[\left(\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 \right) \nabla_y \varphi_{y^\varepsilon}^0 + \left(\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 \right) \varepsilon^{\frac{1}{2}} \nabla_y \phi_j^\varepsilon \right] \\
& + \nabla_y \cdot \left[\left(\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 \right) \nabla_y g^\varepsilon \right] \\
& + \nabla_y \cdot \left[-F_\varepsilon p_0^{y^\varepsilon} \chi\big(y^\varepsilon(\Omega)\big) + p_0^{y^\varepsilon} \chi\big(y^\varepsilon(\Omega)\big) + \varepsilon^{\frac{1}{2}} \left(q^\varepsilon \chi\big(y^\varepsilon(\Omega)\big) - F_\varepsilon q_j^\varepsilon \chi\big(y^\varepsilon(\Omega)\big) \right) \right].
\end{aligned}$$

On the right hand side, $\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 I$ is of order ε, so is $I - F_\varepsilon$. Therefore,

$$\begin{aligned}
& \frac{1}{\varepsilon} \left\| \varepsilon_0 \nabla_y \varphi^\varepsilon - \varepsilon_0 \nabla_y \varphi_{y^\varepsilon}^0 - \varepsilon_0 \varepsilon^{\frac{1}{2}} \nabla_y \phi_{y^\varepsilon}^0 \right\|_{\mathcal{L}^2(\mathbb{R}^N)}^2 \\
\leqslant \; & c \left\| q^\varepsilon - q_j^\varepsilon \right\|_{\mathcal{L}^2(y^\varepsilon(\Omega))}^2 \to 0 \qquad \text{as } \varepsilon \to 0, \; j \to +\infty,
\end{aligned}$$

because

$$\int_{y(\Omega)} \left| q^\varepsilon(y) - q_j^\varepsilon(y) \right|^2 dy = \int_\Omega \frac{1}{J_\varepsilon} \left| q_0^\varepsilon(x) - q_0^j(x) \right|^2 dx \to 0 \quad \text{as } \varepsilon \to 0, \; j \to +\infty.$$

Thus we get (3.45) and the the convergence of term 3.

Now, the only thing left is term4. term4 = termA + termB + termC.

$$\text{termB} = \int_\Omega \frac{H^{-1}}{2} q_0^\varepsilon \cdot q_0^\varepsilon dx \to \int_\Omega \frac{H^{-1}}{2} q_0 \cdot q_0 dx$$

if $\left\| q_0^\varepsilon - q_0 \right\|_{\mathcal{L}^2(\Omega)} \to 0$.

Next, we need to construct $\nabla u^\varepsilon \rightharpoonup \nabla u$ and $q_0^\varepsilon \to q_0$ such that termA converges.

Define $Z_p = \left\{ x \in \Omega : \varepsilon^{\frac{1}{2}} |p_0(x)| \leqslant \varepsilon^s \right\}$ for some s to be decided.

Define $Z_q = \left\{ x \in \Omega : \left| q_0(x) \right| \leqslant \frac{1}{2} Q_\varepsilon \right\}$ for some Q_ε to be decided.

We construct q_0^ε as follows

$$q_0^\varepsilon = \begin{cases} q_0 & \text{if} \quad x \in Z_p \cap Z_q, \\ -\varepsilon^{-\frac{1}{2}} p_0 & \text{if} \quad x \in Z_p^c, \\ 0 & \text{if} \quad x \in Z_q^c \cap Z_p. \end{cases} \tag{3.46}$$

Then $q_0^\varepsilon \to q_0$. Actually,

$$\begin{aligned} \int_\Omega \left| q_0^\varepsilon - q_0 \right|^2 dx &\leqslant \int_{Z_q^c} \left| q_0 \right|^2 dx + \int_{Z_p^c} \left| -\varepsilon^{-\frac{1}{2}} p_0 - q_0 \right|^2 dx \\ &\leqslant 2 \int_{Z_p^c} \frac{1}{\varepsilon} \left| p_0 \right|^2 dx + 2 \int_{Z_q^c \cup Z_p^c} \left| q_0 \right|^2 dx \\ &\leqslant 2 \left[\int_{Z_p^c} \left| p_0 \right|^4 dx \right]^{\frac{1}{2}} \frac{1}{\varepsilon} \left| Z_p^c \right|^{\frac{1}{2}} + 2 \int_{Z_q^c \cup Z_p^c} \left| q_0 \right|^2 dx. \end{aligned} \tag{3.47}$$

If $p_0 \in L^{4+w}$,

$$\left| Z_p^c \right| (\varepsilon^{s-\frac{1}{2}})^{(4+w)} \leqslant \int_{Z_p^c} \left| p_0 \right|^{4+w} dx \to 0.$$

By choosing s small enough such that $(\frac{1}{2} - s)(4+w) > 2$,

$$\left| Z_p^c \right|^{\frac{1}{2}} \varepsilon^{-1} \to 0.$$

Thus the right hand side of (3.47) goes to zero and $q_0^\varepsilon \to q_0$ follows.

We construct the recovery sequence u^ε as the truncation of u for $m = 1$, $M = M_\varepsilon$ to be decided. Assume $M_\varepsilon < \rho_2 \varepsilon^{-1}$, then $\varepsilon \nabla u^\varepsilon < \rho_2$ on the whole domain Ω. If we take $Q_\varepsilon < \varepsilon^{s-\frac{1}{2}}$, then $p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon < \varepsilon^s + \varepsilon^{\frac{1}{2}} Q_\varepsilon$. Now we can use Taylor expansion on termA.

$$
\begin{aligned}
& \frac{1}{\varepsilon^2} \int_\Omega \left[W\big(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon^{\frac{1}{2}} p_0 + \varepsilon q_0^\varepsilon\big) - \frac{\varepsilon H^{-1}}{2} \left(p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon\right) \cdot \left(p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon\right) \right] dx \\
\leqslant \;& \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon + A \nabla u^\varepsilon (p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon)(p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon) + B (p_0 + \varepsilon^{\frac{1}{2}} q_0^\varepsilon)^4 \, dx \\
& + \int_\Omega w_1\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |p_0^\varepsilon|^4 + w_2\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |\nabla u^\varepsilon|^2 \, dx \\
& + \int_\Omega w_3\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |\nabla u^\varepsilon| |p_0^\varepsilon|^2 dx \\
\leqslant \;& \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon \, dx + \int_{Z_q^c \cap Z_p} A \nabla u^\varepsilon p_0 p_0 + B (p_0)^4 \, dx \\
& + \int_{Z_p \cap Z_q} A \nabla u^\varepsilon (p_0 + \varepsilon^{\frac{1}{2}} q_0)(p_0 + \varepsilon^{\frac{1}{2}} q_0) + B (p_0 + \varepsilon^{\frac{1}{2}} q_0)^4 \, dx \\
& + \int_{Z_q^c \cap Z_p} w_1\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0|\big) \, |p_0|^4 + w_2\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0|\big) \, |\nabla u^\varepsilon|^2 \, dx \\
& + \int_{Z_q^c \cap Z_p} w_3\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0|\big) \, |\nabla u^\varepsilon| |p_0|^2 \, dx + \int_{Z_q \cap Z_p} w_3\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |\nabla u^\varepsilon| |p_0^\varepsilon|^2 \, dx \\
& + \int_{Z_q \cap Z_p} w_1\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |p_0^\varepsilon|^4 + w_2\big(\varepsilon |\nabla u^\varepsilon|, \varepsilon^{\frac{1}{2}} |p_0^\varepsilon|\big) \, |\nabla u^\varepsilon|^2 \, dx \\
\leqslant \;& \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon dx + \int_{Z_p} A \nabla u^\varepsilon p_0 p_0 + B (p_0)^4 \, dx \\
& + c \left(M_\varepsilon \varepsilon^s Q_\varepsilon + \varepsilon^{2s} M_\varepsilon \right) + \; c \, \varepsilon^s \|p_0\|_{\mathcal{L}^4(\Omega)}^4 \\
& + c \left(w_1(\varepsilon M_\varepsilon, \varepsilon^s) \|p_0\|_{\mathcal{L}^4(\Omega)}^4 + w_2(\varepsilon M_\varepsilon, \varepsilon^s) M_\varepsilon^2 + w_3(\varepsilon M_\varepsilon, \varepsilon^s) M_\varepsilon \|p_0\|_{\mathcal{L}^2(\Omega)}^2 \right).
\end{aligned}
$$

In the right hand side,

$$
\begin{aligned}
& \int_\Omega \frac{1}{2} C\nabla u^\varepsilon \nabla u^\varepsilon \, dx = \int_{Z_u^c} \frac{1}{2} C\nabla u^\varepsilon \nabla u^\varepsilon \, dx + \int_{Z_u} \frac{1}{2} C\nabla u \nabla u \, dx \\
= & \int_\Omega \frac{1}{2} C\nabla u \nabla u \, dx - \int_{Z_u^c} \frac{1}{2} C\nabla u \nabla u \, dx + \int_{Z_u^c} \frac{1}{2} C\nabla u^\varepsilon \nabla u^\varepsilon \, dx \\
\leqslant & \int_\Omega \frac{1}{2} C\nabla u \nabla u \, dx + c \int_{Z_u^c} \left|\nabla u\right|^2 dx + c \left|Z_u^c\right| \lambda_\varepsilon^2 \\
\to & \int_\Omega \frac{1}{2} C\nabla u \nabla u \, dx,
\end{aligned}
$$

$$
\begin{aligned}
& \int_{Z_p} A\nabla u^\varepsilon p_0 p_0 \, dx \leqslant \int_{Z_p \cap Z_u} A\nabla u p_0 p_0 \, dx + \int_{Z_u^c} \left|A\nabla u^\varepsilon p_0 p_0\right| dx \\
\leqslant & \int_\Omega A\nabla u p_0 p_0 \, dx + c \left(\int_{Z_u^c} \left|\nabla u^\varepsilon\right|^2 dx \right)^{\frac{1}{2}} \left(\int_{Z_u^c} |p_0|^4 \, dx \right)^{\frac{1}{2}} + \int_{Z_u^c \cup Z_p^c} \left|A\nabla u p_0 p_0\right| dx \\
\to & \int_\Omega A\nabla u p_0 p_0 \, dx,
\end{aligned}
$$

and

$$
\int_{Z_p} B(p_0)^4 \, dx \to \int_\Omega B(p_0)^4 \, dx.
$$

Choose

$$
\begin{aligned}
M_\varepsilon &= \min\left\{ \varepsilon^{-\frac{s}{4}}, \varepsilon^{-\frac{1}{4}}, w_2(\varepsilon^{\frac{3}{4}}, \varepsilon^s)^{-\frac{1}{4}}, w_3(\varepsilon^{\frac{3}{4}}, \varepsilon^s)^{-\frac{1}{4}} \right\}, \\
Q_\varepsilon &= \min\left\{ \varepsilon^{-\frac{s}{4}}, w_3(\varepsilon^{\frac{3}{4}}, \varepsilon^s)^{-\frac{1}{4}}, \varepsilon^{s-\frac{1}{2}} \right\},
\end{aligned}
$$

then

$$
\begin{aligned}
& c\left(M_\varepsilon \varepsilon^s Q_\varepsilon + \varepsilon^{2s} M_\varepsilon\right) + \; c\,\varepsilon^s \|p_0\|_{\mathcal{L}^4(\Omega)}^4 \\
& + c\left(w_1(\varepsilon M_\varepsilon, \varepsilon^s) \|p_0\|_{\mathcal{L}^4(\Omega)}^4 + w_2(\varepsilon M_\varepsilon, \varepsilon^s) M_\varepsilon^2 + w_3(\varepsilon M_\varepsilon, \varepsilon^s) M_\varepsilon \|p_0\|_{\mathcal{L}^2(\Omega)}^2 \right) \\
& \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Therefore, we proved that

$$\begin{aligned} \text{termA} &= \frac{1}{\varepsilon^2}\int_\Omega \left[W\big(x, I+\varepsilon\nabla u^\varepsilon, \varepsilon^{\frac{1}{2}}p_0+\varepsilon q_0^\varepsilon\big) - \frac{\varepsilon H^{-1}}{2}\Big(p_0+\varepsilon^{\frac{1}{2}}q_0^\varepsilon\Big)\cdot\Big(p_0+\varepsilon^{\frac{1}{2}}q_0^\varepsilon\Big)\right]dx \\ &\to \int_\Omega \frac{1}{2}C\nabla u\nabla u + A\nabla u p_0 p_0 + B(p_0)^4\,dx. \end{aligned}$$

Putting all the things together and remembering the Euler-Lagrange equation (3.16), we thus complete the construction of the recovery sequences. □

Proof of Theorem 3.4

Again, this is the direct conclusion of Proposition 3.14 and Proposition 3.16 by the definition of Γ-convergence.

□

Now, let us prove the other compactness result.

Proof of Proposition 3.6

Still analyze the energy $\mathcal{F}_r^\varepsilon$ term by term.

$$\begin{aligned} \text{term}\,4 &= \frac{1}{\varepsilon^2}\left[\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}}p_0^\varepsilon) - \frac{\varepsilon H^{-1}}{2}p_0\cdot p_0\,dx\right] \\ &= \frac{1}{\varepsilon^2}\int_\Omega W(x, F_\varepsilon, \varepsilon^{\frac{1}{2}}p_0^\varepsilon) - \frac{\varepsilon H^{-1}}{2}p_0^\varepsilon\cdot p_0^\varepsilon\,dx + \frac{1}{\varepsilon}\int_\Omega \frac{H^{-1}}{2}\left(p_0^\varepsilon\cdot p_0^\varepsilon - p_0\cdot p_0\right)dx \\ &\geqslant \frac{c}{\varepsilon^2}\int_\Omega \big|F_\varepsilon - \mathrm{SO}(N)\big|^2\,dx + \frac{1}{\varepsilon}\int_\Omega \frac{H^{-1}}{2}(p_0^\varepsilon - p_0)\cdot(p_0^\varepsilon - p_0) \\ &\quad + \frac{1}{\varepsilon}\int_\Omega H^{-1}(p_0^\varepsilon - p_0)\cdot p_0\,dx \\ &\geqslant c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|^2_{\mathcal{L}^2(\Omega)} + c\left\|\frac{p_0^\varepsilon - p_0}{\varepsilon^{\frac{1}{2}}}\right\|^2_{\mathcal{L}^2(\Omega)} + \frac{1}{\varepsilon}\int_\Omega H^{-1}p_0\cdot(p_0^\varepsilon - p_0)\,dx. \end{aligned}$$

Above, the first inequality comes from (3.7) in assumption $\mathcal{A}_6$. The second inequality comes from the rigidity theorem and the boundary condition.

$$
\begin{aligned}
\text{term}\,1 &= \frac{1}{\varepsilon}\left[\int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon\, dy - \int_\Omega \nabla_x g \cdot p_0\, dx\right] \\
&= \frac{1}{\varepsilon}\left[\int_\Omega F_\varepsilon^{-T} \nabla_x g \cdot p_0^\varepsilon\, dx - \int_\Omega \nabla_x g \cdot p_0\, dx\right] \\
&= \frac{1}{\varepsilon}\int_\Omega (F_\varepsilon^{-T} - I)\nabla_x g \cdot p_0^\varepsilon\, dx + \frac{1}{\varepsilon}\int_\Omega \nabla_x g \cdot (p_0^\varepsilon - p_0)\, dx \\
&\geqslant -c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} + \frac{1}{\varepsilon}\int_\Omega \nabla_x g \cdot (p_0^\varepsilon - p_0)\, dx.
\end{aligned}
$$

Since $\nabla_x g \in \mathcal{L}^\infty(\mathbb{R}^N)$ and

$$
\int_{\mathbb{R}^N} \left|\frac{I - J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T}}{\varepsilon}\right| dx \geqslant -c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)},
$$

$$
\begin{aligned}
\text{term}\,2 &= \frac{1}{\varepsilon}\left[\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 dx - \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y g^\varepsilon\right|^2 dy\right] \\
&= \frac{\varepsilon_0}{2}\,\frac{1}{\varepsilon}\left[\int_{\mathbb{R}^N} \left|\nabla_x g\right|^2 - J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T} \nabla_x g \cdot \nabla_x g\, dx\right] \\
&= \frac{\varepsilon_0}{2}\,\frac{1}{\varepsilon}\left[\int_{\mathbb{R}^N} (I - J_\varepsilon F_\varepsilon^{-1} F_\varepsilon^{-T})\nabla_x g \cdot \nabla_x g\, dx\right] \\
&\geqslant -c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)}.
\end{aligned}
$$

Finally, recalling $q_0^\varepsilon = \varepsilon^{-\frac{1}{2}}(p_0^\varepsilon - p_0)$ in (3.44), we have

$$
\text{term}\,3 \geqslant -c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} + \frac{1}{\varepsilon}\int_\Omega (p_0^\varepsilon - p_0) \cdot \nabla_x \varphi^0\, dx.
$$

Putting all the inequalities together,

$$\begin{aligned} c &\geqslant \mathcal{F}_r^\varepsilon = \mathrm{term}\,1 + \mathrm{term}\,2 + \mathrm{term}\,3 + \mathrm{term}\,4 \\ &\geqslant c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|^2_{\mathcal{L}^2(\Omega)} + c\left\|\frac{p_0^\varepsilon - p_0}{\varepsilon^{\frac{1}{2}}}\right\|^2_{\mathcal{L}^2(\Omega)} - c\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} \\ &\quad + \frac{1}{\varepsilon}\left[\int_\Omega H^{-1}p_0 \cdot (p_0^\varepsilon - p_0) + \nabla_x \varphi^0 \cdot (p_0^\varepsilon - p_0) + \nabla g \cdot (p_0^\varepsilon - p_0)\, dx\right]. \end{aligned}$$

From Euler-Lagrange equation (3.16), the last term is zero. Hence there exist a constant $\tilde{c}$ such that

$$\left\|\frac{F_\varepsilon - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} \leqslant \tilde{c} \quad \text{and} \quad \left\|\frac{p_0^\varepsilon - p_0}{\varepsilon^{\frac{1}{2}}}\right\|_{\mathcal{L}^2(\Omega)} \leqslant \tilde{c}.$$

□

Chapter 4

Homogenization of the Small-strain Dielectric Model

4.1 Introduction of the Main Results

In Chapter 3, we derived a small-strain model for deformable dielectric elastomers. The electric field $\varphi_e(x)$ satisfies

$$\begin{cases} \nabla_x \cdot \Big[\big(\varepsilon_0 + H\chi(\Omega)\big)\nabla_x \varphi_e\Big] = 0 \quad \text{in } \mathbb{R}^N \backslash \Gamma, \\ \varphi_e = g_0 \qquad \text{on } \Gamma_0, \\ \varphi_e = g_1 \qquad \text{on } \Gamma_1, \\ \varphi_e \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^N), \quad \nabla\varphi_e \in \mathcal{L}^2(\mathbb{R}^N). \end{cases} \tag{4.1}$$

The strain field satisfies

$$\begin{cases} -\nabla \cdot \big(C\nabla u + \tilde{A}\nabla\varphi_e\nabla\varphi_e\big) = 0 & \text{in} \quad \Omega, \\ \big(C\nabla u + \tilde{A}\nabla\varphi_e\nabla\varphi_e\big) \cdot n = f & \text{on} \quad \Gamma_2, \\ u = 0 & \text{on} \quad \Gamma_3. \end{cases} \tag{4.2}$$

For simplicity, from now on, we will just drop the subscription e, denote by φ our electric field with inhomogeneous boundary condition. And we will also drop the tilde on tensor A. Now suppose we have a composite with periodic microstructure. Assume

$H^\varepsilon(x) = H\left(\frac{x}{\varepsilon}\right)$, $C^\varepsilon(x) = C\left(\frac{x}{\varepsilon}\right)$ and $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$. Denote by φ^ε the solution of

$$\begin{cases} \nabla_x \cdot \left[\big(\varepsilon_0 + H^\varepsilon \chi(\Omega)\big)\nabla_x \varphi^\varepsilon\right] = 0 \quad \text{in } \mathbb{R}^N \backslash \Gamma, \\ \varphi^\varepsilon = g_0 \qquad \text{on } \Gamma_0, \\ \varphi^\varepsilon = g_1 \qquad \text{on } \Gamma_1, \\ \varphi^\varepsilon \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^N), \quad \nabla\varphi^\varepsilon \in \mathcal{L}^2(\mathbb{R}^N). \end{cases} \tag{4.3}$$

Denote by $u^\varepsilon(x)$ the solution of

$$\begin{cases} -\nabla \cdot \big(C^\varepsilon \nabla u^\varepsilon + A^\varepsilon \nabla\varphi^\varepsilon \nabla\varphi^\varepsilon\big) = 0 & \text{in} \quad \Omega, \\ \big(C^\varepsilon \nabla u^\varepsilon + A^\varepsilon \nabla\varphi^\varepsilon \nabla\varphi^\varepsilon\big) \cdot n = f & \text{on} \quad \Gamma_2, \\ u^\varepsilon = 0 & \text{on} \quad \Gamma_3. \end{cases} \tag{4.4}$$

The effective property of periodic dielectric composites is studied in this chapter. We will prove first the following result,

Theorem 4.1 *When $\varepsilon \to 0$, φ^ε, the solution of (4.3), two-scale converges to $\varphi^0(x)$, the solution of the following equation*

$$\begin{cases} \nabla_x \cdot \left[\big(\varepsilon_0 + H^0 \chi(\Omega)\big)\nabla_x \varphi^0\right] = 0 \quad \text{in } \mathbb{R}^N \backslash \Gamma, \\ \varphi^0 = g_0 \qquad \text{on } \Gamma_0, \\ \varphi^0 = g_1 \qquad \text{on } \Gamma_1, \\ \varphi^0 \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^N), \quad \nabla\varphi^0 \in \mathcal{L}^2(\mathbb{R}^N), \end{cases} \tag{4.5}$$

in which H^0 is defined as

$$H^0_{ij} = \frac{1}{|Y|}\int_Y H_{ij}(y) - H_{ik}(y)\frac{\partial \hat{\chi}_j(y)}{\partial y_k}\, dy, \tag{4.6}$$

where $\hat{\chi}_j$ is the unit cell solution satisfying

$$\begin{cases} -\nabla_y \cdot \left[\big(\varepsilon_0 I + H(y)\big)\nabla_y \hat{\chi}_j\right] = -\displaystyle\sum_{i=1}^{N} \frac{\partial\big(\varepsilon_0 \delta_{ij} + H_{ij}(y)\big)}{\partial y_i} \quad \text{in} \quad Y, \\ M_Y(\hat{\chi}_j) = 0, \\ \hat{\chi}_j \ \text{ periodic in } \ y. \end{cases} \tag{4.7}$$

Assume further $\nabla\varphi^\varepsilon$ is uniformly bounded in $\mathcal{L}^4(\Omega)$, then we will derive the following homogenized equation for the deformation field.

