Graphs and colors : edge-colored graphs, edge-colorings and proper connections

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To my country that is far far away...

My beloved Argentina.

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Résumé

Dans cette thèse nous étudions différents problèmes de graphes et multigraphes arêtes-coloriés tels que la connexité propre, la coloration forte d'arêtes et les chaînes et cycles hamiltoniens propres. Enfin, nous améliorons l'algorithme connu $O(n^4)$ pour décider du comportement d'un graphe sous opérateur biclique, en étudiant les bicliques dans les graphes sans faux jumeaux. Plus précisément,

- Nous étudions d'abord le nombre k-connexité-propre des graphes, noté $pc_k(G)$, ç'est à dire le nombre minimum de couleurs nécessaires pour colorer les arêtes d'un graphe de façon à ce qu'entre chaque paire de sommets, ils existent k chemins propres intérieurement sommet-disjoints. Nous prouvons plusieurs bornes supérieures pour $pc_k(G)$. Nous énonçons quelques conjectures pour les graphes généraux et bipartis et nous les prouvons dans le cas où $k = 1$.

- Nous étudions l'existence de chaînes et de cycles hamiltoniens propres dans les multigraphes arêtes-coloriés. Nous établissons des conditions suffisantes, en fonction de plusieurs paramètres tels que le nombre d'arêtes, le degré arc-en-ciel, la connexité, etc.

- Nous montrons que l'indice chromatique fort est linéaire au degré maximum pour tout graphe k-dégénéré où, k est fixe. En corollaire, notre résultat conduit à une amélioration des constantes et donne également un algorithme plus simple et plus efficace pour cette famille de graphes. De plus, nous considérons les graphes planaires extérieurs. Nous donnons une formule pour trouver l'indice chromatique fort exact pour les graphes bipartis planaires extérieurs. Nous améliorons également la borne supérieure pour les graphes planaires extérieurs généraux.

- Enfin, nous étudions les bicliques dans les graphes sans faux jumeaux et nous présentons ensuite un algorithme $O(n + m)$ pour reconnaître les graphes convergents et divergents en améliorant l'algorithme $O(n^4)$.

Mots clés: graphes arêtes-coloriés, connexité propre, chaînes et cycles hamiltoniens propres, coloration forte d'arêtes, opérateur biclique.

Abstract

In this thesis, we study different problems in edge-colored graphs and edge-colored multigraphs, such as proper connection, strong edge colorings, and proper hamiltonian paths and cycles. Finally, we improve the known $O(n^4)$ algorithm to decide the behavior of a graph under the biclique operator, by studying bicliques in graphs without false-twin vertices. In particular,

- We first study the k-proper-connection number of graphs, this is, the minimum number of colors needed to color the edges of a graph such that between any pair of vertices there exist k internally vertex-disjoint proper paths. We denote this number $pc_k(G)$. We prove several upper bounds for $pc_k(G)$. We state some conjectures for general and bipartite graphs, and we prove all of them for the case $k = 1$.

- Then, we study the existence of proper hamiltonian paths and proper hamiltonian cycles in edge-colored multigraphs. We establish sufficient conditions, depending on several parameters such as the number of edges, the rainbow degree, the connectivity, etc.

- Later, we show that the strong chromatic index is linear in the maximum degree for any k-degenerate graph where k is fixed. As a corollary, our result leads to considerable improvement of the constants and also gives an easier and more efficient algorithm for this familly of graphs. Next, we consider outerplanar graphs. We give a formula to find exact strong chromatic index for bipartite outerplanar graphs. We also improve the upper bound for general outerplanar graphs from the $3\Delta - 3$ bound.

- Finally, we study bicliques in graphs without false-twin vertices and then we present an $O(n + m)$ algorithm to recognize convergent and divergent graphs improving the $O(n^4)$ known algorithm.

Keywords: edge-colored graphs, proper connection, proper hamiltonian paths and cycles, strong edge-colorings, iterated biclique graph.

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Chapter 1

Introduction

In this thesis we consider edge-colorings and edge-colored graphs. An *edge-coloring* of a graph, is an assignment of colors to the edges of the graph. A *proper edge-coloring* of a graph is an edge-coloring such that adjacent edges have different colors. The natural question is to ask about the minimum number of colors needed in order to color a graph G properly. This number is called *chromatic index* and denoted by $\chi'(G)$. By Vizing's Theorem [107] we know that this number is either $\Delta(G)$ or $\Delta(G)+1$. For several classes of graphs we know exactly the value of $\chi'(G)$, as for example bipartite graphs which have chromatic index equal to $\Delta(G)$. Edge-colorings are interesting not only because of the mathematical point of view, but also because of the many applications they have in real life, for example in scheduling problems and in frequency assignment for fiber optic networks, etc. Therefore, many different types of edge-colorings have been studied over the years. We can cite some of them as strong edge-colorings [6, 23, 33, 35, 45, 61, 62, 81, 85, 99, 100, 104, 105, 112], list edge-colorings [21, 51, 56, 63, 72, 71, 111, 113, 115], interval edge-colorings [7, 64, 92, 93], etc.

An *edge-colored graph* is a graph that its edges have been colored somehow with c different colors. Here, the natural question to ask is, given an edge-colored graph, how can we find (if possible) or guarantee the existence of some subgraphs with certain properties. For example, how to find or guarantee the existence of a hamiltonian cycle that is properly colored. Lot of research has been done in this subject, not only for proper hamiltonian cycles, but also for proper hamiltonian paths, proper trees, proper cycles, rainbow trees, rainbow paths, rainbow cliques, monochromatic cliques, monochromatic cycles, etc. Refer for example to [1, 2, 3, 5, 10, 12, 15, 17, 34, 47, 59, 65, 83, 94, 95, 102, 108, 114] to find some results on the subject.

This thesis is organized as follows. In **chapter two**, we give basic definitions and notation. In **chapter three**, we study the proper connection of graphs. In **chapter four**, we study strong edge-colorings of graphs, in particular in k-degenerate graphs and outerplanar graphs. In **chapter five** and **chapter six**, we study sufficient conditions in edge-colored multigraphs to guarantee the existence of proper hamiltonian paths and cycles, respectively, depending of various parameters such as number of edges, connectivity, rainbow degree, etc. In **chapter seven**, we present a work that started at the beggining of my PhD thesis when I was in Argentina (with Dr. Marina Groshaus as advisor) and finished here in France. This work is the extension of my master thesis and involves several results in bicliques of graphs, in particular, a linear time algorithm to recognize convergent and divergent graphs under the biclique operator. Finally, in **chapter eight**, we present the conclusions of our work.

In what follows we present an introduction of the different problems that we have studied in this thesis. The idea is to introduce them by giving some references to the literature such that the reader can find about their history and applications.

1.1 Proper connection of graphs

Recent works like [49, 108] have considered properly colored subgraphs as opposed to looking at the entire graph. There is even a survey of work concerning alternating cycles [10]. Here *alternating* means the colors of the edges alternate as you traverse the cycle thus making it properly colored. The problem of finding an alternating cycle is precisely the problem of finding a properly colored cycle when only two colors are available.

Similarly, some researchers have considered rainbow colored subgraphs (meaning that every edge has a distinct color). A graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. The *rainbow connection number*, $rc(G)$, as defined in [32], is the minimum number of colors that are needed in order to make G rainbow connected. The rainbow connection number was studied in [27, 31, 37, 69]. Motivated by this, we extend the rainbow connection definition to a proper connection one saying that a graph is k*-proper connected* if any two vertices are connected by k-vertex disjoint paths whose adjacent edges have distinct colors. And define the k*-proper connection number*, $pc_k(G)$, as the minimum number of colors that are needed in order to make G k-proper connected.

About the computational complexity of the problem, there are no results so far. However, there are several results for the rainbow connection problem. We present some of them. Deciding if $rc(G) = 2$ or $rc(G) = k$, for any fixed $k \geq 2$, is NP-Complete [28] and [78] respectively. Checking whether the given coloring makes G rainbow connected is NP-complete [28]. But, the problem is polynomial for complete graphs, trees, paths, cycles and wheels [32].

In this thesis we first study $pc_k(G)$ for bipartite graphs. We prove exact values of $pc_k(G)$ for different bipartite complete graphs and trees. Then, we state the general conjecture that says that if G is $2k$-connected and bipartite with $k \geq 1$, then $pc_k(G) = 2$. We prove that, if true, the conjecture is the best possible since we show a family of $2k-1$-connected bipartite graphs with $pc_k(G) > 2$. Finally, we prove this conjecture for $k = 1$.

Later, we study $pc_k(G)$ in general graphs, starting with the simplest case $k = 1$, i.e., $pc(G)$. We prove several exact values for $pc(G)$ as for example for complete graphs, paths, cycles, etc. Then, we prove the main result of the section, this is, if G is 2-connected, then $pc(G) \leq 3$. This improves Vizing's trivial bound of $\Delta + 1$. We show also that this bound is tight since we present a family of 2-connected graphs with $pc(G) = 3$. Then we present a bound for $pc(G)$ for just connected graphs that uses the maximum degree of a vertex that is an endpoint of a bridge. We state also a general conjecture for $pc_k(G)$ based on the conjecture for bipartite graphs and the result for 2-connected graphs. This is, if G is $2k$-connected with $k \geq 1$, then $pc_k(G) \leq 3$. We remark that we proved this conjecture for $k = 1$. Then, we prove a stronger result for $pc_k(G)$ for complete graphs of order $n \geq 2k$.

Finally, we prove a result concerning the minimum degree of the graph, this is, if G is a connected non-complete graph of order $n \geq 68$ and $\delta(G) \geq \frac{n}{4}$ then $pc(G) = 2$.

All our results lead to efficient algorithms to find such colorings.

We remark that many of the conditions assumed for proper connection are much weaker than those needed to produce upper bounds on the rainbow connection number $rc(G)$. This can be explained by the fact that it takes far fewer colors to make a path properly colored than are needed to make it rainbow colored.

1.2 Strong edge-colorings

A *strong edge-coloring* of a graph G is an edge-coloring such that any two vertices belonging to distinct edges with the same color are not adjacent. The *strong chromatic index*, $\chi'_s(G)$, is the minimum number of colors in a strong edge-coloring of G.

The strong edge-coloring has a long history and has lead to many well known conjectures. Some of the many unsolved conjectures include $\chi'_s(G) \leq 5\Delta^2/4$ for all graphs, $\chi'_s(G) \leq \Delta^2$ for bipartite graphs, and $\chi'_s(G) \leq 9$ for 3-regular planar graphs (see the open problems pages of Douglas West [109] for more details).

Molloy and Reed [85] proved a conjecture by Erdős and Nešetřil (see [45]) that for large Δ, there is a positive constant c such that $\chi'_s(G) \leq (2-c)\Delta^2$. Mahdian [81] proved that for a C_4-free graph G, $\chi'_s(G) \leq (2+o(1))\Delta^2/\ln\Delta$.

For integers $0 \leq \ell \leq k \leq m$, $S_m(k,\ell)$ is the bipartite graph with vertex set $\{x \subseteq [m] \colon |x| = k \text{ or } \ell\}$ and a k-subset x is adjacent to an ℓ-subset y if $y \subseteq x$. Quinn and Benjamin [16] proved that $S_m(k,\ell)$ has strong chromatic index $\binom{m}{k-\ell}$. The Θ-*graph* $\Theta(G)$ of a partial cube G (distance-invariant subgraph of some n-cube), is the intersection graph of the equivalence classes of the *Djoković-Winkler relation* Θ defined on the edges of G such that xy and uv are in relation Θ if $d(x,u)+d(y,v) \neq d(x,v)+d(y,u)$. Šumenjak [68] showed that the strong chromatic index of a tree-like partial cube graph G is at most the chromatic number of $\Theta(G)$.

Faudree, Gyárfás, Schelp and Tuza [46] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \leq \Delta^2$. They mention that this result probably could be improved to a linear function of the maximum degree. Brualdi and Quinn [23] improved the upper bound to $\chi'_s(G) \leq \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [88] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \leq 2\Delta$. Bounds for outerplanar graphs were given recently in [60]. A recent work ([66]) gives an algorithm to find the strong chromatic index of any maximal outerplanar graph, but notice that when you extend the graph to maximal outerplanar, the maximum degree and the chromatic index can increase. Also, Chang and Narayanan [29] proved that $\chi'_s(G) \leq 10\Delta - 10$ for any 2-degenerate graph G, $\chi'_s(G) \leq 8\Delta - 6$ for chordless graphs and proposed, as a conjecture, that there exists an absolute constant c such that for any k-degenerate graph G, $\chi'_s(G) \leq ck^2\Delta$. Thus for fixed k, $\chi'_s(G)$ is linear in Δ.

About the complexity, the strong edge-coloring problem is NP-complete [82]. It is also NP-complete for $4, 5$ or 6 colors in some subclasses of planar graphs of maximum degree 3 [60]. However, is polynomial for some classes of graphs as paths, trees, cycles, chordal graphs [26], graphs of bounded tree-width [103], etc.

One known application of strong edge-colorings is the following: to find a minimum strong edge-coloring ($\chi'_s(G)$) is equivalent to computing an interference-free channel assignment with the fewest channels [14].

In this thesis, we will be focused in improving bounds for k-degenerate graphs and outerplanar graphs. In particular, we prove that if G is k-degenerate graph, then $\chi'_s(G) \leq (4k-1)\Delta - 2k^2 - k + 1$ that improves the conjecture since $\chi'_s(G)$ is linear in Δ and k.

This result implies the two following ones. If G is 2-degenerate graph, then $\chi'_s(G) \leq 7\Delta - 9$ and if G is chordless, then $\chi'_s(G) \leq 5\Delta - 5$, improving both known results. Then we show an $O(n + k\Delta m)$ algorithm to find a coloring for k-degenerate graphs using $(4k-1)\Delta - 2k^2 - k + 1$ colors.

Then, for strong edge-colorings in outerplanar graphs, we define a *puffer graph* or an *n-puffer* as a graph obtained by adding some (possibly empty) pendant edges to each vertex of an n-cycle or adding a common neighbour to two consecutive vertices of the n-cycle. We remark that since the graph is outerplanar, at most one such vertex can be added. We prove several exact and upper bounds for $\chi'_s(G)$ in puffer graphs. Using these results we prove the following: if G is an outerplanar graph, then $\chi'_s(G) = \max\{\max_{uv\in E} d(u) + d(v) - 1, \max_{H\in\mathcal{P}} \chi'_s(H)\}$, where $\mathcal{P}$ is the set of all induced puffer subgraphs of G. If G is also bipartite, then $\chi'_s(G) = \max\{\max_{uv\in E} d(u) + d(v) - 1, \max_{uv\in E(C_4)} d(u) + d(v)\}$ where C_4 is the set of all cycles of length 4 in G. Observe that for outerplanar graphs we obtain an upper bound for $\chi'_s(G)$, while for bipartite outerplanar graphs we have the exact value of $\chi'_s(G)$.

1.3 Proper hamiltonian paths and cycles in edge-colored multigraphs

The research on long colored cycles and paths for edge-colored graphs has given interesting results. Refer to [10, 11, 65] for surveys on related results. From the point of view of applicability, problems arising in molecular biology are often modeled using colored graphs, i.e., graphs with colored edges and/or vertices [95]. Given such an edge-colored graph, original problems translate to extracting subgraphs colored in a specified pattern. The most natural pattern in such a context is that of a proper coloring, i.e., adjacent edges have different colors.

Clearly, the proper hamiltonian path and proper hamiltonian cycle problems are NP-complete in the general case. It is polynomial to find a proper hamiltonian path in c-edge-colored complete graphs, $c \geq 2$ [47]. It is also polynomial to find a proper hamiltonian cycle in 2-edge-colored complete graphs [13]. It is still open to determine the computational complexity for proper hamiltonian cycles, $c \geq 3$ [17]. Many other partial results for edge-colored multigraphs can be found in the survey by Bang-Jensen and Gutin [10].

In this thesis we consider sufficient conditions involving various parameters as the number of edges, rainbow degree, etc., in order to guarantee the existence of properly edge-colored hamiltonian paths and cycles in edge-colored multigraphs. Since, very often the extremal graphs for 2-edge-colored multigraphs are different than those for c-edge-colored multigraphs, $c \geq 3$, we consider our results separately for these two cases. This division is natural since in 2-edge-colored multigraphs proper paths and proper cycles are just alternating, and therefore, the bounds are different.

For proper hamiltonian paths in 2-edge-colored multigraphs we present two main results. The first one involves the number of edges, and the second one involves the number of edges and the rainbow degree. We show that both results are tight. Then, for proper hamiltonian paths in c-edge-colored multigraphs, $c \geq 3$, we show that this problem can be reduced to the existence of proper hamiltonian paths in 3-edge-colored multigraphs and then we present three main results. The first one involves the number of edges. The second one, the number of edges and the connectivity of the graph. The last one, the number of edges and the rainbow degree. Again, all results are the best possible.

About proper hamiltonian cycles in 2-edge-colored multigraphs we prove two tight results involving same parameters as for proper hamiltonian paths. Then, for c-edge-colored multigraphs, $c \geq 3$, we show, as for paths, that looking at 3-edge-colored multigraphs is enough and then we prove two main results. The first one involves the number of edges. The second one, the number of edges and the rainbow degree. We show that both results are the best possible. Finally, we state a conjecture involving the number of edges, the rainbow degree and the 2-connectivity of the graph.

Results involving only degree conditions can be found in [2].

1.4 Bicliques in graphs

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. Let us mention for example the case of line graphs (which are the intersection graphs of the edges of a graph), interval graphs (defined as the intersection graphs of intervals of the real line), and, in particular, clique graphs (defined below) [20, 22, 43, 50, 52, 79, 84].

The *clique graph* of G, denoted by $K(G)$, is the intersection graph of the family of all maximal cliques of G.

Clique graphs were introduced by Hamelink in [57] and characterized by Roberts and Spencer in [101]. It was proved in [4] that the clique graph recognition problem is NP-complete.

As the clique graph construct can be thought of as an operator between graphs, the *iterated clique graph* $K^k(G)$ is the graph obtained by applying the clique operator k successive times. It was introduced by Hedetniemi and Slater in [58]. Much work has been done on the scope of the clique operator, looking at the different possible behaviors. The associated problem is deciding whether an input graph converges, diverges, or is periodic under the clique operator, when k grows to infinity. In general, it is not clear that the problem is decidable. However, partial characterizations have been given for convergent, divergent and periodic graphs, restricted to some classes of graphs. Some of these lead to polynomial time recognition algorithms. For the clique-Helly graph class, graphs which converge to the trivial graph have been characterized in [9]. Cographs, P_4-tidy graphs, and circular-arc graphs are examples of classes where the different behaviors are characterized [36, 73]. Divergent graphs were also considered. For example, in [89], families of divergent graphs are shown. Periodic graphs were studied in [43, 77]. In particular, it is proved that for every integer i, there exist graphs with period i and convergent graphs which converge in i steps. More results about iterated clique graph can be found in [44, 48, 74, 75, 76, 96].

The *biclique graph* of a graph G, denoted by $KB(G)$, is the intersection graph of the family of all maximal bicliques of G. It was defined and characterized in [54]. However, no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construct can be viewed as an operator KB between graphs.

The *iterated biclique graph* $KB^k(G)$, i.e., the graph obtained by applying iteratively the biclique operator KB k times to G was introduced and all possible behaviors were characterized in [53]. It was proven that a graph G is either divergent or convergent, but it is never periodic (with period bigger than 1). In addition, they were given general characterizations for convergent and divergent graphs. These results are based on the fact that if a graph G contains a clique of size at least 5, then $KB(G)$ contains a clique of larger size. Therefore, G diverges. Similarly, if G contains the so-called *gem* or *rocket* as an induced subgraph, then $KB(G)$ contains a clique of size 5, and again, G diverges. Otherwise, it is shown that, after removing false-twin vertices of $KB(G)$, the resulting graph is a clique on at most 4 vertices, in which case, G converges. Moreover, it was proved that if a graph G converges, it converges to the graphs K_1 or K_3, and it does so in at most 3 steps. These results are very different from the ones known for the clique operator. These characterizations leaded to an $O(n^4)$ time algorithm for deciding if a given graph converges or diverges under the biclique operator.

Bicliques have applications in various fields, for example, biology: protein-protein interaction networks [24], social networks: web community discovery [70], genetics [8], medicine [87], information theory [55]. More applications (including some of these) can be found in [80]

In this thesis we continue this work. Using the characterization above and other results we prove that if G has at least 7 bicliques, then G diverges under the biclique operator, i.e., almost every graph is divergent under the biclique operator. Later, based on those results we obtain the main theorem that leads to a linear time algorithm for deciding if a given graph converges or diverges under the biclique operator. Motivated by the fact that false-twin vertices belong to exactly the same bicliques and the successive deletion of them does not change neither the number of bicliques of the graph nor the structure of the biclique graph (and therefore does not change its behavior under the biclique operator), we study this particular class. We prove that given a graph G with no false-twin vertices, if G has at least 13 vertices then G has at least 7 bicliques. Later, we study more general structural properties of bicliques in false-twin free graphs. We prove several small results that imply the following: If G is a K_3-free graph of order $n \geq 4$ without false-twin vertices, then G has at least $\lceil \frac{n}{2} \rceil$ bicliques. We also state a similiar conjecture but for general graphs without false-twin vertices. Finally, we present several results that would help to prove that conjecture.

Chapter 2

Definitions and Notation

In this chapter we introduce the main definitions and notation needed to understand this thesis. We begin by the general notions of graph theory, then about edge-colored graphs and multigraphs, and finally about bicliques in graphs. We use the notation and terminology given by Bondy and Murty in [19]. However, the reader is warned that there may be some differences.

2.1 General graphs

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a non-empty finite set and $E(G)$ is a set of unordered pairs vw with $v, w \in V(G)$ and $v \neq w$. The set $V(G)$ or simply V, is the *vertex set* of G and its elements are called *vertices* of G. The set $E(G)$ or simply E is the *edge set* of G and its elements are called *edges* of G. Given an edge $e = vw$, the vertices v and w are called *endpoints* of e. The *order* of G is the number of vertices of G. We denote, as usual, $n = |V(G)|$ and $m = |E(G)|$, unless otherwise stated. The unique graph of order 1 is called the *trivial* graph. We remark that in our graph definition $vv \notin E(G)$ for $v \in V(G)$, $vw \in E(G)$ if and only if $wv \in E(G)$ for $v, w \in V(G)$ and $|\{vw \in E(G) | v, w \in V(G)\}| \leq 1$. Under these conditions we refer to these graphs as *loopless*, *undirected* and *simple* respectively.

A vertex v is *adjacent* to a vertex w when $vw \in E(G)$. We also say in that case that v is a neighbor of w. A vertex v is *incident* to an edge e when v is an endpoint of e. Two distinct edges e and f are *adjacent* if they have a common endpoint. The *neighborhood* of a vertex v, denoted $N_G(v)$, is the set of all neighbors of v, and the *complement neighborhood* of a vertex v, denoted $\overline{N_G(v)}$, is the set of all non-neighbors of

v. The degree of a vertex v is the cardinality of the set $N_G(v)$ and it is denoted $d_G(v)$. The minimun and maximum values among the degrees of all vertices are denoted $\delta(G)$ and $\Delta(G)$ respectively. If $\delta(G) = \Delta(G)$ then G is a *regular* graph. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. If $N_G[v] = V(G)$ then v is an *universal* vertex while if $N_G(v) = \emptyset$ then v is an *isolated* vertex. We will omit the subscripts in N and d when there is no ambiguity about G. Two vertices v and w are *true-twins*, or simply *twins*, when $N[v] = N[w]$. We refer to v and w as *false-twins* when $N(v) = N(w)$.