Theorem 4.2 *When $\varepsilon \to 0$, u^ε, the solution of (4.4), two-scale converges to $u^0(x)$, the solution of the following equation*

$$\begin{cases} -\nabla \cdot \left(C^H \nabla u^0 + A^H \nabla\varphi^0 \nabla\varphi^0\right) = 0 & \text{in} \quad \Omega, \\ \left(C^H \nabla u^0 + A^H \nabla\varphi^0 \nabla\varphi^0\right) \cdot n = f & \text{on} \quad \Gamma_2, \\ u^0 = 0 & \text{on} \quad \Gamma_3. \end{cases} \tag{4.8}$$

In (4.8), C^H is defined as

$$C^H_{ijkh} = \frac{1}{|Y|} \int_Y C_{ijkh}(y) + C_{ijlm}(y) \frac{\partial \chi_l^{kh}}{\partial y_m}\, dy, \tag{4.9}$$

where χ^{kh} is the solution of the unit cell problem satisfying

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijlm} \dfrac{\partial \chi_l^{kh}}{\partial y_m}\right) = \dfrac{\partial C_{ijkh}}{\partial y_j} \quad \text{in} \quad \Omega, \\ \chi^{kh} \qquad Y - \text{periodic}, \\ M_Y(\chi^{kh}) = 0. \end{cases} \tag{4.10}$$

A^H is defined as

$$A^H_{ijkh} = \frac{1}{|Y|} \int_Y B_{ijkh}(y) + C_{ijlm}(y) \frac{\partial \tilde{\chi}_l^{kh}}{\partial y_m}\, dy, \tag{4.11}$$

in which

$$B_{ijkh} = A_{ijkh} - A_{ijlh}\frac{\partial \hat{\chi}_k}{\partial y_l} - A_{ijkl}\frac{\partial \hat{\chi}_h}{\partial y_l} + A_{ij\alpha\beta}\frac{\partial \hat{\chi}_k}{\partial y_\alpha}\frac{\partial \hat{\chi}_h}{\partial y_\beta}, \tag{4.12}$$

$\tilde{\chi}^{kh} = (\tilde{\chi}_1^{kh}, \dots, \tilde{\chi}_N^{kh})$ *is the solution of the equation*

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijlm}\dfrac{\partial \tilde{\chi}_l^{kh}}{\partial y_m}\right) = \dfrac{\partial B_{ijkh}}{\partial y_j} \quad \text{in} \quad \Omega, \\ \tilde{\chi}^{kh} \qquad Y - \text{periodic}, \\ M_Y(\tilde{\chi}^{kh}) = 0. \end{cases} \tag{4.13}$$

We will use the tool of two-scale convergence. In the following, Section 4.2 is a brief introduction to two-scale convergence. For details, we refer to [3] and [13]. Section 4.3 is the proof of Theorem 4.1. Since the domain for the Maxwell equation (4.3) is unbounded and the matrix $\varepsilon I + H(\frac{x}{\varepsilon})\chi(\Omega)$ in (4.3) is not exactly periodic, we will write down the complete proof, although the method we use here is standard. Section 4.5 is the proof of Theorem 4.2, in which we used the local strong convergence of the electric field. We derived that property in Section 4.4.

4.2 Introduction to Two-scale Convergence

Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ and $Y = [\,0,1\,]^N$ be the closed unit cube. We denote by $\mathcal{C}_\sharp^\infty(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ that are periodic in Y. Then $\mathcal{L}_\sharp^2(Y)$ and $\mathcal{H}_\sharp^1(Y)$ are the completions of $\mathcal{C}_\sharp^\infty(Y)$ with respect to the $\mathcal{L}^2(Y)$ and the $\mathcal{H}^1(Y)$ norms respectively.

Definition 4.1 *A sequence of function v^ε in $\mathcal{L}^2(\mathcal{O})$ is said to two-scale converge to a limit $v^0(x,y)$ belonging to $\mathcal{L}^2(\mathcal{O}\times Y)$ if , for any function $\psi(x,y) \in \mathfrak{D}\big(\mathcal{O}; \mathcal{C}_\sharp^\infty(Y)\big)$, we have*

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}} v^\varepsilon(x)\,\psi\left(x, \frac{x}{\varepsilon}\right)\,dx = \int_{\mathcal{O}}\int_Y v^0(x,y)\,\psi(x,y)\,dydx.$$

Allaire has shown [3] the test functions in the definition above can be enlarged to an "admissible" test function set $\mathcal{A}_{\text{ad}}$. A function $\psi(x,y) \in \mathcal{L}^1(\mathcal{O}\times Y)$, periodic in

y, is an admissible test function if $\psi(x,y)$ is measurable and

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}}\left|\psi\left(x,\frac{x}{\varepsilon}\right)\right|\,dx=\int_{\mathcal{O}}\int_{Y}\left|\psi(x,y)\right|dydx.$$

Allaire has proved [3] that if $\psi(x,y)\in\mathcal{L}^1\big(\mathcal{O};\mathcal{C}_\sharp(Y)\big)$ or $\mathcal{L}^1_\sharp\big(Y;\mathcal{C}(\mathcal{O})\big)$, then $\psi(x,y)\in\mathcal{A}_{\mathrm{ad}}$. If $\psi(x,y)=\psi_1(x)\psi_2(y)$, in which $\psi_1(x)\in\mathcal{L}^s(\mathcal{O})$, $\psi_2(y)\in\mathcal{L}^r_\sharp(Y)$, r and s satisfying $\frac{1}{r}+\frac{1}{s}=\frac{1}{2}$, then we also have $\psi(x,y)\in\mathcal{A}_{\mathrm{ad}}$.

For a two-scale convergence sequence v^ε, we have the following property

Proposition 4.3 *Let v^ε be a sequence of functions in $\mathcal{L}^2(\mathcal{O})$, which two-scale converges to a limit $v^0(x,y)\in\mathcal{L}^2(\mathcal{O}\times Y)$. Then v^ε converges also to $v(x)=\int_Y v^0(x,y)\,dy$ in $\mathcal{L}^2(\mathcal{O})$ weakly. Furthermore, we have*

$$\lim_{\varepsilon\to 0}\|v^\varepsilon\|_{\mathcal{L}^2(\mathcal{O})}\geq\|v^0\|_{\mathcal{L}^2(\mathcal{O}\times Y)}\geq\|v\|_{\mathcal{L}^2(\mathcal{O})}.\tag{4.14}$$

One of the main result in the theory of two-scale convergence is the following proposition.

Proposition 4.4 ([13]) *If $v^\varepsilon(x)$ is a bounded sequence in $\mathcal{H}^1(\mathcal{O})$ that converges weakly to a limit $v(x)$ in $\mathcal{H}^1(\mathcal{O})$, then v^ε two-scale converges to $v(x)$, and there exists a function $v^1(x,y)\in\mathcal{L}^2\big(\mathcal{O};\mathcal{H}^1_\sharp(Y)/\mathbb{R}\big)$ such that , up to a subsequence, ∇v^ε two-scale converges to $\nabla_x v(x)+\nabla_y v^1(x,y)$.*

4.3 The Two-scale Convergence of the Electric Field Potential

In this section, we will prove Theorem 4.1.

Proof Using the same method to get (2.6), we have for each ε, the solution of (4.3) exists and satisfies $\left\|\nabla_x\varphi^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)}\leqslant c$, in which c depends only on $H(y)$, ε_0, the boundary condition, but not on ε. Now, for any integer n, φ^ε is uniformly bounded in $\mathcal{H}^1(B_n\backslash\Gamma)$. According to Proposition 4.4, there exists a function $\varphi^0_n\in\mathcal{H}^1(B_n\backslash\Gamma)$, such that $\varphi^\varepsilon\rightharpoonup\varphi^0_n$ in $\mathcal{H}^1(B_n\backslash\Gamma)$, and another function $\varphi^1_n(x,y)\in$

$\mathcal{L}^2\big(B_n\backslash\Gamma;\mathcal{H}^1_\sharp(Y)/\mathbb{R}\big)$ such that $\nabla\varphi^\varepsilon$ two-scale converges to $\nabla_x\varphi^0_n+\nabla_y\varphi^1_n$. Increase n, we get another φ^0_n and φ^1_n, but they are the same on their common region. In this way, we can find a subsequence φ^ε and two functions $\varphi^0(x)\in\mathcal{D}_1(\mathbb{R}^N\backslash\Gamma)$ and $\varphi^1(x,y)\in\mathcal{L}^2_{\text{loc}}\big(\mathbb{R}^N\backslash\Gamma;\mathcal{H}^1_\sharp(Y)/\mathbb{R}\big)$ such that $\nabla\varphi^\varepsilon\rightharpoonup\nabla\varphi^0$ in $\mathcal{L}^2(\mathbb{R}^N)$, and $\nabla\varphi^\varepsilon$ two-scale converges to $\nabla_x\varphi^0+\nabla_y\varphi^0$ for any test function $\psi(x,y)\in\mathcal{A}_{\text{ad}}$ that has compact support with respect to variable x. Actually, for any compact set $K\subset\mathbb{R}^N$,

$$\begin{array}{cccc}\varphi^\varepsilon & \to & \varphi^0 & \text{in } \mathcal{L}^2(K),\\ \nabla\varphi^\varepsilon & \rightharpoonup & \nabla\varphi^0 & \text{in } \mathcal{L}^2(K),\\ \nabla\varphi^\varepsilon & \text{two-scale} & \text{converges} & \text{to } \nabla_x\varphi^0+\nabla_y\varphi^1.\end{array}$$

From Proposition 4.3, for any compact set K,

$$\big\|\nabla_x\varphi^0+\nabla_y\varphi^1\big\|_{\mathcal{L}^2(K\times Y)}\leqslant\big\|\nabla\varphi^\varepsilon\big\|_{\mathcal{L}^2(K)}\leqslant\big\|\nabla\varphi^\varepsilon\big\|_{\mathcal{L}^2(\mathbb{R}^N)}.$$

Therefore, $\varphi^1\in\mathcal{L}^2\big(\mathbb{R}^N\backslash\Gamma;\mathcal{H}^1_\sharp(Y)/\mathbb{R}\big)$.

φ^ε is the solution of equation (4.3) if for any $\psi\in\mathcal{D}(\mathbb{R}^N\backslash\Gamma)$,

$$\int_{\mathbb{R}^N}\big(\varepsilon_0 I+H^\varepsilon\chi(\Omega)\big)\nabla_x\varphi^\varepsilon\cdot\nabla_x\psi\,dx=0. \tag{4.15}$$

Now for any $\phi^0\in\mathfrak{D}\big(\mathbb{R}^N\backslash\Gamma\big)$, $\phi^1(x,y)\in\mathfrak{D}\big(\mathbb{R}^N\backslash\Gamma;\mathcal{C}^\infty_\sharp(Y)\big)$, using $\phi^0(x)+\varepsilon\phi^1(x,\frac{x}{\varepsilon})$ as a test function in equation (4.15), we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^N}\big(\varepsilon_0 I+H^\varepsilon\chi(\Omega)\big)\nabla_x\varphi^\varepsilon\cdot\left(\nabla_x\phi^0(x)+\nabla_y\phi^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\ &= \int_{\mathbb{R}^N}\varepsilon_0\nabla_x\varphi^\varepsilon\cdot\left(\nabla_x\phi^0(x)+\nabla_y\phi^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\ &\quad+\int_\Omega H^\varepsilon(x)\nabla_x\varphi^\varepsilon\cdot\left(\nabla_x\phi^0(x)+\nabla_y\phi^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx.\end{aligned}$$

Since $\varepsilon\nabla_x\phi^1(x,\frac{x}{\varepsilon}) \to 0$ in $\mathcal{L}^2$ and $\nabla\varphi^\varepsilon$ two-scale converges to $\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \varepsilon_0\nabla_x\varphi^\varepsilon\cdot\left(\nabla_x\phi^0(x)+\nabla_y\phi^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi^1\left(x,\frac{x}{\varepsilon}\right)\right)\,dx\\
\to\quad &\frac{1}{|Y|}\int_{\mathbb{R}^N}\int_Y \varepsilon_0\big(\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)\big)\cdot\big(\nabla_x\phi^0(x)+\nabla_y\phi^1(x,y)\big)\,dydx.
\end{aligned}
$$

As to the other term, first $\nabla\varphi^\varepsilon$ two-scale converges to $\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)$ on the smaller region Ω. Second $H(y)\in\mathcal{L}^\infty_\sharp(Y)$, $\nabla_y\phi^1(x,y)\in\mathfrak{D}\big(\mathbb{R}^N\backslash\Gamma;\mathcal{C}^\infty_\sharp(Y)\big)$, so $H(y)\nabla_y\phi^1(x,y)$ and $H(y)\nabla_x\phi^0(x)$ are both in $\mathcal{L}^1_\sharp\big(Y;\mathcal{C}(\Omega)\big)$, and can be looked as the test functions for the two-scale convergence of $\nabla\varphi^\varepsilon$ on Ω, therefore,

$$
\begin{aligned}
&\int_\Omega H^\varepsilon(x)\nabla_x\varphi^\varepsilon\cdot\left(\nabla_x\phi^0(x)+\nabla_y\phi^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi^1\left(x,\frac{x}{\varepsilon}\right)\right)\,dx\\
\to\quad &\frac{1}{|Y|}\int_\Omega\int_Y H(y)\big(\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)\big)\cdot\big(\nabla_x\phi^0(x)+\nabla_y\phi^1(x,y)\big)\,dydx.
\end{aligned}
$$

Putting these together,

$$
\int_{\mathbb{R}^N}\int_Y\big(\varepsilon_0 I+H(y)\chi(\Omega)\big)\big(\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)\big)\cdot\big(\nabla_x\phi^0(x)+\nabla_y\phi^1(x,y)\big)\,dydx=0 \tag{4.16}
$$

Using exactly the same method in Chapter 9 of [13], it is easy to obtain the existence and the uniqueness of the solution for (4.16).

Choosing $\phi^0\equiv 0$ in (4.16),

$$
-\nabla_y\cdot\Big(\big(\varepsilon_0 I+H(y)\chi(\Omega)\big)\big(\nabla_x\varphi^0(x)+\nabla_y\varphi^1(x,y)\big)\Big)=0. \tag{4.17}
$$

If $x\notin\Omega$, (4.17) becomes

$$
-\nabla_y\cdot\nabla_y\varphi^1(x,y)=0
$$

with $\varphi^1(x,y)$ periodic in y and zero mean on Y. Thus if $x\notin\Omega$, $\varphi^1(x,y)\equiv 0$. However, if $x\in\Omega$, (4.17) is

$$
-\nabla_y\cdot\Big[\big(\varepsilon_0 I+H(y)\big)\nabla_y\varphi^1(x,y)\Big]=\nabla_y\cdot\Big[\big(\varepsilon_0 I+H(y)\big)\nabla_x\varphi^0(x)\Big]. \tag{4.18}
$$

If we define $\widehat{\chi}_j$ as the solution of

$$\begin{cases} -\nabla_y \cdot \left[\big(\varepsilon_0 I + H(y)\big)\nabla_y \hat{\chi}_j\right] = -\displaystyle\sum_{i=1}^{N} \frac{\partial\big(\varepsilon_0\delta_{ij} + H_{ij}(y)\big)}{\partial y_i} \quad \text{in} \quad Y, \\ M_Y(\hat{\chi}_j) = 0, \\ \hat{\chi}_j \ \text{ periodic } \ \text{in} \ \ y, \end{cases} \tag{4.19}$$

φ^1 can be expressed as

$$\varphi^1(x,y) = -\chi(\Omega)\sum_{j=1}^{N} \hat{\chi}_j(y)\frac{\partial \varphi^0}{\partial x_j} + \tilde{\varphi}(x). \tag{4.20}$$

Next, choosing $\phi^1(x,y) \equiv 0$ in (4.16),

$$-\nabla_x \cdot \int_Y \big(\varepsilon_0 I + H(y)\chi(\Omega)\big)\big(\nabla_x \varphi^0(x) + \nabla_y \varphi^1(x,y)\big)\, dy = 0. \tag{4.21}$$

Plugging (4.20) into (4.21), the equation for $\varphi^0(x)$ is then

$$-\nabla_x \cdot \left[\big(\varepsilon_0 I + H^0\chi(\Omega)\big)\, \nabla_x \varphi^0(x)\right] = 0, \tag{4.22}$$

where

$$H^0_{ij} = \frac{1}{|Y|}\int_Y H_{ij}(y) - H_{ik}(y)\frac{\partial \hat{\chi}_j(y)}{\partial y_k}\, dy. \tag{4.23}$$

$\square$

4.4 The Local Strong Convergence of the Electric Field

Now, assume that $\nabla\varphi^\varepsilon$ is locally uniformly bounded on Ω in $\mathcal{L}^4$ norm, then $\nabla\varphi^\varepsilon$ converges strongly on the smaller region Ω. This fact is very important in the derivation of the homogenized equation for the deformation field. Since this strong convergence is not for the entire domain, but only valid locally on Ω, we can not use the similar results in the theory of two-scale convergence. Here, we will prove it by combining the local fluctuation estimation with two-scale convergence.

Proposition 4.5 *Assume there exist a compact set $K \supset \overline{\Omega}$ and a constant c, such that $\left\|\nabla_x \varphi^\varepsilon\right\|_{\mathcal{L}^4(K)} < c$, then*

$$\nabla \varphi^\varepsilon - \nabla \varphi^0 - \nabla_y \varphi^1 \left(x, \frac{x}{\varepsilon}\right) \to 0 \qquad \text{in } \mathcal{L}^2(\Omega). \tag{4.24}$$

Proof Construct function $\xi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ satisfying $\operatorname{supp}(\xi) \subset K$ and $\xi(x) \equiv 1$ in Ω. First we prove that there exists a constant c such that for any $\Theta \in \left(\mathfrak{D}(K \backslash \Gamma)\right)^N$,

$$\limsup_{\varepsilon \to 0} \left\|\xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \nabla\hat{\chi}_i^\varepsilon \Theta_i\right\|_{\mathcal{L}^2(K)} \leqslant c \left\|\nabla_y \varphi^1 + \nabla_y \hat{\chi}_i \Theta_i\right\|_{\mathcal{L}^2(\Omega \times Y)}, \tag{4.25}$$

where $\hat{\chi}_i^\varepsilon(x) = \chi(\Omega)\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)$. In fact,

$$\begin{aligned}
&\varepsilon_0 \left\|\xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \nabla\hat{\chi}_i^\varepsilon\Theta_i\right\|^2_{\mathcal{L}^2(K)} \\
\leqslant\ & \int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\left(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0 + \chi(\Omega)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i\right) \cdot \\
& \quad \left(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0 + \chi(\Omega)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i\right) dx \\
=\ & \int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\left(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0\right) \cdot \left(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0\right) dx \\
& + \int_\Omega \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i \cdot \nabla\hat{\chi}_j\left(\frac{x}{\varepsilon}\right)\Theta_j\, dx \\
& + 2\int_\Omega \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\left(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0\right) \cdot \nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i\, dx.
\end{aligned}$$

Note that $H(y)\hat{\chi}_i(y)\Theta_i\Theta_j$ can be the test function for the two-scale convergence of $\hat{\chi}_j\left(\frac{x}{\varepsilon}\right)$ and φ^ε. Therefore,

$$\begin{aligned}
& \int_\Omega \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i \cdot \nabla\hat{\chi}_j\left(\frac{x}{\varepsilon}\right)\Theta_j\, dx \\
\to\ & \int_\Omega\int_Y \left(\varepsilon_0 I + H(y)\right)\nabla\hat{\chi}_i(y)\,\Theta_i \cdot \nabla\hat{\chi}_j(y)\Theta_j\, dy dx
\end{aligned}$$

and

$$
\begin{aligned}
& 2\int_\Omega \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0)\cdot\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i\,dx \\
\rightarrow\ & 2\int_\Omega\int_Y \big(\varepsilon_0 I + H(y)\big)\,\xi\,(\nabla\varphi^0 + \nabla_y\varphi^1 - \nabla\varphi^0)\cdot\nabla\hat{\chi}_i(y)\Theta_i\,dydx \\
=\ & 2\int_\Omega\int_Y \big(\varepsilon_0 I + H(y)\big)\,\xi\nabla_y\varphi^1\cdot\nabla\hat{\chi}_i(y)\Theta_i\,dydx.
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
& \int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0)\cdot(\xi\nabla\varphi^\varepsilon - \xi\nabla\varphi^0)\,dx \\
=\ & \int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\varphi^\varepsilon\cdot\xi^2(\nabla\varphi^\varepsilon - \nabla\varphi^0)\,dx \\
& -\int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\varphi^0\cdot\xi^2(\nabla\varphi^\varepsilon - \nabla\varphi^0)\,dx \\
=\ & -\int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\varphi^\varepsilon\cdot\nabla\xi(\varphi^\varepsilon - \varphi^0)2\xi\,dx \qquad (4.26) \\
& -\int_K \left(\varepsilon_0 I + H\left(\frac{x}{\varepsilon}\right)\chi(\Omega)\right)\nabla\varphi^0\cdot\xi^2(\nabla\varphi^\varepsilon - \nabla\varphi^0)\,dx \\
\rightarrow\ & -\int_\Omega\int_Y \big(\varepsilon_0 I + H(y)\chi(\Omega)\big)\nabla\varphi^0\cdot\xi^2\nabla_y\varphi^1(x,y)\,dydx \qquad (4.27) \\
=\ & \int_\Omega\int_Y \big(\varepsilon_0 I + H(y)\big)\nabla_y\varphi^1\cdot\xi^2\nabla_y\varphi^1\,dydx. \qquad (4.28)
\end{aligned}
$$

Above, (4.26) comes from equation (4.15). (4.27) is because $\varphi^\varepsilon - \varphi^0 \to 0$ in $\mathcal{L}^2(K)$. (4.28) comes from (4.18).

Putting these together, we obtain

$$
\begin{aligned}
& \varepsilon_0 \left\| \xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \nabla\hat{\chi}_i^\varepsilon\Theta_i \right\|^2_{\mathcal{L}^2(K)} \\
\leqslant \; & \int_\Omega \int_Y \big(\varepsilon_0 I + H(y)\big)\nabla_y\varphi^1 \cdot \xi^2 \nabla_y\varphi^1 \, dydx \\
+ \; & 2\int_\Omega \int_Y \big(\varepsilon_0 I + H(y)\big)\, \xi\nabla_y\varphi^1 \cdot \nabla\hat{\chi}_i(y)\Theta_i \, dydx \\
+ \; & \int_\Omega \int_Y \big(\varepsilon_0 I + H(y)\big)\nabla\hat{\chi}_i(y)\Theta_i \cdot \nabla\hat{\chi}_j(y)\Theta_j \, dydx \\
\leqslant \; & c \left\| \nabla_y\varphi^1 + \nabla_y\hat{\chi}_i\Theta_i \right\|_{\mathcal{L}^2(\Omega\times Y)}.
\end{aligned}
$$

Now consider

$$
\begin{aligned}
& \left\| \xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \chi(\Omega)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\nabla_i\varphi^0 \right\|^2_{\mathcal{L}^2(K)} \\
\leqslant \; & \left\| \xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \chi(\Omega)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i \right\|^2_{\mathcal{L}^2(K)} \\
& + \left\| -\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i + \nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\nabla_i\varphi^0 \right\|^2_{\mathcal{L}^2(\Omega)} \\
\leqslant \; & \left\| \xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \chi(\Omega)\nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\Theta_i \right\|^2_{\mathcal{L}^2(K)} \\
& + \sum_i \left\| -\Theta_i + \nabla_i\varphi^0 \right\|^2_{\mathcal{L}^4(\Omega)} \left\| \nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right) \right\|^2_{\mathcal{L}^4(\Omega)} \\
\leqslant \; & c \left\| \nabla_y\varphi^1 + \nabla_y\hat{\chi}_i\Theta_i \right\|_{\mathcal{L}^2(\Omega\times Y)} + c\sum_i \left\| -\Theta_i + \nabla_i\varphi^0 \right\|^2_{\mathcal{L}^4(\Omega)} && (4.29) \\
\leqslant \; & c \left\| -\nabla_y\hat{\chi}_i(y)\nabla_i\varphi^0 + \nabla_y\hat{\chi}_i(y)\Theta_i \right\|_{\mathcal{L}^2(\Omega\times Y)} + c\sum_i \left\| -\Theta_i + \nabla_i\varphi^0 \right\|^2_{\mathcal{L}^4(\Omega)} && (4.30) \\
\leqslant \; & c\sum_i \left\| -\Theta_i + \nabla_i\varphi^0 \right\|^2_{\mathcal{L}^4(\Omega)}.
\end{aligned}
$$

Above, (4.29) comes from the estimate (4.25) and the $\mathcal{L}^4$ boundedness of $\hat{\chi}_i(y)$. (4.30) comes from (4.20).

If we choose $\Theta_i \to \nabla_i \varphi^0$ in $\mathcal{L}^4(\Omega)$, then

$$\left\| \xi(x)\nabla\varphi^\varepsilon - \xi(x)\nabla\varphi^0 + \nabla\hat{\chi}_i\left(\frac{x}{\varepsilon}\right)\nabla_i\varphi^0 \right\|^2_{\mathcal{L}^2(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

$\xi(x) = 1$ in Ω, so

$$\left\| \nabla\varphi^\varepsilon(x) - \nabla\varphi^0(x) - \nabla_y\varphi^1\left(x, \frac{x}{\varepsilon}\right) \right\|^2_{\mathcal{L}^2(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

$\square$

With this result, we can prove the two-scale convergence of the Maxwell stress. Consider the electromechanical coupling tensor $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$, $A(y)$ is Y periodic in y and is $\mathcal{L}^\infty$ bounded. For any test function $v(x,y) \in \left(\mathcal{C}_0^\infty(\Omega; \mathcal{H}^1_\sharp(Y)/\mathbb{R})\right)^N$, we have

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla\varphi^\varepsilon \nabla\varphi^\varepsilon v\left(x, \frac{x}{\varepsilon}\right) dx \\
= & \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla\varphi^\varepsilon \left(\nabla\varphi^\varepsilon - \nabla_x\varphi^0 - \nabla_y\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right) v\left(x, \frac{x}{\varepsilon}\right) dx \\
& + \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla\varphi^\varepsilon \left(\nabla_x\varphi^0 + \nabla_y\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right) v\left(x, \frac{x}{\varepsilon}\right) dx \\
= & \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla\varphi^\varepsilon \left(\nabla_x\varphi^0 + \nabla_y\varphi^1\left(x, \frac{x}{\varepsilon}\right)\right) v\left(x, \frac{x}{\varepsilon}\right) dx \qquad (4.31) \\
\to & \frac{1}{|Y|}\int_\Omega\int_Y A(y)\big(\nabla_x\varphi^0 + \nabla_y\varphi^1(x,y)\big)\big(\nabla_x\varphi^0 + \nabla_y\varphi^1(x,y)\big)v(x,y)\, dydx,
\end{aligned}$$

in which (4.31) comes from Proposition 4.5. The last step is because $\nabla\varphi^0\nabla\varphi^0$ two-scale converges to itself and we can use $A(y)\nabla\hat{\chi}_j\nabla\widehat{\chi}_i v(x,y)$ as the test function for this convergence, because $A(y)\nabla\hat{\chi}_j\nabla\widehat{\chi}_i v(x,y) \in \mathcal{L}^1_\sharp\big(Y; \mathcal{C}(\bar{\Omega})\big)$.

4.5 The Two-scale Convergence of the Deformation Field

Before the proof of Theorem 4.2, we will give some basic notation, assumption and existence result in the homogenization theory of elasticity as in [13] first.

In the component form, equation (4.2) is

$$\begin{cases} \dfrac{\partial}{\partial x_j}\left(C_{ijkh}\dfrac{\partial u_k}{\partial x_h} + A_{ijkh}\dfrac{\partial \varphi}{\partial x_k}\dfrac{\partial \varphi}{\partial x_h}\right) = 0 & \text{in} \quad \Omega, \\ \left(C_{ijkh}\dfrac{\partial u_k}{\partial x_h} + A_{ijkh}\dfrac{\partial \varphi}{\partial x_k}\dfrac{\partial \varphi}{\partial x_h}\right) n_j = f_i & \text{on} \quad \Gamma_2, \\ u = 0 & \text{on} \quad \Gamma_3. \end{cases} \tag{4.32}$$

Above, we used the classical notation about fourth order tensors as follows. If $C = (c_{ijkh})$ is a fourth order tensor, $\xi = (\xi_{ij})$, $\xi^1 = (\xi^1_{ij})$ are square matrices, we set

$$\begin{cases} C\xi = \big((C\xi)_{ij}\big) = \big((c_{ijkh}\xi_{kh})_{ij}\big), \\ C\,\xi\,\xi^1 = c_{ijkh}\,\xi_{ij}\,\xi^1_{kh}, \\ |\xi| = \left(\sum_{ij=1}^N \xi^2_{ij}\right)^{\frac{1}{2}}. \end{cases}$$

Definition 4.2 *Let α, $\beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and let Ω be an open set of $\mathbb{R}^N$. We denote by $M_e(\alpha_1, \alpha_2, \Omega)$ the set of the fourth order tensor $C = (c_{ijkh})$ which satisfies*

$$\begin{cases} c(x)_{ijkh} \in \mathcal{L}^\infty(\Omega) & \forall\ i, j, k, h = 1, \ldots, N, \\ c_{ijkh} = c_{ijhk} = c_{khij} & \forall\ i, j, k, h = 1, \ldots, N,\ \forall\ x \in \Omega, \\ \alpha_1 |\,\xi\,|^2 \leqslant C\,\xi\,\xi & \forall\ \textit{symmetric matrix}\ \xi, \\ |\,C\xi\,| \leqslant \alpha_2 |\,\xi\,| & \forall\ \textit{matrix}\ \xi. \end{cases}$$

As in the classical elasticity theory, introduce the linearized strain tensor e defined by

$$e(u) = \big(e_{ij}(u)\big), \qquad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Then for any $C \in M_e(\alpha_1, \alpha_2, \Omega)$,

$$\big(Ce(u)\big)_{ij} = c_{ijkh}\,e_{kh}(u) = c_{ijkh}\,\frac{\partial u_k}{\partial x_h} = C\nabla u. \tag{4.33}$$

Define space V by

$$V = \left\{ v \,\middle|\, v \in \mathcal{H}^1(\Omega),\ \gamma(v) = 0 \ \text{ on } \ \Gamma_3 \right\}.$$

Set $\mathcal{V} = (V)^N$ and equip $\mathcal{V}$ with norm

$$\|v\|_{\mathcal{V}} = \left(\sum_{i=1}^{N} \|\nabla v_i\|^2_{\mathcal{L}^2(\Omega)} \right)^{\frac{1}{2}},$$

then $\mathcal{V}$ is a Hilbert space with inner product

$$(u, v)_{\mathcal{V}} = \sum_{i=1}^{N} \big(\nabla u_i, \nabla v_i\big)_{\mathcal{L}^2(\Omega)} \qquad \forall\, u,\ v \ \text{ in } \ \mathcal{V}.$$

Observe that $\mathcal{V}' = (V')^N$.

Now, assume $C = (C_{ijkh}) \in M_e(\alpha_1, \alpha_2, \Omega)$ and $f = (f_1, \ldots, f_N) \in \big(\mathcal{H}^{-\frac{1}{2}}(\Gamma_2)\big)^N$, thanks to equation (4.33), the weak form for equation (4.4) is

$$\int_\Omega C(x)\, e(u)\, e(v)\, dx + \int_\Omega A(x) \nabla\varphi \nabla\varphi \nabla v\, dx = \langle f, v \rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N, (\mathcal{H}^{1/2}(\Gamma_2))^N}, \qquad (4.34)$$

for any $v \in \mathcal{V}$.

Define bilinear form by

$$\mathfrak{L}_u(u, v) = \int_\Omega C(x)\, e(u)\, e(v)\, dx.$$

Then $\mathfrak{L}_u(u, v)$ is a bounded bilinear map because $C(x) \in \mathcal{L}^\infty(\Omega)$.

On the other hand, since $C(x) \in M_e(\alpha_1, \alpha_2, \Omega)$, we get

$$\alpha_1 \int_\Omega \big|e(v)\big|^2\, dx \leqslant \mathfrak{L}_u(v, v), \qquad \forall\ v \in \mathcal{V}.$$

From the second Kohn inequality,

$$\mathfrak{L}_u(v, v) \geqslant \alpha_1 \int_\Omega \big|e(v)\big|^2\, dx \geqslant c\, \|v\|^2_{\mathcal{H}^1(\Omega)}.$$

In addition, $f \in \left(\mathcal{H}^{-\frac{1}{2}}(\Gamma_2)\right)^N \subset \mathcal{V}'$, and $A(x)\nabla\varphi\nabla\varphi \in \mathcal{L}^2 \subset \mathcal{V}'$, hence the Lax-Milgram theorem applies, a unique solution exists and satisfies

$$\|u\|_{\mathcal{V}} \leqslant c\left(\big\|f\big\|_{(\mathcal{H}^{-1/2}(\Gamma_2))^N} + \big\|A(x)\nabla\varphi(x)\nabla\varphi(x)\big\|_{\mathcal{L}^2(\Omega)}\right). \tag{4.35}$$

Now, let us prove Theorem 4.2 in this framework.

Proof Since $\nabla_x\varphi^\varepsilon$ is locally uniformly bounded in $\mathcal{L}^4$ norm, from (4.35), u^ε is uniformly bounded in $\left(\mathcal{H}^1(\Omega)\right)^N$. Thus we can find a subsequence and a function u^0 such that $u^\varepsilon \rightharpoonup u^0$ in $\left(\mathcal{H}^1(\Omega)\right)^N$. Moreover, there exists $u^1(x,y) \in \mathcal{L}^2\left(\Omega; \mathcal{H}^1_\sharp(Y)/\mathbb{R}\right)^N$ such that up to a subsequence, ∇u^ε two-scale converges to $\nabla_x u^0 + \nabla_y u^1(x,y)$. Now consider

$$v^0(x) \in \left(\mathcal{C}_0^\infty(\Omega)\right)^N \quad \text{and} \quad v^1(x,y) \in \left(\mathcal{C}_0^\infty(\Omega; \mathcal{H}^1_\sharp(Y)/\mathbb{R})\right)^N,$$

we have $v(x) = v^0(x) + \varepsilon v^1(x, \frac{x}{\varepsilon}) \in \left(\mathcal{H}^1_0(\Omega)\right)^N$. Using this as a test function for equation (4.34), we get

$$\int_\Omega C^\varepsilon(x)\, e(u^\varepsilon)\, e(v^\varepsilon)\, dx + \int_\Omega A^\varepsilon(x)\nabla\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla v^\varepsilon\, dx = \langle f, v^\varepsilon\rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N, (\mathcal{H}^{1/2}(\Gamma_2))^N}. \tag{4.36}$$

First,

$$\int_\Omega C^\varepsilon(x)\, e(u^\varepsilon)\, e(v^\varepsilon)\, dx = \int_\Omega C^\varepsilon(x)\, e(v^\varepsilon)\, e(u^\varepsilon)\, dx.$$

Second, $e(u^\varepsilon)_{ij} = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right)$ two-scale converges to

$$\frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i} + \frac{\partial u_i^1}{\partial y_j} + \frac{\partial u_j^1}{\partial y_i}\right) = \left(e_x(u^0)\right)_{ij} + \left(e_y(u^1)\right)_{ij},$$

where

$$\left(e_x(u^0)\right)_{ij} = \frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i}\right), \qquad \left(e_y(u^1)\right)_{ij} = \frac{1}{2}\left(\frac{\partial u_i^1}{\partial y_j} + \frac{\partial u_j^1}{\partial y_i}\right).$$

Since $C(y) \in \left(\mathcal{L}^\infty(Y)\right)^{N^2}$, $C^\varepsilon e(v^\varepsilon)$ can be used as the test function for the two-scale convergence of $e(u^\varepsilon)$, we then get

$$\lim_{\varepsilon\to 0}\int_\Omega C^\varepsilon(x)\,e(u^\varepsilon)\,e(v^\varepsilon)\,dx = \frac{1}{|Y|}\int_\Omega\int_Y C(y)\big(e_x(u^0)+e_y(u^1)\big)\big(e_x(v^0)+e_y(v^1)\big)\,dydx.$$

We already know from the end of Section 4.4 that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla\varphi^\varepsilon\nabla\varphi^\varepsilon\nabla v\left(x,\frac{x}{\varepsilon}\right)\,dx\\
= &\ \frac{1}{|Y|}\int_\Omega\int_Y A(y)\big(\nabla_x\varphi^0+\nabla_y\varphi^1(x,y)\big)\big(\nabla_x\varphi^0+\nabla_y\varphi^1(x,y)\big)\\
&\qquad\qquad \big(\nabla_x v^0(x)+\nabla_y v^1(x,y)\big)\,dydx.
\end{aligned}$$

In addition,

$$\lim_{\varepsilon\to 0}\langle\, f, v^\varepsilon\rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N,(\mathcal{H}^{1/2}(\Gamma_2))^N} = \langle\, f, v^0\rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N,(\mathcal{H}^{1/2}(\Gamma_2))^N}.$$

Hence by passing to the limit in equation (4.36) as $\varepsilon\to 0$, we finally get

$$\begin{aligned}
&\frac{1}{|Y|}\int_\Omega\int_Y C(y)\big(e_x(u^0)+e_y(u^1)\big)\big(e_x(v^0)+e_y(v^1)\big)\,dydx\\
&+\frac{1}{|Y|}\int_\Omega\int_Y A(y)\big(\nabla_x\varphi^0+\nabla_y\varphi^1(x,y)\big)\big(\nabla_x\varphi^0+\nabla_y\varphi^1(x,y)\big)\\
&\qquad\qquad \big(\nabla_x v^0(x)+\nabla_y v^1(x,y)\big)\,dydx\\
= &\ \langle\, f, v^0\rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N,(\mathcal{H}^{1/2}(\Gamma_2))^N}.
\end{aligned}\tag{4.37}$$

Let us show that (4.37) is a variational equation in the space

$$\mathcal{H}_u := \big[\mathcal{H}^1(\Omega)\big]^N\times\big[\mathcal{L}^2\big(\Omega;\mathcal{H}^1_\sharp(Y)/\mathbb{R}\big)\big]^N,$$

and that the hypotheses of the Lax-Milgram theorem are fulfilled. Indeed, endowing the space $\mathcal{H}_u$ with the norm

$$\big\|V\big\|^2_{\mathcal{H}_u} = \big\|v^0\big\|^2_{(\mathcal{H}^1(\Omega))^N} + \big\|v^1\big\|^2_{(\mathcal{L}^2(\Omega;\mathcal{H}^1_\sharp(Y)/\mathbb{R}))^N},\qquad \forall\, V=(v^0,v^1)\in\mathcal{H}_u,$$

the bilinear form defined by

$$\mathfrak{L}_u(U,V) = \frac{1}{|Y|}\int_\Omega\int_Y C(y)\big(e_x(u^0)+e_y(u^1)\big)\big(e_x(v^0)+e_y(v^1)\big)\,dydx$$

is then continuous on $\mathcal{H}_u$.

Since $C(y) \in M_e(\alpha_1, \alpha_2, Y)$,

$$\mathfrak{L}_u(U,U) \geqslant \frac{\alpha}{|Y|}\int_\Omega\int_Y \sum_{i,j=1}^{N}\big(e_x(u^0)_{ij}+e_y(u^1)_{ij}\big)^2\,dydx. \tag{4.38}$$

Observe that

$$\begin{aligned}
&\int_\Omega\int_Y \sum_{i,j=1}^{N}\big(e_x(u^0)_{ij}+e_y(u^1)_{ij}\big)^2\,dydx \\
= \;& \big\|e_x(u^0)\big\|^2_{\mathcal{L}^2(\Omega)}+\big\|e_y(u^1)\big\|^2_{\mathcal{L}^2(\Omega\times Y)}+2\int_\Omega\int_Y \sum_{i,j=1}^{N} e_x(u^0)_{ij}\,e_y(u^1)_{ij}\,dydx.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& 2\int_\Omega\int_Y \sum_{i,j=1}^N e_x(u^0)_{ij}\, e_y(u^1)_{ij}\, dydx \\
= \; & \frac{1}{2}\sum_{i=1}^N \int_\Omega\int_Y \sum_{j=1}^N \left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)\frac{\partial u_i^1}{\partial y_j}\, dydx \\
& + \frac{1}{2}\sum_{j=1}^N \int_\Omega\int_Y \sum_{i=1}^N \left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)\frac{\partial u_j^1}{\partial y_i}\, dydx \\
= \; & \frac{1}{2}\sum_{i=1}^N \int_\Omega\int_Y \sum_{j=1}^N \frac{\partial}{\partial y_j}\left(\left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)u_i^1\right) dydx \\
& + \frac{1}{2}\sum_{j=1}^N \int_\Omega\int_Y \sum_{i=1}^N \frac{\partial}{\partial y_i}\left(\left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)u_j^1\right) dydx \\
= \; & \frac{1}{2}\sum_{i=1}^N \int_\Omega\int_{\partial Y} \sum_{j=1}^N \left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right) u_i^1 n_j\, dS_y dx \\
& + \frac{1}{2}\sum_{j=1}^N \int_\Omega\int_{\partial Y} \sum_{i=1}^N \left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right) u_j^1 n_i\, dS_y dx \\
= \; & 0.
\end{aligned}
$$

Above, we used the periodicity of $u^1(x,y)$ with respect to variable y.

Therefore, (4.38) becomes

$$
\begin{aligned}
\mathfrak{L}_u(U,U) \; & \geqslant \; \frac{\alpha}{|Y|}\left(\left\|e_x(u^0)\right\|^2_{\mathcal{L}^2(\Omega)} + \left\|e_y(u^1)\right\|^2_{\mathcal{L}^2(\Omega\times Y)}\right) \\
& \geqslant \; c\left(\left\|u^0\right\|^2_{(\mathcal{H}^1(\Omega))^N} + \left\|u^1\right\|^2_{(\mathcal{L}^2(\Omega;\mathcal{H}^1_\sharp(Y)/\mathbb{R})^N}\right) \\
& = \; c\left\|U\right\|^2_{\mathcal{H}_u}.
\end{aligned}
$$

This gives the coerciveness of $\mathfrak{L}_u(U,U)$. Furthermore, the following mappings are linear and continuous on $\mathcal{H}_u$,

$$\begin{aligned}
1: \quad (v^0, v^1) \quad &\to \langle f, v^0\rangle_{(\mathcal{H}^{-1/2}(\Gamma_2))^N, (\mathcal{H}^{1/2}(\Gamma_2))^N};\\
2: \quad (v^0, v^1) \quad &\to \int_\Omega \int_Y A(y)\big(\nabla_x\varphi^0 + \nabla_y\varphi^1(x,y)\big)\big(\nabla_x\varphi^0 + \nabla_y\varphi^1(x,y)\big)\\
&\qquad\qquad \big(\nabla_x v^0(x) + \nabla_y v^1(x,y)\big)\, dydx.
\end{aligned}$$

Now, we can apply the Lax-Milgram theorem to obtain the existence and the uniqueness of $(u^0, u^1) \in \mathcal{H}_u$ as a solution of (4.37). Choosing first $v^0 \equiv 0$ and after $v^1 \equiv 0$, we see that equation (4.37) is equivalent to the following problem,

$$\begin{cases}
-\nabla_y \cdot \big(C(y)\nabla_y u^1\big) = \nabla_y \cdot \big(C(y)\nabla_x u^0 + A(y)(\nabla_x\varphi^0 + \nabla_y\varphi^1)(\nabla_x\varphi^0 + \nabla_y\varphi^1)\big),\\
-\nabla_x \cdot \displaystyle\int_Y C(y)(\nabla_x u^0 + \nabla_y u^1)\, dy = \nabla_x \cdot \int_Y A(y)(\nabla_x\varphi^0 + \nabla_y\varphi^1)(\nabla_x\varphi^0 + \nabla_y\varphi^1)\, dy,\\
\left[\displaystyle\int_Y C(y)(\nabla_x u^0 + \nabla_y u^1) + A(y)(\nabla_x\varphi^0 + \nabla_y\varphi^1)(\nabla_x\varphi^0 + \nabla_y\varphi^1)\, dy\right] n = f \quad \text{ on } \Gamma_2,\\
u = 0 \qquad \text{on } \Gamma_3.
\end{cases} \tag{4.39}$$

Recalling that

$$\nabla_y\varphi^1 = -\sum_{j=1}^N \nabla_y\widehat{\chi}_j\, \frac{\partial\varphi^0}{\partial x_j},$$

then

$$\begin{aligned}
& A(y)(\nabla_x\varphi^0 + \nabla_y\varphi^1)(\nabla_x\varphi^0 + \nabla_y\varphi^1)\\
= \;& A(y)_{ij\alpha\beta}\left(\frac{\partial\varphi^0}{\partial x_\alpha} + \frac{\partial\varphi^1}{\partial y_\alpha}\right)\left(\frac{\partial\varphi^0}{\partial x_\beta} + \frac{\partial\varphi^1}{\partial y_\beta}\right)\\
= \;& A_{ij\alpha\beta}\left(\frac{\partial\varphi^0}{\partial x_\alpha} - \frac{\partial\hat{\chi}_k}{\partial y_\alpha}\frac{\partial\varphi^0}{\partial y_k}\right)\left(\frac{\partial\varphi^0}{\partial x_\beta} - \frac{\partial\hat{\chi}_h}{\partial y_\beta}\frac{\partial\varphi^0}{\partial y_h}\right) \qquad (4.40)\\
= \;& A_{ij\alpha\beta}\frac{\partial\varphi^0}{\partial x_\alpha}\frac{\partial\varphi^0}{\partial x_\beta} - A_{ij\alpha\beta}\frac{\partial\hat{\chi}_h}{\partial y_\beta}\frac{\partial\varphi^0}{\partial y_h}\frac{\partial\varphi^0}{\partial x_\alpha} - A_{ij\alpha\beta}\frac{\partial\widehat{\chi}_k}{\partial y_\alpha}\frac{\partial\varphi^0}{\partial y_k}\frac{\partial\varphi^0}{\partial x_\beta} + A_{ij\alpha\beta}\frac{\partial\hat{\chi}_k}{\partial y_\alpha}\frac{\partial\widehat{\chi}_h}{\partial y_\beta}\frac{\partial\varphi^0}{\partial y_k}\frac{\partial\varphi^0}{\partial x_h}.
\end{aligned}$$