A graph H is a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If also $E(H) = \{vw \in E(G) | v, w \in V(H)\}$, then H is an *induced* subgraph of G. For each $S \subseteq V(G)$, the subgraph of G induced by S is the unique induced subgraph of G whose vertex set is S. We denote by $G[S]$ the subgraph of G induced by V. A subgraph of G whose vertex set is $V(G)$ is called a *spanning subgraph.* For each $S \subseteq V(G)$, we denote by $G - S$ the subgraph of G induced by $V(G) - S$. If $S = \{v\}$, we write shortly $G - v$. Similarly, for each $F \subseteq E(G)$, we denote by $G - F$ the spanning subgraph of G with edge set $E(G) - F$. If $F = \{e\}$, we write shortly $G - e$. A graph is *k-degenerate*, if every subgraph has a vertex of degree at most k.

Two graphs G and H are *isomorphic* if there is a one-to-one mapping f between $V(G)$ and $V(H)$ such that $vw \in E(G)$ if and only if $f(v)f(w) \in E(H)$. The mapping f is referred to as an *isomorphism* between G and H. For a graph H, the graph G is $H - free$ if no induced subgraph of G is isomorphic to H.

The *complement* of a graph G, denoted by $\overline{G}$, is the graph that has the same vertices as G and such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *walk* in a graph G is a sequence $v_1 \dots v_k$ of vertices such that v_i is adjacent to v_{i+1}, for every $1 \leq i < k$. Such a walk is said to be a walk *between* v_1 and v_k (or *joining* v_1 with v_k). The vertices v_i and v_{i+1} are said to be *consecutive* in the walk and v_iv_{i+1} is said to be an *edge* of the walk. The *length* of the walk is the number $k - 1$ of edges of the walk. A *closed walk* is a walk that joins a vertex with itself. A *path* is a walk formed by pairwise distinct vertices and it is denoted by P_k. A *cycle* is a closed walk $v_1 \dots v_k v_1$ where $k \geq 3$ and $v_1 \dots v_k$ is a path. We denote it by C_k. A *hamiltonian path* (*cycle*) is a path (cycle) containing all vertices of the graph.

A graph is *connected* if it contains a path between any two of its vertices. A *disconnected* graph is a graph that is not connected. A *connected component*, or simply a *component*, is a maximal connected subgraph. A graph is *k-connected* if for any set S of $k - 1$ vertices, $G - S$ is connected. The *connection number* of a graph is the minimum value k that there exists a set of vertices S of size k such that $G - S$ is disconnected and for every set of vertices S' of size $k-1$, $G - S'$ remains connected. We denote it by $\kappa(G)$. Similarly, A graph is *k-edge-connected* if for any set F of $k-1$ edges, $G - F$ is connected. A vertex v of a connected graph G such that $G - v$ is disconnected is called *cut vertex*, similarly a set of vertices S of a connected graph G such that $G - S$ is disconnected is called *cut set*. A edge e of a connected graph G such that $G - e$ is disconnected is called *bridge*.

A *forest* is a graph without cycles. A *tree* is a connected graph without cycles. A *leaf* of a tree is a vertex of degree 1.

The *distance* of two vertices v and w in a graph G, denoted by $d_G(v, w)$, is the minimum among the lengths of all the paths between v and w. The distance of v and w is *infinity* when there is no path joining v with w. The $\max_{u,v\in V} d(u, v)$ is called the *diameter* of G.

A *chord* of a cycle is an edge that joins two non-consecutive vertices of the cycle. Those cycles that have no chords are called *chordless*. A graph is *chordless* if every cycle is chordless.

A *clique* in a graph G is a set of pairwise adjacent vertices. An *independent set* is a set of pairwise non-adjacent vertices. The complete graph of order n is denoted by K_n and K_3 is referred to as a *triangle*.

A graph G is *bipartite* when there is a partition of $V(G)$ into two non-empty sets V_1, V_2 of $V(G)$ such that both V_1 and V_2 are independent sets. The partition V_1, V_2 is called a *bipartition* of $V(G)$, and we denote it by V_1, V_2 . If each vertex in V_1 is adjacent to all the vertices in V_2 , then G is a *complete bipartite graph*. The complete bipartite graph with bipartition $V1, V2$ is denoted by $K_{|V1|,|V2|}$.

A *matching* in a graph is a set of pairwise non-adjacent edges. Given a graph of order n, for n even, a *perfect matching* is a matching M of G such that $|M| = \frac{n}{2}$ and for n odd, an *almost perfect matching* is a matching M of G such that $|M| = \frac{n-1}{2}$. A path or cycle is said to be *compatible* with a matching M if the edges of the path or the cycle are alternatively in M and not in M. We assume that, if the path is not hamiltonian, it starts and ends with edges in M. An *induced matching* M in G is a matching such that $G[V(M)] = M$. That is, the subgraph of G induced by the vertices of M is M itself.

A graph G is *outerplanar*, if it has a planar embedding in which all vertices are incident to the infinite face. We define a special outerplanar graph and we call it *puffer graph* or an *n-puffer* as a graph obtained by adding some (possibly empty) pendant edges to each vertex of an n-cycle or adding a common neighbour to two consecutive vertices of the n-cycle. Notice that since the graph is outerplanar, at most one such vertex can be added.

Depending on whether the parity of the cycle is even or odd, they are respectively called *even puffer* and *odd puffer*.

2.2 Edge-colored graphs

Let $C = \{1, 2, \cdots, c\}$ be a set of $c \geq 2$ colors. A *c-edge-coloring* of a graph G is a mapping between C and $E(G)$ such that each of the c colors if assigned to at least one edge of G. A *proper c-edge-coloring* is a *c-edge-coloring* such that every pair of adjacent edges have different colors. The smallest positive integer c such that G admits a proper c-edge-coloring is known as the *chromatic index* of G and is denoted $\chi'(G)$. A *color class* (in an edge-coloring) is the set of all edges which receive the same color.

If H is a subgraph of G, then $N^i_H(v)$ denotes the set of vertices of H adjacent to v with an edge of color i. Whenever H is isomorphic to G, we write $N^i(x)$ instead of $N^i_G(v)$. The *colored i-degree* of a vertex v, denoted by $d^i(v)$, is the cardinality of $N^i(v)$. The *rainbow degree* of a vertex v, denoted by $rd(v)$, is the number of different colors on the edges incident to v. The *rainbow degree* of a graph, denoted by $rd(G)$, is the minimum rainbow degree among all vertices of G. An edge with endpoints v and w is denoted by vw, and its color by $c(vw)$. A vertex is *monochromatic* if it has all its incident edges of the same color.

A path or cycle in an edge-colored graph is said to be *properly edge-colored* (or *proper*), if every two adjacent edges differ in color. A *proper hamiltonian path* (*cycle*) is a proper path (cycle) containing all vertices of the graph. An edge-colored graph G is *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. We define the *k-proper connection number* of a k-connected graph G, denoted by $pc_k(G)$, as the smallest number of colors that are needed in order to make G k-proper connected. Similarly, An edge-colored graph G is *k-proper edge-connected* if any two vertices are connected by k internally pairwise edge-disjoint proper paths. We define the *k-proper edge-connection number* of a k-edge-connected graph G, denoted by $pec_k(G)$, as the smallest number of colors that are needed in order to make G k-proper edge-connected. An edge-colored graph is *connected* if the underlying non-colored graph is connected.

Given a colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1, v_s, we denote by $start(P)$ the color of the first edge in the path, i.e. $c(v_1v_2)$, and by $end(P)$ the last color, i.e. $c(v_{s-1}v_s)$. If P is just the edge v_1v_s then $start(P) = end(P) = c(v_1v_s)$.

A proper edge-coloring is a *strong edge-coloring*, if every color class is an induced matching in G. In other words, the distance between any two edges having the same color is at least two. The minimum positive integer k such that G admits a strong k-edge-coloring is called the *strong chromatic index* of G denoted $\chi'_s(G)$.

2.3 Edge-colored multigraphs

A *multigraph* is a loopless, undirected graph, i.e., the condition of being simple is dropped (several edges between the same pair of vertices are allowed). Let $C = \{1, 2, \cdots, c\}$ be a set of $c \geq 2$ colors. A *c-edge-coloring* of a multigraph is an edge-coloring such that every edge is colored with one color and no two parallel edges joining the same pair of vertices have the same color. We denoted it by G^c.

A *rainbow complete multigraph* is the one having all possible colored edges between any pair of vertices.

We use two families of multigraphs without proper hamiltonian paths. First, let $H^2_{k,k+3}$ denote a 2-edge-colored multigraph on $2k+3$ vertices, $k \geq 1$, defined as follows. Consider a complete red graph on k vertices and join it with red edges to an independent set on $k+3$ vertices. Finally, superpose a complete blue graph on $2k+3$ vertices. For the second family, let $H^c_{k,k+2}$ denote a c-edge-colored multigraph on $2k+2$ vertices, $k \geq 1$ and $c \geq 3$. Consider a rainbow complete graph on k vertices and join it with edges of all possible colors to an independent set on $k+2$ vertices.

Finally we use two families of multigraphs without proper hamiltonian cycles. Let $H^2_{k,k+2}$ denote a 2-edge-colored multigraph on $2k+2$ vertices, $k \geq 2$. Consider a complete red graph on k vertices and join it with red edges to an independent set on $k+2$ vertices. Finally, superpose a complete blue graph on $2k+2$ vertices. Let $H^c_{k,k+1}$ denote a c-edge-colored multigraph on $2k+1$ vertices, $k \geq 2$ and $c \geq 3$. Consider a rainbow complete graph on k vertices and join it with edges of all possible colors to an independent set on $k+1$ vertices.

2.4 Bicliques in graphs

A *biclique* is a maximal complete bipartite induced subgraph of G. A *diamond* is a complete graph with 4 vertices minus an edge. A *gem* is an induced path with 4 vertices plus an universal vertex. A *rocket* is a complete graph with 4 vertices and a vertex adjacent to two of them.

Given a family of sets $\mathcal{A}$, the *intersection graph* of $\mathcal{A}$ has as vertices the set of $\mathcal{A}$ and the edges correspond to the pairs of sets from $\mathcal{A}$ with a non-empty intersection. We remark that any graph is an intersection graph [106].

The *clique graph* of G, denoted by $K(G)$, is the intersection graph of the family of all maximal cliques of G. The *biclique graph* of a graph G, denoted by $KB(G)$, is the intersection graph of the family of all maximal bicliques of G.

Let F be any graph operator. Given a graph G, the iterated graph under the operator F^k is defined iteratively as follows: $F^0(G) = G$ and for $k \geq 1$, $F^k(G) = F^{k-1}(F(G))$. We say that a graph G diverges under the operator F whenever $\lim_{k\to\infty} |V(F^k(G))| = \infty$. We say that a graph G converges under the operator F whenever $\lim_{k\to\infty} F^k(G) = F^m(G)$ for some m. We say that a graph G is periodic under the operator F whenever $F^k(G) = F^{k+s}(G)$ for some k, s, $s \geq 2$.

The *iterated biclique graph* $KB^k(G)$, is the graph obtained by applying iteratively the biclique operator KB k times to G.

In the thesis we will use the terms convergent or divergent meaning convergent or divergent under the biclique operator KB.

By convention, we arbitrarily say that the trivial graph K_1 is convergent under the biclique operator (observe that this remark is needed, since the graph K_1 does not contain bicliques).

Chapter 3

Proper Connection of Graphs

This chapter is organized as follows: In Section 3.1 we study $pc_k(G)$ for bipartite graphs. We state a conjecture, prove several small results and finally we prove the conjecture for $k = 1$, that is, for $pc(G)$. In Section 3.2, we study $pc(G)$ for general graphs and prove non-trivial bounds, improving Vizing's trivial bound of $\Delta + 1$. Then, motivated by both of these sections, we state a conjecture regarding $pc_k(G)$ for general graphs. In Section 3.3 we prove a bound concerning the minimum degree of G.

3.1 Bipartite graphs

In this section, we study proper connection numbers in bipartite graphs. We state a general conjecture for $pc_k(G)$ where G is a bipartite graph with some specific connectivity that depends on k. Following that, we show that this conjecture is best possible in the sense of connectivity. Later, we prove some results for specific classes of graphs such as complete bipartite graphs with lower connectivity assumptions than that which is required for the conjecture. Then, we prove that the conjecture is true for complete bipartite graphs. Finally, we study the case $k = 1$ and obtain results for trees and other graphs depending on their connectivity. We end the section by obtaining, as main result, the proof of the conjecture for the special case $k = 1$ and some corollaries stemming from it.

Conjecture 3.1.1. *If G is a $2k$-connected bipartite graph with $k \geq 1$, then $pc_k(G) = 2$.*

If true, Conjecture 3.1.1 is the best possible in the sense of connectivity. In the following we present a family of bipartite graphs which are $(2k-1)$-connected with the property that $pc_k(G) > 2$. It is also clear that we cannot exchange the vertex connectivity for edge connectivity since it is easy to find graphs with connectivity 1 which have edge connectivity $2k$.

Consider the complete bipartite graph $G = K_{p,q}$ with $p = 2k - 1$ ($k \geq 1$) and $q > 2^p$ where $G = V \cup W$, $V = \{v_1, v_2, \ldots, v_p\}$ and $W = \{w_1, w_2, \ldots, w_q\}$. Clearly, G is $(2k-1)$-connected. We will show that $pc_k(G) > 2$.

Proposition 3.1.2. *Let $p = 2k - 1$ ($k \geq 1$) and $q > 2^p$. Then $pc_k(K_{p,q}) > 2$.*

Proof. Suppose that $pc_k(G) = 2$ and consider a k-proper connected coloring of G with 2 colors. For each vertex $w_i \in W$, there exists a p-tuple $C_i = (c_1, c_2, \ldots, c_p)$ so that $c(v_j w_i) = c_j$ for $1 \leq j \leq p$. Therefore, each vertex $w_i \in W$ has 2^p different ways of coloring its incident edges using 2 colors. Since $q > 2^p$, there exist at least two vertices $w_i, w_j \in W$ such that $C_i = C_j$. As $pc_k(G) = 2$, there exist k internally disjoint proper paths in G between w_i, w_j. Using this, we will arrive to a contradiction. First, observe that one of these paths between w_i, w_j (say P) must have only one intermediate vertex $v_l \in V$ since otherwise, if all the paths have at least two intermediate vertices in V, we would have $|V| \geq 2k$, which is a contradiction. Hence, as $C_i = C_j$ we have $c(v_l w_i) = c(v_l w_j)$ and therefore the path P is not properly colored, leading to a contradiction. □

Based on the previous result we prove the following.

Theorem 3.1.3. *Let $G = K_{n,3}$ then*

$$pc_2(G) = \begin{cases} 2 & \text{if } 3 \leq n \leq 6 \\ 3 & \text{if } 7 \leq n \leq 8 \\ \lceil \sqrt[3]{n} \rceil & \text{if } n \geq 9 \end{cases}$$

Proof. It is easy to check that $pc_2(G) = 2$ for $3 \leq n \leq 6$ and $pc_2(G) = 3$ for $7 \leq n \leq 8$. Now let $n \geq 9$. We will give a 2-proper coloring of G using $c = \lceil \sqrt[3]{n} \rceil$ colors and we will also show that this is the best possible. Consider the bipartition of $G = V \cup W$ such that $|V| = n$ and $|W| = 3$. Let $V = \{v_1, ..., v_n\}$ and $W = \{w_1, w_2, w_3\}$. For each vertex $v_i \in V$, we consider a 3-tuple $C_i = (c_1, c_2, c_3)$ so that $c(v_i w_j) = c_j$ for $1 \leq j \leq 3$. Therefore, each vertex $v_i \in V$ has c^3 different ways of coloring its incident edges using c colors. We then color the edges of G as follows. If $c \geq 4$ then we color the edges of $(c-1)^3$ vertices of V with all the different triples of $c - 1$ colors and, for the remaining vertices, we choose different triples but this time using the c^{th} color. If $c = 3$, we just choose different triples of colors but starting with the $c!$ colorings with all three colors different. By this coloring we have that for each pair of vertices $v_i, v_j \in V$ we have that $C_i \neq C_j$ for all $1 \leq i \neq j \leq n$.

Before proving that this coloring is 2-proper, it is easy to see that G cannot be colored to make it 2-proper connected using fewer than c colors by following the same argument as in Proposition 3.1.2. That is, if we use fewer than c colors, there must exist at least two vertices $v_i, v_j \in V$ such that $C_i = C_j$, a contradiction.

Now consider two vertices $v_i, v_j \in V$ and we would like show the existence of 2-proper paths between them. Since $C_i \neq C_j$, we know that at least one of the three colors is different. If two or three are different, then we have 2-proper paths of the form $v_i w_k v_j$ and $v_i w_l v_j$ such that $c(v_i w_k) \neq c(v_j w_k)$ and $c(v_i w_l) \neq c(v_j w_l)$. Suppose now that exactly one of the three colors is different, say c_1 without losing generality, then $v_i w_1 v_j$ is a proper path. For the second path, there exists a vertex $v_k \in V$ such that, by construction of the coloring, $c(v_i w_2) \neq c(v_k w_2)$, $c(v_j w_3) \neq c(v_k w_3)$ and $c(v_k w_2) \neq c(v_k w_3)$. Therefore $v_i w_2 v_k w_3 v_j$ is a proper path between v_i and v_j.

Next consider $w_i, w_j \in W$, it is clear that there exist two vertices $v_k, v_l \in V$ such that C_k and C_l have both colors different to w_i, w_j. Therefore $w_i v_k w_j$ and $w_i v_l w_j$ are proper paths. Finally, we consider the case where $v_i \in V$ and $w_j \in W$. The edge $v_i w_j$ provides a trivial proper path. For the second path, simply choose other appropriate vertices $v_k \in V$ and $w_l \in W$ such that $v_i w_l v_k w_k$ results in a proper path. These vertices exist by the constructed coloring of G. As no cases are left, the theorem holds. □

Now we prove the conjecture for complete bipartite graphs.

Theorem 3.1.4. *Let $G = K_{n,m}$, $m \geq n \geq 2k$ for $k \geq 1$. Then $pc_k(G) = 2$*

Proof. Take the bipartition of $G = A \cup B$. Then split each set A and B into the sets A_1, A_2, B_1, B_2 such that $|A_i|, |B_i| \geq k$ for $i = 1, 2$. This is clearly possible since $|A|, |B| \geq 2k$. Now color the graph in the following way. Put $c(vw) = 1$ for all $v \in A_1$ and $w \in B_1$, and for all $v \in A_2$ and $w \in B_2$. Finally put color 2 to the rest of the edges, that is, $c(vw) = 2$ for all $v \in A_1$ and $w \in B_2$, and for all $v \in A_2$ and $w \in B_1$ (see Figure 3.1). Now we prove that this coloring produces k proper paths between each pair of vertices of G. First, consider two vertices $v, w \in A_1$ (an identical argument holds for pairs in other sets). Since the cardinality of each set is at least k, we form k proper paths $v b_1 a_2 b_2 w$ choosing $b_1 \in B_1, a_2 \in A_2$ and $b_2 \in B_2$. If $v \in A_1$ and $w \in A_2$ (similarly for $v \in B_1$ and $w \in B_2$) we have at least $2k$ proper paths formed as vbw for each choice of $b \in B$. The final case is when $v \in A_1$ and $w \in B_1$ (that is, v and w are adjacent). Here we have at least $k + 1$ proper paths, as follows. One path is simply the edge vw while the k that remain are of the form $v b_2 a_2 w$ for each choice of $b_2 \in B_2$ and $a_2 \in A_2$. This completes the proof. □

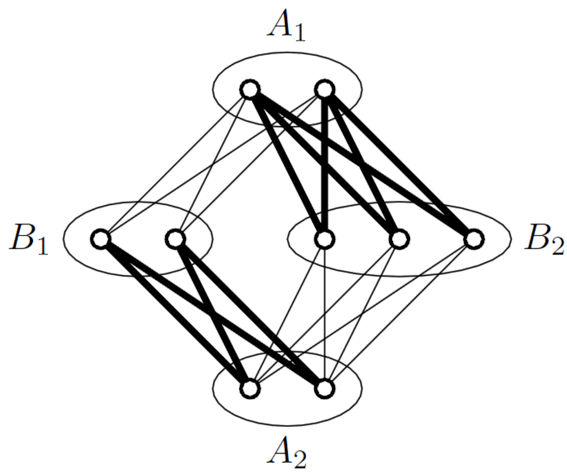

Figure 3.1: Coloring of $K_{4,5}$. Normal edges represent color 1 and bold edges color 2.

Now we will study the case $k = 1$, that is $pc(G)$. By König's Bipartite Theorem [67] we have that the edge chromatic number is Δ for bipartite graphs and therefore Δ is a trivial upper bound for $pc(G)$ for any bipartite graph G. Then, we obtain this trivial corollary.

Corollary 3.1.5. *If G is a tree then $pc(G) = \Delta$.*

The following theorem is the main result of the section. It improves upon the upper bound of Δ by König to the best possible whenever the graph is bipartite and 2-edge-connected.

Theorem 3.1.6. *Let G be a graph. If G is bipartite and 2-connected then $pc(G) = 2$ and there exists a 2-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exists two paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

Given a 2-connected graph G, let G_1 be an instance of the graph $G \setminus P$ where P is the set of internal vertices of the last ear of an ear decomposition of a G. Similarly, if the graph is 2-edge-connected, there is a (closed) ear decomposition in which an ear may attach to the previous structure at a single vertex. Therefore, using the same argument, one could easily show the result also holds for a 2-edge-connected graph G.

Proof. Suppose G is 2-connected and bipartite and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected).

For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply an even cycle and we alternate colors on the edges.

Let P be the last ear added where the ends u and v of P are in G_1 and all internal vertices of P are in $G \setminus G_1$. Since G is minimally 2-edge-connected, we know that the length of P is at least 2. By induction on the number of ears, we obtain a 2-coloring of G_1 so that G_1 has the strong property. Color P with alternating colors. Let C be the proper cycle of G such that $C = PP'$ where P' is the appropriate proper path in G_1 between u and v. Clearly this path exists, since in G_1 we have the strong property.

Finally we show that this coloring of G is proper connected with the strong property. Every pair of vertices in C has the strong property since C is an alternating even cycle. Also, by induction, every pair of vertices in G_1 has the strong property. Let $x \in G \setminus C$ and let $y \in P$. The pair xu has the strong property so there exists a path Q_u from x to u so that xQ_uuPy forms a proper path Q'_u. Similarly the pair xv has the strong property so there exists a path Q_v from x to v so that xQ_vvPy is a proper path Q'_v. Since C is a proper cycle, Q'_u and Q'_v must have different colors on the edges incident to y. Note also that, since G is bipartite, the parity of the length of Q'_u is the same as the parity of the length of Q'_v. Hence, Q'_u and Q'_v must also have different colors on the edges incident to x. This shows that x and y have the strong property, thereby completing the proof. □

As a result of Theorem 3.1.6 we obtain the following corollary.

Corollary 3.1.7. *Let G be a graph. If G is 3-connected, then $pc(G) = 2$ and there exists a 2-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that start$(P_1) \neq$ start(P_2) and end$(P_1) \neq$ end(P_2).*

Proof. By [91], any 3-connected graph has a spanning 2-connected bipartite subgraph. Then the result holds by Theorem 3.1.6 □

3.2 General graphs

We begin this section by studying $pc(G)$ for a general graph G. We show some easy results for specific classes such as complete graphs and cycles. Following this, we prove a result analogous to that obtained in the previous section for 2-connected graphs but using 3 colors instead of 2. We also show that this bound is sharp by presenting a 2-connected graph for which 2 colors are not enough to make it proper connected. As a main result of the section, we state an upper bound for $pc(G)$ for general graphs that can be possibly reached as we saw in the previous section. Based on the results of 2-connected graphs we extend Conjecture 3.1.1 to general graphs and finally we prove this for complete graphs.

By Vizing's Theorem [107], we have that the edge chromatic number of any graph is at most $\Delta + 1$ and therefore $\Delta + 1$ is a trivial upper bound for $pc(G)$ for any graph G. First we present some easy results.

Fact 3.2.0.1. *A graph G has $pc(G) = 1$ if and only if G is complete.*

By using alternating colors, it is easy to see that any path of length at least 2 and any cycle of length at least 4 has proper connection number 2. Also it is clear that the addition of an edge to G cannot increase $pc_k(G)$.

Fact 3.2.0.2. *For $n \geq 3$, $pc(P_n) = 2$ and if $n \geq 4$, $pc(C_n) = 2$. Furthermore, pc_k is monotone decreasing with respect to edge addition.*

We present now the following proposition.

Proposition 3.2.1. *If $pc(G) = 2$ then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$.*

Proof. Let u, w be two neighbors of v in G. Since we have assumed there is a 2-coloring of G so that G is properly connected, there is a properly colored path P from u to w in G. Color the edge uv so that $c(uv) \neq start(P)$ and color vw so that $c(vw) \neq end(P)$. Since every vertex of G has a properly colored path to a vertex of P, every vertex has a properly colored path to v through either u or w, thereby completing the proof. □

The following theorem improves the Vizing's $\Delta + 1$ upper bound whenever the graph is 2-connected. This result is a natural extension of Theorem 3.1.6.

Theorem 3.2.2. *Let G be a graph. If G is 2-connected, then $pc(G) \leq 3$ and there exists a 3-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

As in Theorem 3.1.6, we note that an edge-connected version of this result is immediate from the proof.

Proof. Suppose G is a 2-connected graph and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected). For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply a cycle and we properly color the edges with at most 3 colors.

Let P be the last ear added in an ear decomposition of G and let G_1 be the graph after removal of the internal vertices of P. Since G is assumed to be minimally 2-connected, we know that P has at least one internal vertex. Let u and v be the vertices of $P \cap G_1$ so $P = uu_1u_2 \ldots u_pv$.

By induction, there is a 3-coloring of G_1 which is proper connected with the strong property. Color the edges of G_1 as such.

Within this coloring, there exist two paths P_1 and P_2 from u to v such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$. If possible, we properly color the path P so that $c(uu_1) \notin \{start(P_1), start(P_2)\}$ and $c(u_pv) \notin \{end(P_1), end(P_2)\}$. Note that this is always possible if either P has at least 2 internal vertices or $\{start(P_1), start(P_2)\} \cup \{end(P_1), end(P_2)\} = \{1, 2, 3\}$. It will become clear that this is the easier case so we assume this is not the case, namely that P has only one internal vertex x and $\{start(P_1), start(P_2)\} \cup \{end(P_1), end(P_2)\} = \{1, 2\}$.

Color the edge xu with color 3 and xv with color 2 (supposing that $end(P_2) = 2$). We will show that this coloring of G is proper connected with the strong property. For any pair of vertices in G_1, there is a pair of proper paths connecting them with the strong property by induction. Since $P \cup P_1$ forms a proper cycle, any pair of vertices in this cycle also have the desired paths. Let $y \in G_1 \setminus P_1$ and note that our goal is to find two proper paths from x to y with the strong property.

Since y and u are both in G_1, there exist a pair of paths P_{u_1} and P_{u_2} starting at y and ending at u with the strong property. Similarly, there exist two paths P_{v_1} and P_{v_2} starting at y and ending at v with the strong property. Since these paths have the strong property, we know that $Q_1 = xuP_{u_i}y$ (note that the implied orientation on P_{u_i} is reversed when traversing the path from u to y) is a proper path for some $i \in \{1, 2\}$ (suppose $i = 1$) and similarly $Q_2 = xvP_{v_j}y$ is a proper path for some $j \in \{1, 2\}$ (suppose $j = 1$). These paths form the desired pair if $end(Q_1) \neq end(Q_2)$ so suppose $start(P_{v_1}) = start(P_{u_1})$.

Next consider walk $R_1 = xuP_1vP_{v_2}y$ and the path $R_2 = Q_2$. If R_1 is a path, then R_1 and R_2 are the desired pair of paths since $end(P_1) \neq c(xv) = end(P_{v_2})$, meaning that R_1 is a proper walk. Hence, suppose R_1 is not a path and let z be the vertex closest to y on P_{v_2} which is in $P_1 \cap P_{v_2}$. Now if the path $R_1' = xuP_1zP_{v_2}y$ is a proper path, then R_1' and R_2 are the desired pair of paths so we may assume that $end(uP_1z) = start(zP_{v_2}y)$.

Finally we show that the paths $S_1 = xvP_1zP_{v_2}y$ and $S_2 = Q_1 = xuP_{u_1}y$ are proper paths from x to y with the strong property. Certainly, as noted above, S_2 is a proper path. Also, S_1 is a proper path since P_1 is proper so $end(vP_1z) \neq end(uP_1z) = start(zP_{v_2}y)$. Finally since $end(zP_{v_2}y) = start(P_{v_2}) \neq start(P_{v_1}) = start(P_{u_1})$, we see that S_1 and S_2 have the strong property. □

It is important to mention that there exist 2-connected graphs with $pc(G) = 3$ and therefore the bound obtained by Theorem 3.2.2 is reached. Now we give an example (see Fig. 3.2) of such a graph and prove why two colors are not enough.

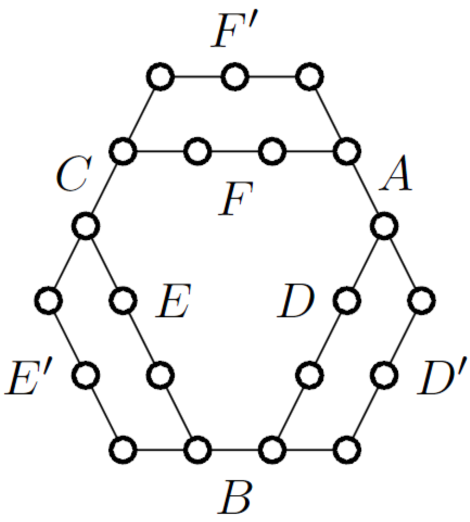

Figure 3.2: Smallest 2-connected graph with $pc(G) = 3$

Proposition 3.2.3. *Any graph G consisting of an even cycle with the addition of three ears creating disjoint odd cycles such that each uninterupted segment has at least 4 edges has $pc(G) = 3$.*

The assumption that each uninterupted segment has length at least 4 is mostly for convenience. Note that the graph G (in Figure 3.2) does not satisfy this condition but it can still be shown that $pc(G) = 3$ by a similar argument.

Proof. By Theorem 3.2.2, we know that $pc(G) \leq 3$ so it suffices to show that $pc(G) \neq 2$. Suppose we have a 2-coloring of G which is properly connected. Label the segments of G

as in Figure 3.2. Note that we may assume there are no three edges in a row of the same color within an uninterupted segment since we could switch the color of the middle edge (making that subsegment alternating) without disturbing the proper connectivity.

We would first like to show the segments A, B and C are all alternating. If two of these segments are not alternating, suppose A and B, then any vertex in D cannot be properly connected to any vertex of C so this is clearly not the case. This means that at most one segment, suppose A, is non-alternating. Suppose the edges uv and vw have the same color for some $u, v, w \in A$ (see Figure 3.3). There must exist a proper path from u to w so suppose there is such a path using the segments $FCEBD$. Since the following argument does not rely on the parity of this path, this assumption, as opposed to using any of D', E' or F', does not lose any generality.

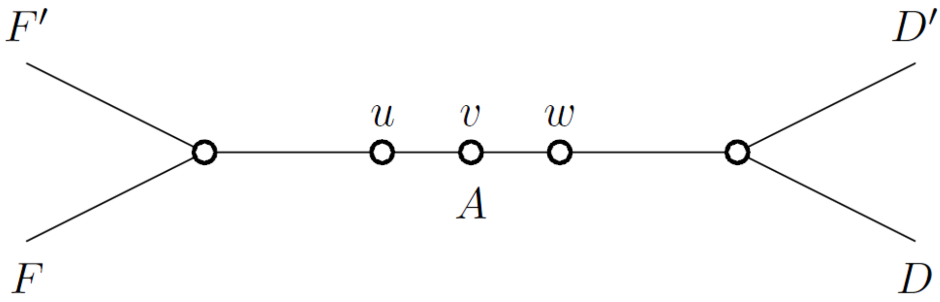

Figure 3.3: Placement of vertices 1.

Let x be a vertex in the interior of B. We already know there is a proper path from x to v using D. Since $D \cup D'$ forms an odd cycle, there can be no proper path from x to v through D'. Let $y \in E'$. In order for y to have a proper path to w, it must use the segments BD (as opposed to BD') and similarly to reach u, it must use CF (as opposed to CF'). Since $E \cup E'$ forms an odd cycle, and yet y can reach both u and w, we know that the edges on either side of y must have the same color. This holds for all $y \in E'$, clearly a contradiction. Therefore we know that A, B and C are all alternating segments.

Next we would like to show that at least one of D or D' must be alternating (and similarly at least one of E or E' and one of F or F'). Suppose D and D' are both non-alternating. Let v be an interior vertex in D which has two edges of the same color and let y be a vertex of D' with two edges of the same color. Let u and w be the neighbors of v and let x and z be the neighbors of y (see Figure 3.4). Clearly there can be at most one pair (in this case D and D') in which neither segment is alternating since there must be an alternating path from u to w and it must pass through the other segments. Also, there can be no other pairs of adjacent monochromatic edges within D and D' since u, v and w (likewise x, y and z) must have alternating paths out of the segment and we have assumed that there are no three edges of the same color in a row. Note that, in the figure, possibly $x = a$, $u = a$, $z = b$ or $w = b$.

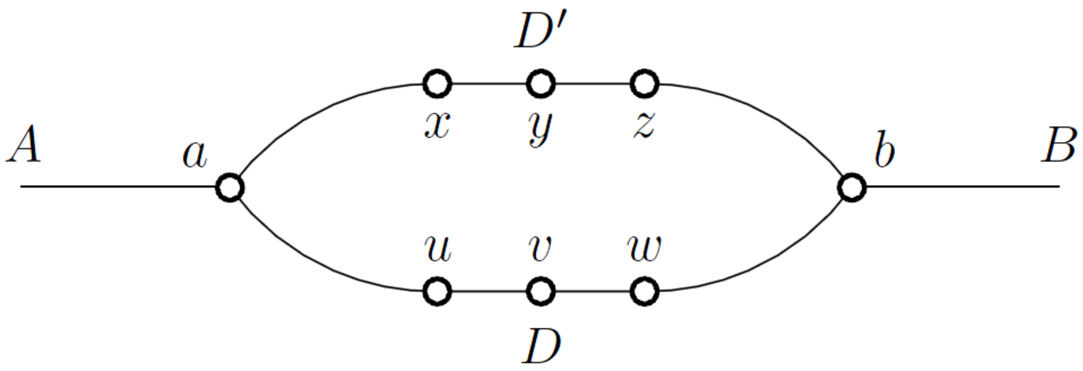

Figure 3.4: Placement of vertices 2.

Let $Q = D \cup D'$ and let a and b be the vertices in $D \cap D' \cap A$ and $D \cap D' \cap B$ respectively. If we let $c \in C$, then each of u, w, x and z must have an alternating path to c. Suppose the edge of A incident to a has color 1. Then both edges incident to a in Q must have color 2. This means that both edges of Q which are incident to a must be the same color (and similarly both edges of Q incident to b must have the same color). Therefore, there are exactly 4 vertices in Q for which both edges of Q have the same color. Unless $x = a$ (or possibly $z = b$, $u = a$ or $w = b$), this means that Q is even, a contradiction. Suppose $x = a$ so, in order for $z \neq b$ to have a proper path to w, we must also have $w = b$, meaning that $u \neq a$ and z so again Q is even for a contradiction. Hence, we know that at least one of D or D' must be alternating (and similarly for the other odd ears). Without loss of generality, suppose D, E and F are all alternating.

Our next goal is to show that $Q = A \cup B \cup C \cup D \cup E \cup F$ forms an alternating cycle (with the possible replacement of D with D', E with E' or F with F'). As we have shown, the only places where we can have a problem is at the intersections so let a and b be (as before) the end-vertices of D (the same argument may be applied for E or F) and suppose a is between two edges of the same colors (suppose color 1) on Q. Let u, v, w

be the neighbors of a with $u \in A$, $v \in D'$ and $w \in D$ so we have assumed the edges au and aw both have color 1 (see Figure 3.5 where the darker edges represent edges that must have color 1). In order for an alternating path to get from u to w, we must either use $D' \cup D$ or Q (with the possible replacements noted above). If the path uses D', then $D \cup D'$ forms an alternating (and hence even) cycle, a contradiction. Hence, we may assume there is an alternating path from u to w through $BECFA$ (recall again that E may be replaced with E' or F with F' in this argument).

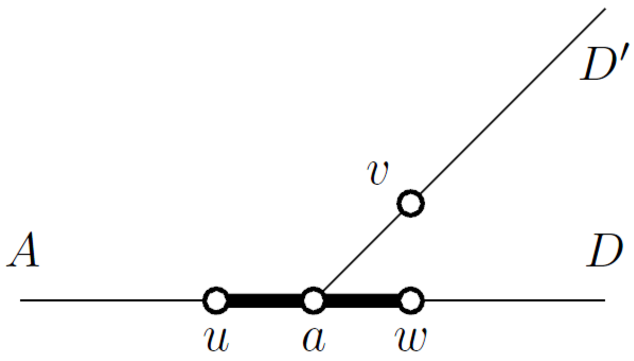

Figure 3.5: Placement of vertices 3.

Let $x \in E'$. There is an alternating path from u to x and from w to x. Since $E \cup E'$ forms an odd cycle but x has an alternating path through B (to get to w) and through C and A (to get to u), we know that x must have two edges of the same color within E'. Since x was chosen arbitrarily, this is clearly a contradiction. This means that Q is an alternating (and hence even) cycle.

Now we simply consider one vertex in each of D', E' and F'. Since these ears form odd cycles, there exists a vertex in each segment from which (and to which) an alternating path can only go one direction on Q. By the pigeon hole principle, at least two of them must go the same direction, meaning there is no alternating path between them. This completes the proof of Proposition 3.2.3. □

If the diameter is small, then the proper connection number is also small. More formally, we get the following result.

Theorem 3.2.4. *If $diam(G) = 2$ and G is 2-connected, then $pc(G) = 2$.*

Proof. If G is 3-connected, Corollary 3.1.7 implies that $pc(G) = 2$ so we may assume $\kappa(G) = 2$. Let $C = \{c_1, c_2\}$ be a (minimum) 2-cut of G and let $H_1, \ldots, H_t$ be the components of $G \setminus C$. Order components so that there is an integer $0 \leq s \leq t$ such that every vertex of H_i is adjacent to both c_1 and c_2 for $i > s$. Note that if $s = 0$, we have all edges from C to $G \setminus C$ so G contains a spanning 2-connected bipartite graph and by Theorem 3.1.6, $pc(G) = 2$. For each component H_i with $i \leq s$, define subsets $H_{i,1} = N(c_1) \cap H_i$ and $H_{i,2} = N(c_2) \cap H_i$. Since each component is connected and C is a minimum cut, there must be an edge from $H_{i,1}$ to $H_{i,2}$. Let $e_i = v_{i,1}v_{i,2}$ be one such edge in each component H_i. Define the graph $G_0 = G[C \cup (\bigcup_{i=1}^{s}\{v_{i,1}, v_{i,2}\})]$. This graph is 2-connected and bipartite so $pc(G_0) = 2$ and notice that $|G_0| = 2 + 2s$. Let G_1 be a subgraph of G obtained by adding a vertex to G_0 which has at least 2 edges into G_0. Furthermore, let G_i be a subgraph of G obtained by adding a vertex to G_{i-1} which has at least 2 edges into G_{i-1}. By Proposition 3.2.1, $pc(G_i) = 2$ for all i. We claim that there exists such a sequence of subgraphs of G such that $G_{n-(2+2s)}$ is a spanning subgraph of G. In order to prove this, suppose that G_i is the largest such subgraph of G and suppose there exists a vertex $v \in G \setminus G_i$. Certainly every vertex which is adjacent to both c_1 and c_2 is in G_i. This means $v \in H_j$ for some $1 \leq j \leq s$. Since H_j is connected, there exists a path from $v_{i,1}$ to v within H_j. Let w be the first vertex on this path which is not in G_i. Since $diam(G) = 2$, we know that w must be adjacent to at least one vertex of C. This means that $d_{G_i}(w) \geq 2$ so we may set $G_{i+1} = G_i \cup w$ for a contradiction. This completes the proof. □

Finally we prove an upper bound for $pc(G)$ for general graphs which is best possible as we saw before.

Theorem 3.2.5. *Let G be a connected graph. Consider $\widetilde{\Delta}(G)$ as the maximum degree of a vertex which is an endpoint of a bridge in G. Then $pc(G) \leq \widetilde{\Delta}(G)$ if $\widetilde{\Delta}(G) \geq 3$ and $pc(G) \leq 3$ otherwise.*

Proof. Let $B_1, B_2, \ldots B_s$ be the blocks of G with at least 3 vertices. For each block of B_i we have the following cases.

- B_i is bipartite or 3-connected: Then by Theorem 3.1.6 and Corollary 3.1.7, B_i can be colored with 2 colors having the strong property. We color B_i in such a way.
- $\kappa(B_i) = 2$: Then by Theorem 3.2.2, B_i can be colored with 3 colors having the strong property. We color B_i in such a way.

It is easy to see that G is proper connected if there are no more uncolored edges in G since each B_i has the strong property. Thus, suppose that there remain uncolored edges in G. It is clear that these edges induce a forest F in G. We color them as follows. Take one of the blocks, say B_1, which contains a vertex $v \in B_1$ which is incident with some uncolored edges. Clearly, v is an endpoint of a bridge in G. We color these uncolored edges incident to v with different colors starting with color $rd_{B_1}(v) + 1$. Then, we have that $rd_G(v) \leq \widetilde{\Delta}(G)$. We do the same for the rest of the vertices incident to bridges in B_1. Then, we extend our coloring for each tree going out from B_1 in a Breadth First Search (BFS) way, coloring its edges with different colors (observe from Corollary 3.1.5 that $rd_G(w) \leq \widetilde{\Delta}(G) \leq \Delta$ for each vertex w in the interior of a tree) until we reach the rest of the blocks. And finally, for each of these blocks (in this order), we repeat the previous step. Before proving that this coloring makes G proper connected, it is important to mention that, if we reach a block B_i with some color $c \geq rd_{B_i}(w) + 1$, and the corresponding vertex, say w, of B_i has more than $c - rd_{B_i}(w)$ uncolored incident edges, then, when we color these edges, we do not repeat color c. Also, it is important to remark that, by coloring F in this way, we have that in any path that traverses some block from one tree in F to another, at least one of the colors before or after traversing the block is not used in the block.

We now prove that G is proper connected. Let v, w be vertices of G. It is clear that if both belong to the same block B_i, then there exists a proper path between them and the same happens if they belong to the same tree outside the blocks. If $v \in B_i$, $w \in B_j$ and $B_i \cap B_j = \{u\}$, then there exist two paths P_1, P_2 between v and u in B_i, and two paths P_3, P_4 between u and w in B_j with the strong property. Suppose without losing generality that $\mathit{end}(P_1) \neq \mathit{start}(P_3)$ and $\mathit{end}(P_2) \neq \mathit{start}(P_4)$, then we obtain the paths P_1P_3 and P_2P_4 between v and w. It is clear that $\mathit{start}(P_1P_3) \neq \mathit{start}(P_2P_4)$ and $\mathit{end}(P_1P_3) \neq \mathit{end}(P_2P_4)$ since $\mathit{start}(P_1P_3) = \mathit{start}(P_1) \neq \mathit{start}(P_2) = \mathit{start}(P_2P_4)$ and $\mathit{end}(P_1P_3) = \mathit{end}(P_3) \neq \mathit{end}(P_4) = \mathit{end}(P_2P_4)$. Therefore, these paths are proper. Now, if $B_i \cap B_j = \emptyset$ and there is a tree T in F such that $B_i \cap T = \{u_1\}$ and $B_j \cap T = \{u_2\}$, we form a proper path between v and w as follows. Let P_1 be the unique (proper) path in the tree T between u_1 and u_2. Let P_2 be the proper path in B_i between v and u_1 such that $\mathit{end}(P_2) \neq \mathit{start}(P_1)$. This path exists since we have the strong property in each block. Analogously, let P_3 be the proper path in B_j between u_2 and w such that $\mathit{end}(P_1) \neq \mathit{start}(P_3)$. Finally the path $P = P_2P_1P_3$ is proper between v and w. The same idea applies if v is in a block B_i and w is in a tree T in F such that $B_i \cap T = \{u\}$.

The idea also applies in the case that v is in a tree T_i in F, w is in a tree T_j in F and there is a block B such that $T_i \cap B = \{u_1\}$ and $T_j \cap B = \{u_2\}$. Finally, the result holds by induction on the number of trees and blocks between vertices v and w using the remark stated before to guarantee the paths always traverse the blocks. Therefore, $pc(G) \leq \widetilde{\Delta}(G)$ if $\widetilde{\Delta}(G) \geq 3$ and $pc(G) \leq 3$ otherwise. □

To end the section, based on the Theorem 3.2.2 and the previous section, we extend the Conjecture 3.1.1 to general graphs.

Conjecture 3.2.6. *If G is a $2k$-connected graph with $k \geq 1$, then $pc_k(G) \leq 3$.*

This conjecture is proved for $k = 1$ in Theorem 3.2.2. Now we prove a stronger result for complete graphs.

Theorem 3.2.7. *Let $G = K_n$, $n \geq 4$, and $k > 1$. If $n \geq 2k$ then $pc_k(G) = 2$*

Proof.

- Case $n = 2p$ for $p \geq 2$:

  Take a hamiltonian cycle $C = v_1v_2 \ldots v_{2p}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. Color the rest of the edges using color 1. It is clear that there are $p \geq k$ edges with color 2. We will prove that this coloring gives us k proper paths between each pair of vertices of G. Take two vertices v, w such that $c(vw) = 2$. This edge colored with color 2 is one proper path between v and w. Now, since there are at least other $p - 1 \geq k - 1$ edges colored with color 2 and the rest of the edges are colored with color 1, we have at least $k - 1$ proper paths between v and w using these edges. That is, for each vertices v', w' such that $c(v'w') = 2$ we form the proper path $vv'w'w$. The case where $c(vw) = 1$ is similar.