If we define

$$B_{ijkh} = A_{ijkh} - A_{ijlh}\frac{\partial \hat{\chi}_k}{\partial y_l} - A_{ijkl}\frac{\partial \hat{\chi}_h}{\partial y_l} + A_{ij\alpha\beta}\frac{\partial \hat{\chi}_k}{\partial y_\alpha}\frac{\partial \hat{\chi}_h}{\partial y_\beta}, \tag{4.41}$$

and let $\chi^{kh} = (\chi_1^{kh}, \ldots, \chi_N^{kh})$ be the solution of

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijlm}\dfrac{\partial \chi_l^{kh}}{\partial y_m}\right) = \dfrac{\partial C_{ijkh}}{\partial y_j} \quad \text{in} \quad \Omega, \\ \chi^{kh} \quad Y-\text{periodic}, \\ M_Y(\chi^{kh}) = 0, \end{cases} \tag{4.42}$$

$\tilde{\chi}^{kh} = (\tilde{\chi}_1^{kh}, \ldots, \tilde{\chi}_N^{kh})$ be the solution of

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijlm}\dfrac{\partial \tilde{\chi}_l^{kh}}{\partial y_m}\right) = \dfrac{\partial B_{ijkh}}{\partial y_j} \quad \text{in} \quad \Omega, \\ \tilde{\chi}^{kh} \quad Y-\text{periodic}, \\ M_Y(\tilde{\chi}^{kh}) = 0, \end{cases} \tag{4.43}$$

then $u^1(x,y)$ can be expressed as

$$u^1(x,y) = \chi^{kh}(y)\frac{\partial u_k^0}{\partial x_h}(x) + \tilde{\chi}^{kh}(y)\frac{\partial \varphi^0}{\partial x_k}(x)\frac{\partial \varphi^0}{\partial x_h}(x). \tag{4.44}$$

Plugging (4.44) into the second equation of (4.39), we get

$$-\frac{\partial}{\partial x_j}\left[\frac{1}{|Y|}\int_Y C_{ijlm}(y)\left(\frac{\partial u_l^0}{\partial x_m} + \frac{\partial \chi_l^{kh}}{\partial y_m}\frac{\partial u_k^0}{\partial x_h} + \frac{\partial \tilde{\chi}_l^{kh}}{\partial y_m}\frac{\partial \varphi^0}{\partial x_k}\frac{\partial \varphi^0}{\partial x_h}\right) + B_{ijkh}\frac{\partial \varphi^0}{\partial x_k}\frac{\partial \varphi^0}{\partial x_h}\,dy\right] = 0.$$

Now, define

$$C_{ijkh}^H = \frac{1}{|Y|}\int_Y C_{ijkh}(y) + C_{ijlm}(y)\frac{\partial \chi_l^{kh}}{\partial y_m}\,dy \tag{4.45}$$

and

$$A_{ijkh}^H = \frac{1}{|Y|}\int_Y B_{ijkh}(y) + C_{ijlm}(y)\frac{\partial \tilde{\chi}_l^{kh}}{\partial y_m}\,dy, \tag{4.46}$$

the homogenized equation for $u^0(x)$ is then

$$\begin{cases} -\nabla \cdot \left(C^H \nabla u^0 + A^H \nabla \varphi^0 \nabla \varphi^0\right) = 0 & \text{in} \quad \Omega, \\ \left(C^H \nabla u^0 + A^H \nabla \varphi^0 \nabla \varphi^0\right) \cdot n = f & \text{on} \quad \Gamma_2, \\ u^0 = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{4.47}$$

$\square$

Chapter 5

Numerical Results of Lamination

5.1 Basic Formula for Laminates

In this chapter, we will apply our reduced model to one of the idealized microstructures: the laminated and sequentially laminated structures. The simple microstructure of laminates allows us a simple computation for the effective properties. The motivation of this study is threefold. First, in small deformation elasticity and electrostatics, in the linear (e.g., [32]) and the nonlinear (e.g., [14, 21]) regimes, these materials are extremal. Thus the results for these microstructures provide straightforward estimates for the possible range of actuation strains, energy densities and breakdown fields. Second, the understanding of the lamination results shed some light on more complicated microstructures. Finally, we note that in plane strain conditions, laminated structures can provide good approximation for fiber composites. Thus for example, these results can be used to determine the response of EAPCs with carbon nanotubes inclusions.

We note that this chapter is similar to the independent work of deBotton et al. [15].

Let us now consider a laminate composite made out of two electroactive polymer materials as in Figure 5.1. One of the constituent material has a relatively low electric and elastic modulus and is referred as the compliant material. The other one has a relatively high electric and elastic modulus and is referred as the stiff material. We denote by θ the lamination angle. It is the angle between the interface of the two

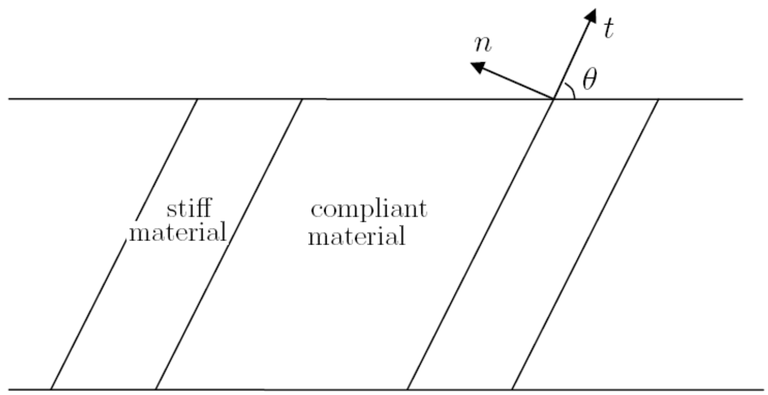


Figure 5.1: Two dimensional laminates.

constituent materials and the positive direction of the x_1 axis. The tangential vector of the interface is $t = (t_1, t_2) = (\cos\theta, \sin\theta)$, and the normal vector of the interface is $n = (n_1, n_2) = (-\sin\theta, \cos\theta)$. We take the distribution of material properties as

$$\begin{aligned} \text{Dielectric constant} \qquad & H_{ij} = H^c_{ij}\chi^c + H^s_{ij}\chi^s, \\ \text{Elastic modulus} \qquad & C_{ijkl} = C^c_{ijkl}\,\chi^c + C^s_{ijkl}\,\chi^s, \\ \text{Electromechanical coupling} \qquad & A_{ijkl} = A^c_{ijkl}\,\chi^c + A^s_{ijkl}\,\chi^s, \end{aligned} \tag{5.1}$$

where χ^c is the characteristic function of the region occupied by the compliant material and χ^s is the characteristic function of the region occupied by the stiff material. Note the tensor H here is $\varepsilon_0 I + H$ in the previous chapters. The volume fractions are λ^c and λ^s respectively. $\lambda^s = 1 - \lambda^c$.

The reduced model in Chapter 3 gives the equation of the electric field

$$\nabla \cdot (H\nabla\varphi) = 0 \tag{5.2}$$

subject to $\langle\nabla\varphi\rangle = \bar{E}$. The solution for (5.2) with H in (5.1) is piecewise constant. Denote the electric field $\nabla\varphi$ in the two materials by E^c and E^s respectively, equation

(5.2) reduces to

$$\begin{cases} \lambda^c E^c + \lambda^s E^s = \bar{E}, \\ (E^c - E^s) \cdot t = 0, \\ (H^c E^c - H^s E^s) \cdot n = 0. \end{cases} \tag{5.3}$$

The physical meanings of the above equations are clear. The first equation says that the average electric field is $\bar{E}$. The second equation comes from the fact that the electric field is continuous along the interface. The third equation says that the electric displacement is continuous across the interface.

Equation (5.3) has four unknowns with four equations and the solution can be verified to be

$$\begin{aligned} E^c &= B^c \bar{E} = \left(I - nn^T + ant^T + cnn^T H^s\right) \bar{E}, \\ E^s &= B^s \bar{E} = \left(I - nn^T + ant^T + cnn^T H^c\right) \bar{E}, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} a &= -\frac{n^T \left(\lambda^s H^c + \lambda^c H^s\right) t}{n^T \left(\lambda^s H^c + \lambda^c H^s\right) n}, \\ c &= \frac{1}{n^T \left(\lambda^s H^c + \lambda^c H^s\right) n}. \end{aligned} \tag{5.5}$$

Thus the average electric displacement is

$$\begin{aligned} \overline{HE} &= \lambda^c H^c E^c + \lambda^s H^s E^s \\ &= \lambda^c H^c B^c \bar{E} + \lambda^s H^s B^s \bar{E} = \left(\lambda^c H^c B^c + \lambda^s H^s B^s\right) \bar{E}. \end{aligned} \tag{5.6}$$

Therefore, the effective dielectric constant for this laminate is

$$\bar{H} = \lambda^c H^c B^c + \lambda^s H^s B^s. \tag{5.7}$$

With the electric field solved, the equation for the elastic field u is

$$\nabla \cdot (C\nabla u + A\nabla\varphi\nabla\varphi) = 0 \tag{5.8}$$

subject to $\langle \nabla u \rangle = \bar{\epsilon}$. Again, the above equation has a piecewise constant solution. If we denote the strain tensor by ϵ^c and ϵ^s in the two materials, (5.8) reduces to

$$\begin{cases} \lambda^c \epsilon^c + \lambda^s \epsilon^s = \bar{\epsilon}, \\ t^T \left(\epsilon^c - \epsilon^s\right) t = 0, \\ \left(C^c \epsilon^c + A^c E^c E^c - C^s \epsilon^s - A^s E^s E^s\right) n = 0. \end{cases} \tag{5.9}$$

The first equation specifies the average strain. The second equation is the kinematic compatibility. The third equation gives the traction continuity. From the first equation of (5.9),

$$\epsilon^s = \frac{\bar{\epsilon} - \lambda^c \epsilon^c}{\lambda^s}. \tag{5.10}$$

Plugging (5.10) together with (5.4) into the second and the third equation of (5.9), we get the equation for ϵ^c,

$$\begin{cases} t^T \epsilon^c t = t^T \bar{\epsilon}\, t, \\ \left(\lambda^s C^c + \lambda^c C^s\right) \epsilon^c n = C^s \bar{\epsilon}\, n - \lambda^s \left(A^c B^c \bar{E} B^c \bar{E} - A^s B^s \bar{E} B^s \bar{E}\right) n. \end{cases} \tag{5.11}$$

In Cartesian coordinates the strain and stress tensor are represented as 2×2 symmetric matrices. To solve (5.11) conveniently, we represent them as three dimensional vectors, where each element of the vector is related to a corresponding element of the matrix. Accordingly the constitutive relation linking the stress and strain components can be rewritten as

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \sqrt{2}\,\tau_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}\,C_{1112} \\ C_{2211} & C_{2222} & \sqrt{2}\,C_{2212} \\ \sqrt{2}\,C_{1211} & \sqrt{2}\,C_{1222} & 2\,C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \sqrt{2}\,\epsilon_{12} \end{pmatrix}. \tag{5.12}$$

Remark 5.1 *Above, we give a way to transform a fourth order tensor into a second order tensor. For the elastic modulus, since it has a symmetry property as* $C_{ijkl} = C_{jikl} = C_{klij}$*, we get a symmetric matrix. But for the electromechanical coupling tensor, we only have* $A_{ijkl} = A_{jikl} = A_{ijlk}$*, its corresponding matrix could be nonsymmetric.*

For any 2×2 matrix ξ, denote by $\check{\xi}$ the vector created from ξ as follows

$$\check{\xi} = \begin{pmatrix} \xi_{11} \\ \xi_{22} \\ \frac{1}{\sqrt{2}}\left(\xi_{12} + \xi_{21}\right) \end{pmatrix}.$$

Then if we let T be the row vector generated from matrix $t\,t^T$, the first equation in (5.11) can be rewritten as

$$T\,\check{\epsilon}^c = T\,\check{\bar{\epsilon}}, \tag{5.13}$$

where $\check{\epsilon}$ and $\check{\bar{\epsilon}}$ is the column vector form of tensor ϵ and $\bar{\epsilon}$.
Define

$$\begin{aligned} D^{(i)}_{mn} &= A^s_{ijkl}\,B^s_{km}\,B^s_{ln}\,n_j - A^c_{ijkl}\,B^c_{km}\,B^c_{ln}\,n_j, \\ W^{(i)}_{kl} &= \left(\lambda^s\,C^c_{ijkl} + \lambda^c\,C^s_{ijkl}\right)n_j, \\ V^{(i)}_{kl} &= C^s_{ijkl}\,n_j, \end{aligned} \tag{5.14}$$

and denote by $d^{(i)}$, $w^{(i)}$, $v^{(i)}$ the row vector form of matrix $D^{(i)}$, $W^{(i)}$ and $V^{(i)}$ respectively. We now construct matrices using these row vectors as follows:

$$R = \begin{pmatrix} T \\ w^{(1)} \\ w^{(2)} \end{pmatrix}, S = \begin{pmatrix} T \\ v^{(1)} \\ v^{(2)} \end{pmatrix}, P = \begin{pmatrix} 0 \\ \lambda^s d^{(1)} \\ \lambda^s d^{(2)} \end{pmatrix}.$$

Note that 0 is a 3 component vector in the expression of P.

Using these new notations, equation (5.11) can be rewritten as

$$R\,\check{\epsilon}^c = S\,\check{\bar{\epsilon}} + P\tilde{E} \tag{5.15}$$

with $\tilde{E}$ the column vector form of matrix $\bar{E}\bar{E}^T$. Clearly, the solution for equation (5.15) is

$$\check{\epsilon}^c = R^{-1}S\,\check{\bar{\epsilon}} + R^{-1}P\tilde{E}. \tag{5.16}$$

Let Q be the matrix form of the fourth order tensor

$$\lambda^c\,A^c_{ijkl}\,B^c_{km}\,B^c_{ln} + \lambda^s\,A^s_{ijkl}\,B^s_{km}\,B^s_{ln},$$

the average total stress is then

$$\begin{aligned}
\check{\bar{\tau}} &= \lambda^c\,\check{\tau}^c + \lambda^s\,\check{\tau}^s \\
&= \lambda^c\,\check{C}^c\,\check{\epsilon}^c + \lambda^s\,\check{C}^s\,\check{\epsilon}^s + Q\tilde{E} \\
&= \left[\check{C}^s + \lambda^c\left(\check{C}^c - \check{C}^s\right)R^{-1}S\right]\check{\bar{\epsilon}} \\
&\quad + \left[\lambda^c\left(\check{C}^c - \check{C}^s\right)R^{-1}P + Q\right]\tilde{E}.
\end{aligned} \tag{5.17}$$

Thus we obtain the effective tensor $\bar{C}$ and $\bar{A}$ as

$$\begin{aligned}
\check{\bar{C}} &= \check{C}^s + \lambda^c\left(\check{C}^c - \check{C}^s\right)R^{-1}S, \\
\check{\bar{A}} &= \lambda^c\left(\check{C}^c - \check{C}^s\right)R^{-1}P + Q.
\end{aligned} \tag{5.18}$$

Now, assume an external electric field E_{ext} is applied on this composite. Let $\check{E}_{\text{ext}}$ be the column vector for matrix $E_{\text{ext}}E^T_{\text{ext}}$. The electric field induced strain for the laminate is then given by:

$$\text{strain} = -(\check{\bar{C}})^{-1}\,\check{\bar{A}}\,\check{E}_{\text{ext}}. \tag{5.19}$$

We can compute the effective property for any rank-n laminates by iteration.

5.2 Numerical Results

Consider a rank-1 laminate made out of a homogeneous stiff material (2) with a high dielectric constant and a homogeneous compliant material (1) with a low dielectric constant. Assume that both materials have the same electrostatic strain response (i.e., the ratios of the shear to the dielectric modulus are the same). For numerical studies, we consider the following values consistent with experimental observations as in [15]:

$$\begin{aligned} h^{(1)} &= 10\varepsilon_0, & h^{(2)} &= 1000\varepsilon_0; \\ \alpha^{(1)} &= 10^8\,\mathrm{MPa}, & \alpha^{(2)} &= 10^8\,\mathrm{MPa}; \\ \mu^{(1)} &= 10\,\mathrm{MPa}, & \mu^{(2)} &= 1000\,\mathrm{MPa}\,. \end{aligned} \tag{5.20}$$

Above ε_0 is the free-space permittivity ($8.85 \times 10^{-12}\,\mathrm{Fm}^{-1}$), α is the Lame modulus, μ is the shear modulus. We take the Lame modulus (thus the bulk modulus) very large, so that the materials are nearly incompressible.

As mentioned in Zhang et al. [42], a large external electric field is needed to gain a practically meaningful strain. A simple computation shows that for the above pure materials, the longitudinal strain is 2.212% under an external field of 100 MV/m.

Now, let us examine the longitudinal field-induced strain for the rank-1 laminate. Assume in formula (5.19) that the external electric field is in the $[0, 1]$ direction. We are considering the longitudinal strain in the vertical direction, the $[1, 0]$ direction. Figure 5.2 shows the longitudinal field-induced strain ϵ_{11} as a function of the lamination angle θ and the compliant material volume fraction $\lambda^{(1)}$. we see that the maximum longitudinal strain is achieved at $\lambda^{(1)} = 0.5$, $\theta = 0.66\pi$ and has a value of 2.4768%. This means we can achieve as much as a 12% increase of the longitudinal strain compared to each of the constituent material by making a rank-1 laminate. We also observe that the longitudinal strain depends significantly on the lamination angle θ, but little on the volume fraction of the compliant material. To be accurate, the longitudinal strain depends on the relative angle between the applied electric field

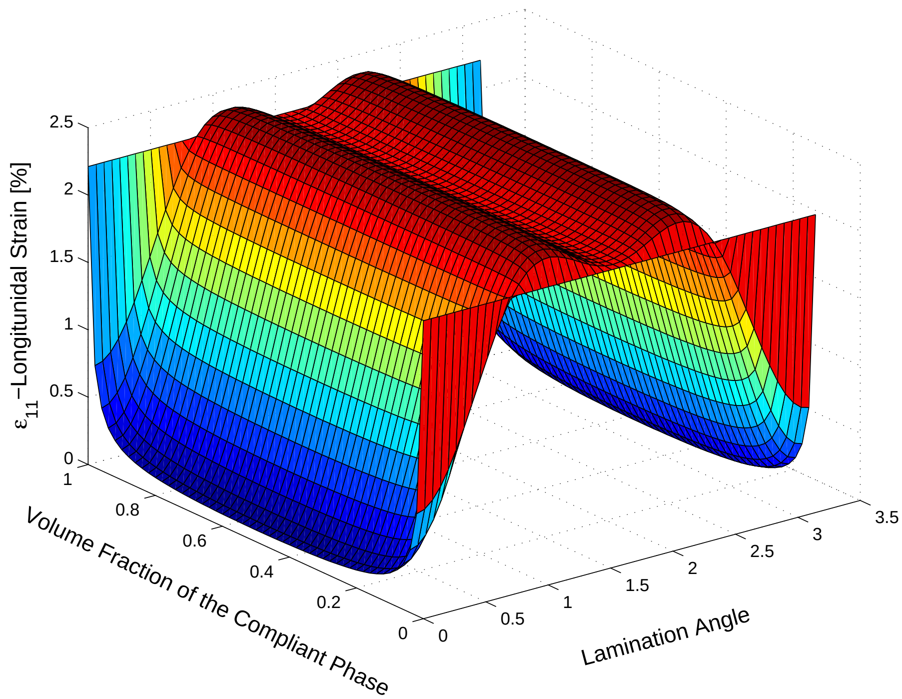


Figure 5.2: Effective longitudinal field-induced strain for rank-1 laminates

and the interface. If we fix the interface at $\theta = 0$, apply an electric field of different directions to the laminates, we get exactly the same strain in the electric field perpendicular direction as Figure 5.2. This means the rank-1 laminate is highly anisotropic. If we apply the electric field in the right direction, we can gain a large strain. In some other directions however, we get very small strains, even much smaller than the constituent materials.

While $\lambda^{(1)}$ and $\lambda^{(2)}$ are the volume fractions of material (1) and (2) in a composite respectively, we introduce new notations λ_n^c and λ_n^s when we consider multirank laminates. Assume a rank-n laminate is composed of two materials (homogeneous or heterogeneous). One of them has a relatively small elastic and electric modulus. We denote its volume fraction by λ_n^c. The other one has a relatively large elastic and electric modulus. We denote its volume fraction by λ_n^s. We extend this notation to rank-1 laminates by letting $\lambda_1^c = \lambda^{(1)}$ and $\lambda_1^s = \lambda^{(2)}$. Further, we define θ_n to be the lamination angle of a rank-n laminate.

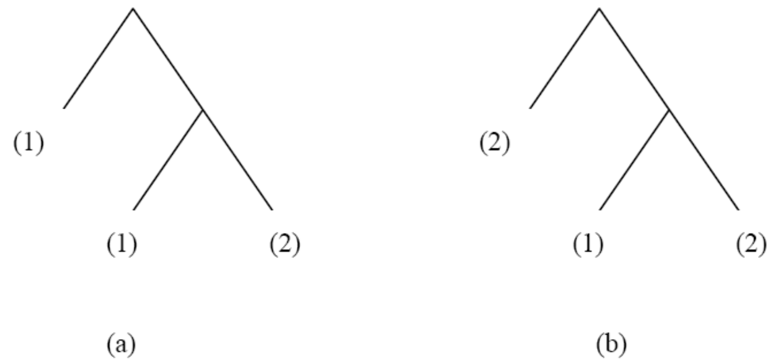


Figure 5.3: Two ways to construct a rank-2 laminate.

Now consider a rank-2 laminate. There are two ways to make the laminate. One is to laminate the rank-1 composite with the pure compliant material (1), as shown in (a) of Figure 5.3, the other is to laminate the rank-1 composite with the pure stiff material (2), as shown in (b) of Figure 5.3. For choice (a), no matter how we choose the volume fraction and the angle, we can get at most 2.72% for the longitudinal strain. However for choice (b), if we laminate the rank-1 composite ($\lambda_1^c = 0.98$, $\theta_1 = 0.66\pi$) with the pure stiff material, a very high longitudinal strain of 8.124% (increased 267.27%) can be achieved at $\lambda_2^c = 0.04$, $\theta_2 = 0.24$. Figure 5.4 is the longitudinal strain for the rank-2 material as a function of the lamination angle and the volume fraction for the compliant material.

Observe that a much larger longitudinal strain can be obtained from the rank-2 laminate compared to the rank-1 laminate for the same amount of compliant material at $\lambda^{(1)} = 0.04$. Further, the rank-1 composite depends significantly on the lamination angle, but little on the volume fraction of the compliant material. In contrast, the rank-2 laminates depend on the lamination angle as well as the volume fraction of the compliant material. We now explain the underlying reason for these differences.

Inside the laminates, the electric fields satisfy equation (5.3). This results in fluctuation of the electric field. The electric field in the compliant material is quite different from that in the stiff material. Figure 5.5 is the norm of the electric field in the compliant material for rank-1 laminates. We can see that the electric field in the

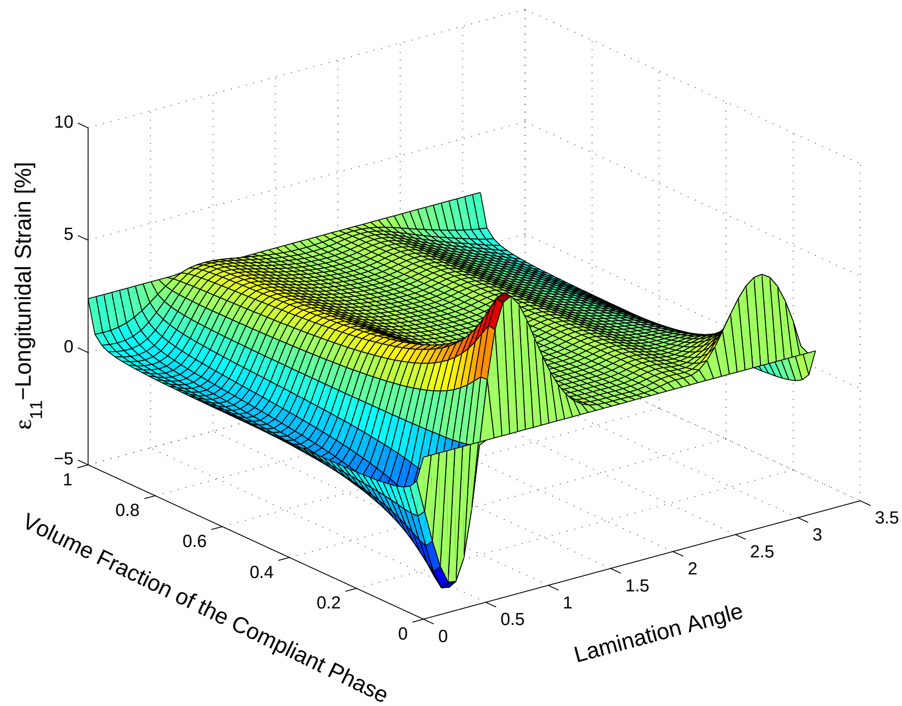


Figure 5.4: Effective longitudinal field-induced strain for rank-2 laminates

compliant material gains a very large increase when λ^c is very small. The direction of the electric field also changes with the angle of the laminates. However, the electric field in the stiff material behaves quite differently. Figure 5.6 is the norm of the electric field in the stiff material. To explain the difference between the electric fields in the compliant material and the stiff material, let us look at (5.4). The main contribution for each electric field comes from the last term in (5.4). For the compliant material this term is approximately

$$\frac{h^s}{\lambda^s h^c + \lambda^c h^s} \sim \frac{1}{\lambda^s \frac{h^c}{h^s} + \lambda^c}. \tag{5.21}$$

Because of the large ratio of the dielectric constants of the two materials, when λ^c is small, this term becomes very large. Meanwhile, the contribution of this term to the

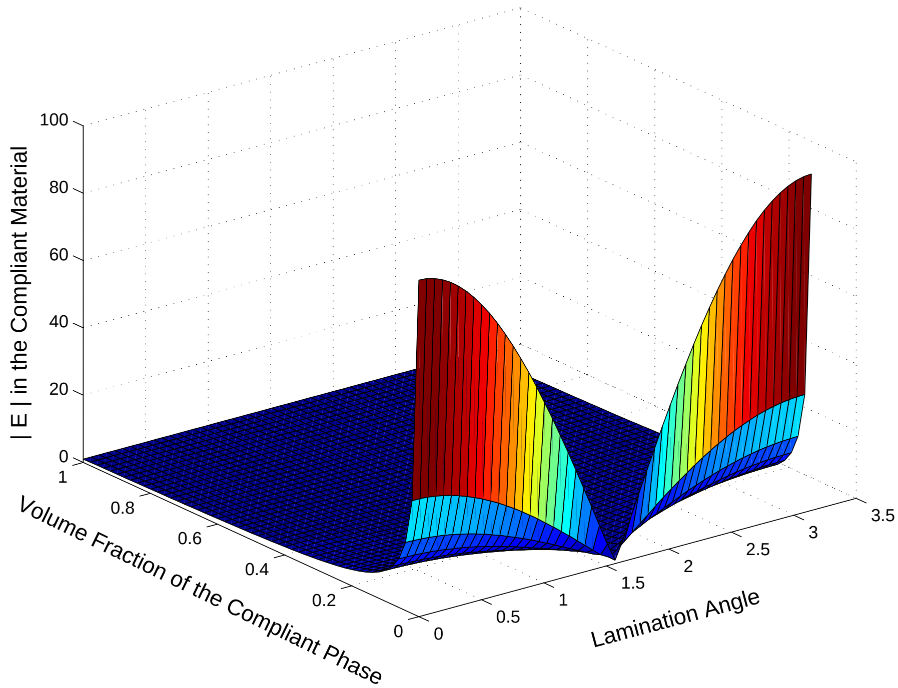


Figure 5.5: The norm of the electric field in the compliant material for rank-1 laminates

electric field in the stiff material is approximately

$$\frac{h^c}{\lambda^s h^c + \lambda^c h^s} \sim \frac{1}{\lambda^c \frac{h^s}{h^c} + \lambda^s}. \tag{5.22}$$

Again, because of the large value of $\frac{h^s}{h^c}$, this term almost contributes nothing to the electric field in the stiff material. Thus the electric field in the stiff material is always smaller than the average electric field.

In the expression of the effective coupling (5.18), the second term is

$$Q = \lambda^c A^c E^c E^c + \lambda^s A^s E^s E^s. \tag{5.23}$$

This is the average of the Maxwell stress. As we have shown, the electric field in the compliant material increases significantly when λ^c is small. However, since there are two small factors in front of the product of $E^c E^c$, the average Maxwell stress behaves the same way as the effective compliance modulus. Figure 5.7 shows the 11

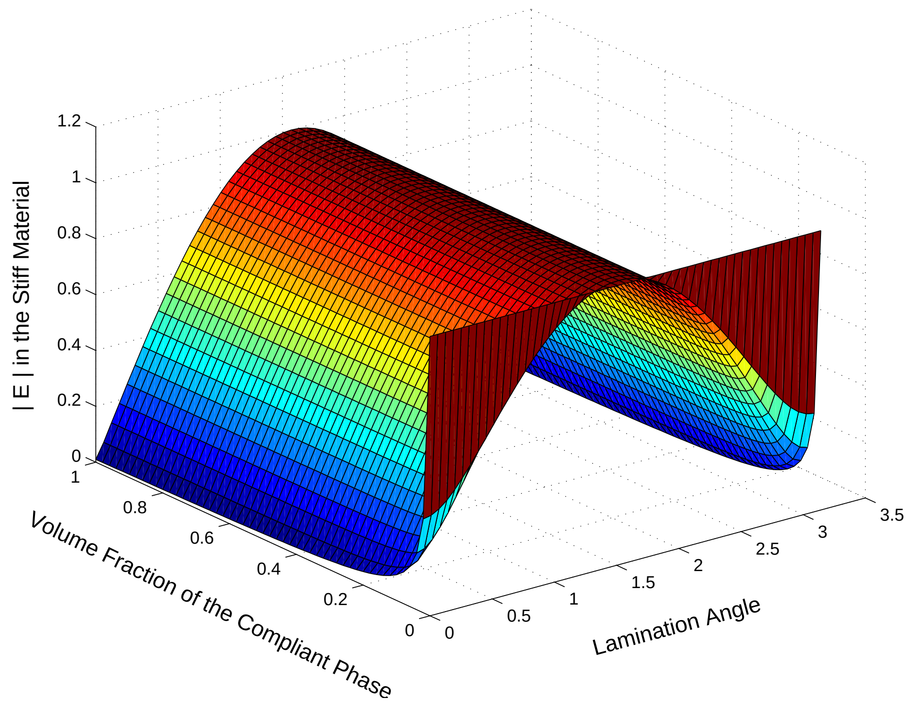


Figure 5.6: The norm of the electric field in the stiff material for rank-1 laminates

component of the average Maxwell stress and the 1111 component of the effective compliance. Clearly the average Maxwell stress and the elastic modulus behave in the same way qualitatively. Physically, when λ^c becomes smaller, the Maxwell stress becomes bigger. But the whole material becomes stiffer. Therefore, this part of the contribution to the effective longitudinal strain does not change very much with the volume fraction of the compliant material.

The first term in (5.18) is the contribution from the fluctuation of the Maxwell stress as we can see from its equation (5.9) (or equation (5.11)). Basically, the difference of the Maxwell stress in the two materials provides a stress for the small scale deformation in the laminate. The force in the compliant material on any surface with normal vector n is

$$-\lambda^s\big(A^c E^c E^c - A^s E^s E^s\big)n. \tag{5.24}$$

Let η be the angle between this force and the normal vector n, Figure 5.8 gives the norm and the angle η of the force as a function of the lamination angle θ and the

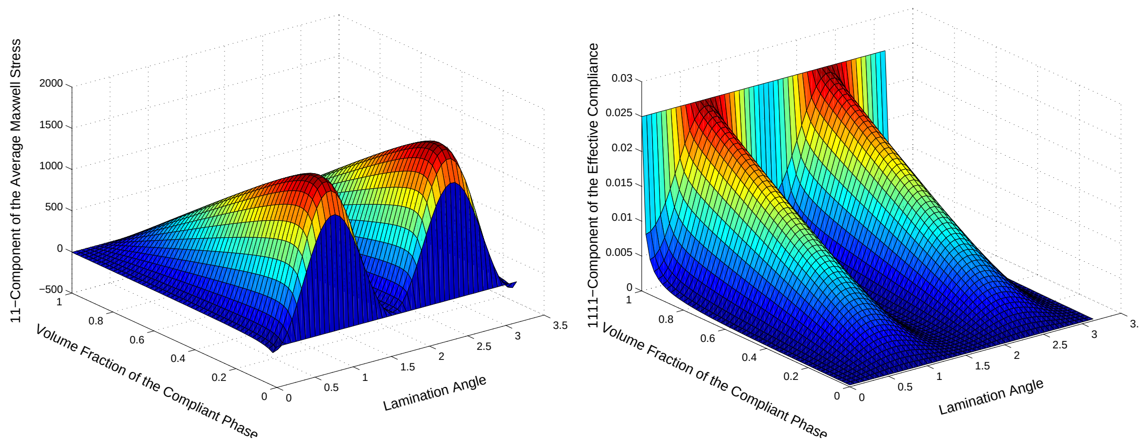


Figure 5.7: The 11 component of the average Maxwell stress and the 1111 component of the effective compliance in rank-1 laminates.

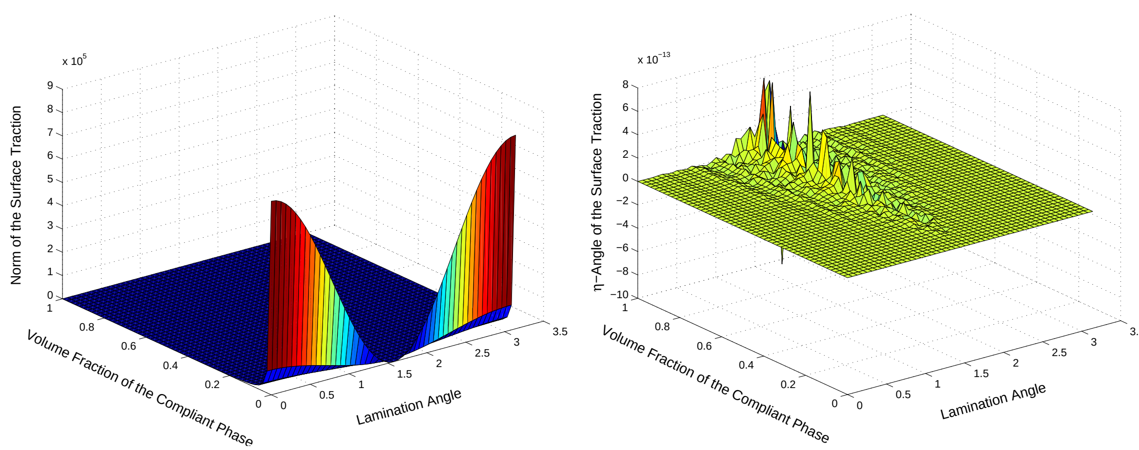


Figure 5.8: The norm and the angle of the surface traction in rank-1 laminates.

volume fraction λ^c. In the first figure, the force becomes very large when λ^c is small. This is natural, because now we do not have a small factor λ^c in front of the big product $E^c E^c$. However, in the rank-1 laminates, the small scale deformation caused by this force turns out to be very small. The answer lies in the angle η of the force.

From the second picture of Figure 5.8, the direction of the force is exactly the same with the normal vector of the interface. Actually,

$$
\begin{aligned}
& A^c E^c E^c n \\
= \;& h^c E^c \otimes E^c n = h^c E^c \left(E^c\right)^T n \\
= \;& h^c \left(E^c \cdot n\right) \left(I - nn^T + a\, nt^T + c\, nn^T H^s\right) \bar{E} \\
= \;& h^c \left(E^c \cdot n\right) \bar{E} + h^c \left(E^c \cdot n\right) \left(a\, t^T \bar{E} + c\, n^T H^s \bar{E} - n^T \bar{E}\right) n \\
= \;& h^c h^s c \left(n \cdot \bar{E}\right) \bar{E} + h^c \left(E^c \cdot n\right) \left(a\, t^T \bar{E} + c\, n^T H^s \bar{E} - n^T \bar{E}\right) n.
\end{aligned} \tag{5.25}
$$

Similarly,

$$
\begin{aligned}
& A^s E^s E^s n \\
= \;& h^s E^s \otimes E^s n = h^s E^s \left(E^s\right)^T n \\
= \;& h^s \left(E^s \cdot n\right) \left(I - nn^T + a\, nt^T + c\, nn^T H^c\right) \bar{E} \\
= \;& h^s \left(E^s \cdot n\right) \bar{E} + h^s \left(E^s \cdot n\right) \left(a\, t^T \bar{E} + c\, n^T H^c \bar{E} - n^T \bar{E}\right) n \\
= \;& h^s h^c c \left(n \cdot \bar{E}\right) \bar{E} + h^s \left(E^s \cdot n\right) \left(a\, t^T \bar{E} + c\, n^T H^c \bar{E} - n^T \bar{E}\right) n.
\end{aligned} \tag{5.26}
$$

Deducting (5.26) from (5.25), we end up a vector along n direction. Now the force is along n direction, but the first equation in (5.11) says that there is no strain in t direction. In addition, the material is incompressible. Putting these together, it is difficult for this big force to contribute anything to the strain in the compliant material and the stiff material.

Thus, in the rank-1 laminate, the main contribution of the longitudinal strain comes from the product of the average Maxwell stress and the effective compliance modulus. From the above analysis, it does not depend on the volume fraction very much, but depends on the direction of the interface. The contribution from the small scale oscillation is small. Now, look at the rank-2 laminate. In this laminate, one of the materials is the rank-1 composite with 2% stiff material. This laminate is soft and has a low dielectric constant. Thus, we still have high contrast between the dielectric

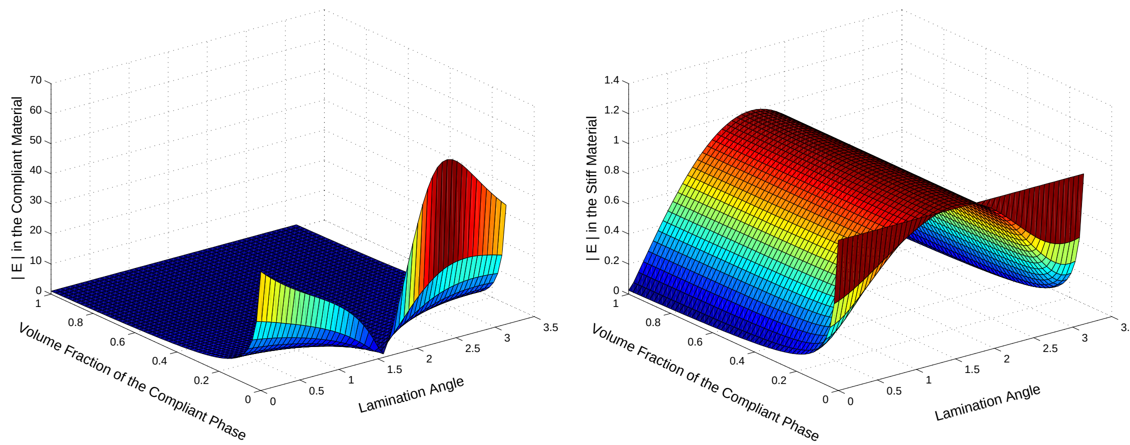


Figure 5.9: The norm of the electric field in the compliant material and the stiff material for rank-2 laminates.

modulus of the two materials. Therefore, as shown in Figure 5.9, the norm of the electric field in the compliant material and the stiff material still behaves the same way as in the rank-1 laminates. So are the average Maxwell stress and the effective compliance (Figure 5.10). The contribution of the average Maxwell stress term to the longitudinal strain is shown in Figure 5.11. The qualitative property of this part is the same as that in the rank-1 composites.

However, the contribution from the small scale oscillation is quite different in the rank-2 laminates. Figure 5.12 shows the norm and the angle η of the force on the interface in rank-2 laminates. From the first figure, the norm of the surface traction increases significantly when λ^c is small just as in the rank-1 laminates. However, the direction of the force is no longer the same as in the rank-1 case because of two reasons. First, the dielectric constant is no longer isotropic for the rank-1 laminate. Second, the electromechanical coupling for the rank-1 laminate is not related to its dielectric constant anymore. Now, if we choose the right lamination angle, there is a big shear stress in the compliant material and the compliant material will generate a large shear strain under this big shear stress. Thus the fluctuation of the Maxwell stress creates a small scale deformation field in the laminates. This is a major contribution to the longitudinal strain.

Clearly, in the rank-2 laminate, between the two terms composing the effective

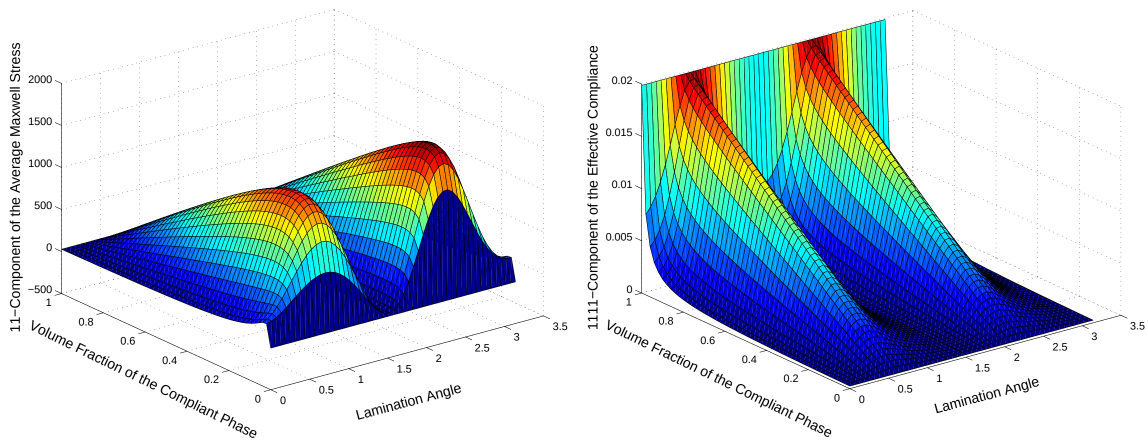


Figure 5.10: Average Maxwell stress and effective compliance in rank-2 laminates.

longitudinal strain, the contribution from the small scale oscillation is much larger than that from the average Maxwell stress. But to generate the small scale deformation, using a compliant material with an anisotropic dielectric constant is very crucial. We did not get a big increase in the longitudinal strain in the rank-1 laminate, but we create a compliant material with anisotropic electric property, which has a relatively bigger component in the direction of the external electric field. This reminds us to use an anisotropic material in the rank-1 laminates. Actually, if we use a compliant material with dielectric constant as $H^{(1)}_{11} = 10\epsilon_0$, $H^{(1)}_{22} = 20\epsilon_0$, the longitudinal strain for the rank-1 laminates is shown in Figure 5.13. We gain a strain as large as 12.5952% when $\lambda^c_1 = 0.02, \theta_1 = 0.78\pi$ in this case. Of course, it is very crucial to choose the right lamination angle as well as the volume fraction.

In the previous rank-2 material, the ratio of the stiff material is about 96%. It is not a practical material to use. However, this composite can be used as a stiff material to laminate with the pure compliant material again. The longitudinal strain for such rank-3 laminates is shown in Figure 5.14. Again the longitudinal strain does not depend on the volume fraction significantly for the same reason as in the rank-1 laminates and we gain a longitudinal strain of 10.241% at $\lambda^c_3 = 0.9$ and $\theta_3 = 0.52\pi$. This material is a compliant material.

Now, laminate this compliant material with the pure stiff material again, and we

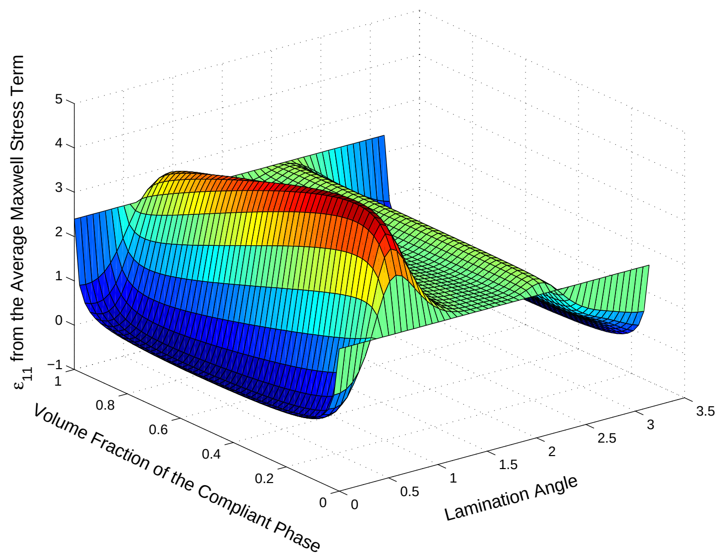


Figure 5.11: Contribution of the average Maxwell stress term to the effective longitudinal strain

get rank-4 laminates. The effective longitudinal strain is shown in Figure 5.15. Again, for the same reason as in rank-2 laminates, a dramatic increase in the longitudinal strain is achieved at $\lambda_4^c = 0.06$, $\theta_4 = 0.16$. With the strain increased to 29.632%, that is 13.4 times of the strain generated by the pure materials. This material is a stiff material again. And we can laminate it with the pure compliant material. If we keep doing this alternate lamination, the effective longitudinal stain increases very quickly. Figure 5.16 shows the strain as a function of rank n. The stars are the effective longitudinal strain as a function of the rank n, the bottom line is the function $f(n) = 3^{\frac{n+1}{2}}$. It appears that the strain becomes unbounded with increasing n.