- Case $n = 2p - 1$ for $p \geq 2$:

  Take a hamiltonian cycle $C = v_1v_2 \ldots v_{2p-1}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. We have p edges with color 1 and $p - 1$ edges with color 2 so far since $c(v_1v_2) = 1$ and $c(v_1v_{2p-1}) = 1$. Now, put $c(v_2v_{2p-1}) = 2$, $c(v_1v_3) = 2$, $c(v_1v_{2p-2}) = 2$ and for each edge with color 2, different from v_2v_3 and $v_{2p-2}v_{2p-1}$, choose one of the endpoints, say v', and put $c(v_1v') = 2$ (see Fig. 3.6). Finally, color the rest of the edges with color 1. We now show that this coloring gives k proper paths between each pair of vertices v and w of G. First, take $v = v_1$ and $w = v_2$ (or similarly taking $w = v_{2p-1}$). We have the edge v_1v_2 and the path $v_1v_{2p-1}v_2$. Now since $n = 2p - 1 \geq 2k$ we have at least $(p - 1) - 2 \geq k - 2$ edges in the cycle C with color 2 different from v_2v_3 and $v_{2p-2}v_{2p-1}$ and therefore we form the following $k - 2$ proper paths between v_1 and v_2 of the form $v_1v'v_2$ where v' is an endpoint of each of these edges such that $c(v_1v') = 2$. Now take $v = v_1$

and $w = v_3$ (analog taking $w = v_{2p-2}$). This case is similar to the previous except changing the second formed path to $v_1v_2v_3$. Suppose now that $v = v_1$ and $w = w'$ with $w' \notin \{v_2, v_3, v_{2p-2}, v_{2p-1}\}$. We take the edge v_1w' and now, since there are at least $i(p-1) - 1 \geq k - 1$ edges in the cycle C with color 2 with endpoints different from v', we form the following $k - 1$ proper paths between v_1 and w' of the form $v_1v'w'$ where v' is an endpoint of each of these edges such that $c(v_1v') = 2$. The rest of the cases are similar to those described before in the case $n = 2p$ forming most of the proper paths with length 3.

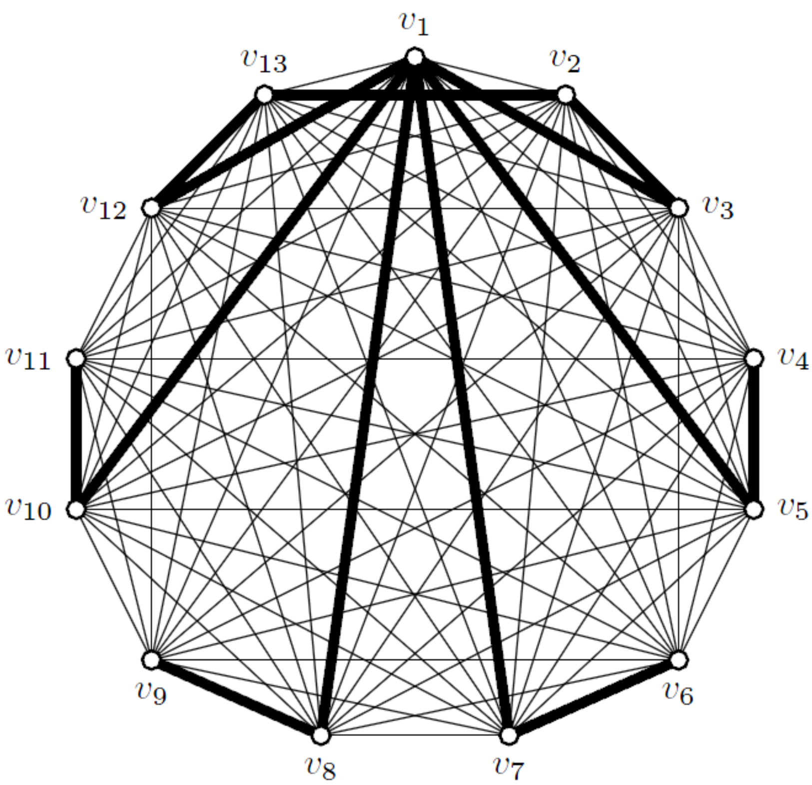

Figure 3.6: Coloring of K_{13}. Normal edges represent color 1 and bold edges color 2.

□

3.3 Minimum degree

In this section, we prove a result concerning minimum degrees. For this, we will make use of the following theorems.

Theorem 3.3.1 ([40]). *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n-1}{2}$, then G has a hamiltonian path. Moreover, if $\delta(G) \geq n/2$, then G has a hamiltonian cycle. Also, if $\delta(G) \geq \frac{n+1}{2}$, then G is hamilton-connected.*

Theorem 3.3.2 ([110]). *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n+2}{2}$ then G is panconnected meaning that, between any pair of vertices in G, there is a path of every length from 2 up to $n-1$.*

Theorem 3.3.3 ([90]). *Let G be a 3-connected graph with n vertices and $\delta(G) \geq n/4+2$. Then, for any longest cycle C in G, every component of $G-C$ has at most two vertices.*

Theorem 3.3.4 ([42]). *Let G be a connected graph with n vertices and $\delta(G) \geq n/3$. Then one of the following holds:*

(i) G contains a hamiltonian path.

(ii) For any longest cycle C of G, $G-C$ has no edge.

Also we use the following easy fact as a matter of course.

Fact 3.3.4.1. *Every 2-connected graph G with $\delta(G) \geq 2$ is either hamiltonian or contains a cycle C with at least $2\delta(G)$ vertices.*

For this statement, we use the following notation. For a path $P = v_1 v_2 \cdots v_\ell$, we let $endpoints(P) = \{v_1, v_\ell\}$.

Lemma 3.3.5. *The following graphs H_i, for $(i = 1, 2, \ldots, 6)$, have $pc(H_i) = 2$.*

(1) The graph H_1 obtained from a path P with $|P| \geq 2$ and $m \geq 0$ isolated vertices $v_1, \ldots, v_m$ by joining each v_i for $(i \leq m)$ within P with at least two edges.

(2) The graph H_2 obtained from a path P with $|P| \geq 1$ and even cycle C by identifying exactly one vertex (i.e., $|P \cap C| = 1$).

(3) The graph H_3 obtained from H_2 and $m \geq 0$ isolated vertices $v_1, \ldots, v_m$ by joining each v_i for $(i \leq m)$ with at least two edges to either $P-C$ or $C-P$ in H_2.

(4) The graph H_4 obtained from an even cycle C and two paths P_1 and P_2 by identifying an end of each path to a vertex of C. As in H_3, we may also join vertices each with at least 2 edges to either a path P_i or C.

(5) The graph H_5 obtained from the union of two disjoint cycles which are connected by two disjoint paths to form a 2-connected graph. Furthermore, we may also add vertices each with at least 2 edges to this structure.

(6) The graph H_6 obtained from H_5 by removing an edge from one of the cycles. Again we may add vertices each with at least 2 edges to this structure.

Proof. One can easily get a 2-coloring of H_i which forces $pc(H_i) = 2$ for $i = 1, 2, \dots, 6$. For example, as for H_1, by Fact 3.2.0.2 and Proposition 3.2.1, there is a 2-coloring of H_1 that is properly connected. □

We are now ready to prove our main result.

Theorem 3.3.6. *If G is a connected non-complete graph with $n \geq 68$ vertices and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

Proof. If $\kappa(G) \geq 3$, then by Corollary 3.1.7, we have $pc(G) = 2$. So we may assume that $\kappa(G) = 1$ or 2. We divide the proof into two cases according to the value of $\kappa(G)$.

Case 1: $\kappa(G) = 1$.

Let v be a cutvertex of G and let $C_1, \dots, C_\ell$ be the components of $G \setminus v$ such that $|C_1| \leq \dots \leq |C_\ell|$. By the minimum degree condition, we see that $\ell = 2$ or 3 and $|C_1| \geq n/4$. We further divide the proof into two subcases:

Subcase 1.1: $\ell = 2$.

In this case note that $|C_1| \leq (n-1)/2$ and, by the minimum degree condition, $|C_2| \leq 3n/4 - 1$. Utilizing Theorem 3.3.1 and the minimum degree condition, it is easy to check that $\langle \{v\} \cup C_1 \rangle$ contains a hamiltonian path P_1 such that $v \in endpoints(P_1)$. If $\kappa(C_2) \geq 3$, then let C be a longest cycle of C_2. Since G is connected, there is a path P' from v to C. Now $H = P_1 \cup P' \cup C$ satisfies the conditions of H_2 in Lemma 3.3.5. This means that $pc(H) = 2$. By Theorem 3.3.3, every component of $C_2 \setminus C$ has at most 2 vertices. By the minimum degree condition and since we assume $n \geq 12$, for each $x \in C_2 \setminus H$, we have $|E(x, H)| \geq \frac{n}{4} - 1 \geq 2$. Hence, G contains a spanning subgraph which satisfies the properties of H_3 in Lemma 3.3.5 so $pc(G) = 2$. Thus we may assume that $\kappa(C_2) = 1$ or 2. Let S be a cutset in C_2 with $1 \leq |S| \leq 2$. By the minimum degree condition, it is easy to check that there are exactly two components C_{21}, C_{22} with $|C_{21}| \leq |C_{22}|$ in $C_2 - S$. Note that $n/4 - |S| \leq |C_{21}| \leq |C_{22}| \leq (3n/4 - 1) - |S| - (n/4 - |S|) = n/2 - 1$ because $\delta(G) \geq n/4$ and $|C_{21}| \leq (3n/4 - 1 - |S|)/2 = 3n/8 - (|S| + 1)/2 \leq 3n/8 - 1$. Hence by Theorem 3.3.1, C_{21} contains a hamiltonian cycle C'_{21}. Since $\delta(C_{22}) \geq n/4 - 3$, C_{22} is either hamiltonian or contains a cycle C'_{22} with $|C'_{22}| \geq n/2 - 6$. Now take a path P_2 with $v \in endpoints(P_2)$ so that

(1) P_2 contains a longer segment of C'_{2j} for each $j = 1, 2$, and subject to condition (1),

(2) $|P_2|$ is as large as possible.

By the choice of P_2, note that $P_2 \cap S \neq \emptyset$. Let P be a path joining P_1 and P_2 at the common vertex v. Then, utilizing P and the assumption $\delta(G) \geq n/4$, we will find a spanning subgraph which has a property of H_1 in Lemma 3.3.5. In order to show this, we need only show that each vertex in $G \setminus P$ has at least 2 edges to P. As previously discussed, we know that all vertices in C_1 have at least 2 edges to P_1 so we need only check vertices $x \in C_2 \setminus P_2$. If $x \in C_{21}$ then since $|P \cap C_{21}| \geq |C_{21}|/2$ and $|C_{21}| \leq 3n/8 - 1$, by the minimum degree condition, x has at least $n/4 - 3n/16 \geq 2$ edges to P since $n \geq 32$. For $x \in C_{22}$, we know $|C_{22}| \leq n/2 - 2$ and either C_{22} is hamiltonian or contains a cycle of length at least $n/2 - 6$. In either case, the same arguments easily show that x has at least 2 edges to P, meaning that $pc(G) = 2$.
Subcase 1.2: $\ell = 3$.
In this case, by the minimum degree condition, we see that $n/4 \leq |C_1| \leq (n-1)/3 \leq |C_3| \leq n/2 - 1$, and $|C_2| \leq 3n/8 - 1/2$. Hence by Theorem 3.3.1, each C_i with $i = 1, 2$ is hamilton-connected. Also, by the minimum degree condition and since $n \geq 36$, we see that $\delta(C_i) \geq (|C_i| + 2)/2$ for $i = 1, 2$ so for any vertex $z \in C_i$, $C_i - z$ is hamilton-connected. By Theorem 3.3.1, C_3 is hamiltonian so it contains a spanning path P with $v \in endpoints(P)$. If $|E(v, C_i)| \geq 2$ holds for $i = 1$ or 2 (suppose $i = 1$), then we can find an even cycle C in $C_1 \cup v$ such that $v \in C$ and $|C_1| \leq |C| \leq |C_1| + 1$. Using a hamiltonian path of C_2 ending at v, together with the path P and the even cycle C, we can easily find a spanning subgraph which satisfies the property of H_3 in Lemma 3.3.5, and hence $pc(G) = 2$. Thus we may assume that $|E(v, C_1)| = |E(v, C_2)| = 1$. This implies $|C_1| \geq n/4 + 1$, because there is a vertex of C_1 which is not adjacent to v. Then we get $|C_3| \leq n/2 - 3$ so $\delta(C_3) \geq n/4 - 1 \geq (|C_3| + 1)/2$. If $|C_3|$ is odd, then by Theorem 3.3.1, C_3 is hamiltonian connected. Hence, we can find an even cycle using all of C_3 and v and a single path through v using all of C_1 and $C2$. This provides a spanning subgraph satisfying the properties of H_3 in Lemma 3.3.5. If $|C_3|$ is even, then $\delta(C_3) \geq \lceil \frac{|C_3|+1}{2} \rceil = \frac{|C_3|+2}{2}$ so, by Theorem 3.3.2, C_3 is panconnected. Thus we can find an even cycle through $v \cup C_3$ which avoids exactly 1 vertex of C_3 again easily providing a subgraph satisfying the conditions of H_3 in Lemma 3.3.5. This shows that $pc(G) = 2$ and completes the proof of this case.

Case 2: $\kappa(G) = 2$.
Let u and v be a minimum cutset of G. Again we let $C_1, C_2, \ldots, C_\ell$ be the components of $G \setminus \{u, v\}$ with $|C_i| \leq |C_j|$ for $i \leq j$ and break the rest of the argument into cases based on the value of ℓ. Note that, since $\delta(G) \geq n/4$, we have $2 \leq \ell \leq 4$.
Subcase 2.1: $\ell = 4$
Since $\delta(G) \geq n/4$, we know that $n/4 - 1 \leq |C_1| \leq (n-2)/4 \leq |C_4| \leq n/4 + 1$. This means that $\delta(C_i) \geq |C_i| - 2$ for all i. The graph G is 2-connected so there are two independent edges from $\{u, v\}$ to each component C_i. With $n \geq 26$, we see that $|C_i| \geq 6$ so the minimum degree condition $\delta(C_i) \geq |C_i| - 2$ implies, by Theorem 3.3.2, that each component C_i is panconnected. This means that, if $|C_3 \cup C_4|$ is even, we may find a cycle through $\{u, v\} \cup C_3 \cup C_4$ using all the vertices, and if $|C_3 \cup C_4|$ is odd, we may find a similar cycle which misses exactly one vertex $w \in C_4$. This cycle, along with a spanning path of $u \cup C_1 \cup C_2$ and possibly w provides a spanning subgraph of G satisfying the properties of H_3 from Lemma 3.3.5, meaning that $pc(G) = 2$.
Subcase 2.2: $\ell = 3$
Since $\delta(G) \geq n/4$, we have $n/4 - 1 \leq |C_1| \leq |C_2| \leq (5n-4)/12$ and $\delta(C_i) \geq n/4 - 2$ so $\delta(C_i) \geq \frac{3|C_i|+1}{5} - 2$ for $i = 1, 2$. Since $n \geq 23$, this implies that $\delta(C_i) \geq \frac{|C_i|+1}{2}$ for $i = 1, 2$ so C_1 and C_2 are both hamiltonian-connected by Theorem 3.3.1. This means we may create a single cycle D_{12} using all of $C_1 \cup C_2$. If $\kappa(C_3) \geq 2$, then let D_3 be a longest cycle in C_3. Since $\delta(C_3) \geq \frac{n}{4} - 2$, we know $|D_3| \geq \min\{|C_3|, \frac{n}{2} - 4\}$. In either case every vertex of C_3 has at least 2 edges to H_3. Now since G is 2-connected, there exist two disjoint paths from D_{12} to D_3 meaning there is a spanning subgraph of G satisfying the conditions of the graph H_5. By Lemma 3.3.5, we have $pc(G) = 2$. If $\kappa(C_3) = 1$, then by Theorem 3.3.4, there is a spanning path P of C_3. The vertices u and v must each have at least one edge to P so $P \cup D_{12}$ forms a spanning subgraph of G satisfying the conditions of the graph H_6 in Lemma 3.3.5. Hence, $pc(G) = 2$.
Subcase 2.3: $\ell = 2$
If C_1 and C_2 are both 3-connected, then by Corollary 3.1.7, there is a 2-coloring of each with that strong property. Along with these colorings, we also color all edges between $\{u, v\}$ and C_i with color i. This coloring clearly shows that $PC(G) = 2$ so we may assume that at least one component C_i has $1 \leq \kappa(C_i) \leq 2$. Next we will suppose that $1 \leq \kappa(C_i) \leq 2$ for both $i = 1, 2$. In this case, by the minimum degree condition and the fact that G is 2-connected, we may easily show that each component is hamiltonian connected (since n is large) so G is hamiltonian. This means $pc(G) = 2$.

Finally, if we suppose C_1 is 3-connected while $1 \leq \kappa(C_2) \leq 2$, each possible case contains a large (almost spanning) subgraph with the properties of H_4 from Lemma3.3.5, meaning that $pc(G) = 2$. This completes the proof of Theorem 3.3.6. □

The minimum degree condition is best possible. To see this, we construct the following graph. Let G_i be a complete graph with $n/4$ vertices for $i = 1, 2, 3, 4$, and take a vertex $v_i \in G_i$ for each $1 \leq i \leq 4$. Let G be a graph obtained from $G_1 \cup G_2 \cup G_3 \cup G_4$ by joining v_1 and v_j with an edge for each $2 \leq j \leq 4$. Then the resulting graph G is connected and it has $\delta(G) = n/4 - 1$ and $pc(G) = 3$.

Chapter 4

Strong Edge-Colorings

This chapter is organized as follows: In Section 4.1 we improve known bounds for strong edge-colorings in k-degenerate graphs and chordless graphs. Then we give a polynomial time algorithm to find such colorings. In Section 4.2, we improve known bounds for outerplanar graphs and also, we give polynomial time algorithms to find the colorings.

4.1 k-degenerate graphs

In this section, we prove the general result for k-degenerate graphs. In the following, a vertex of degree k is called a *k-vertex*. Vertices of degree at most k and at least k, are respectively called *k^--vertex* and *k^+-vertex*. An edge incident to a 1-vertex is called a *pendant edge*. We may use *partial coloring* or *partial strong edge-coloring* to denote a strong edge-coloring of a subset of the edges of G. Given a partial coloring, the *colored degree* of a vertex x denoted $cd(x)$, is the number of colored edges incident to x. The following easy lemma from [29] shows a nice structural property of k-degenerate graphs.

Lemma 4.1.1 ([29]). *If G is a k-degenerate graph, then there is some $v \in V(G)$ such that at least $\max\{1, d(v) - k\}$ of its neighbors are k^--vertices.*

This implies the following facts.

Fact 4.1.1.1.

(1) We can construct any k-degenerate graph from the trivial graph, by adding edges pq such that at most k neighbors of p have degree more than k, and degree of q is at most k (in the present graph after adding e).

(2) *There is an ordering of edges, such that we can construct any k-degenerate graph from the empty graph, by adding edges in that order where each edge added satisfies the above property in the current graph.*

Now, we state the main theorem of the section.

Theorem 4.1.2. *Let G be k-degenerate, $k \geq 2$. Then, $\chi'_s(g) \leq (4k-1)\Delta - 2k^2 - k + 1$.*

Proof. We use induction on the number of colored edges of a partial coloring of G. Let $\mathcal{B} = \{1, 2, \ldots, (4k-1)\Delta - 2k^2 - k + 1\}$ be the set of colors. We assume that, there is a partial strong edge-coloring of t edges added according to the order given above, such that the coloring satisfies the following.

(1) The partial coloring is a strong edge-coloring of the colored subgraph, and draws its colors from $\mathcal{B}$.

(2) For every edge e incident to a vertex of colored degree at most $k-1$, having the color say c, no edge f at a distance at most one in the original graph is colored c. Note that, while the distance can be through a non-colored edge, both e and f are part of the partial coloring.

Base cases can be easily verified. We now extend the coloring to $t+1$ edges and show that the induction hypothesis still holds. Since we add edges according to the order given by Fact 4.1.1.1.(2), we have an uncolored edge $e = vw$ such that,

(1) At most k neighbors of v have high colored degree ($> k$).

(2) Colored degrees of all other neighbors of v except w are at most k and $cd(w) \leq k-1$.

Lemma 4.1.3. *Let $e = vw$ be an edge chosen to be colored according to the degeneracy order given by Fact 4.1.1.1.(2). The number of colored edges within distance 1 from e is at most $(4k-1)\Delta - 2k^2 - k$.*

For any vertex x, let $\alpha(x)$ denote the number of colored edges xy with $cd(y) \geq k+1$. Similarly, let $\beta(x)$ be the number of uncolored edges xy with $cd(y) \geq k$.

Proof. Consider the vertex v. By the choice of e, we have $\alpha(v) + \beta(v) \leq k$. If not, once $cd(v)$ becomes k, (which definitely happens in the coloring process since we have more than $k - \alpha(v)$ uncolored edges), the remaining uncolored edges incident to v violate the degeneracy order as both endpoints of these edges have colored degree at least k. Thus, if $cd(v) \geq k$, then $\beta(v) = 0$. Therefore, in the degeneracy ordering (Lemma 4.1.1) this will not happen. See Figure 4.1 for an illustration. Thus the number of colored edges at distance 1 from e through vertex v can be at most

$$\alpha(v)\Delta + \beta(v)(\Delta - 1) + (d(v) - (\alpha(v) + \beta(v)) - 1)\, k \leq k\Delta + (\Delta - k - 1)k$$

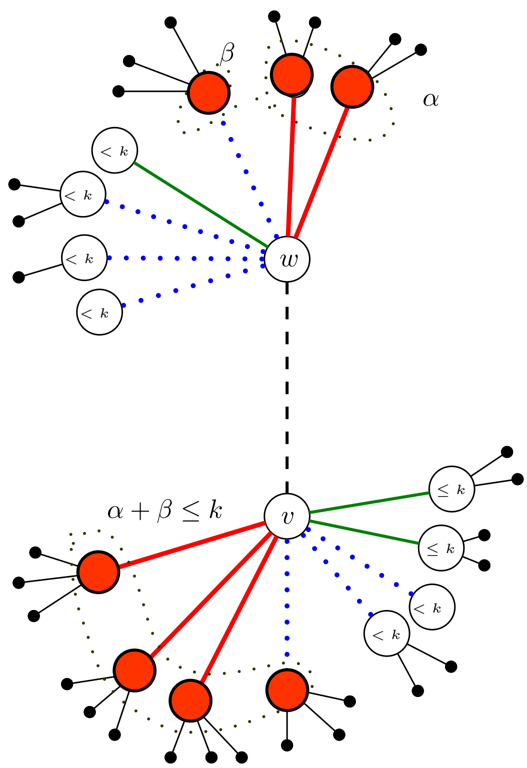

Figure 4.1: Local structure of a partial coloring when edge vw is selected. The dotted edges are yet to be colored and the shaded vertices potentially have colored degree more than k.