The alternate lamination as shown in Figure 5.17 is very crucial in this process. If we keep laminating the rank-n composite with the compliant material, we will not get any increase of the strain after several steps. The reason is as follows. The effective longitudinal strain depends very much on the fluctuation of the electric field, which is generated by the high contrast of the dielectric constants. If we keep doing lamination with the compliant material, we will end up laminating two compliant materials with low dielectric constants eventually. Therefore we do not get the fluctuation of the

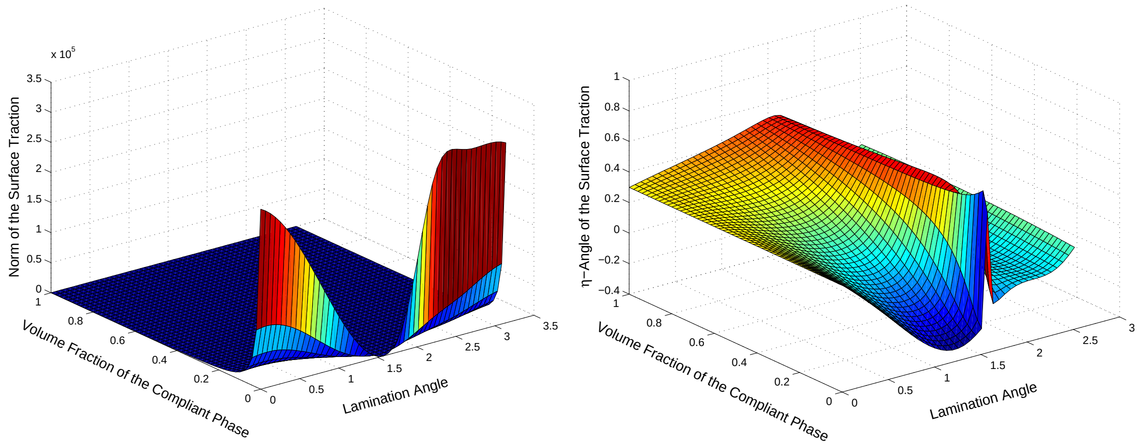


Figure 5.12: The norm and the angle of the surface traction in rank-2 laminates.

electric field any more.

During the alternate lamination, the same pattern occurs. Whenever we laminate the hard composite material with the pure compliant material, the effective longitudinal strain behaves similarly to the rank-3 laminates. It depends significantly on the lamination angle, but little on the volume fraction. Whenever we laminate the soft composite material with the pure stiff material, the effective longitudinal strain behaves similarly to rank-2 laminates. We gain a dramatic increase in the strain, and it depends both on the volume fraction and the lamination angle.

This exciting potentially unbounded strain is only a theoretical result. In practice, it is hard to make multirank laminates. Besides, there is always a breakdown field for each material. From the above analysis, we gain a very high longitudinal strain because a very high electric field is generated inside the compliant material. Actually, in the above rank-2 laminate, the electric field in the compliant material is about 10 times of the external electric field. Therefore, the effective breakdown field for the composite decreases. In the above process, each time we gain a dramatic increase of about 300% in the effective longitudinal strain, the breakdown field decreases by 1/10.

Clearly, if both materials have their own breakdown fields, no matter how we construct the microstructure, the average longitudinal strain can never exceed the

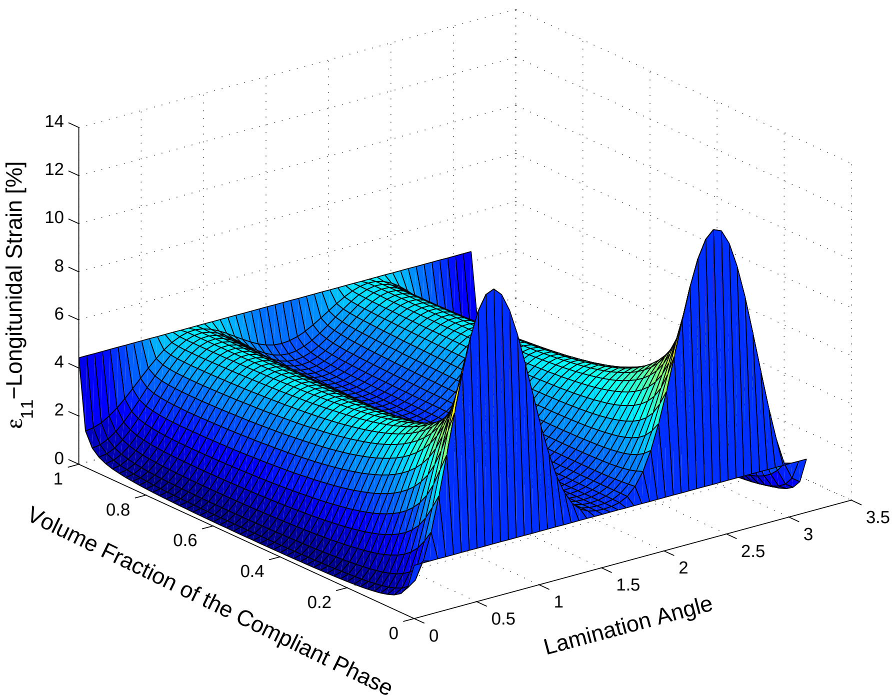


Figure 5.13: Effective longitudinal strain of the rank-1 laminates composed by an isotropic stiff material and an anisotropic compliant material.

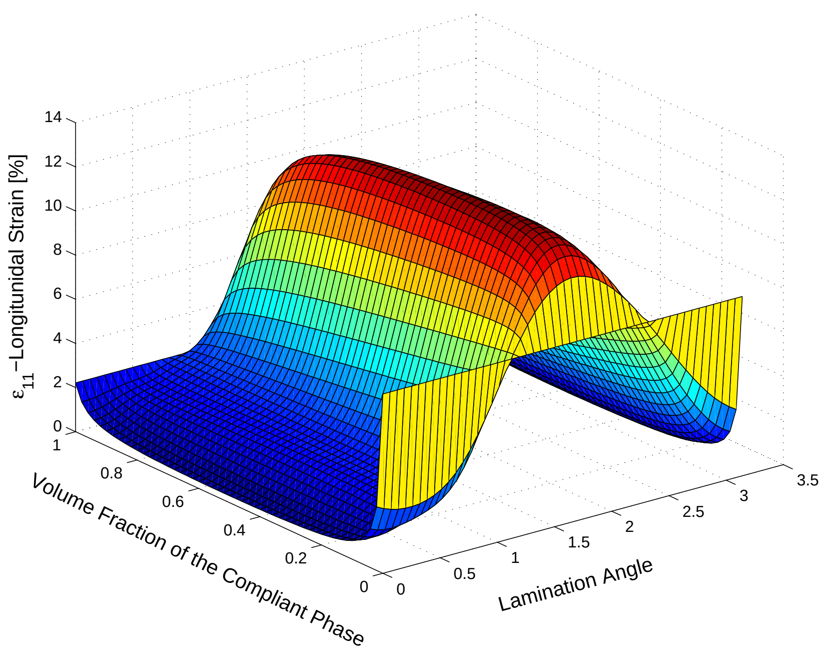


Figure 5.14: The effective longitudinal strain for rank-3 laminates

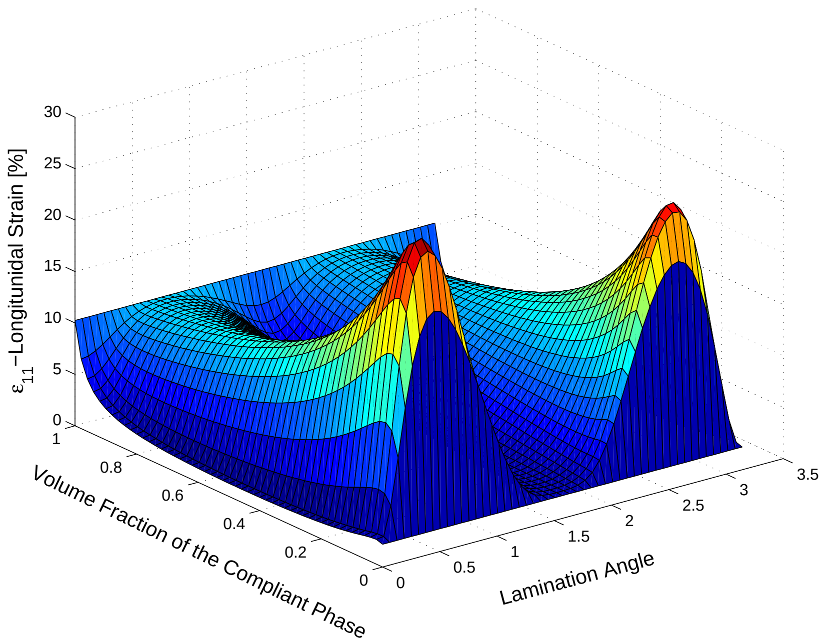


Figure 5.15: The effective longitudinal strain for rank-4 laminates

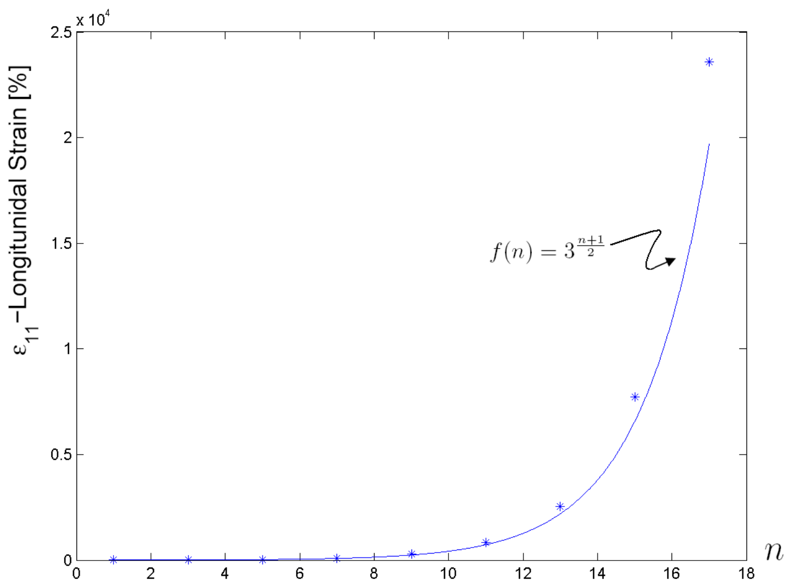


Figure 5.16: The longitudinal strain as a function of the rank.

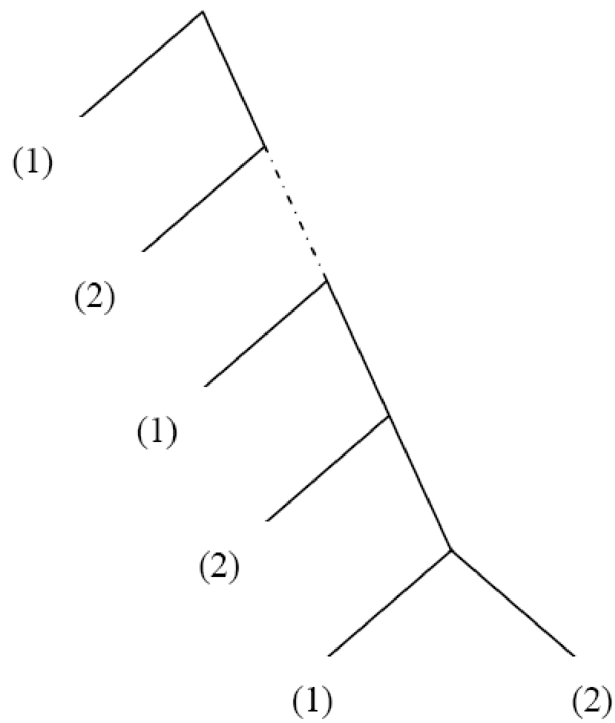


Figure 5.17: Alternate lamination to get an infinite strain

maximum value of the longitudinal strains that the two materials can achieve under their own breakdown fields.

To illustrate the importance of the high contrast of the dielectric constants, we compute two extreme cases. In the first one, we choose two materials with $h^{(1)} = 10\varepsilon_0$, $h^{(2)} = 10h^{(1)}$ and $\mu^{(1)} = 10\,\text{MPa}, \mu^{(2)} = 10\mu^{(1)}$. Figure 5.18 gives the longitudinal strain for the corresponding rank-2 laminates. We get a maximum strain only to 2.4571%.

In the second case, we laminate two materials with $h^{(1)} = 10\varepsilon_0, h^{(2)} = 1000h^{(1)}$ and $\mu^c = 10\,\text{MPa}, \mu^{(2)} = 1000\mu^{(1)}$. The effective strain is shown in Figure 5.19. In rank-1 laminates, the maximum strain is 2.48%, achieved at $\lambda_1^c = 0.52, \theta_1 = 0.34\pi$. In rank-2 laminates, the maximum strain is 38.3807%, achieved at $\lambda_2^c = 0.02, \theta_2 = 0.2\pi$ and $\lambda_1^c = 0.98, \theta_1 = 0.66\pi$.

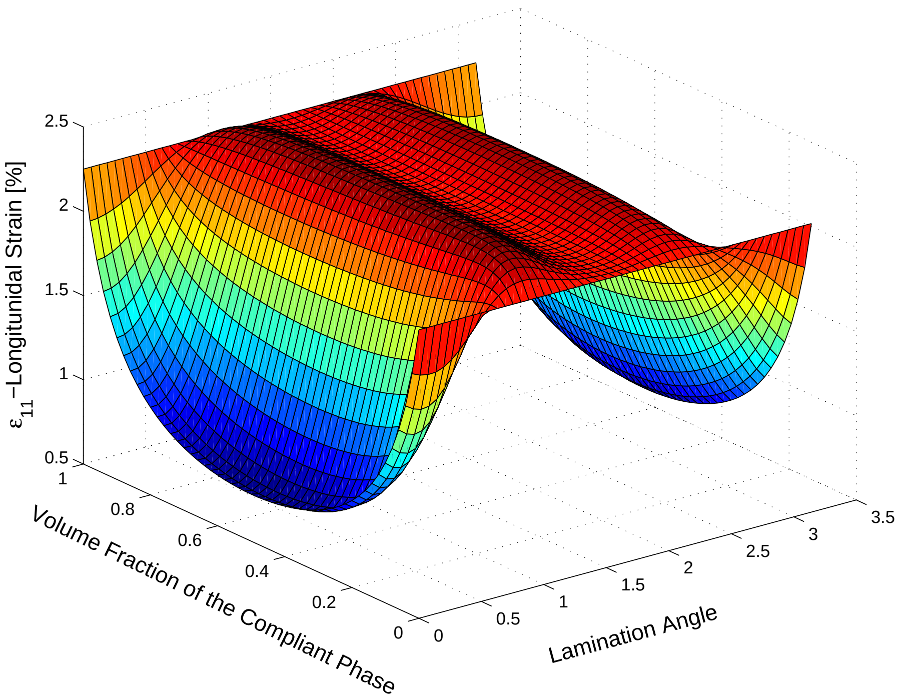
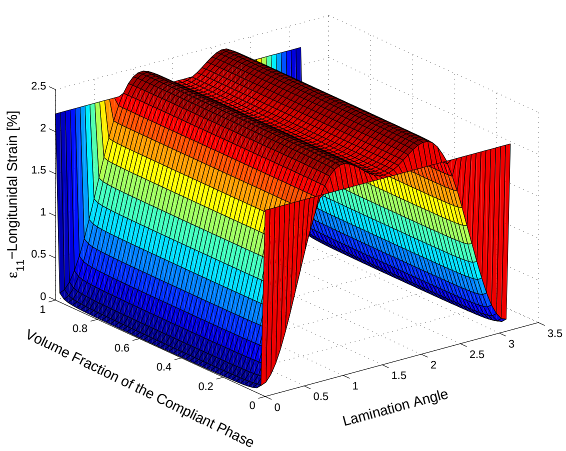
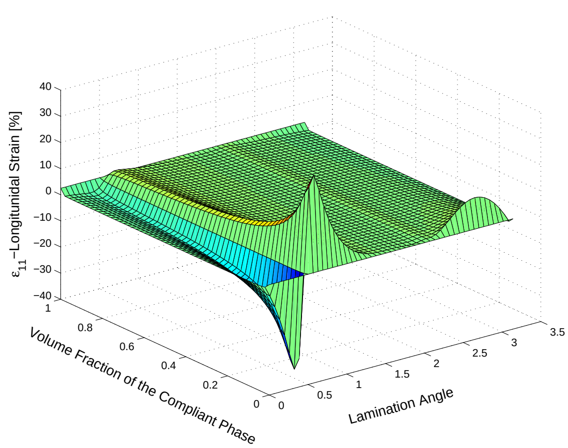


Figure 5.18: The longitudinal strain of rank-2 laminates with low contrast of dielectric modulus.



(a)



(b)

Figure 5.19: The longitudinal strain of the rank-1 and rank-2 laminates with high contrast of dielectric modulus.

Chapter 6

Numerical Results for Particulate Composites

In Chapter 4, we derived the effective property for the dielectric composites with periodic microstructure. (4.45) and (4.46) give the effective elastic modulus and the effective eletromechanical coupling coefficient. If we apply a 100 MV/m external electric field in $[1,0]$ direction on the composites, the effective longitudinal strain in $[0,1]$ direction is

$$C^{-1}_{22ij}\, A^H_{ij11}. \tag{6.1}$$

From (4.46)

$$A^H_{ij11} = \frac{1}{|Y|}\int_Y B_{ij11}(y) + C_{ijlm}(y)\frac{\partial \tilde{\chi}^{11}_l}{\partial y_m}\, dy, \tag{6.2}$$

where

$$B_{ij11} = A_{ij11} - A_{ijl1}\frac{\partial \hat{\chi}_1}{\partial y_l} - A_{ij1l}\frac{\partial \hat{\chi}_1}{\partial y_l} + A_{ij\alpha\beta}\frac{\partial \hat{\chi}_1}{\partial y_\alpha}\frac{\partial \hat{\chi}_1}{\partial y_\beta}. \tag{6.3}$$

As we can see from (4.19), the physical meaning of $\hat{\chi}_1$ is the small scale electric field in the unit cell generated by the oscillation of the dielectric constant if the average electric field is $[1,0]$. Thus from (4.40), B_{ij11} is the distribution of the Maxwell stress in the unit cell if the average electric field is $[1,0]$. $\tilde{\chi}^{11}$ is then the deformation field in the unit cell caused by the oscillation of the Maxwell stress B_{ij11} as we can see from (4.43). Therefore, A^H_{ij11} is composed of two terms; the first term is the average Maxwell stress in the unit cell, the second term is the contribution from the small scale oscillation of the Maxwell stress.

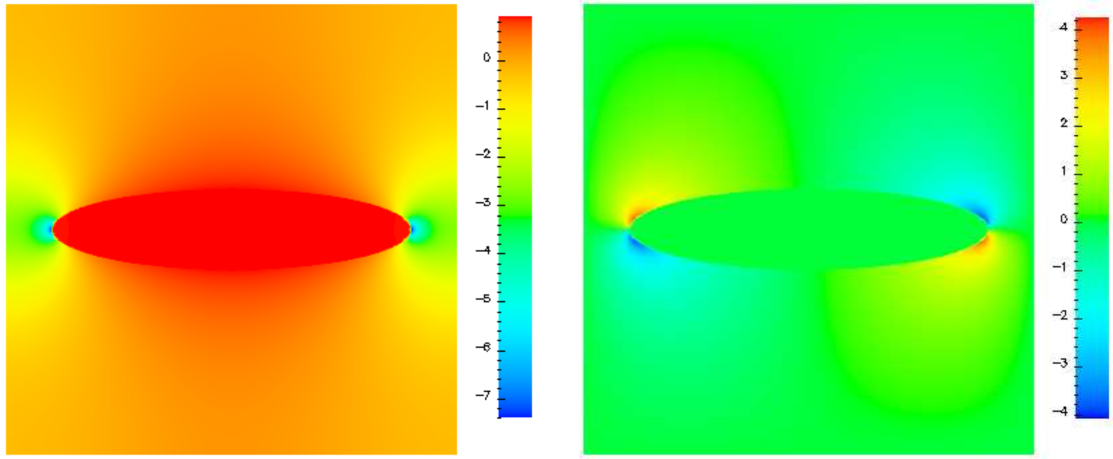
Before the more general computation of particulate composites, we use this formula to compute the effective strain of the lamination geometry. Numerical experiments show that the results given by the finite element computation match with the results in chapter 5 for at least 6 digits. This provides a simple test for our finite element program.

Consider a unit cell $[0,1]\times[0,1]$ and a composite with a microstructure as follows. In the unit cell, the stiff material inclusion is surrounded by the compliant material. The stiff inclusion has a geometry of ellipsoid. The properties of the two materials are similar to those in the laminate computation. We take

$$\begin{aligned} h^{(1)} &= 10\varepsilon_0, & h^{(2)} &= 1000\varepsilon_0; \\ \alpha^{(1)} &= 5000\,\mathrm{MPa}, & \alpha^{(2)} &= 5000\,\mathrm{MPa}; \\ \mu^{(1)} &= 10\,\mathrm{MPa}, & \mu^{(2)} &= 1000\,\mathrm{MPa}\,. \end{aligned} \tag{6.4}$$

Choose the axis lengths of the ellipsoid as $a = 0.40$, $b = 0.09$. The volume fraction for the stiff material is about $\lambda^{(2)} = 0.1131$. By the numerical computation, the effective longitudinal strain is 3.31%. Compared with the 2.209% strain of the pure compliant material, this gives an almost 50% increase. Figure 6.1 shows the electric field caused by the oscillation of the dielectric constant in the unit cell. Figure 6.1(a) is the first component of the electric field. Figure 6.1(b) is the second component of the electric field. Clearly, the oscillation of the dielectric constant causes a fluctuation of the electric field in the unit cell. The electric field in the stiff material is small. However, there are two hot spots near the ends of the ellipsoid in the compliant material. Because of this high electric field, the compliant material near that area is squeezed significantly, as we can see from (a) of Figure 6.2. At the same time, because the material is incompressible, it is then hardly stretched in the $[0,1]$ direction as shown in (b) of Figure 6.2.

Keep the volume fraction and increase the length ratio of the axis of the ellipsoid. Let $a = 0.48, b = 0.075$, we get an effective longitudinal strain of 7.29%. It is more than three times of the original strain. This means the longer the long axis of the

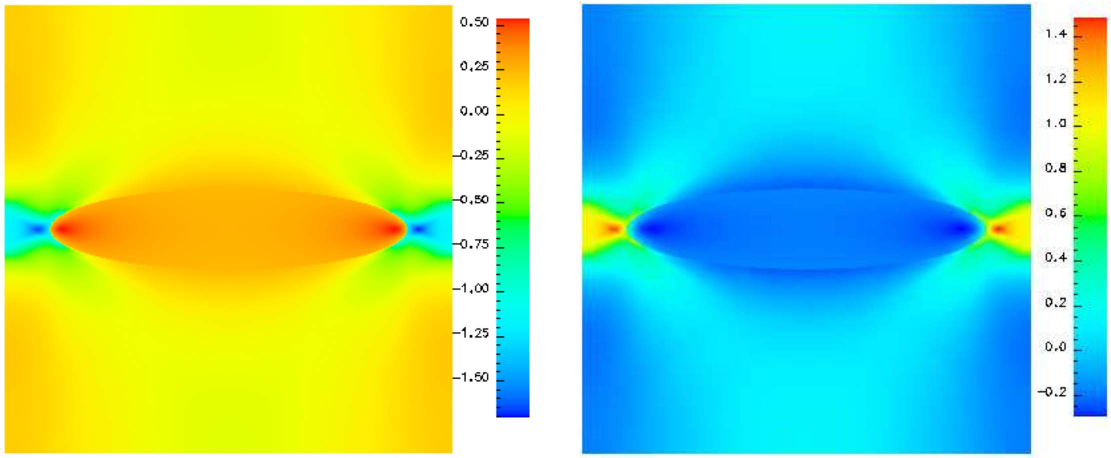


(a) (b)

Figure 6.1: The electric field caused by the oscillation of the dielectric constant.

ellipsoid, the bigger the effective longitudinal strain. The reason behind this can be seen very clearly from Figure 6.3 and 6.4. In Figure 6.3, the longer the long axis of the ellipsoid, the narrower the region between every two ellipsoids, the higher the electric field in that region due to the same reason as shown in (5.21) and (5.22). In Figure 6.4, the higher the electric field in the compliant material, the harder the soft material is squeezed in $[1, 0]$ direction and stretched in $[0, 1]$ direction. Thus we get a larger effective longitudinal strain.