Now, consider w. Since $cd(w) \leq k-1$, $\alpha(w) \leq k-1$ and similar to the previous case, we have that $\alpha(w) + \beta(w) \leq k$. The number of colored edges at distance 1 from e through vertex w is at most $\alpha(w)\Delta + \beta(w)(\Delta-1) + (d(w) - (\alpha(w) + \beta(w)) - 1)\,(k-1) \leq (k-1)\Delta + \Delta - 1 + (\Delta - k - 1)(k-1)$. Summing up, the maximum number of colored edges within distance 1 from e is upper bounded by $k\Delta + (\Delta - k - 1)k + (k-1)\Delta + \Delta - 1 + (\Delta - k - 1)(k-1) = (4k-1)\Delta - 2k^2 - k$. □

Now, since $\mathcal{B}$ has strictly more colors, we can always find one color from $\mathcal{B}$ to color the edge e such that the partial coloring remains a strong edge-coloring. We only need to verify that the second condition of the induction hypothesis holds after the coloring.

This follows easily, since we ensure that the color selected for e is different from the colors used at any edges incident to any neighbor of v and w. Thus, we can use induction and the result follows.

□

By substituting $k = 2$, as a corollary to the above, we get

Corollary 4.1.4. *If G is 2-degenerate, then $\chi'_s(G) \leq 7\Delta - 9$.*

which improves the earlier upper bound. We also improve the bounds for chordless graphs as follows.

Theorem 4.1.5. *If G is chordless, then $\chi'_s(G) \leq 5\Delta - 5$.*

Proof. We know that, every minimally 2-connected graph is 2-degenerate, and their minimum degree is 2 [18]. A result from Plummer [97] states that a minimally 2-connected graph does not contain chords (chordless) and every 2-connected chordless graph is minimally 2-connected. Using the results from Plummer [97] and Dirac [41], the following lemma has been shown in [29, 86]

Lemma 4.1.6 ([86])**.** *Every chordless graph G contains some vertex x such that at least $d(x) - 1$ of its neighbors are 2^--vertices.*

The rest of the proof uses essentially the same arguments as for the above proof of k-degenerate graphs. The only difference is that, we can always find a vertex having at most one high degree neighbor. Thus, for each newly added edge, the maximum number of unavailable colors is $\Delta + 2(\Delta - 2) + \Delta + \Delta - 2 \leq 5\Delta - 6$. Thus, setting $\mathcal{B} = \{1, 2, \ldots, 5\Delta - 5\}$, the induction steps goes through and the result follows. □

4.1.1 Algorithmic aspects

In this section, we sketch an algorithm to color any $k-degenerate$ graph with $(4k-1)\Delta - 2k^2 - k + 1$ colors. The algorithm follows from the proof arguments of Theorem 4.1.2. As in the case of the proof, the algorithm consists of two phases. The first phase, where we compute an ordering of the edges according to which the edges are selected for coloring, and the second phase, where the actual coloring takes place.

First, we identify the sets of vertices $S_1, S_2, \ldots, S_\ell$, such that $S_i \cap S_j = \emptyset$, $\cup_{i=1}^{\ell} S_i = V(G)$, and for all $v \in S_j$, we have $d(v) \leq k$ in the graph $G - \cup_{i=1}^{j-1} S_i$. Finding these sets can be done in $O(n+m)$ operations as follows. We assume that the adjacency list of the input graph is given. It is easy to see that we can find the set S_1 in $O(n)$ time. Then we can update the degrees of all vertices in $G - S_1$ just looking at the neighbors of the vertices in S_1. While doing this update, if a vertex in $G - S_1$ has degree lower than or equal to k, we add it to S_2. This can be done in $\sum_{v \in S_1} d(v)$. We proceed similarly with S_2 to find S_3, and so on until we find the ℓ sets. It follows that finding the sets $S_i, 1 \leq i \leq \ell$, can be done in $O(n + \sum_{v \in V} d(v)) = O(n+m)$. Now, we explain how to compute the required ordering of edges using the sets S_i. Let us denote by $E(X, Y)$ the set of edges having one endpoint in the set X and the other in set Y. Observe that the desired ordering is given by the edges $E(S_l, S_l), E(S_l, \cup_{i=1}^{l-1} S_i)$, $E(S_{l-1}, S_{l-1})$, $E(S_{l-1}, \cup_{i=1}^{l-2} S_i)$, $E(S_{l-2}, S_{l-2}), \ldots, E(S_2, S_2)$, $E(S_2, S_1), E(S_1, S_1)$. It is easy to check that this ordering verifies Fact 4.1.1.1.(2). The ordering can be computed easily by checking for each edge the sets where the endpoints belong, which can be done in $O(m)$.

Using the above ordering, we can color the edges using the $(4k-1)\Delta - 2k^2 - k + 1$ colors. Associated to each vertex, we keep a boolean array of size $(4k-1)\Delta - 2k^2 - k + 1$, where the index corresponds to color. Initially, all the entries are initialised to 0. Every time we color an edge $e = uv$, we select the first color available to both u and v, and then we update the entries corresponding to this color from 0 to 1 for all the neighbors of u and v (in order to guarantee the strong edge coloring condition). We can find an available color in $O(k\Delta)$ time. The update operation need to be applied to at most $d(u) + d(v)$ vertices, and can be done in $O(d(u) + d(v)) = O(\Delta)$ time. Since we do this for each edge in the ordering, we obtain the complexity $O(k\Delta m)$.

Finally, the total complexity of the algorithm is bounded by $O(n + k\Delta m)$.

4.2 Outerplanar graphs

In this section, we give the exact value of $\chi'_s(G)$ for bipartite outerplanar graphs and we also improve the known bound for the general outerplanar case. We introduce some more definitions and notation. A *block* is a maximal connected component without a cut-vertex. A *block decomposition* of a graph G is a partition of G into its blocks. Notice that, each component is either a maximally 2-connected subgraph or a single edge. We define an *end block* of a graph as a 2-*connected* block that contains a unique cut-vertex which separates it from all the other 2-connected blocks (if exists). Notice that this definition of end block differs from the standard notion in order to address some specific issues.

An *ear* in G to a subgraph H is a simple path P on at least three vertices with endpoints in H such that (1). none of the internal vertices of P are contained in H, and (2). P forms an induced cycle with H. An *ear decomposition* of a 2-connected subgraph is a partition of its edges into a sequence of ears, where the first ear is an induced cycle. It is easily seen that, for a 2-connected outerplanar graph, there is an ear decomposition where each ear contains at least one internal vertex and the endpoints of every ear are adjacent in the preceding graph (if not, the outerplanarity property is affected). Further, notice that, when the graph is bipartite outerplanar, any added ear has an even number of internal vertices. Any such ear (which forms an induced cycle) together with the edges incident to it forms a puffer graph we defined earlier. Thus, we first show an upper bound for the puffer graphs.

4.2.1 Puffer graphs

Note that, to compute the strong chromatic index we suppose that the puffer graph only has pendant edges (no common neighbors forming a triangle) since we can always split any common neighbor of adjacent vertices of the cycle to two pendant edges which does not affect the upper bound.

The following lemma gives bounds for the puffer graphs.

Lemma 4.2.1. *Let G be a puffer graph. Let C be its cycle, then we have the following according to the cycle length $|C|$.*

(1) $\chi'_s(G) = d(u) + d(v) + d(w) - 3$ *if* $|C| = 3$, $u, v, w \in C$.

(2) $\chi'_s(G) = \max_{uv \in E(C)} d(u) + d(v)$, *if* $|C| = 4$.

(3) $\chi'_s(G) = 5$ *if* $G = C_5$.

(4) $\chi'_s(G) = \max_{u \in C} d(u) + 2$ *if* $|C| = 5$ *and either only a single vertex or exactly two vertices at distance* 2 *have pendant edges.*

(5) $\chi'_s(G) = \max_{uv \in E(C)} d(u) + d(v) - 1$ *if* $|C| = 5$, *at least one vertex has at most* 1 *pendant edge, and not in cases* 3) *and* 4).

(6) If $|C| = 5$ *and every vertex has at least* 2 *pendant edges, let* u, v *be the vertices where the* $\max_{u_1 v_1 \in E(C)} d(u_1) + d(v_2)$ *is reached and let* x, y *and* z *be the rest of the*

vertices. Call $\eta = \lceil \frac{d(x)+d(y)+d(z)-d(u)-d(v)-3}{2} \rceil$. *Then,*

$$\chi'_s(G) \leq \begin{cases} d(u)+d(v)-1 \text{ if } d(u)+d(v) \geq d(x)+d(y)+d(z)-3 \\ d(u)+d(v)-1+\eta, \text{ otherwise} \end{cases}$$

(7) If $G = C_k$, $k \geq 6$, *then*

$$\chi'_s(G) = \begin{cases} 3 \text{ if } k \equiv 0 (\mathrm{mod} 3) \\ 4 \text{ otherwise} \end{cases}$$

(8) $\chi'_s(G) = \max_{uv \in E(C)} d(u)+d(v)-1$ *if* $G \neq C$, $|C| \geq 6$ *and even.*

(9) Let u, v *be the vertices where the* $\max_{u_1v_1 \in E(C)} d(u_1)+d(v_2)$ *is reached and let* x, y *and* z *be the consecutive vertices of* C *not considering* u *and* v *where the* $\min_{s_1,s_2,s_3 \in C} d(s_1)+d(s_2)+d(s_3)$ *is attained. Let* $\eta = \lceil \frac{d(x)+d(y)+d(z)-d(u)-d(v)-3}{2} \rceil$. *Then, if* $G \neq C_7$ *and* $|C| \geq 9$ *is odd,*

$$\chi'_s(G) \leq \begin{cases} d(u)+d(v)-1 \text{ if } d(u)+d(v) \geq d(x)+d(y)+d(z)-3 \\ d(u)+d(v)-1+\eta \text{ otherwise} \end{cases}$$

And same bounds plus 1 *if* $|C| = 7$.

Proof. The proof is trivial for statements 1) through 4). Statement 5) can also be easily verified.

For 6) we note that every vertex of the cycle has at least 2 pendant edges. Let $uvxyz$ be the vertices of the C_5 in a cyclic order. We color the cycle with colors 1 to 5 starting at the edge uv. Then we color one uncolored incident edge of each vertex with the only possible color among the used ones (keeping the strong coloring property). Thus we have $d(u)+d(v)-6$ uncolored edges incident to u and v, and $d(x)+d(y)+d(z)-9$ uncolored edges incident to x, y and z. Suppose that $d(u)+d(v)-6 \geq d(x)+d(y)+d(z)-9$, i.e., $d(u)+d(v) \geq d(x)+d(y)+d(z)-3$. We use $d(u)-3$ new colors to color the uncolored edges at u and $d(v)-3$ new colors for the ones at v. We remark that this is the only possibility to keep the strong coloring property. Clearly, $d(x) \leq d(u)$ and $d(z) \leq d(v)$. We color the uncolored edges at x and z respectively from the set of colors used at u and v respectively. Since $d(x)+d(y)+d(z)-3 \leq d(u)+d(v)$, we notice that there are enough colors left to color the edges incident to y. Since we use only $d(u)+d(v)-1$ colors, the bound is optimal in this case.

Now, suppose that $d(u)+d(v) < d(x)+d(y)+d(z)-3$. As before, we color the $d(u)-3$ edges at u and the $d(v)-3$ edges at v with new colors. Now, we introduce an additional η new colors, and color as many edges incident to both x and z (one can verify that both x and z have at least η uncolored edges in this case). Then for the remaining edges at x we use at most $d(x)-3-\eta$ colors used at u and for the ones at z use at most $d(z)-3-\eta$ colors used at v. As before, it is not difficult to see that we have enough colors left to color the edges incident to y.

For statement 7), we color the cycle in the following way. If $k \equiv 0(\text{mod}3)$ we use colors $1, 2$ and 3 repeatedly for the cycle and we are done. If $k \equiv 1(\text{mod}3)$, we color one edge with color 4 and then repeatedly with colors $1, 2$ and 3. Finally, if $k \equiv 2(\bmod 3)$, we color the first 5 edges with colors $4, 1, 2, 3, 4$ and then repeatedly with colors $1, 2$ and 3.

For 8), let $C = v_1v_2v_3 \dots v_{2k}v_1$ be the even cycle of the puffer graph for $k \geq 3$. Let $u, v \in C$ be two adjacent vertices which attain $max_{u_1u_2 \in E(C)} d(u_1) + d(u_2)$. We use a total of $d(u) + d(v) - 1$ colors to color the whole graph. First, we use at most 5 colors for the edges of the cycle as follows. Color the edges of the cycle $v_1v_2, v_2v_3, \dots, v_{2k-4}v_{2k-3}$ repeatedly with colors $1, 2$ and 3. Then color the edges v_1v_{2k} and $v_{2k-3}v_{2k-2}$ with color 4. If the edge $v_{2k-5}v_{2k-4}$ have color 2 then change the color of the edge v_2v_3 to color 5. Now color the edges $v_{2k-1}v_{2k}$ with the same color as v_2v_3 and the edge $v_{2k-2}v_{2k-1}$ with the same color as $v_{2k-5}v_{2k-4}$. Observe that the cycle is strong edge-colored with the property that for every odd vertex on it, there are two available colors to use on their uncolored incident edges among the 5 already used colors (to see this, notice that both edges of the cycle at distance 1 from these vertices are always colored the same). So, color those uncolored edges (if there are) incident to the odd vertices of the cycle with these two available colors.

To color the rest of the uncolored edges, first suppose without loss of generality that $u \in C$ is an odd numbered vertex. Now, introduce a set of new colors A, where $|A| = d(u) - 4$, and for each odd vertex on the cycle, color its uncolored incident edges with colors from A using the least permissible color. Then do the same for each even vertex on the cycle using another set of new colors B, where $|B| = d(v) - 2$. Finally, suppose that for some vertex v_i on the cycle we have some uncolored edges left after using all the colors from its set. We assume without loss of generality that i is even. Since we have uncolored edges at v_i, this means that $d(v_i) > d(v)$. Observe that $d(v_i) + d(v_{i+1}) \leq d(u) + d(v)$ and $d(v_i) + d(v_{i-1}) \leq d(u)+d(v)$. This implies that $max\{d(v_{i+1}), d(v_{i-1})\} \leq d(u)+d(v)-d(v_i)$ and therefore we have used at most $d(u)+d(v)-d(v_i)$ colors from B to color the edges incident to v_{i+1} and v_{i-1}. It follows that we may use the remaining $d(v)-(d(u)+d(v)-d(v_i)) = d(v_i)-d(u)$ colors from B to color the edges incident to v_i. Same argument follows when there is an odd vertex with its degree greater than $d(u)$. We remark that in total we use $|A|+|B|+5 = d(u)+d(v)-1$ colors to give this strong edge-coloring which is optimal since $d(u)+d(v)-1$ is also a lower bound.

Finally, we show 9). Let $C = v_1v_2v_3 \ldots v_{2k-1}v_1$ be the odd cycle of the puffer graph for $k \geq 4$. First, we use at most 5 colors for the edges on the cycle in the following manner. Without lose of generality suppose that $v_1 = x$, $v_2 = y$ and $v_3 = z$. Start to color the edges on the cycle from the edge v_1v_2 with colors 1,2 and 3 repeatedly till the edge $v_{2k-7}v_{2k-6}$. Then, color the edges $v_{2k-1}v_1$ with color 3, $v_{2k-2}v_{2k-1}$ with color 2, $v_{2k-6}v_{2k-5}$ and $v_{2k-3}v_{2k-2}$ with color 4 and $v_{2k-4}v_{2k-3}$ with color 3. Now, if the edge $v_{2k-8}v_{2k-7}$ (v_1v_7 in the special case when $k = 4$) has color 3 then use color 5 for the edges $v_{2k-8}v_{2k-7}$ (v_1v_7 for $k = 4$) and $v_{2k-5}v_{2k-4}$, otherwise color the edge $v_{2k-5}v_{2k-4}$ with the same color as the edge $v_{2k-8}v_{2k-7}$ (v_1v_7 for $k = 4$).

Observe that the cycle is strong edge-colored with the property that, if we consider vertices v_1, v_2 as a single vertex (say v_2) (to emulate the even case), for every alternate vertex on the cycle (originally numbered even) (except for one in the case of $k = 4$) there are two available colors to use on their uncolored incident edges among the 5 already used colors. Now, color those uncolored edges (if there are) incident to the even vertices of the cycle with these available colors. In the special case of $k = 4$, one of the vertices have only one available color and therefore we will need one new color more.

Suppose without loss of generality that u has $d(u)-4$ uncolored edges. First, if $d(u)+d(v) \geq d(x)+d(y)+d(z)-3$, introduce two sets of new colors A and B of sizes $d(u)-4$ and $d(v)-2$ respectively. Now, color the rest of the edges as in 8), considering x and y as a single vertex. Clearly, this leads to a strong edge-coloring of G following same arguments as in 6) and 8). We use $d(u)+d(v)$ colors if $k = 4$, and $d(u)+d(v)-1$ colors otherwise. Second, suppose that $d(u)+d(v) < d(x)+d(y)+d(z)-3$. We use η (as defined earlier) new colors to color a subset of η edges incident to each of v_1 and v_3. Again, it is easily seen that $d(v_1 = x)$ and $d(v_3 = z)$ are at least η using the assumed inequalities. Finally to color the rest of the edges, we proceed as in the first case. As before, the coloring is strong by 6) and 8). We use $d(u)+d(v)+\eta$ colors if $k = 4$, and $d(u)+d(v)-1+\eta$ colors otherwise. □

Theorem 4.2.2. *Let G be an outerplanar graph. Then* $\chi'_s(G) = \max\{\max_{uv\in E} d(u) + d(v) - 1, \max_{H\in\mathcal{P}} \chi'_s(H)\}$, *where $\mathcal{P}$ is the set of all induced puffer subgraphs of G. If G is* *also bipartite, then*

$$\chi'_s(G) = \max\{\max_{uv \in E} d(u) + d(v) - 1, \max_{uv \in E(C_4)} d(u) + d(v)\}$$

where C_4 *is the set of all cycles of length* 4 *in* G*.*

Proof. We observe that given an ear decomposition of an outerplanar graph, every ear together with the edges incident to it forms a puffer graph. Adding ears in the order, a new ear joins two adjacent vertices. Since only the edges incident to 2 adjacent vertices of the new ear are precolored, we note that we can simply extend the coloring to the new puffer graph (as the precolored edges all get distinct colors in both cases). The upper bound for outerplanar graphs is now clear by maximising over all puffer graphs and over all pairs of adjacent vertices (the latter is a trivial lower bound). When the graph is bipartite, this gives the exact value as the upper bound matches the trivial lower bound. □

The proof itself gives the algorithm to obtain such a coloring, and it is easy to see that it takes sub-quadratic time.

Chapter 5

Proper Hamiltonian Paths in Edge-Colored Multigraphs

This chapter is organized as follows: In Section 5.1 we present some preliminary results that will be useful for the main results. In Section 5.2 we study proper hamiltonian paths in 2-edge-colored multigraphs. Finally, in Section 5.3 we study proper hamiltonian paths in c-edge-colored multigraphs, for $c \geq 3$. We remark that this division is because of proper hamiltonian paths in 2-edge-colored multigraphs are just alternating paths in 2 colors, therefore the results are different of those for $c \geq 3$ colors.

5.1 Preliminary results

Lemma 5.1.1. *Let G be a simple non-colored graph on $n \geq 14$ vertices. If $m \geq \frac{(n-3)(n-4)}{2} + 4$ and for every vertex x, $d(x) \geq 1$, then G has a matching M of size $|M| \geq \lceil \frac{n-2}{2} \rceil$.*

Proof. We can assume that the graph is connected, otherwise it is easy to see that the only possible case to analyze is when G has two components, the first one is two adjacent vertices and the other has $n-2$ vertices and at least $\frac{(n-3)(n-4)}{2} + 3$ edges. Then, the result follows from a theorem in [25].

Let G be a graph of order n. Let M be a maximum matching in G and let U be the set of unmatched vertices. We shall prove that $|U| \leq 2$ by contradiction.

Since M is a maximum matching, neither we can add any new edge to M nor we can replace a set of edges in M in order to obtain a new matching in G which is larger than M.

Necessarily, U is an independent set since otherwise you could add a new edge to M to have a new matching that is larger which contradicts M being maximum. First, let us deal with the case where G has odd order.

We shall prove the result depending on the maximum matching between the independent set $U = \{u_1, u_2, u_3\}$ and the induced subgraph $G - U$.

We will count the number of edges in $E(\overline{G})$, i.e., those edges which cannot be present in $E(G)$ because otherwise the matching M would not be maximum.

Since $m = |E(G)| \geq \frac{(n-3)(n-4)}{2} + 4$, then $|E(\overline{G})| \leq 3n - 10$.

Let us study the odd case. For n even, the proof runs parallel.

Since the graph is connected, we shall prove the result depending on the size of a maximum matching $\widetilde{M}$ between the independent set $U = \{u_1, u_2, u_3\}$ and the induced subgraph $G - U$.

Case 1: If $|\widetilde{M}| = 1$, there is a unique vertex in $N(U)$ and therefore that there are at least three vertices, u_1, u_2, u_3, of degree 1 in $V(G)$. This leads to a contradiction since the number of edges in $E(G)$ would be at most $\frac{(n-3)(n-4)}{2} + 3$ (the edges which are present in a complete graph on $n - 3$ vertices and three more which connect the vertices of degree one to the complete graph), which contradicts the hypothesis of the lemma.

Case 2: If $|\widetilde{M}| = 2$, then there are two vertices $v_1, v_2 \in G - U$, $v_1 \neq v_2$, such that $u_1v_1, u_2v_2, u_3v_2 \in E(G)$ and $N(\{u_2, u_3\}) \subseteq \{v_1, v_2\}$, since otherwise we would have a larger matching between the sets U and $G - U$. And therefore, all edges of type u_2v and u_3v, for any $v \in G - U - \{v_1, v_2\}$, are in $E(\overline{G})$. Also, $\{u_1, u_2, u_3\}$ is an independent set, so there are at least 3 more edges in $E(\overline{G})$.

Since M is a perfect matching in $G - U$, the vertices v_1 and v_2 are extremities of some edge in M. If $v_1v_2 \in M$, then we can replace v_1v_2 by u_1v_1 and u_2v_2 contradicting that M is maximum. Thus, there exist $w_1, w_2 \in G - U - \{v_1, v_2\}$, $w_1 \neq w_2$, such that $v_1w_1, v_2w_2 \in M$.

Necessarily, $w_1w_2 \in E(\overline{G})$. Otherwise we can replace $\{v_1w_1, v_2w_2\}$ by $\{u_1v_1, u_2v_2, w_1w_2\}$ and obtain a larger matching.

The edge u_1w_2 is also in $E(\overline{G})$ because otherwise we can replace v_2w_2 in M by u_1w_2 and u_2v_2 in order to get a larger matching.

So far we have at least $2(n - 5) + 3 + 1 + 1 = 2n - 5$ edges in $E(\overline{G})$.

There are $\frac{n-7}{2}$ edges, $e_1, e_2, ..., e_{\frac{n-7}{2}}$, in $M - \{v_1w_1, v_2w_2\}$. We shall study the conditions given by the possible connections between the edges e_i, for $i = 1, ..., \frac{n-7}{2}$, and the set $\{w_1, w_2\}$. Necessarily, $|N_{e_i}(\{w_1, w_2\})| \leq 2$, $\forall i = 1, 2, ..., \frac{n-7}{2}$. Otherwise, we can find a matching of size two, say $\{f_{i1},\ f_{i2}\}$, connecting e_i and $\{w_1, w_2\}$ and we can replace

$\{e_i, v_1w_1, v_2w_2\}$ by $\{u_1v_1,\ u_2v_2,\ f_{i1}, f_{i2}\}$. We can assure then that there are at least $n-7$ new edges in $E(\overline{G})$. By adding up these edges to the $2n-5$ edges we have obtained before, we already count $3n-12$ edges in $E(\overline{G})$.

We shall study two different subcases.

First of all, if $N(v_1)\cap\{u_2, u_3, w_2\} = \emptyset$, then 3 new edges are shown to be in $E(\overline{G})$. By adding up these new edges to the ones we have counted previously, we have $3n-12+3 = 3n-9$ edges in $E(\overline{G})$. This number is larger than $3n-10$, which was the maximum number of edges in $E(\overline{G})$ allowed from the lemma's hypothesis and the result follows.

Otherwise, if $N(v_1)\cap\{u_2, u_3, w_2\} \neq \emptyset$, then $u_1w_1 \in E(\overline{G})$ because, if for example $u_2v_1 \in E(G)$, we can replace v_1w_1 in M by $\{u_1w_1, u_2v_1\}$, so one more edge is in $E(\overline{G})$. Now if u_2v_1 or u_3v_1 are in $E(G)$ then all edges of type u_1v, for any $v \in G-U-\{v_1, v_2\}$, are in $E(\overline{G})$ since $|\widetilde{M}| = 2$. Therefore both u_2v_1 and u_3v_1 are missing, or $n-5-2 = n-7$ (since we have already counted u_1w_1 and u_1w_2) edges are missing. In the first case we arrive to $3n-9$, a contradiction, in the second case we have that $3n-11+n-7 = 4n-18 > 3n-10$ and again a contradiction.

Case 3: If $|\widetilde{M}| = 3$, then there are three distinct vertices $v_1, v_2, v_3 \in G-U$, such that $u_iv_i \in E(G)$ for $i = 1, 2, 3$. Since there is a perfect matching in $G-U$, the vertices v_1, v_2 and v_3 are extremities of some edge in M. If $v_iv_j \in M$, then we can replace v_iv_j by u_iv_i and u_jv_j contradicting that M is maximum. Thus, there exist three distinct vertices $w_1, w_2, w_3 \in G-U-\{v_1, v_2, v_3\}$, such that $v_1w_1, v_2w_2, v_3w_3 \in M$.

Necessarily, $w_iw_j \in E(\overline{G})$ for all $i, j = 1, 2, 3$. Otherwise we can replace v_iw_i and v_jw_j by u_iv_i, u_jv_j and w_iw_j and obtain a larger matching. In the same way, $u_iw_j \in E(\overline{G})$ for all $i, j = 1, 2, 3$, $i \neq j$, because otherwise we can replace v_jw_j by $\{u_jv_j, u_jv_j\}$. Since U is an independent set, thus there are at least $3+3+6 = 12$ edges in $E(\overline{G})$.

If $u_iw_i \in E(G)$ then $N(v_i)\cap\{u_j, w_j\} = \emptyset$ for $j \neq i$. Otherwise, if for example $u_jv_i \in E(G)$, we can replace v_iw_i in M by u_iw_i and u_jv_i. Thus, there are 3 more edges in $E(\overline{G})$.

Now, there are $\frac{n-9}{2}$ edges, $e_1, e_2, ..., e_{\frac{n-9}{2}}$, in $M-\{v_1w_1, v_2w_2, v_3w_3\}$. We shall study the conditions given by the possible connections between the edges e_i, for $i = 1, ..., \frac{n-9}{2}$, and the sets $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$. Necessarily $|N_{e_i}(W)| \leq 3$, $\forall\ i = 1, 2, ..., \frac{n-9}{2}$. Otherwise, we can find a matching of size two, say $\{f_{ij},\ f_{ik}\}$, $j, k \in \{1, 2, 3\}$, connecting e_i and $\{w_j, w_k\}$, and we can replace $\{e_i, v_jw_j,\ v_kw_k\}$ by $\{u_jv_j,\ u_kv_k,\ f_{ij},\ f_{ik}\}$. Also $|N_{e_i}(U)| \leq 3$, $\forall\ i = 1, 2, ..., \frac{n-9}{2}$. Otherwise, we can find a matching of size two, say $\{g_{ij},\ g_{ik}\}$, $j, k \in \{1, 2, 3\}$, connecting e_i and $\{u_j, u_k\}$, and we can replace e_i by $\{\ g_{ij},\ g_{ik}\}$. We can assure then that there are at least $2\cdot 3\cdot \frac{n-9}{2} = 3n-27$ new edges in $E(\overline{G})$.

By adding up these edges to the 15 edges we have obtained before, we already count

$3n-12$ edges in $E(\overline{G})$.

Recall that we have denoted p_i and q_i the extremities of e_i. Without loss of generality, let us assume that if $|N_{e_i}(u_j)| = 1$, $\forall\ i = 1, 2, ..., \frac{n-9}{2}$ and $j = 1, 2, 3$, then $N_{e_i}(u_j) = p_i$. Thus either $|N_{e_i}(U)| \leq 1$ or $N_{q_i}(U) = \emptyset$ for all $i = 1, 2, 3$. Otherwise we can find a matching of size 2 between U and e_i and we again can find a larger matching.

Now, for $n \geq 15$, there are at least 3 edges in $M - \{v_1w_1, v_2w_2, v_3w_3\}$. If $N_{e_i}(U) = \emptyset$, then 3 more edges are in $E(\overline{G})$ and by adding up these new edges to the $3n-12$ that have already found and the result follows. If $|N_{e_i}(U)| = 1$, then either there exists $j \neq i$ such that $N_{e_i}(U) \neq N_{e_j}(U)$ or $N_{e_i}(U) = N_{e_j}(U)$ for all $i \neq j$. In the first case $q_iq_j \in E(\overline{G})$ and either p_1u_1 or p_1q_2 is in $E(\overline{G})$. We again have 3 new edges in $E(\overline{G})$ and the proof is finished. In the later case all edges connecting the edges in $M - \{v_1w_1, v_2w_2, v_3w_3\}$ and U have a common vertex, say u_1. Again, at least 3 more edges are in $E(\overline{G})$.

Just one case is left to be studied: if $N_{q_i}(U) = \emptyset$, then $q_iq_j \in E(\overline{G})$ for all $i = 1, 2, 3$ and we also get 3 new edges in $E(\overline{G})$.

The proof is now finished. □

Lemma 5.1.2. *Let G^c be a 2-edge-colored multigraph on $n \geq 14$ vertices. Suppose that for every vertex x in G^c, $rd(x) = 2$. If $m \geq (n-3)(n-4)+3n-2$, then G^c has two matchings M^r and M^b on colors, say red and blue, such that $|M^r| = \lfloor \frac{n}{2} \rfloor$ and $|M^b| \geq \lceil \frac{n-2}{2} \rceil$.*

Proof. Let us denote $E^r(G^c)$ and $E^b(G^c)$ the set of edges colored in red and blue, of sizes $|E^r(G^c)| = m^r$ and $|E^b(G^c)| = m^b$, respectively. Observe that, as for every vertex x in G^c, $rd(x) = 2$, we have that $d^i(x) \geq 1$ for $i \in \{r, b\}$. Observe also that $m^i \geq \frac{(n-3)(n-4)}{2} + 4$ for $i \in \{r, b\}$, since this threshold is tight when the multigraph is complete on one of the colors.

Let us see the case when n is odd. By Lemma 5.1.1, there exist two matchings M^r and M^b size $\frac{n-1}{2}$, so the result follows straightforward.

Let us see now the case when n is even. Then, again by Lemma 5.1.1, there exist two matchings, M^r and M^b, of size at least $\frac{n-2}{2}$. We shall prove the result by contradiction. Let us consider the monochromatic subgraphs in color r and b and suppose that $|M^r| = |M^b| = \frac{n-2}{2}$. Let $U = \{u_1, u_2\}$ denote the independent set of unmatched vertices in M^r. The vertices u_1 and u_2 are connected to the edges in M^r. We claim that there exist two distinct vertices v_1, v_2 in $V(G) - \{u_1, u_2\}$ such that $u_1v_1, u_2v_2 \in E^r(G^c)$. Otherwise, if $N^r(\{u_1, u_2\}) = v_1$, this vertex is the extremity of some edge v_1w_1 in M^r and then we distinguish two cases. First, if $N^r(w_1) = v_1$, we have three distinct vertices of degree one which leads us to a contradiction with the total number of edges. Second, there exists

$w_2 \in N^r(w_1) - \{v_1\}$ and v_2, such that $v_2w_2 \in M^r$. Then we can replace $\{v_1w_1, v_2w_2\}$ by $\{u_1v_1, w_1w_2\}$. After this permutation, we have a new matching on same size but u_1 is replaced by v_2. So, as we claimed, there always exist two distinct vertices v_1, v_2 in $V(G) - \{u_1, u_2\}$ such that $u_1v_1,\ u_2v_2 \in E^r(G^c)$ (u_2 and v_2 are those vertices in the case we have just seen).

Now, the edge v_1v_2 is not in M^r, since otherwise, we can replace it by $\{u_1v_1, u_2v_2\}$ and get a larger matching in color r, contradicting that M^r is maximum.

So, there are two vertices w_1 and w_2 in $V(G) - \{u_1, u_2, v_1, v_2\}$ such that $v_1w_1,\ v_2w_2 \in M^r$. Observe that there can be at most two edges connecting the endpoints of any edge in M^r to the set $\{u_1, u_2\}$, i.e., there are at least two missing edges for each edge in M^r, in total $2\frac{n-2}{2}$. Similarly, for w_1 and w_2, we have same constraints and therefore, this also means two missing edges for each edge in $M^r - \{v_1w_1, v_2w_2\}$, in total $2\frac{n-6}{2}$. Otherwise, suppose that there is an edge v_3w_3 with three edges between its endpoints to the vertices w_1 and w_2, so we can replace v_1w_1 and v_2w_2 by say w_1v_3, w_2w_3, v_1u_1 and v_2u_2 to obtain a perfect matching. Finally, as u_1 and u_2 are independent, the edge u_1u_2 is missing. If we sum up these numbers, there are at least $(n-2)+(n-6)+1 = 2n-7$ missing edges in color red.

Same reasoning can be done with the matching M^b to obtain $2n-7$ blue missing edges. So, the total number of missing edges in colors red and blue is $4n-14$. Since the complement of G^c has edge set of size less than or equal to $3n-10$, for $n \geq 6$ we have contradiction. and therefore, the result holds. □

Lemma 5.1.3. *Let G^c, $c \geq 2$ be a connected c-edge-colored multigraph. Suppose that G^c contains a proper path $P = x_1y_1x_2y_2 \dots x_py_p$, $p \geq 2$, such that each edge x_iy_i is red. If G^c does not contain a proper cycle C with vertex set $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$, then there are at least $(c-1)(2p-2)$ missing edges in G^c.*

Proof. Let $P = x_1y_1x_2y_2 \dots x_py_p$ be a proper path, $p \geq 2$, such that each edge x_iy_i is red. Let blue be another color.

The blue edge x_1y_p can not be in G^c, otherwise $C = x_1y_1 \dots x_py_px_1$ is a proper cycle.

Suppose that the blue edges x_1x_i are present in G^c, for $i = 2, \dots, p$. Then, the blue edges $y_{i-1}y_p$ can not be in G^c, otherwise we have the proper cycle $C = x_1x_i \dots y_py_{i-1} \dots x_1$ that contradicts our hypothesis. Therefore for each edge $y_{i-1}x_i$ in the path, either the blue edge x_1x_i or the blue edge $y_{i-1}y_p$ is missing. So there are $\frac{2p-2}{2}$ blue missing edges.

Now, suppose that the blue edges x_1y_i are present in G^c, for $i = 3, \dots, p-2$. Then, the blue edges $x_{i+1}y_p$ cannot be in G^c at same time as x_iy_{i+1}, $y_{i-1}x_{i+2}$ or $y_{i-1}y_{i+1}$,

x_ix_{i+2}, otherwise we have the following proper cycles: $x_1y_ix_iy_{i+1}x_{i+1}y_p \dots x_{i+2}y_{i-1} \dots x_1$ or $x_1y_ix_ix_{i+2} \dots y_px_{i+1}y_{i+1}y_{i-1} \dots x_1$. The minimum in this case corresponds to one missing edge $x_{i+1}y_p$ for each edge $y_{i-1}x_i$ in the path, for $i = 2, \dots, p-1$. Therefore, there are $\frac{2p-6}{2}$ blue missing edges.

For the moment we have $2p-3$ blue missing edges. To obtain the last missing edge suppose that the blue edge x_2y_p is present in G^c. Then, it can not be at same time with x_1y_2, y_1x_3 or x_1x_3, y_1y_2, otherwise we obtain the proper cycles $C = x_1y_2 \dots x_2y_p \dots x_3y_1x_1$ or $C = x_1x_3 \dots y_px_2 \dots y_2y_1x_1$. The minimum in this case corresponds to one missing edge x_2y_p. We remark that the blue edges x_2y_p, y_1y_2 and y_1x_3 were not counted before. The edges x_1x_3 and x_1y_2 were supposed to exist, otherwise, to obtain the last missing edge we consider the symmetric case, i.e., using the blue edge $y_{p-1}x_1$ (if exists).

In total there are $\frac{2p-2}{2} + \frac{2p-6}{2} + 2 = (2p-2)$ blue missing edges in G^c. As we have $c-1$ colors different from red, that gives us $(c-1)(2p-2)$ missing edges as desired.

Note that this number of missing edges is the same as in the simplest case, this is, if all edges different from red x_1x_i and x_1y_i, for $i = 2, \dots, p$ are not present in G^c. □

Lemma 5.1.4. *Let G^c be a connected c-edge-colored multigraph, $c \geq 2$. Let M be a matching of G^c in one color, say red, of size $|M| \geq \lceil \frac{n-2}{2} \rceil$. Let $P = x_1y_1x_2y_2 \dots x_py_p$, $p \geq 2$, be a longest proper path compatible with M. Then we have the following cases:*

- ***(1)*** *n is even, $|M| = \frac{n}{2}$ and $2p < n$*
  - ***(1a)*** *If there is no proper cycle C such that $V(C) = V(P)$, then there are at least $(n-2+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-4)(c-1)$ for $p = \frac{n-2}{2}$.*
  - ***(1b)*** *If there is a proper cycle C such that $V(C) = V(P)$, then there are at least $(2pn-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-4)(c-1)$ for $p = \frac{n-2}{2}$.*
- ***(2)*** *n is odd, $|M| = \frac{n-1}{2}$ and $2p < n-1$*
  - ***(2a)*** *If there is no proper cycle C such that $V(C) = V(P)$, then there are at least $(n-3-p+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-6)(c-1)$ for $p = \frac{n-3}{2}$.*
  - ***(2b)*** *If there is a proper cycle C such that $V(C) = V(P)$, then there are at least $(2pn-2p-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-6)(c-1)$ for $p = \frac{n-3}{2}$.*

- ***(3)*** *n is even, $|M| = \frac{n-2}{2}$ and $2p < n-2$*
  - ***(3a)*** *If there is no proper cycle C such that $V(C) = V(P)$, then there are at least $(n-4-2p+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-8)(c-1)$ for $p = \frac{n-4}{2}$.*
  - ***(3b)*** *If there is a proper cycle C such that $V(C) = V(P)$, then there are at least $(2pn-4p-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-8)(c-1)$ for $p = \frac{n-4}{2}$.*

Proof. Before starting the proof, we remark that the edges x_1y_1 and x_py_p are of color red. Otherwise, we can easily extend the path by adding an edge of the matching to P.

Suppose first that n is even and $2p < n$. Since M has size $\frac{n}{2}$ there are $\frac{n-2p}{2}$ red edges outside P. Let us denote these edges by e_i for $i = 1, \ldots, \frac{n-2p}{2}$. Suppose there is no proper C cycle such that $V(C) = V(P)$. Let blue be an another color. By Lemma 5.1.3 there are $(2p-2)$ blue missing edges. As the path is maximum, we cannot extend P having an edge e_i neither at the beginning nor at the end of it, then there are no blue edges between the vertices x_1, y_p and the edges e_i. Therefore, there are $4\frac{n-2p}{2}$ blue missing edges. Finally, As we cannot add any edge e_i in-between the path then there at most 2 blue edges between the edges e_i and the edges y_ix_{i+1}, $i = 1, \ldots, p-1$. So, there are $2\frac{n-2p}{2}\frac{2p-2}{2}$ blue missing edges different from red. Adding up and simplifying all these numbers and having $c-1$ colors different from red, we arrive that there are $(n-2+pn-2p^2)(c-1)$ missing edges in G^c different from red. If we search the minimum value of this function we arrive to $(2n-4)(c-1)$ for $p = \frac{n-2}{2}$ and case **(1a)** holds. Now if there is a proper cycle C such that $V(C) = V(P)$ then there cannot exist any edge at all different from red between all vertices of C and the edges e_i and therefore there are $2\frac{n-2p}{2}2p(c-1) = (2pn-4p^2)(c-1)$ missing edges different from red. Again, minimizing the function we obtain the same result as above and case **(1b)** holds.

Suppose now that n is odd, $M = \frac{n-1}{2}$ and $2p < n-1$, or n is even, $M = \frac{n-2}{2}$ and $2p < n-2$. In both cases, same arguments as before apply just replacing n with $n-1$ or $n-2$ respectively, in the number of missing edges. This is because we have $n-1$ matched vertices and one non-matched vertex for the first case and $n-2$ matched vertices and two non-matched vertices for the second one. □

Lemma 5.1.5. *Let G be a connected non-colored simple graph on n vertices, $n \geq 9$. If $m \geq \frac{(n-2)(n-3)}{2} + 3$, then G has a matching M of size $|M| = \lfloor \frac{n}{2} \rfloor$.*

Proof. By a theorem in [25], a 2-connected graph on $n \geq 10$ vertices and $m \geq \frac{(n-2)(n-3)}{2} + 5$ edges, has a hamiltonian cycle. So, if we add a new vertex v to G, joined to all its vertices, we have that $G + \{v\}$ has $m \geq \frac{(n-2)(n-3)}{2} + 3 + n = \frac{(n-1)(n-2)}{2} + 5$ edges. Therefore, $G + \{v\}$ has a hamiltonian cycle, i.e., G has a hamiltonian path and this implies that there exists a matching M in G of size $|M| = \lfloor \frac{n}{2} \rfloor$. □

5.2 2-edge-colored multigraphs

In this section we study the existence of proper hamiltonian paths in 2-edge-colored multigraphs. We present two main results. The first one involves the number of edges, and the second one involves the number of edges and the rainbow degree. Both results are tight.

Theorem 5.2.1. *Let G^c be a 2-edge-colored multigraph on $n \geq 8$ vertices. If $m \geq (n-2)(n-3) + 2(n-2) + 2$, then G^c has a proper hamiltonian path.*

Proof. The proof is by induction on n. For $n = 8, 9$, by a tedious analysis, the result can be shown. Suppose now that $n \geq 10$. Observe that $E(\overline{G^c}) \leq 2n - 4$. By a Theorem of [2], if for every vertex $v \in G^c$ we have that $d^r(v) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(v) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper hamiltonian path. Suppose then than there exists a vertex v such that $d^r(v) \leq \lceil \frac{n+1}{2} \rceil - 1$.

Suppose first that there exist two neighbors of v, say u and w, such that $c(vu) = r$ and $c(vw) = b$. We construct then a new multigraph G'^c by replacing the vertices v, u and w with a new vertex z such that $N^r(z) = N^r_{G^c - \{v,u,w\}}(w)$ and $N^b(z) = N^r_{G^c - \{v,u,w\}}(u)$. We remark that $N^r_{G^c - \{v,u\}}(w)$ and $N^r_{G^c - \{v,w\}}(u)$ cannot be both empty at the same time, otherwise $E(\overline{G^c}) \geq n - 3 + n - 3 + n - (\lceil \frac{n+1}{2} \rceil - 1) > 2n - 4$. A contradiction in the total number of edges. So, by this we remove at most $n - 1$ blue edges and $\lceil \frac{n+1}{2} \rceil - 1$ red edges from v, $n - 3$ red edges from u, $n - 3$ blue edges from w, and one red and one blue between u and w. Therefore G'^c has at least $(n-2)(n-3) + 2(n-2) + 2 - (n-1) - (\lceil \frac{n+1}{2} \rceil - 1) - (n-3) - (n-3) - 2$ edges. This number is greater or equal than $(n-4)(n-5) + 2(n-4) + 2$, i.e., the number of edges required to have a proper hamiltonian path in a graph on $n - 2$ vertices. So, by the inductive hypothesis G'^c has a proper hamiltonian path P. Finally, as we have chosen the appropriate edges to remove at u and w it is easy to extend P to a proper hamiltonian path for G^c.

Suppose now that there does not exist two neighbors of v, say u and w, such that $c(vu) = r$ and $c(vw) = b$. So, we have two possible cases. First case is when v has one only neighbor w in both colors. It is easy to observe that $G^c - v$ has $(n-2)(n-3) + 2(n-2) = (n-1)(n-2)$ edges, i.e., it is a rainbow complete multigraph. Therefore, we have any possible proper hamiltonian path, in particular the one that starts at w and then we easily extend the path to G^c. For the second case, v has just neighbors in one color, say b. Observe that for every vertex $w \neq v$, w has a red neighbor different of v. Otherwise, if there is a vertex w without red neighbors, we have that $E(\overline{G^c}) \geq n-1+n-2 > 2n-4$, a contradiction. Now, suppose first that v has at most $n-2$ blue neighbors. So, take a neighbor w of v and remove all its blue incident edges. Remove then v from G^c and call this graph G'^c. In G'^c, w is monochromatic in red, and G'^c has at least $(n-2)(n-3) + 2(n-2) + 2 - (n-2) - (n-2)$ edges. This number is exactly $(n-3)(n-4)+2(n-3)+2$, i.e., the number of edges required to have a proper hamiltonian path in a graph on $n-1$ vertices. Then, by inductive hyphothesis in G'^c we obtain a proper hamiltonian path that clearly starts at w since it was monochromatic. Therefore we have a proper hamiltonian path for G^c. Finally, if v has $n-1$ blue neighbors, suppose that one neighbor w of v has at most $n-3$ blue neighbors, then we proceed as before, we remove v from the graph and we remove all blue incident edges to w. This graph G'^c has at least $(n-2)(n-3)+2(n-2)+2-(n-1)-(n-3) = (n-3)(n-4)+2(n-3)+2$ edges. Again we obtain a proper hamiltonian path for G'^c and therefore a proper hamiltonian path for G^c (since w is monochromatic in red). Now, every vertex w has $n-2$ neighbors, i.e., the graph $G^c - v$ is complete in blue. Call this graph G'^c. Now, G'^c has at least $(n-2)(n-3) + 2(n-2) + 2 - (n-1) = n^2 - 4n + 5$ edges and since this is bigger than $(n-3)(n-4) + 2(n-3) + 2$ by inductive hypothesis we have a proper hamiltonian path in G'^c. Now, if $n-1$ is odd, one of the extremities of the path is red and therefore we trivially add v to the path. If $n-1$ is even, both extremities have the same color. If they are red we are done. Otherwise, remove all the blue edges from G'^c, this new (red) graph has $n-1$ vertices and at least $n^2 - 4n + 5 - \frac{(n-1)(n-2)}{2} = \frac{(n-2)(n-3)}{2} + 1$ edges therefore by a theorem in [25], it has a hamiltonian path P. Now, since G'^c is complete in blue we can use those blue edges along with P to form a proper hamiltonian P' for G'^c that starts and ends with color red. Finally, we can trivially add v to P'.