Another factor that affects the longitudinal strain significantly is the distance between the inclusions. Figure 6.5 and 6.6 is the electric field and the deformation field in the unit cell if the distance between two $a = 0.48, b = 0.075$ ellipsoids is $\frac{1}{3}$. We get a longitudinal strain of only 5.6% in this case. The reason is as follows: when the distance of every two ellipsoids becomes smaller, the material becomes very hard to be squeezed. If we look at (b) of Figure 6.6, the compliant material between the two ellipsoids is being pushed up and pulled down simultaneously, these two effects cancel each other and weaken the squeezing effect. Table 6.1 is the longitudinal strain for different distances between the ellipsoids. The strain increases with respect to the distance at first, then goes down at a certain distance. This is reasonable, because once the distance is too large, part of the compliant material between the ellipsoids is out of the squeezing zone. Thus the longitudinal strain will decrease with the increase



(a) (b)

Figure 6.2: ϵ_{11} and ϵ_{22} caused by the fluctuation of the Maxwell stress.

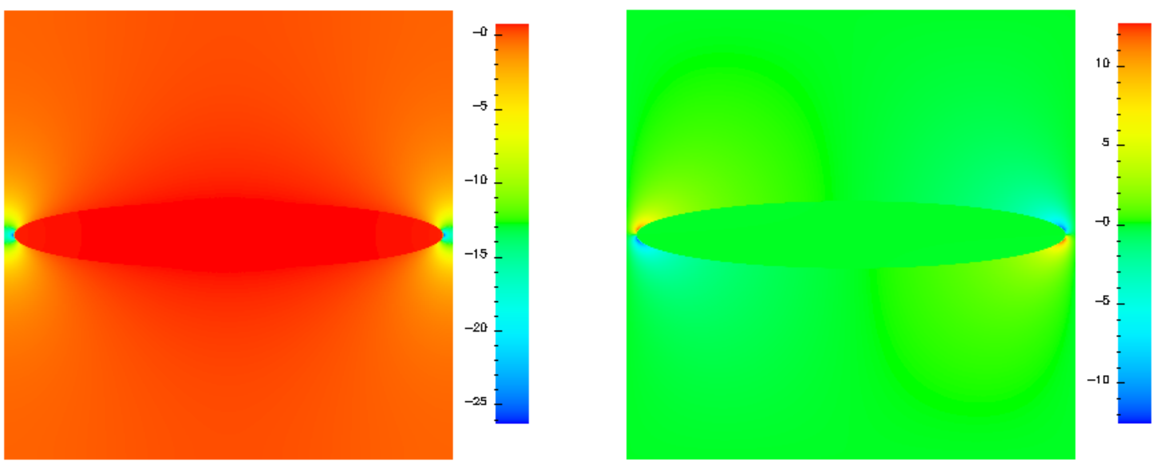


Figure 6.3: The electric field caused by the oscillation of the dielectric constant.

of the distance after that.

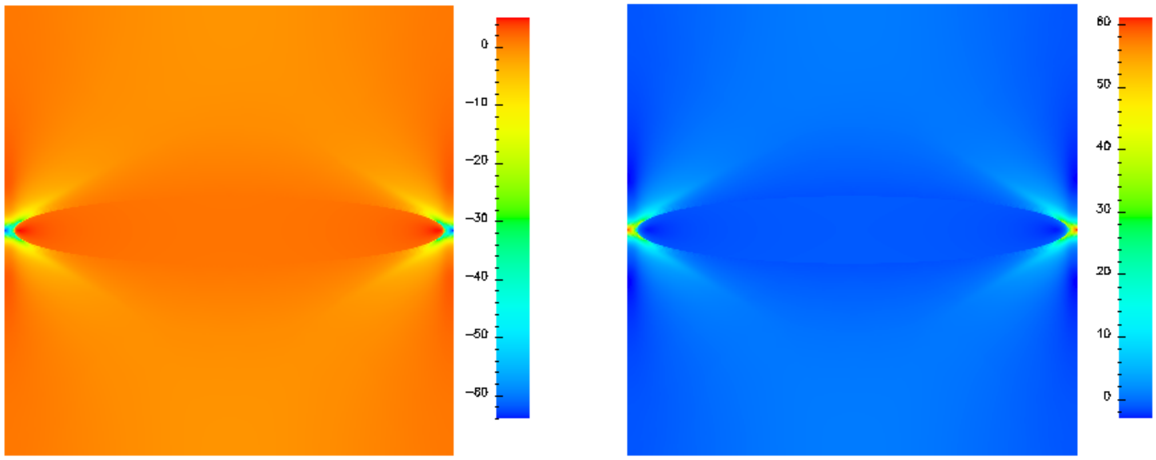


Figure 6.4: ϵ_{11} and ϵ_{22} caused by the fluctuation of the Maxwell stress.

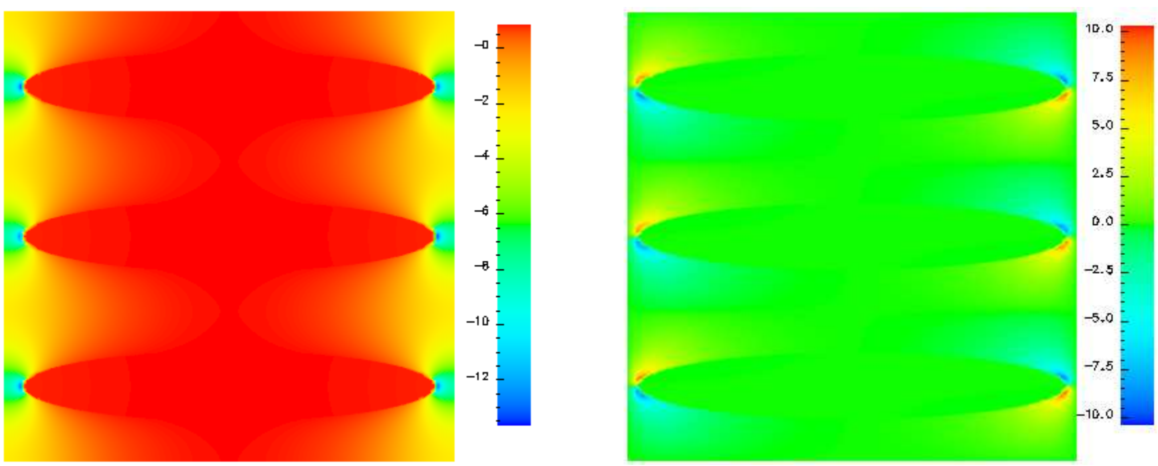
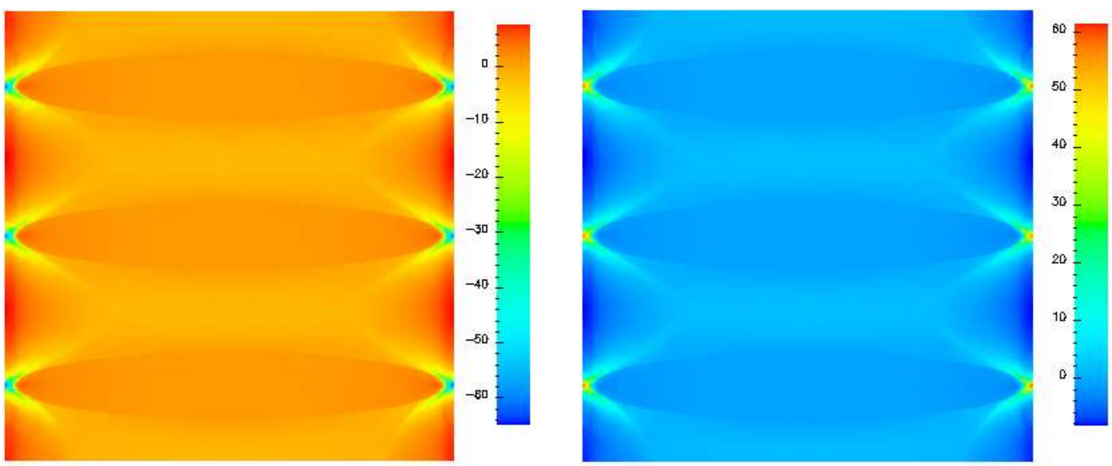


Figure 6.5: The electric field caused by the oscillation of the dielectric constant.

distance	0.25	0.333	0.5	0.7	1.0	1.2
strain	3.75	5.60	5.90	6.72	7.29	6.34

Table 6.1: The longitudinal strain as a function of the distances between inclusions



(a) (b)

Figure 6.6: ϵ_{11} and ϵ_{22} caused by the fluctuation of the Maxwell stress.

Chapter 7

Conclusion

Electroactive polymers (EAPs) offer unique capabilities in the form of flexible and lightweight actuators. However, current EAPs suffer from either high operation field (such as SRI dielectric elastomer) or inadequate strain level (such as the electrostrictive PVDF terpolymers, transverse strain of $\sim 5\%$). Recent experimental work suggests that this limitation can be overcome by making composites of flexible and high dielectric modulus materials. One approach is to make electroactive polymer composites (EAPC) by combining an elastomer with a high dielectric or even conductive material [43, 22, 23]. This thesis provides a theoretical study of such composites.

The first accomplishment is the rigorous derivation in Chapter 3 of an approximate model of deformable dielectric EAPs which preserves the quadratic elelctrostrictive coupling between strain and electric field. We start from a finite deformation model of a deformable dielectric EAPs [37] and perform an asymptotic development under the assumption that the strain is of order ε and the electric field is of order $\varepsilon^{\frac{1}{2}}$. We show that the leading order energy is the electrostatic energy assuming no deformation. The next order correction is the elastic energy under the forcing induced by the electrostatic fields through the Maxwell stress. The analysis is carried out using the notion of Γ-convergence and Γ-development.

The resulting model is described by two equations. The first is the classical equation of electrostatics. The second is linear elastostatics where the forcing is determined by the solution to the first. Notice that the coupling is one way: the electric field affects the deformation but not the other way. Further, both equations are linear; however the forcing term in the elastostatic equation depends nonlinearly on the electric field. Finally, the result shows why the electrostrictive coupling is necessarily weak: the leading order energy is independent of deformation.

The second accomplishment of the thesis is a rigorous periodic homogenization theory in Chapter 4. The analysis used the two-scale convergence method developed by Allaire [3]. Recall that the elastic forcing depends quadratically on the electric field. Consequently the overall deformation depends on the square of the electric fluctuation. This can be very large in a heterogeneous material and accounts for the unexpected experimental observations. This result is consistent with that of Li [27, 26] and [15].

This result is complemented with a study of laminate composites. Numerical computations and analysis reveal several important facts about the laminate composites. First, a larger longitudinal strain can be obtained by laminating compliant material and stiff material with high dielectric contrasts. Second, between the two terms composing the effective electromechanical coupling, the contribution from the oscillation of the Maxwell stress plays a key role to enhance the effective longitudinal strain. The mechanism is as follows. The composites have different dielectric properties which cause the fluctuation of the electric field. A much larger electric field is then generated in the soft material. On the other hand, the electric field together with the Maxwell stress in the hard material are relatively small. Thus we get a big jump of the Maxwell stress between the two materials. This provides a force for a small scale deformation field which contributes significantly to the effective longitudinal strain. While the fluctuation of the electric field provides a mechanism to increase the longitudinal strain, it also decreases the effective breakdown field. Another suggestion is that an infinite longitudinal strain can be achieved by making sequential laminates.

Finally the thesis describes numerical experiments on particulate composites with periodic microstructures. For different geometries, we compute the unit cell solutions, and then the effective longitudinal strain using the effective properties derived in Chapter 4. Numerical results shows that we can get a much larger effective longitudinal strain from fiber like inclusions. The mechanism here is still the oscillation of the electric field as in the laminate case. The difference is that there is a strong squeeze-stretch effect besides the shear stress. This squeeze-stretch effect is not present in the laminate composite and maybe the reason for the larger strain of the particulate composite than the laminate composite.

While this thesis gave a theoretical framework for the dielectric elastomers, it is by no means the end of the story. One promising direction is to incorporate the breakdown field. As we analyzed in Chapter 5, if both materials have their own breakdown fields, no matter how we construct the microstructure, the effective longitudinal strain can never exceed the maximum value of the longitudinal strains that the two materials can achieve under their own breakdown fields. However, this is a very loose bound. Thus the maximum effective longitudinal strain and the optimum microstructure under the constraint of breakdown fields is an interesting and open problem. A second open problem is the validity of the assumptions that lead to the approximate model in this thesis.

Appendix A

Small-strain Model for Piezoelectric Material

A.1 Assumptions and Results

For piezoelectric elastomers, we assume the energy density function W satisfies $\mathcal{A}_1$ and $\mathcal{A}_5$ in Chapter 3 plus the following conditions:

$\mathcal{B}_1$. Assume that there exists a constant c independent of x, F and p_0, such that

$$W(x, I, p_0) \geqslant c p_0 \cdot p_0 \tag{A.1}$$

and

$$W(x, F, 0) \geqslant c\mathrm{dist}(F, \mathrm{SO}(N))^2. \tag{A.2}$$

$\mathcal{B}_2$. Denote

$$\left.\frac{\partial^2 W}{\partial p_0 \partial F}\right|_{\substack{F=I\\ p_0=0}} = A, \qquad \left.\frac{\partial^2 W}{\partial p_0^2}\right|_{\substack{F=I\\ p_0=0}} = H^{-1}, \qquad \left.\frac{\partial^2 W}{\partial F^2}\right|_{\substack{F=I\\ p_0=0}} = C.$$

We assume

$$\frac{1}{2}C(\nabla u)^2 + \frac{1}{2}H^{-1}p_0^2 + A\nabla u\, p_0 \geqslant 0, \tag{A.3}$$

for any ∇u and p_0. In addition, the functional

$$\mathcal{I}_1^0 := \int_\Omega \frac{1}{2}C(\nabla u)^2 + \frac{1}{2}H^{-1}p_0^2 + A\nabla u\, p_0\, dx \tag{A.4}$$

is lower semicontinuous.

$\mathcal{B}_3$. Assume there exist constant ρ_1 and ρ_2 such that for $|p_0| \leqslant \rho_1$, $|G| \leqslant \rho_2$, we have

$$\begin{aligned} &\Big|\, W(x, I+G, p_0) - \frac{H^{-1}}{2} p_0 \cdot p_0 - \frac{1}{2} CGG - AGp_0 \,\Big| \\ \leqslant\ & a(x)\, w_1\big(|G|, |p_0|\big)\, |p_0|^2 + b(x)\, w_2\big(|G|, |p_0|\big)\, |G|^2. \end{aligned}$$

Here $w_1, w_2 \to 0$ as $|p_0| \to 0$, $|G| \to 0$ monotonically, $a(x), b(x) \in \mathcal{L}^1(\Omega)$.

From the scale analysis for piezoelectric material in Section 2.4.2, if both the electric field and the deformation field are of order ε, then the electric energy and the elastic energy are all of order ε^2. We rescale the energy by ε^2 and define

$$\begin{aligned} \mathcal{I}^\varepsilon \ \ &:= \ \ \frac{1}{\varepsilon^2} \int_\Omega W(x, F^\varepsilon, \varepsilon p_0^\varepsilon\,)\, dx + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \big|\nabla_y \varphi^\varepsilon\big|^2\, dy \\ &\quad - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \big|\nabla_y g^\varepsilon\big|^2\, dy + \int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon\, dy, \end{aligned} \tag{A.5}$$

where φ^ε satisfies

$$\begin{cases} \nabla_y \cdot \big[-\varepsilon_0 \nabla_y (\varphi^\varepsilon + g) + p^\varepsilon\, \chi\big(y^\varepsilon(\Omega)\big)\big] = 0 \quad \text{in} \quad \mathbb{R}^N \backslash y^\varepsilon(\Gamma), \\ \varphi^\varepsilon \in \mathcal{D}\big(\mathbb{R}^N \backslash y^\varepsilon(\Gamma)\big). \end{cases} \tag{A.6}$$

Define $\mathcal{I}^0$ to be

$$\begin{aligned} \mathcal{I}^0 \ \ &:= \ \ \int_\Omega \frac{1}{2} C(\nabla u)^2 + \frac{1}{2} H^{-1} p_0 \cdot p_0 + A \nabla u p_0\, dx \\ &\quad + \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \big|\nabla \varphi_0\big|^2\, dx + \int_\Omega \nabla g \cdot p_0\, dx - \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \big|\nabla g\big|^2\, dx, \end{aligned} \tag{A.7}$$

in which φ^0 is the solution of

$$\begin{cases} \nabla_x \cdot \big[\, -\varepsilon_0 \nabla_x (\varphi^0 + g) + p_0\, \chi(\Omega) \,\big] = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Gamma, \\ \varphi^0 \in \mathcal{D}(\mathbb{R}^N \backslash \Gamma). \end{cases} \tag{A.8}$$

Then we have the following theorem:

Theorem A.1 *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_5$ and $\mathcal{B}_1$ to $\mathcal{B}_3$, then the functional $\mathcal{I}^\varepsilon$ Γ-converges to functional $\mathcal{I}^0$ under the norm $\mathcal{W}^{1,t}$ for y^ε and the $\mathcal{L}^2(\Omega)$ weak norm for p_0^ε.*

In addition, we have the following compactness result.

Proposition A.2 *Under the same condition as in Theorem A.1, assume further the material is pined on part of the boundary. Then if the energy functional $\mathcal{I}^\varepsilon$ is bounded, there exists a constant c, such that*

$$\left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{\mathcal{L}^2(\Omega)} < c \quad \textit{and} \quad \left\| p_0^\varepsilon - p_0 \right\|_{\mathcal{L}^2(\Omega)} < c.$$

In addition, we have $F^\varepsilon \to I \,\text{in}\, \mathcal{L}^t(\Omega)$.

As a direct corollary of Theorem A.1 and Proposition A.2, we have

Theorem A.3 *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_5$ and $\mathcal{B}_1$ to $\mathcal{B}_3$. If y^ε, p_0^ε satisfies*

$$\mathcal{I}^\varepsilon(y^\varepsilon, p_0^\varepsilon) \leq \inf_{\substack{z^\varepsilon \in \mathcal{W}^{1,t}(\Omega) \\ q_0^\varepsilon \in \mathcal{L}^2(\Omega)}} \mathcal{I}^\varepsilon(z^\varepsilon, q_0^\varepsilon) + \varepsilon,$$

$u^\varepsilon = \frac{1}{\varepsilon}(y^\varepsilon - x)$ weakly convergent to u in $\mathcal{W}^{1,t}(\Omega)$ and p_0^ε weakly convergent to p_0 in $\mathcal{L}^2(\Omega)$, then

1. $\mathcal{I}^0(u, p_0) \leqslant \mathcal{I}^0(v, q_0), \ \ \forall\, v \in W^{1,2}(\Omega), \ q_0 \in \mathcal{L}^2(\Omega)$;

2. $\lim_{\varepsilon \to 0} \mathcal{I}^\varepsilon(x + \varepsilon u^\varepsilon, p_0^\varepsilon) = \mathcal{I}^0(u, p_0)$.

A.2 The Γ-limit of the energy functional

Proposition A.4 (Lower bound of the energy functional) *Assume Ω satisfies the strong local Lipschitz condition and belongs to class $\mathfrak{B}^{r_0}$. Suppose the energy* *density W satisfies conditions $\mathcal{A}_1$ to $\mathcal{A}_5$ and $\mathcal{B}_1$ to $\mathcal{B}_3$. Then for any sequences $y^\varepsilon \to x$ in $\mathcal{W}^{1,t}(\Omega)$, $\nabla u^\varepsilon = \frac{1}{\varepsilon}\nabla(y^\varepsilon - x) \rightharpoonup \nabla u$ in $\mathcal{L}^2(\Omega)$ and $p_0^\varepsilon \rightharpoonup p_0$ in $\mathcal{L}^2(\Omega)$, we have*

$$\lim_{\varepsilon\to 0} \mathcal{I}^\varepsilon \geqslant \mathcal{I}^0.$$

where $\mathcal{I}^\varepsilon$ is defined in (A.5) and $\mathcal{I}^0$ is defined in (A.7).

Proof In this proof, we take $g^\varepsilon(y)$ as $g^\varepsilon\big(y^\varepsilon(x)\big) = g(x)$, the same as in the proof of Proposition 3.8 and 3.14.

Exactly as in the proof of Proposition 3.14, we still have

$$\int_{y^\varepsilon(\Omega)} \nabla_y g^\varepsilon \cdot p^\varepsilon \, dy \to \int_\Omega \nabla_x g \cdot p_0 \, dx \qquad \text{as } \varepsilon \to 0, \tag{A.9}$$

$$-\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_y g^\varepsilon\big|^2 \, dy \to -\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \big|\nabla_x g\big|^2 \, dx \qquad \text{as } \varepsilon \to 0, \tag{A.10}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \nabla_y \varphi^\varepsilon \cdot \nabla_y \varphi^\varepsilon \, dy \geqslant \int_{\mathbb{R}^N} \nabla_x \varphi^0 \cdot \nabla_x \varphi^0 \, dx. \tag{A.11}$$

The only thing we need to prove is that for any $u^\varepsilon \rightharpoonup u$ in $\mathcal{H}^1(\Omega)$ and $p_0^\varepsilon \rightharpoonup p_0$ in $\mathcal{L}^2(\Omega)$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega W(x, F^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \geqslant \int_\Omega \frac{1}{2} C\nabla u \nabla u + \frac{1}{2} H^{-1} p_0 \cdot p_0 + A\nabla u \, p_0 \, dx. \tag{A.12}$$

Applying the truncation lemma to u^ε with $m = 1$, $M = M_\varepsilon$ yields truncation function v^ε, which satisfies

$$\big|\nabla v^\varepsilon\big| \leqslant \lambda_\varepsilon, \qquad 1 \leqslant \lambda_\varepsilon \leqslant M_\varepsilon,$$

and

$$\lambda_\varepsilon^2 \,\big|Z_u^c\big| \leqslant \frac{c}{\ln(M_\varepsilon)}, \qquad \text{where } Z_u = \big\{\, x : u^\varepsilon(x) = v^\varepsilon(x) \,\big\}.$$

Now, define another set Z_p for some $Q_\varepsilon \to +\infty$ to be decided later by,

$$Z_p = \Big\{\, x \in \Omega : |p_0^\varepsilon(x)| \leqslant \frac{1}{2} Q_\varepsilon \,\Big\}.$$

Denote by Z_u^c, Z_p^c the complements of Z_u, Z_p in Ω respectively.