Since we covered all cases, the proof is finished. □

Theorem 5.2.1 is the best possible for $n \geq 8$. In fact, consider a rainbow complete 2-edge-colored multigraph on $n-2$ vertices for n odd. Add two new vertices x_1 and x_2. Then add the red edge x_1x_2 and all red edges between $\{x_1, x_2\}$ and the complete graph. Although the resulting graph has $(n-1)(n-2) + 2(n-2) + 1$ edges, it has no proper hamiltonian path, since at least one of the vertices x_1 or x_2 cannot be attached to any such path. Indeed, for n odd, the first and last edge of any proper hamiltonian path must differ in colors. If $n = 5, 7$, Theorem 5.2.1 does not hold for the graphs $H^2_{k,k+3}$, $k = 1, 2$.

Theorem 5.2.2. *Let G^c be a 2-edge-colored multigraph on $n \geq 14$ vertices. Suppose that for every vertex x in G^c, $rd(x) = 2$. If $m \geq (n-3)(n-4) + 3n - 2$, then G^c has a proper hamiltonian path.*

Proof. Let us suppose that G^c has not a proper hamiltonian path. We will show that $E(\overline{G^c})$ has more than $3n - 10$ edges, i.e., G^c has less than $(n-3)(n-4) + 3n - 2$ edges, contradicting the hypothesis of the theorem.

We distinguish between two cases depending on the parity of n.

- *Case A*: n is even. By Lemma 5.1.2, G^c has two matchings M^r, M^b, such that $|M^r| = \frac{n}{2}$ and $|M^b| \geq \frac{n-2}{2}$. Take the longest proper paths, say, $P = x_1y_1x_2y_2 \dots x_py_p$ and $P' = x'_1y'_1x'_2y'_2 \dots x'_{p'}y'_{p'}$ compatibles with M^r and M^b, respectively. Observe that as the length of P (P') is odd then both its first and last edges are on the same color. It follows that, since $|M^r| = \frac{n}{2}$, $c(x_1y_1) = c(x_py_p) = r$. Otherwise, we can easily extend P by adding an edge of M^r. It follows that the edges x_iy_i are red, $i = 1, \dots, p$. Similarly, we may suppose that $c(x'_1y'_1) = c(x'_{p'}y'_{p'}) = b$. Indeed if $c(x'_1y'_1) = c(x'_{p'}y'_{p'}) = r$, then either x'_1 or $y'_{p'}$ is the endpoint of an edge of $M^b - P'$ and therefore to obtain a path longer that P' compatible with M^b, a contradiction to the definition of P'.

  Notice now that if $2p = n$ or $2p' = n$, then we are finished. In addition, if $2p < n-2$ or $2p' < n-2$, then by Lemma 5.1.4, there are at least $2n-4$ blue missing edges and $2n-8$ red ones. This gives a total of $4n-12 > 3n-10$ missing edges, a contradiction. Consequently, in what follows we may suppose that $2p = 2p' = n - 2$.

  Suppose first that there exists a proper cycle C in G^c such that $V(C) = V(P)$. Let e be the red edge of M^r in $G^c - C$. If there exists a blue edge, say e', between an endpoint of e and C, we can easily obtain a proper hamiltonian path considering e, e' and a segment of C of length $n - 3$ in the appropriate direction. Otherwise, as the graph is connected, all edges between the endpoints of e and C are red. Now, as $rd(G^c) = 2$, there must exist a blue edge, say e', parallel to e and therefore we can obtain a proper hamiltonian path just as before but starting with e'.

  Next suppose that there exists no proper cycle C in G^c such that $V(C) = V(P)$. By Lemma 5.1.4, there are at least $2n - 4$ blue missing edges. Consider now the path P' and let v_1, w_1 be the vertices in $G^c - P'$. It is clear that if there exists a blue edge joining v_1 and w_1, then by symmetry on the colors there are at least $2n-4$ red missing edges. This gives a total of $4n-8$ missing edges, a contradiction. Otherwise, assume that there is no blue edge between v_1 and w_1. In this case we will count red missing edges assuming that we cannot extend P' to a proper hamiltonian path. If there exists no cycle C' in G^c such that $V(C') = V(P')$ then by Lemma 5.1.3 there are $2p' - 2 = n - 4$ red missing edges. By summing up, we obtain $2n - 4 + n - 4 = 3n - 8 > 3n - 10$ missing edges, a contradiction. Finally, assume that there exists a proper cycle C' in G^c such that $V(C') = V(P')$. Set $C = c_1c_2 \dots c_{2p'}c_1$ where $c(c_ic_{i+1}) = r$ for $i = 1, 3, \dots, 2p' - 1$. If there are 3 or more red edges between $\{v_1, w_1\}$ and $\{c_i, c_{i+1}\}$, for some $i = 1, 3, \dots, 2p' - 1$, then either the edges v_1c_i and w_1c_{i+1}, or v_1c_{i+1} and w_1c_i are red. Suppose v_1c_i and w_1c_{i+1} are red. In this case, the path $v_1y'_ix'_i \dots y'_{i+1}x'_{i+1}w_1$ is a hamiltonian one. Otherwise, if there are no 3 or more red edges between $\{v_1, w_1\}$ and $\{c_i, c_{i+1}\}$, for all $i = 1, 3, \dots, 2p' - 1$, then there are $2\frac{2p'-2}{2} = n - 4$ red missing edges. If we sum up we obtain a total of $2n - 4 + n - 4 = 3n - 8 > 3n - 10$ missing edges, a contradiction.

- *Case B: n* is odd. By Lemma 5.1.2, G^c has two matchings M^r, M^b, such that $|M^r| = |M^b| = \frac{n-1}{2}$. As in *Case A*, we consider the longest proper paths P and P' compatibles with the matchings M^r and M^b respectively. Suppose first that $2p < n-1$ and $2p' < n-1$. By Lemma 5.1.4, there are at least $2n-6$ blue missing edges and $2n-6$ red ones. We obtain a total of $4n-12 > 3n-10$ missing edges, a contradiction.

  Suppose next $2p = 2p' = n - 1$ (the cases where $2p < n - 1$ and $2p' = n - 1$, or $2p = n - 1$ and $2p' < n - 1$ are similar). In the rest of the proof, because of symmetry, we will consider only the path P since same arguments may be applied as well for P'. In this case we will count blue missing edges assuming that we cannot extend P to a proper hamiltonian path. Now, let v be the unique vertex in $G^c - P$. It is clear that if there is a proper cycle C in G^c such that $V(C) = V(P)$, we can trivially obtain a proper hamiltonian path since the graph is connected, i.e., there is at least one edge between v and C. So, as there is no proper cycle C in G^c such that $V(C) = V(P)$, we have by Lemma 5.1.3, that there are $2p - 2 = n - 3$ blue missing edges. If there exists a blue edge between x_1 and x_i, for some $i = 2, \dots, p$, then it cannot exist the blue edge vy_{i-1}, otherwise we obtain the proper hamiltonian path $vy_{i-1}\dots x_1x_i\dots y_p$. We can complete the argument in a similar way, if both edges y_py_i and vx_{i+1}, $i = 1,\dots,p-1$ exist in G^c and are on color blue. Note that since there is no proper cycle C in G^c such that $V(C) = V(P)$, we cannot have at the same time the blue edges x_1x_i and y_py_{i-1}, $i = 2,\dots,p$. Therefore, there are $\frac{2p-2}{2} = \frac{n-3}{2}$ blue missing edges. If we make the sum and multiply it by two (since the same number of red missing edges is obtained with P'), we conclude that there are $2(n-3+\frac{n-3}{2}) = 3n-9$ missing edges, a contradiction.

Since we covered all the cases, the theorem is proved. □

Theorem 5.2.2 is the best possible for $n \geq 14$. Indeed, for n odd, $n \geq 14$, consider a complete blue graph, say A, on $n-3$ vertices. Add 3 new vertices v_1, v_2, v_3 and join them to a vertex v in A with blue edges. Finally, superpose the obtained graph with a complete red graph on n vertices. Although the resulting 2-edge-colored multigraph has $(n-3)(n-4)+3n-3$ edges, it has no proper hamiltonian path since one of the vertices v_1, v_2, v_3 cannot belong to such a path. For $n = 7, 9, 11, 13$, it is easy to see that the graphs $H^c_{k,k+3}$, $k = 2, 3, 4, 5$, are exceptions for Theorem 5.2.2.

5.3 c-edge-colored multigraphs, $c \geq 3$

Finally, in this section we study the existence of proper hamiltonian paths in c-edge-colored multigraphs, for $c \geq 3$. We present three main results. The first one involves the total number of edges. The second one, the total number of edges and the connectivity of the graph. The last one, the total number of edges and the rainbow degree. All results are best possible.

First, we will present a result that allows us to consider just the case $c = 3$.

Lemma 5.3.1. *Let G^c be a c-edge-colored connected multigraph on n vertices, $c \geq 3$ and $m \geq c\ f(n)+1$ edges. There exists one color c_j such that if we color its edges with another used color and we delete parallel edges with the same color, then the resulting $(c-1)$-edge-colored multigraph is connected and has $m' \geq (c-1)\ f(n)+1$ edges, such that if G^{c-1} has a proper hamiltonian path then G^c has one too. Moreover, if $rd(G^c) = k$, then $rd(G^{c-1}) = k-1$ for $1 \leq k \leq c$.*

Proof. Let c_i denote the color i, for $i = 1,\dots,c$, in G^c, and denote by $|c_i|$ the number of edges with color i. Let c_j be the color with less number of edges. Color the edges on color c_j with another used color, say c_l, and delete (if necessary) parallel edges with that color. Call this graph G^{c-1}. By this, we delete at most $|c_j|$ edges. It is clear that this graph is connected since we delete just parallel edges. Also, if is G^{c-1} has a proper hamiltonian path, then, this path is also proper hamiltonian in G^c but maybe with some edges on color c_j (in the case that they have been recolored with c_l). Observe also that if $rd(G^c) = k$ then $rd = (G^{c-1}) = k-1$ since only the color c_j disappeared. We will show now that $m' \geq (c-1)\ f(n)+1$. We have two cases. First, if $|c_j| > f(n)$, then clearly $m' \geq (c-1)\ f(n)+1$ edges since for all i, $|c_i| > f(n)$. For the second case, we have that $|c_j| \leq f(n)$. Now, $m = \sum_{i=1}^{c} |c_i| \geq c\ f(n)+1$ and therefore $\sum_{i=1,i\neq j}^{c} |c_i| \geq c\ f(n)-|c_j|+1 = (c-1)\ f(n)+f(n)-|c_j|+1$. This last expression is greater or equal than $(c-1)\ f(n)+1$ since $f(n)-|c_j| \geq 0$. Finally, we have that G^{c-1} has $m' \geq (c-1)\ f(n)+1$ edges as desired. □

Theorem 5.3.2. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 2$ and $c \geq 3$. If $m \geq \frac{c(n-1)(n-2)}{2}+1$, then G^c has a proper hamiltonian path.*

Proof. By Lemma 5.3.1 we can assume that $c = 3$ and $m \geq \frac{3(n-1)(n-2)}{2}+1$. Furthermore, cases $n \leq 7$ can be checked by exhaustive methods. Assume so, $n \geq 8$. Since there exists one color, say red, such that the number of red edges are at least $\frac{(n-1)(n-2)}{2}+1$ then by a theorem in [25], there is a hamiltonian red path and therefore a perfect or almost perfect matching M^r. Take the longest proper path $P = x_1y_1x_2y_2\ldots x_py_p$ compatible with M^r. So we have that, $c(x_iy_i) = r$ for $i = 1,\ldots,p$. As $m \geq \frac{3(n-1)(n-2)}{2}+1$, $|E(\overline{G^c})| \leq 3n-4$. Now, we will distinguish between two cases depending on the parity of n.

Case A: n even. Clearly $|M^r| = \frac{n}{2}$. By contradiction suppose that $2p < n$, otherwise we are finished. Assume first that there is no proper cycle C in G^c such that $V(P) = V(C)$. By Lemma 5.1.4, there are at least $2(n-2+pn-2p^2)$ missing edges different from red and therefore the inequality $2(n-2+pn-2p^2) \leq 3n-4$ must be satisfied. This inequality does not hold for $n \geq 8$. Therefore, we have a contradiction with the number of edges of G^c. Assume next that there is a proper cycle C in G^c such that $V(P) = V(C)$. Again by Lemma 5.1.4, $2(2pn-4p^2) \leq 3n-4$ must be satisfied, and as before this is never possible for $n \geq 8$.

Case B: n odd. Clearly $|M^r| = \frac{n-1}{2}$.

Assume first that there is a proper cycle C in G^c such that $V(P) = V(C)$. If $2p = n-1$ then we can obtain a proper hamiltonian path by just taking any edge between the only vertex outside C and C and then following C in the appropriate direction. Otherwise by Lemma 5.1.4 there are $2(2pn-2p-4p^2)$ missing edges different from red and therefore $2(2pn-2p-4p^2) \leq 3n-4$ must hold and this never happens for $n \geq 9$.

Assume next that there is no proper cycle C in G^c such that $V(P) = V(C)$. If $2p < n-1$ the again by Lemma 5.1.4 the inequality $2(n-2+pn-2p^2) \leq 3n-4$ must be satisfied. However this is not possible for $n \geq 9$. It remains to handle the case $2p = n-1$. Let v be the unique vertex of $G^c - P$. We have the following cases depending on the degree and neighbors of v.

Subcase B1: All edges incident to v are in the same color, say red. We have these situations. First, $d(v) \leq n-2$, consider the $G^c - \{v\}$ and delete from a neighbor of v, say w, all edges in two colors in order to have w monochromatic not in red. Call this graph G'^c. Observe that we can always do this since, it is impossible to have 2 monochromatic vertices. So, by this, we delete at most $n-2+n-2+n-2 = 3n-6$ edges. It is easy to see that the graph G'^c on $n-1$ vertices has at least $\frac{(n-1)(n-2)}{2} + 1 - (3n-6) = \frac{(n-2)(n-3)}{2} + 1$, then, by even case, we have a proper hamiltonian path P' in G'^c. Since w is monochromatic not in red, w is either in the beginning or in the end of P' and therefore it is trivial to add v to P' in order to find a proper hamiltonian path in G^c. If this does not hold, we have that $d(v) = n-1$. Therefore the graph $G^c - \{v\}$ on $n-1$ vertices has at least $\frac{(n-1)(n-2)}{2} + 1 - (n-1) \geq \frac{(n-2)(n-3)}{2} + 1$, then, again, we have a proper hamiltonian path P' in G'^c. Now, if the path either starts or ends with a color different from red, we trivially add v to the path. If not, both of them finish with red. Now, if we can take any parallel edge of these without losing the property of being properly colored we have again that it is easy to add v to the path. Otherwise, we have, without losing generality, the degree in some color, say c_1 of the first vertex of the path, say w, is at most $n-3$. So, we are in the same case as the first one, since we take the graph $G^c - \{v\}$ and we delete from w the edges in that color c_1 and in another color in order to have w monochromatic not in red. By this, we delete $n-1+n-2+n-3 = 3n-6$ edges and finally, the result follows exactly as in the first situation.

Subcase B2: There exist two distinct edges incident to v, say vx and vy, such that $c(vx) = c_1 \neq c_2 = c(vy)$. Assume first that $x = y$. This case is analog to the last one just taking the graph $G^c - \{v\}$ and deleting from w edges in the appropriate two colors in order to have it monochromatic. By this we delete $3 + 2(n-2) = 2n+1$ edges and we have that $\frac{(n-1)(n-2)}{2} + 1 - (2n+1) \geq \frac{(n-2)(n-3)}{2} + 1$ for $n \geq 7$.

Assume next that $x \neq y$. We will prove the result by induction. If we cannot attach v to the path P we have that there are 4 missing edges in colors different from red between v and the vertices x_1, y_p. Also, we have at most 2 edges different from red between v and the edges $y_i x_{i+1}$ of the path. Therefore there are $2\frac{n-3}{2}$ missing edges. Adding up all this, we conclude that the degree of v in colors different from red is at most $n-3$. So, if we replace the three vertices v, x, y to a new one, say v', such that $N^{c_1}(v') = N^{c_1}_{G^c-\{v,x,y\}}(y)$, $N^{c_2}(v') = N^{c_2}_{G^c-\{v,x,y\}}(x)$ and $N^{c_3}(v') = N^{c_3}_{G^c-\{v,x,y\}}(x) \cap N^{c_3}_{G^c-\{v,x,y\}}(y)$. So, by this we delete at most $5n-10$ edges. Now, it is easy to see that the new 3-edge-colored multigraph on $n-2$ vertices, say G'^c, has at least $\frac{(n-1)(n-2)}{2}+1-(5n-10) \geq \frac{(n-3)(n-4)}{2}+1$, for $n \geq 5$, therefore, by inductive hypothesis, we have a proper hamiltonian path P' in G'^c. Now, because of the way we have chosen the edges to delete at x and y, it is easy to obtain from P' a proper hamiltonian path for G^c.

Since we covered all the cases, the theorem is proved. □

Theorem 5.3.2 is the best possible. Indeed, consider a rainbow complete graph on $n-1$ vertices with c colors and add a new isolated vertex x. The resulting graph, although it has $c\frac{(n-1)(n-2)}{2}$ edges, contains no proper hamiltonian path since it is not connected.

Notice that in the above theorem there is no condition guaranteeing the connectivity of the underlying graph. Next result adds this condition.

Theorem 5.3.3. *Let G^c be a connected c-edge-colored multigraph on n vertices, $n \geq 9$ and $c \geq 3$. If $m \geq \frac{c(n-2)(n-3)}{2} + n$, then G^c has a proper hamiltonian path.*

Proof. By Lemma 5.3.1 it is enough to prove the theorem for $c = 3$.

The proof is by induction on n. The cases $n = 9, 10$ can be shown by a tedious case analysis. We have two cases, depending on whether G^c contains a monochromatic vertex or not.

Case a: There exists a monochromatic vertex in G^c. Let x be a monochromatic vertex. Let y be a neighbor of x. If y is also monochromatic then the graph $G^c - \{x, y\}$ has $n-2$ vertices and at least $\frac{3(n-2)(n-3)}{2} + n - (2n-3) = \frac{3(n-2)(n-3)}{2} - n + 3$ edges, i.e., it is almost rainbow complete and therefore it has a proper hamiltonian cycle. Then, it is easy to add x and y to the cycle to obtain a proper hamiltonian path in G^c.

Suppose then that y is not monochromatic and that xy is of color b.

Let us replace the vertices x and y by a new vertex, say z, such that $N^b(z) = N^r(z) = \emptyset$ and $N^g(z) = N^g(y)$ (or $N^b(z) = N^g(z) = \emptyset$ and $N^r(z) = N^r(y)$).

Observe that, if the resulting multigraph on $n-1$ vertices is connected, (we show this later) and has enough edges for the induction hypothesis to hold, by induction it contains a proper hamiltonian path. Since z is monochromatic, it can only be the endpoint of the path. Therefore, using the blue edge xy, we can extend the path to a proper hamiltonian one in the initial multigraph.

We now count the maximum number of edges that may be deleted by the contraction. We delete all the edges incident to x, which is at most $n-1$, and the edges of color b

and say g (or r) incident to y, at most $2(n-2)$. This gives us a total of $3n-5$ edges. We now show that we can choose x and y for the contraction process, such that we delete at most $3\frac{(n-2)(n-3)}{2} + n - 3\frac{(n-3)(n-4)}{2} - n + 1 = 3n-8$ edges, necessary for the induction hypothesis to hold.

We have the following cases, depending on $d^b(x)$. First of all, note that if $d^b(x) \leq n-4$, we delete at most $n-4+2(n-2) = 3n-8$ edges from x and any selected neighbor y of x and we are done. Further, from a theorem in [2], if $d^i(z) \geq \lceil \frac{n}{2} \rceil \; \forall z \in V(G^c - \{x\}); i \in \{r, g, b\}$, then $G^c - \{x\}$ has a proper hamiltonian cycle. This would imply a proper hamiltonian path in G. Thus, we may assume that there exists some vertex $w \in G^c - \{x\}$ such that $d^i(w) < \lceil \frac{n}{2} \rceil$ for some $i \in \{r, g, b\}$.

Case $d^b(x) = n-1$. Observe that $w \in N^b(x)$. In this case, the contraction process deletes $n-1$ edges from x, and at most $n - 2 + \frac{n}{2} - 1$ from w, much less than $3n-8$ for $n > 10$.

Case $d^b(x) = n-2$. Let w be the only vertex not adjacent to x. Now, if there is a vertex y adjacent to x such that $d^{r,b}_{G^c-x}(y) \leq 2n-6$ or $d^{r,g}_{G^c-x}(y) \leq 2n-6$. We choose x and y for the contraction process. Otherwise, for all y adjacent to x we have that $d^{r,b}_{G^c-x}(y) \geq 2n-5$ and $d^{r,g}_{G^c-x}(y) \geq 2n-5$. Therefore, by the theorem in [2], $G^c - \{x, z\}$ has a proper hamiltonian cycle. Now, since $d^b(x) = n-2$ it is easy to add x and w to the cycle to form a proper hamiltonian path for G^c.

Case $d^b(x) = n-3$. If there is a vertex y adjacent to x such that $d^{r,b}_{G^c-x}(y) \leq 2n-5$ or $d^{r,g}_{G^c-x}(y) \leq 2n-5$. We choose x and y for the contraction process. Otherwise, for all y adjacent to x we have that $d^{r,b}_{G^c-x}(y) \geq 2n-4$ and $d^{r,g}_{G^c-x}(y) \geq 2n-4$, i.e., the graph $G^c - x$ is raibow complete and therefore, it has a proper hamiltonian cycle. Finally, we trivially add x to the cycle to form a proper hamiltonian path for G^c.

Case b: There is no monochromatic vertex in G^c. First suppose that there exists a vertex x such that $|N(x)| = 1$. Pick x and some vertex $y \in N(x)$, for the contraction. We delete at most 3 edges at x and $2n-4$ at y, which guarantees the induction hypothesis.

In what follows, we suppose that $|N(x)| \geq 2$ for all $x \in G^c$. We describe another contraction process, but now between 3 vertices. Consider a vertex x, $y, z \in N(x)$ and suppose that $c(xy) = b$ and $c(xz) = r \neq c(xy)$. For the contraction, replace x, y and z by a new vertex say s, such that $N^r(s) = N^r_{G^c-\{x,y,z\}}(y)$, $N^b(s) = N^b_{G^c-\{x,y,z\}}(z)$ and $N^g(s) = N^g_{G^c-\{x,y,z\}}(y) \cap N^g_{G^c-\{x,y,z\}}(z)$.