Assume

$$Q_\varepsilon < \rho_1 \varepsilon^{-1}, \qquad M_\varepsilon < \rho_2 \varepsilon^{-1},$$

then

$$\begin{aligned} \left|\varepsilon \nabla u^\varepsilon\right| = \left|\varepsilon \nabla v^\varepsilon\right| \leqslant \varepsilon M_\varepsilon < \rho_2 \qquad & \text{on set } Z_u, \\ \left|\varepsilon p_0^\varepsilon\right| \leqslant Q_\varepsilon \varepsilon < \rho_1 \qquad & \text{on set } Z_p. \end{aligned}$$

From assumption $\mathcal{B}_3$

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_\Omega W(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \\ = \; & \frac{1}{\varepsilon^2} \int_{Z_u \cap Z_p} W(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \\ & + \frac{1}{\varepsilon^2} \int_{Z_u^c \cup Z_p^c} W(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \\ \geqslant \; & \int_{Z_u \cap Z_p} \frac{1}{2} C \nabla v^\varepsilon \nabla v^\varepsilon + \frac{1}{2} H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon + A \nabla v^\varepsilon p_0^\varepsilon \, dx \\ & - \int_{Z_u \cap Z_p} a(x)\, w_1\big(\varepsilon |\nabla v^\varepsilon|, \varepsilon |p_0^\varepsilon|\big)\, |p_0^\varepsilon|^2 + b(x)\, w_2\big(\varepsilon |\nabla v^\varepsilon|, \varepsilon |p_0^\varepsilon|\big)\, |\nabla v^\varepsilon|^2 \, dx \\ \geqslant \; & \int_{Z_u \cap Z_p} \frac{1}{2} C \nabla v^\varepsilon \nabla v^\varepsilon + \frac{1}{2} H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon + A \nabla v^\varepsilon p_0^\varepsilon \, dx \\ & - c\, \big(w_1(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, Q_\varepsilon^2 + w_2(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, M_\varepsilon^2 \big). \end{aligned}$$

Choose

$$\begin{aligned} M_\varepsilon &= \min\big\{ \varepsilon^{-\frac{1}{2}}, w_2(\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}})^{-\frac{1}{4}}, |Z_p^c|^{-\frac{1}{4}} \big\}, \\ Q_\varepsilon &= \min\big\{ \varepsilon^{-\frac{1}{2}}, w_1(\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}})^{-\frac{1}{4}} \big\}, \end{aligned}$$

then $w_1(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, Q_\varepsilon^2 + w_2(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, M_\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

Thus we get

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega W(x, I+\varepsilon\nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\,dx \\
\geqslant\ &\int_\Omega \frac{1}{2}C\nabla v^\varepsilon\nabla v^\varepsilon\,dx + \int_{Z_u\cap Z_p}\frac{1}{2}H^{-1}p_0^\varepsilon\cdot p_0^\varepsilon\,dx \\
&+\int_{Z_u\cap Z_p} A\nabla v^\varepsilon p_0^\varepsilon\,dx - \int_{Z_u^c\cup Z_p^c}\frac{1}{2}C\nabla v^\varepsilon\nabla v^\varepsilon\,dx.
\end{aligned}
$$

Observe that

$$
\int_{Z_p^c}\frac{1}{2}C\nabla v^\varepsilon\nabla v^\varepsilon\,dx \leqslant c\left|Z_p^c\right|M_\varepsilon^2 \leqslant c\left(|Z_p^c|\right)^{\frac{1}{2}} \to 0, \qquad \text{as } \varepsilon\to 0,
$$

and

$$
\int_{Z_u^c}\frac{1}{2}C\nabla v^\varepsilon\nabla v^\varepsilon\,dx \leqslant c\,\lambda_\varepsilon^2\left|Z_u^c\right| \leqslant \frac{c}{\ln M_\varepsilon} \to 0.
$$

On the other hand, we have $p_0^\varepsilon\,\chi(Z_u\cap Z_p)\rightharpoonup p_0$. Actually for any $h(x)\in(\mathcal{L}^2(\Omega))^N$,

$$
\begin{aligned}
&\int_\Omega h(x)\cdot p_0(x)\,dx = \lim_{\varepsilon\to 0}\int_\Omega h(x)p_0^\varepsilon(x)\,dx \\
=\ &\lim_{\varepsilon\to 0}\int_{Z_u\cap Z_p} p_0^\varepsilon(x)\cdot h(x)\,dx + \lim_{\varepsilon\to 0}\int_{Z_p^c\cup Z_u^c} p_0^\varepsilon(x)\cdot h(x)\,dx,
\end{aligned}
$$

$$
\int_{Z_p^c\cup Z_u^c} p_0^\varepsilon(x)\cdot h(x)\,dx \leqslant \left\|p_0^\varepsilon\right\|_{L^2(\Omega)}\left\|h(x)\right\|_{L^2(Z_p^c\cup Z_u^c)} \to 0, \qquad \text{as } \varepsilon\to 0.
$$

Hence

$$
\int_\Omega h(x)\cdot p_0(x)\,dx = \lim_{\varepsilon\to 0}\int_{Z_u\cap Z_p} p_0^\varepsilon(x)\cdot h(x)\,dx,
$$

which means $p_0^\varepsilon\,\chi(Z_u\cap Z_p)\rightharpoonup p_0$. From this and the weak convergence of ∇v^ε to ∇u, together with the lower semicontinuity property in (A.4), we get

$$
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega W(x, F^\varepsilon, \varepsilon p_0^\varepsilon)\,dx \geqslant \int_\Omega \frac{1}{2}C\nabla u\nabla u + \frac{1}{2}H^{-1}p_0\cdot p_0 + A\nabla u p_0\,dx. \tag{A.13}
$$

Putting (A.9), (A.10), (A.11), and (A.13) together, we have $\lim_{\varepsilon\to 0}\mathcal{I}^\varepsilon \geqslant \mathcal{I}^0$. □

Proposition A.5 (The existence of the recovery sequences) *Under the same* *condition with Proposition A.4, there exist a sequence* $y^\varepsilon \to x$ *in* $\mathcal{W}^{1,t}(\Omega)$, $\nabla u^\varepsilon = \frac{1}{\varepsilon}\nabla(y^\varepsilon - x) \rightharpoonup \nabla u$ *in* $\mathcal{L}^2(\Omega)$ *and a sequence* $p_0^\varepsilon \rightharpoonup p_0$ *in* $\mathcal{L}^2(\Omega)$*, such that*

$$\lim_{\varepsilon\to 0} \mathcal{I}^\varepsilon = \mathcal{I}^0.$$

where $\mathcal{I}^\varepsilon$ *is defined in (A.5) and* $\mathcal{I}^0$ *is defined in (A.7).*

Proof First, (A.9) and (A.10) still hold for any sequences $\nabla u^\varepsilon \rightharpoonup \nabla u$ in $\mathcal{L}^2(\Omega)$ and p_0^ε weakly convergent to p_0 in $\mathcal{L}^2(\Omega)$. Next, let us prove that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y \varphi^\varepsilon\right|^2 dy = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_x \varphi^0\right|^2 dx \qquad \text{as } \varepsilon \to 0, \tag{A.14}$$

if $p_0^\varepsilon \to p_0$ in $\mathcal{L}^2(\Omega)$.

Denote $\varphi_j^\varepsilon\big(y^\varepsilon(x)\big) = \varphi_j^0(x)$, $\varphi_j^0 \in \mathcal{D}(\mathbb{R}^N\backslash\Gamma)$ is the solution of (3.23). Let us derive the equation for $\nabla_y \varphi_j^\varepsilon$ in the y-space for each fixed ε and j. Actually, $\forall \psi(y) \in \mathfrak{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, define $\psi\big(y(x)\big) = \psi_x^\varepsilon(x)$, then $\psi_x^\varepsilon(x) \in \mathcal{D}(\mathbb{R}^N\backslash\Gamma)$. This is because $\psi_x^\varepsilon(x) \in \mathcal{L}^2(\mathbb{R}^N)$, $\nabla_x\psi_x^\varepsilon(x) = F_\varepsilon \nabla_y \psi(y) \in \mathcal{L}^2(\mathbb{R}^N)$ and $\psi_x^\varepsilon(\Gamma) = \psi\big(y^\varepsilon(\Gamma)\big) = 0$.
Now we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y \varphi_j^\varepsilon \cdot \nabla_y \psi(y)\, dy &= \int_{\mathbb{R}^N} \varepsilon_0 \nabla_x \varphi_j^0 \cdot \nabla_x \psi_x^\varepsilon \, dx \\
&= \int_{\mathbb{R}^N} -\varepsilon_0 \nabla_x g \cdot \nabla_x \psi_x^\varepsilon + p_0^j \cdot \nabla_x \psi_x^\varepsilon \, dx \\
&= \int_{\mathbb{R}^N} \left(-\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y g^\varepsilon \cdot \nabla_y \psi + F_\varepsilon p_j^\varepsilon \cdot \nabla_y \psi \right) dy,
\end{aligned}$$

where $p_j^\varepsilon\big(y(x)\big) = \frac{p_0^j(x)}{J(x)}$.

Since $\mathfrak{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$ is dense inside $\mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, $\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y \varphi_j^\varepsilon \in \mathcal{D}^{-1}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, $F_\varepsilon p_j^\varepsilon \in \mathcal{D}^{-1}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$, the above equality holds for any $\psi(y) \in \mathcal{D}\big(\mathbb{R}^N\backslash y^\varepsilon(\Gamma)\big)$. Thus the equation for $\nabla_y \varphi_j^\varepsilon$ is

$$\nabla_y \cdot \left[\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y \varphi_j^\varepsilon + \frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y g^\varepsilon - F_\varepsilon p_j^\varepsilon \, \chi\big(y^\varepsilon(\Omega)\big) \right] = 0. \tag{A.15}$$

Denote by ϕ_j^ε the solution of

$$\begin{cases} \nabla_y \cdot \big[-\varepsilon_0 \nabla_y \phi_j^\varepsilon - \varepsilon_0 \nabla g^\varepsilon + p_j^\varepsilon\, \chi\big(y^\varepsilon(\Omega)\big)\big] = 0 \quad \text{in} \quad \mathbb{R}^N \backslash y^\varepsilon(\Gamma), \\ \phi_j^\varepsilon \in \mathcal{D}\big(\mathbb{R}^N \backslash y^\varepsilon(\Gamma)\big), \end{cases} \tag{A.16}$$

then the equation for $\varphi^\varepsilon - \phi_j^\varepsilon$ is

$$\begin{cases} \nabla_y \cdot \big[\, \varepsilon_0 \nabla_y \varphi^\varepsilon - \varepsilon_0 \nabla_y \phi_j^\varepsilon + (p_j^\varepsilon - p^\varepsilon)\, \chi\big(y^\varepsilon(\Omega)\big)\, \big] = 0 \quad \text{in} \quad \mathbb{R}^N \backslash y^\varepsilon(\Gamma), \\ \varphi^\varepsilon - \phi_j^\varepsilon \in \mathcal{D}\big(\mathbb{R}^N \backslash y^\varepsilon(\Gamma)\big). \end{cases} \tag{A.17}$$

Thus

$$\begin{aligned} & \big\|\nabla_y \varphi^\varepsilon - \nabla_y \phi_j^\varepsilon\big\|_{\mathcal{L}^2(\mathbb{R}^N)} \\ \leqslant\ & \big\|p^\varepsilon - p_j^\varepsilon\big\|_{\mathcal{L}^2(y^\varepsilon(\Omega))} \leqslant c\, \big\|p_0^\varepsilon - p_0^j\big\|_{\mathcal{L}^2(\Omega)} \\ \leqslant\ & c\,\big(\, \big\|p_0^\varepsilon - p_0\big\|_{\mathcal{L}^2(\Omega)} + \big\|p_0 - p_0^j\big\|\,\big) \\ \to\ & 0 \qquad\quad \text{as } \varepsilon \to 0 \text{ and } j \to +\infty. \end{aligned}$$

On the other hand, the equation for $\phi_j^\varepsilon - \varphi_j^\varepsilon$ is

$$\begin{aligned} & \nabla_y \cdot (\varepsilon_0 \nabla_y \phi_j^\varepsilon - \varepsilon_0 \nabla_y \varphi_j^\varepsilon) \\ =\ & \nabla_y \cdot p_j^\varepsilon\, \chi\big(y^\varepsilon(\Omega)\big) - \varepsilon_0 \nabla_y \cdot \nabla_y g^\varepsilon - \varepsilon_0 \nabla_y \cdot \nabla_y \varphi_j^\varepsilon \\ & + \nabla_y \cdot \left[\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y \varphi_j^\varepsilon + \frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T \nabla_y g^\varepsilon - F_\varepsilon p_j^\varepsilon\, \chi\big(y^\varepsilon(\Omega)\big) \right] \\ =\ & \nabla_y \cdot \left[\left(\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 I \right) \nabla_y \varphi_j^\varepsilon + \left(\frac{\varepsilon_0}{J_\varepsilon} F_\varepsilon F_\varepsilon^T - \varepsilon_0 I \right) \nabla_y g^\varepsilon + (I - F_\varepsilon)\, p_j^\varepsilon\, \chi\big(y^\varepsilon(\Omega)\big) \right]. \end{aligned}$$

The right hand side of the above equation goes to zero as $\varepsilon \to 0$ for any j.

Now

$$
\begin{aligned}
&\left\|\nabla_y \varphi^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \leqslant \left\|\nabla_y \phi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} + \left\|\nabla_y \phi_j^\varepsilon - \nabla_y \varphi^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \\
\leqslant\ & \left\|\nabla_y \phi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} + c\left(\left\|p_0^\varepsilon - p_0\right\|_{\mathcal{L}^2(\Omega)} + \left\|p_0 - p_0^j\right\|_{\mathcal{L}^2(\Omega)}\right) \\
\leqslant\ & \left\|\nabla_y \varphi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} + \left\|\nabla_y \varphi_j^\varepsilon - \nabla_y \phi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} + c\left(\left\|p_0^\varepsilon - p_0\right\|_{\mathcal{L}^2(\Omega)} + \left\|p_0 - p_0^j\right\|_{\mathcal{L}^2(\Omega)}\right) \\
\leqslant\ & \left\|\nabla_x \varphi_j^0\right\|_{\mathcal{L}^2(\mathbb{R}^N)} + \left(\left\|\nabla_y \varphi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} - \left\|\nabla_x \varphi_j^0\right\|_{\mathcal{L}^2(\mathbb{R}^N)}\right) + \left\|\nabla_y \varphi_j^\varepsilon - \nabla_y \phi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \\
&+ c\left(\left\|p_0^\varepsilon - p_0\right\|_{\mathcal{L}^2(\Omega)} + \left\|p_0 - p_0^j\right\|_{\mathcal{L}^2(\Omega)}\right).
\end{aligned}
$$

In the right hand side, for each fixed j, as $\varepsilon \to 0$

$$
\begin{aligned}
&\left\|\nabla_y \varphi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} - \left\|\nabla_x \varphi_0^j\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \to 0, \\
&\left\|\nabla_y \varphi_j^\varepsilon - \nabla_y \phi_j^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \to 0, \\
&\left\|p_0^\varepsilon - p_0\right\|_{\mathcal{L}^2(\Omega)} \to 0.
\end{aligned}
$$

Next, let $j \to +\infty$, $\left\|p_0 - p_0^j\right\|_{\mathcal{L}^2(\Omega)} \to 0$. Thus we get $\left\|\nabla_y \varphi^\varepsilon\right\|_{\mathcal{L}^2(\mathbb{R}^N)} \leqslant \left\|\nabla_y \varphi^0\right\|_{\mathcal{L}^2(\mathbb{R}^N)}$. This means as long as $\left\|p_0^\varepsilon - p_0\right\|_{\mathcal{L}^2(\Omega)} \to 0$,

$$
\lim_{\varepsilon \to 0} \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_y \varphi^\varepsilon\right|^2 dy = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^N} \left|\nabla_x \varphi^0\right|^2 dx \qquad \text{as } \varepsilon \to 0.
$$

Now, The only thing left is to find a sequence $u^\varepsilon \rightharpoonup u$ in $\mathcal{H}^1(\Omega)$ and $p_0^\varepsilon \to p_0$ in $\mathcal{L}^2(\Omega)$, such that

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega W(x, F^\varepsilon, \varepsilon p_0^\varepsilon)\, dx = \int_\Omega \frac{1}{2} C \nabla u \nabla u + \frac{1}{2} H^{-1} p_0 \cdot p_0 + A \nabla u p_0 \, dx.
$$

Define $Z_p = \left\{ x : |p_0(x)| \leqslant \frac{1}{2} Q_\varepsilon \right\}$ for some Q_ε to be decided.

We construct p_0^ε as

$$
p_0^\varepsilon = \begin{cases} p_0 & \text{if} \quad x \in Z_p, \\ 0 & \text{if} \quad x \in Z_p^c. \end{cases}
$$

Then $p_0^\varepsilon \to p_0$ in $\mathcal{L}^2(\Omega)$ since

$$\int_\Omega \left|p_0^\varepsilon - p_0\right|^2 dx \;=\; \int_{Z_p^c} \left|p_0\right|^2 dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

We construct the recovery sequence u^ε as the truncation of u for $m = 1$, $M = M_\varepsilon$ to be decided. Assume $M_\varepsilon < \rho_2 \varepsilon^{-1}$ and $Q_\varepsilon < \rho_1 \varepsilon^{-1}$, then $\varepsilon \nabla u^\varepsilon < \rho_2$ and $\varepsilon p_0^\varepsilon < \rho_1$ on the whole domain Ω. Now we can use Taylor expansion.

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega W(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \\
\leqslant\;& \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon + \frac{1}{2} H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon + A \nabla u^\varepsilon p_0^\varepsilon\, dx \\
&+ \int_\Omega a(x)\, w_1\left(\varepsilon\left|\nabla u^\varepsilon\right|, \varepsilon\left|p_0^\varepsilon\right|\right)\, \left|p_0^\varepsilon\right|^2 + b(x)\, w_2\left(\varepsilon\left|\nabla u^\varepsilon\right|, \varepsilon\left|p_0^\varepsilon\right|\right) \left|\nabla u^\varepsilon\right|^2 dx \\
\leqslant\;& \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon + \frac{1}{2} H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon + A \nabla u^\varepsilon p_0^\varepsilon\, dx \\
&+ c\, \left(w_1(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, Q_\varepsilon^2 + w_2(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, M_\varepsilon^2 .\right).
\end{aligned}$$

Take

$$\begin{aligned}
M_\varepsilon &= \min\left\{\varepsilon^{-\frac{1}{2}}, w_2(\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}})^{-\frac{1}{4}}\right\}, \\
Q_\varepsilon &= \min\left\{\varepsilon^{-\frac{1}{2}}, w_1(\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}})^{-\frac{1}{4}}\right\},
\end{aligned}$$

then

$$w_1(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, Q_\varepsilon^2 + w_2(\varepsilon M_\varepsilon, \varepsilon Q_\varepsilon)\, M_\varepsilon^2 \to 0 \qquad \text{as } \varepsilon \to 0.$$

Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W(x, I + \varepsilon \nabla u^\varepsilon, \varepsilon p_0^\varepsilon)\, dx \\
\leqslant\;& \lim_{\varepsilon\to 0} \int_\Omega \frac{1}{2} C \nabla u^\varepsilon \nabla u^\varepsilon + \frac{1}{2} H^{-1} p_0^\varepsilon \cdot p_0^\varepsilon + A \nabla u^\varepsilon p_0^\varepsilon\, dx \\
=\;& \int_\Omega \frac{1}{2} C \nabla u \nabla u + \frac{1}{2} H^{-1} p_0 \cdot p_0 + A \nabla u p_0\, dx. \qquad \text{(A.18)}
\end{aligned}$$

(A.18) comes from the fact $p_0^\varepsilon \to p_0$ and $\nabla u^\varepsilon \to \nabla u$. Actually,

$$\int_\Omega \left|\nabla u^\varepsilon - \nabla u\right|^2 dx = \int_{Z_u^c} \left|\nabla u^\varepsilon - \nabla u\right|^2 dx$$

$$\leqslant \; 2\int_{Z_u^c} \left|\nabla u^\varepsilon\right|^2 + \left|\nabla u\right|^2 dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

This completes the proof. □

A.3 The Compactness of the Electric and Elastic Fields

Now, let us prove the compactness result for the piezoelectric material.

Proof of Proposition A.2

In this part, we take $g^\varepsilon(y)$ the same as in the proof of Proposition 3.5 and 3.6. That is, assume there exist Ω_y^0 and Ω_y^1 such that $\Omega_y^0 \cap \Omega_y^1 = \varnothing$, $\Omega_y^0 \supset y^\varepsilon(\Gamma_0)$, $\Omega_y^1 \supset y^\varepsilon(\Gamma_1)$ for all ε. Define function $g(y) \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ as a fixed function in y-space, satisfying $g(y) \equiv g_0$ in Ω_y^0 and $g(y) \equiv g_1$ in Ω_y^1. Let $g^\varepsilon(y) = g(y)$ for each ε. Then the quantity

$$-\frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \nabla g(y) \cdot \nabla g(y)\, dy$$

is a constant. As in the proof of Proposition 3.5, if $\mathcal{I}^\varepsilon \leqslant c$, there exists a constant $\tilde{c}$

such that

$$
\begin{aligned}
\tilde{c} \;&\geqslant\; \mathcal{I}^{\varepsilon} + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y g\right|^2 dy \\
&=\; \frac{1}{\varepsilon^2}\int_{\Omega} W(x, F^{\varepsilon}, \varepsilon p_0^{\varepsilon})\, dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^N} \left|\nabla_y \varphi^{\varepsilon}\right|^2 dy + \int_{y^{\varepsilon}(\Omega)} \nabla_y g \cdot p^{\varepsilon}\, dy \\
&\geqslant\; \frac{1}{\varepsilon^2}\int_{\Omega} W(x, F^{\varepsilon}, \varepsilon p_0^{\varepsilon})\, dx - c\left\|\nabla_y g\right\|_{L^{\infty}(\mathbb{R}^N)} \left\|p_0^{\varepsilon}\right\|_{L^2(\Omega)} \\
&\geqslant\; \frac{1}{2}\left[\frac{1}{\varepsilon^2}\int_{\Omega} W(x, F^{\varepsilon}, 0)\, dx + \frac{1}{\varepsilon^2}\int_{\Omega} W(x, I, \varepsilon p_0^{\varepsilon})\, dx\right] \\
&\quad\; - c\left\|\nabla_y g\right\|_{L^{\infty}(\mathbb{R}^N)} \left\|p_0^{\varepsilon}\right\|_{L^2(\Omega)} \\
&\geqslant\; \frac{1}{2}\left[\frac{c_1}{\varepsilon^2}\left\|F^{\varepsilon} - I\right\|_{L^t}^{t} + \frac{c_2}{\varepsilon^2}\left\|\varepsilon p_0^{\varepsilon}\right\|_{L^2(\Omega)}^{2}\right] - c\left\|\nabla_y g\right\|_{L^{\infty}(\mathbb{R}^N)} \left\|p_0^{\varepsilon}\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Thus there exists a constant c such that

$$
\begin{aligned}
&\left\|p_0^{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} \leqslant c, \\
&\left\|\frac{F^{\varepsilon} - I}{\varepsilon}\right\|_{\mathcal{L}^2(\Omega)} \leqslant c, \\
&F^{\varepsilon} \to I \qquad \text{in } \mathcal{L}^t(\Omega).
\end{aligned}
$$

$\square$

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