If the resulting multigraph has a proper hamiltonian path, it is easy to obtain a proper hamiltonian path for the initial graph since we chose the appropriate edges to delete at

y and z. Let us count now the maximum number of edges that may be deleted in the contraction process.

We now need to select x, y and z so that we delete at most $3\frac{(n-2)(n-3)}{2}+n-(3\frac{(n-4)(n-5)}{2}+n-2) = 6n-19$ edges, necessary for the induction hypothesis to hold.

Since we delete at most $3n-6$ edges incident to y and z in $G^c - x$, if the degree of x is at most $6n-19-(3n-6) = 3n-13$, the hypothesis holds. If not, we show that there exist x,y and z such that the total number of deleted edges in the contraction process is less than or equal to $6n-19$.

In what follows we show how to find the desired triplet. We have two cases, depending on the parity of n.

Let E^i be the set of edges in color i and suppose that the color b maximizes $|E^i|$. The monochromatic subgraph in color b has at least $m^b \geq \frac{(n-2)(n-3)}{2} + \lceil \frac{n}{3} \rceil$ edges. We distinguish two possibilities. If this subgraph is connected, then by Lemma 5.1.5 there is a perfect matching for n even and almost perfect matching for n odd. Otherwise, there is a matching of size $\frac{n-2}{2}$ for n even and $\frac{n-1}{2}$ if n is odd. Let M^b denote the maximum matching in E^b.

- n is even. Let $P = x_1y_1x_2y_2\ldots x_py_p$ be the longest proper path compatible with M^b. It is easy to check that $|P| \geq 4$, otherwise there are not enough edges in G^c. Suppose now that there is a proper cycle C such that $V(C) = V(P)$. We have the following cases.

  (1) $|M^b| = \frac{n}{2}$. By Lemma 5.1.4, we can check that $|P| \geq n-2$, otherwise there is a contradiction with the total number of edges of the graph. Now, if $|P| = n-2$, we trivially add the edge of the matching outside the path since P also defines a proper cycle and the graph is connected. If $|P| = n$, P is a proper hamiltonian path and we are done.

  (2) $|M^b| = \frac{n-2}{2}$. By Lemma 5.1.4, we can check that $|P| \geq n-4$, otherwise, as before, there is a contradiction with the number of edges of the graph. Suppose first that $|P| = n-4$. Let xy be the edge of the matching outside P. Clearly, there are no edges in color g or r between xy and the cycle otherwise we would have a longest proper path compatible with the matching. Then, we have that $d^{r,g}(x) \leq 6$ and therefore $d(x) \leq 6+n-1 < 3n-13$. So we take x with any two neighbors in different colors for the contraction process and we are done. Suppose now that $|P| = n-2$. Let x, y be the unmatched vertices.

For every edge $y_i x_{i+1}$ on the cycle we can have at most 4 edges in colors r and g between their endpoints and the vertices x, y. Otherwise, we can add x and y to the cycle in order to obtain a proper hamiltonian path. Therefore, without losing of generality $d^{r,g}(x) \leq 2\frac{n-2}{2} + 2$. Same observation applies for the blue edges $x_i y_i$. This is, we can have at most 2 blue edges between their endpoints and the vertices x, y. Then $d^b(x) \leq \frac{n-2}{2} - 1$. Summing up we have $d(x) \leq n + \frac{n-2}{2} - 1 < 3n - 13$. Thus, we can take x for the contraction process.

Let us suppose now that there is no proper cycle C such that $V(C) = V(P)$. Consider the following cases as we did before.

(1) $|M^b| = \frac{n}{2}$. By Lemma 5.1.4 we can check that $|P| \geq n-4$. If P is hamiltonian we are done. Otherwise, there is at least one edge in $M^b \setminus (P \cap M^b)$. Clearly, there are no edges in color r and g from the extremities of P to the edges in $M^b \setminus (P \cap M^b)$. Now, for each each edge outside the matching, there are at most 4 edges between their endpoints and the endpoints of the edges $y_i x_{i+1}$, otherwise we can obtain a longer path. Therefore, taking the vertex, say x, outside P with minimum degree, we have that $d^{r,g}(x) \leq 2\frac{n-6}{2} + 6$, for $|P| = n-4$, and $d^{r,g}(x) \leq 2\frac{n-4}{2} + 2$, for $|P| = n-2$. In both cases, if we consider $d^b(x) = n-1$, we get $d(x) \leq 3n - 13$ and we take it for the contraction.

(2) $|M^b| = \frac{n-2}{2}$. For $|P| = n-4$ the proof is similar as previous case. Suppose then that $|P| = n-2$. Let v, w be the unmatched vertices. We try to add these vertices to the path either at the extremities or between vertices y_i, x_{i+1}. Suppose first that we cannot add any of them. Then $d^{r,g}(v) \leq 2\frac{n-4}{2} + 2$. Summing up this with at most $n-2$ blue edges we obtain $d(v) \leq 2n-4 < 3n-13$ and we take v for the contraction. Suppose last that we can add v but we cannot add w. If v was added at one extremity of the path we obtain that $d(w) \leq 2\frac{n-4}{2} + 1 + n - 2 = 2n - 5 < 3n - 11$. If v was added between the path, we have $d(w) \leq 2\frac{n-6}{2} + 2 + n - 2 = 2n - 6 < 3n - 11$. In both cases we can take w for the contraction.

- n is odd and therefore $|M^b| \geq \frac{n-1}{2}$. Let $P = x_1 y_1 x_2 y_2 \ldots x_p y_p$ be the longest proper path compatible with M^b. It is easy to check that $|P| \geq 4$, otherwise there are not enough edges in G^c. Then we distinguish two cases:

(1) There is a proper cycle C such that $V(C) = V(P)$. By Lemma 5.1.4 we can check that $|P| \geq n-5$. If $|P| = n-1$ we are done since is trivial to add

the unmatched vertex to the cycle to obtain a proper hamiltonian path. If $n-5 \leq |P| \leq n-3$, as in the even case, we can use for the contraction any vertex incident to any edge of the matching outside the path, since there are no red and green edges at all between the edges of the matching outside the path and the path.

(2) There is no proper cycle C such that $V(C) = V(P)$. Again, by Lemma 5.1.4 we can check that $|P| \geq n-5$. This case is also similar to the even case. For $n-5 \leq |P| \leq n-3$, we try to extend P with the edges of the matching outside the path. Since this is not possible we can chose one matched vertex outside the path to do the contraction. For $|P| = n-1$, we try to extend P with the unique unmatched vertex. If this is not possible, we can use it for the contraction and we are done.

Since we cover all cases, we can always find three appropriate vertices to make the contraction process. We now check the connectivity of the resulting multigraph after the contraction of 2 and of 3 vertices.

- We contract two vertices x, y, where x is monochromatic and y is a neighbor of x, to a vertex s. Suppose that the graph is disconnected. It can be easily shown that the graph has two components with 1 vertex and $n-2$ vertices respectively. Otherwise, there is a contradiction on the total number of edges of the graph. Observe first that the isolated vertex cannot be s unless the x and y are both monochromatic, but this case was solved independently. Then, as we have the choice at y of which colors to delete, s cannot be isolated and therefore this case cannot occur. Suppose now that a vertex z is the isolated vertex. Now, as we delete at most $3n-8$ edges for the contraction process, we have at least $3n-1$ edges in $E(\overline{G^c})$. Then, since the graph is disconnected, there are no edges between both components and therefore $3n-9$ more edges in $E(\overline{G^c})$. Summing up we obtain $6n-10$ edges in $E(\overline{G^c})$. A contradiction.

- We contract three vertices x, y, z to a vertex s. Suppose that the graph is disconnected. Again, this graph has exactly two components with 1 vertex and $n-3$ vertices respectively. If the isolated vertex z is not s we have $3n-12$ edges in $E(\overline{G^c})$ since there are no edges between both components. Now, in the contraction process we deleted at most $6n-19$ edges, therefore there are $3n+1$ more edges in in $E(\overline{G^c})$. Again, if we sum up we obtain $6n-11$ edges in $E(\overline{G^c})$. Another contradiction.

Suppose finally that $z = s$. As we deleted at most $6n-19$ edges, the new graph has at least $\frac{3(n-2)(n-3)}{2}+n-6n+19 = \frac{3(n-4)(n-5)}{2}+n-2$ edges in the component of $n-3$ vertices. As we will see in next chapter, this component has a proper hamiltonian cycle (Theorem 6.2.2) and therefore it is trivial to add the three contracted vertices to obtain a proper hamiltonian path for the initial graph.

Now as the connectivity is proved the theorem holds. □

Theorem 5.3.3 is the best possible. Indeed, consider a rainbow complete graph on $n-2$ vertices with c colors and add two new vertices x and y. Add now the edge xy and also all edges between y and the complete graph, all on a same color. The resulting graph, although it has $c\frac{(n-2)(n-3)}{2}+n-1$ edges, it contains no proper hamiltonian path, as x cannot belong to such a path.

Theorem 5.3.4. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 11$ and $c \geq 3$. Assume that for every vertex x of G^c, $rd(x) = c$. If $m \geq \frac{c(n-2)(n-3)}{2}+2c+1$, then G^c has a proper hamiltonian path.*

Proof. By Lemma 5.3.1 it is enough to prove the theorem for $c = 3$. As $m \geq \frac{3(n-2)(n-3)}{2}+7$ then $E(\overline{G^c}) \leq 6n-16$. The proof will be done by construction of a proper hamiltonian path or, if it is not possible, by a reduction to Theorem 5.3.3, i.e., to a connected 3-edge-colored multigraph on $n' \geq 9$ vertices and $m' \geq \frac{3(n'-2)(n'-3)}{2}+n'$ that has a proper hamiltonian path. We will do this reduction by contracting 2 or 3 vertices depending on if there exists a vertex x in G^c, such that $|N^{r,g,b}(x)| = 1$ or not.

- There exists a vertex $x \in G^c$ such that $|N^{r,g,b}(x)| = 1$. Let y be the neighbor of x. We replace x and y by a new vertex s, such that $N^r(s) = N^r(y)$ and $N^b(s) = N^g(s) = \emptyset$. Clearly, we delete 3 edges between x and y and at most $2(n-2)$ edges in color b and g incident to y. This gives us a total of $2n-1$ deleted edges. For the reduction hypothesis we can delete at most $\frac{3(n-2)(n-3)}{2}+7-\frac{3(n-3)(n-4)}{2}-n+1 = 2n-1$ edges. Observe that the resulting multigraph on $n-1$ vertices is connected since as $rd(x) = 3$, we can choose at y which two colors to delete, therefore since the original graph is connected it is impossible that in all 3 possible choices the graph after contraction would be disconnected. So, as we deleted $2n-1$ edges and the graph is connected, by Theorem 5.3.3, it has a proper hamiltonian path. Now, since s is monochromatic, it can only be the endpoint of the path. We can replace back s with x and y using the edge xy in color b and find the proper hamiltonian path for the initial multigraph.

- There is no vertex $x \in G^c$, such that $|N^{r,g,b}(x)| = 1$. Let us suppose that there are 3 vertices x, y and z, such that xy is in color b, and xz in color r. Then, we replace x, y and z by a new vertex s, such that $N^r(s) = N^r_{G^c-\{x,y,z\}}(y)$, $N^b(s) = N^b_{G^c-\{x,y,z\}}(z)$ and $N^g(s) = N^g_{G^c-\{x,y,z\}}(y) \cap N^g_{G^c-\{x,y,z\}}(z)$. Clearly, we delete at most $3(n-1)$ edges incident to x, $n-3$ edges in b incident to y, $n-3$ edges in r incident to z, $n-3$ in color g incident to y and z, and 3 edges between y and z. In total we delete at most $6n-9$ edges. Only $\frac{3(n-2)(n-3)}{2} + 7 - \frac{3(n-4)(n-5)}{2} - n + 2 = 5n - 12$ edges can be deleted to make the reduction. Therefore, we need to find a vertex x such that its total degree is less than or equal to $2n-6$.

Suppose without losing generality that $|E^b| \geq |E^r| \geq |E^g|$, then $|E^b| \geq \frac{3(n-2)(n-3)}{2} + 3$. Since the subgraph in color b is connected because of the rainbow degree of the vertices, we have by Lemma 5.1.5 that there is a matching M^b such that $|M^b| = \frac{n}{2}$ for n even and $|M^b| = \frac{n-1}{2}$ for n odd. Let $P = x_1y_1x_2y_2 \ldots x_iy_i \ldots x_py_p$ be the longest proper path compatible with M^b. The proof is divided in two cases depending on the parity of n.

(1) n is even.

- There is a proper cycle C such that $V(C) = V(P)$. By Lemma 5.1.4 we can check that $|P| \geq n-2$, otherwise we have a contradiction with the number of edges. This case is trivial since, either P is a proper hamiltonian path or we can add the unique edge of the matching outside P directly to the cycle to obtain a proper hamiltonian path since the graph is connected.
- There is no proper cycle C such that $V(C) = V(P)$. By Lemma 5.1.4 we can check that $|P| \geq n-4$. Let x be the vertex in $M^b \setminus (M^b \cap P)$ with minimum degree. Clearly, there cannot be edges in colors r and g between $M^b \setminus (M^b \cap P)$ and the extremities of P. Also, there can be at most 4 edges in colors r and g from each edge in $M^b \setminus (M^b \cap P)$ and the edges $y_{i-1}x_i$ in P. Then, we can conclude that there are at most $2\frac{2p-2}{2}$ edges in color r and g, between x and the vertices in P.

  Suppose now that there is one parallel edge in color r or g between the blue edge of the matching xy, then using edges xx_i, yy_i or yx_i, xy_i in color b, we can add the edge xy in color r or g into P. Since this contradicts our hypothesis we can conclude that there are 2 missing edges in color b from xy to each edge x_iy_i in P. Now, since there are $\frac{2p}{2}$ edges x_iy_i we conclude that the vertex x has $d^b(x) \leq \frac{2p}{2} + (n-2p-1)$. In total we have that $d(x) \leq 3(n-2p-1) + \frac{2p}{2} + (2p-2) = 3n-3p-5$, and for $|P| = n-2$, this is less than or equal to $2n-6$ and we can make the contraction with x. If $|P| = n-4$ there are $6n-20$ missing edges in colors r and g. Suppose now that there are no edges in colors r and g between x and y, and the other edge of the matching outside the path. We obtain then, 4 missing edges more in colors r and g and therefore $6n-16$. Observe now that if we can replace the edges xy and the other edge of the matching outside P, say uv, in M^b by xu and yv, then there are missing the 4 parallel edges in color r and g, obtaining $6n-12$. A contradiction. Otherwise if not, there are 2 more missing edges and we obtain $6n-14$. Again a contradiction. Finally, if $|P| = n-2$ and there are no parallel edges in r and g at xy, we have that $d(x) \leq 2n-5$ and $d(y) \leq 2n-5$. We can use one of these vertices unless both inequalities become equalities. In this case we try to replace the edges x_iy_i in P by x_ixy_i and x_jyy_j, $i \neq j$. Suppose that we have the edges xx_i in colors r and g, xy_i in color b, yx_j in colors r and g, and yy_j in color b. Then we have a proper hamiltonian path $P' = x_1y_1 \dots x_ixy_ix_{i+1} \dots x_jyy_j \dots x_py_p$. Otherwise at least one of those edges is missing and therefore either $d(x) \leq 2n-6$ or $d(y) \leq 2n-6$ as desired.

(2) n is odd.

- There is a proper cycle C such that $V(C) = V(P)$. By Lemma 5.1.4 we can check that $|P| \geq n-5$. The case $|P| = n-1$ is trivial. Clearly, there are no edges in colors r and g between $M^b \setminus (M^b \cap P)$ and P. Therefore taking any vertex x in $M^b \setminus (M^b \cap P)$ we obtain that $d(x) \leq 4+n-1 = n+3$, for $|P| = n-3$, and $d(x) \leq 8+n-1 = n+7$, for $|P| = n-5$. In both cases, this is less than $2n-6$ so we can do the contraction with x.
- There is no proper cycle C such that $V(C) = V(P)$. By Lemma 5.1.4 we can check that $|P| \geq n-7$. Suppose first $|P| < n-1$. Let xy and uv be two edges in $M^b \setminus (M^b \cap P)$ where x has the minimum degree. There cannot be edges in colors r and g to the extremities of P and there can be only 4 edges in colors r and g between $M^b \setminus (M^b \cap P)$ and $y_{i-1}x_i$ in P. Suppose now that there exists an edge xy in color r or g, then we can use the edges in color b between xy and x_iy_i to extend P. Since it is not possible we conclude that there are only $\frac{2p}{2} + (n-2p-1)$ edges in color b at x. For $|P| = n-7$

and $|P| = n - 5$, we have a contradiction in the total number of edges. If $|P| = n-3$, then $d(x) \leq \frac{2p}{2} + (2p-2) + 3(n-2p-1) = 3n-3p-5 \leq 2n-6$ and we use x for the contraction.
Suppose there are no parallel edges in colors r and g at the edges in $M^b \setminus (M^b \cap P)$. If $|P| = n - 7$ then there are 6 missing edges. Now, if we can replace the three blue edges in $M^b \setminus (M^b \cap P)$ to another different three blue edges, we are, either in the previous case or there are 6 more parallel missing edges and by Lemma 5.1.4 there are $8n - 48 - 6 - 6 = 8n - 36 > 6n - 16$ missing edges. A contradiction. Otherwise there are at least 5 missing edges in color b and again by Lemma 5.1.4 there are $8n - 48 - 6 - 5 = 8n - 37$ missing edges. Again a contradiction.
If $|P| = n - 5$ there are 4 parallel edges r and g missing at the edges in $M^b \setminus (M^b \cap P)$ and by Lemma 5.1.4 there are $6n - 26$ more missing edges in color r and g. Therefore $6n - 22$. Now, if we cannot replace the blue edges xy and uv with the blue edges xu, yv there are at least 2 missing edges in color b, otherwise we would miss more parallel edges. Also, there are 4 more missing edges in color b between the extremities of the edges in $M^b \setminus (M^b \cap P)$ and the unmatched vertex, since otherwise we can construct 4 different matchings and therefore 8 more missing edges. In conclusion we have at least 6 missing edges in color b and $6n - 22$ in colors r and g, that gives us a total of $6n - 16$ missing edges. To find the last one to get a contradiction suppose that there are the parallel edges x_1y_1 and x_py_p in colors r and g, then we can make a proper cycle of length $n - 1$: $x_1y_1 \dots x_py_pvyxux_1$, if we also have the edges, y_pv and ux_1 in color b, xu and yv in color r or g. Now, as it is trivial to attach the unmatched vertex to the proper cycle there is at least one of those edges missing and therefore a contradiction.
If $|P| = n - 3$, we have as before that $d_P^{r,g}(x) \leq n - 5$. As always, there are no parallel edges to the edge xy in colors r and g. Now, if there are no edges between xy and the unmatched vertex z in color r and g, then $d(x) \leq 2n - 6$ as desired for the contraction. Otherwise, if there are edges in colors r and g to the vertex z, then there cannot be the edges in color b parallel to them since we could replace xy by one of those and thus come back to the previous case. Therefore, we have that $d(x) \leq 2n - 5$ and $d(y) \leq 2n - 5$. Then, we can choose for the contraction, either x

or y, unless both inequalities become equalities. Now, if we suppose that $c(xz) = r$, we contract the three vertices x, y and z such that we delete at most $2n-5$ edges at x, $n-3$ blue edges at y, $n-3$ red edges at z, $n-3$ green edges with common enpoints at y and z and 2 edges, one red and one green between y and z. We remark that the blue edge yz is not present. By this, we delete at most $5n-12$ edges, i.e., the required number for the contraction.

Finally, suppose that $|P| = n-1$. Let z be the unmatched vertex. Then, for same reasons as before, $d^{r,g}(z) \leq n-3$. If we suppose that there are edges in color r and g between z and y_i, then there cannot be the edges $x_i z$ in color b, otherwise a hamiltonian path can be found. Suppose there are no such edges then, by Lemma 5.1.3, either x_1 or y_p have degree in colors r and g at most $n-3$. If there is at least one of the edges zy_1 or zx_p in color b, then we can replace z by either x_1 or y_p. In both cases, we arrive that $d(x_1) \leq 2n-6$ or $d(y_p) \leq 2n-6$, otherwise $d(z) \leq 2n-6$ and the contraction can be done.

Since all cases were covered, there always exists a vertex x such that $d^{r,g,b}(x) \leq 2n-6$ and the contraction can be done. Therefore by Theorem 5.3.3 we obtain a proper hamiltonian path in this new graph. Then, it is simple to extend this path to a proper hamiltonian one in the initial graph because of the choice of the edges to delete at the contracted vertices.

We will check now the connectivity of the resulting multigraph after the contraction of 3 vertices. Suppose we contract three vertices x, y, z to a vertex s. Suppose that the graph is disconnected. This graph has exactly two components with 1 vertex and $n-3$ vertices respectively. If the isolated vertex z is not s we have $3n-12$ edges in $E(\overline{G^c})$ since there are no edges between both components. Now, in the contraction process we deleted at most $5n-12$ edges, therefore there are $4n-6$ more edges in in $E(\overline{G^c})$. Summing up we obtain $7n-18$ edges in $E(\overline{G^c})$. A contradiction. Suppose finally that $z = s$. As we deleted at most $5n-12$ edges, the new graph has at least $\frac{3(n-2)(n-3)}{2} + 7 - 5n + 12 = \frac{3(n-4)(n-5)}{2} + n - 2$ edges in the component of $n-3$ vertices. As in the previous theorem, this component has a proper hamiltonian cycle (Theorem 6.2.2) and therefore, it is trivial to add the three contracted vertex to obtain a proper hamiltonian path for the initial graph.

Now as the connectivity is proved the theorem holds. □

Theorem 5.3.4 is the best possible for $n \geq 11$. In fact, consider a rainbow complete multigraph, say A, on $n-2$ vertices. Add 2 new vertices v_1, v_2 and then join them to a vertex v of A with all possible colors. The resulting c-edge-colored multigraph has $\frac{c(n-2)(n-3)}{2} + 2c$ edges and clearly has no proper hamiltonian path. If $n = 6, 8$, the graphs $H^c_{k,k+2}$, $k = 2, 3$, are exceptions for Theorem 5.3.4.

Chapter 6

Proper Hamiltonian Cycles in Edge-Colored Multigraphs

As in Chapter 5, we divide this chapter in two sections. In Section 6.1 we study proper hamiltonian cycles in 2-edge-colored multigraphs and in Section 6.2 we study proper hamiltonian cycles in c-edge-colored multigraphs, for $c \geq 3$. Again, this division is because, since proper cycles in 2-edge-colored multigraphs are just alternating, proper hamiltonian cycles only can exist if n is even (condition not required when $c \geq 3$).

6.1 2-edge-colored multigraphs

In this section we study the existence of proper hamiltonian cycles in 2-edge-colored multigraphs. We present two main results. The first one involes the number of edges and the second one, the rainbow degree and the number of edges. Both results are tight.

Lemma below is established in view of Theorem 6.1.2

Lemma 6.1.1. *Assume that G^c contains a proper cycle C, of length at most $n-2$ and that there exists a red edge xy in $G^c - C$. If $d^b_C(x) + d^b_C(y) > |C|$, then G^c has a proper cycle of length $|C|+2$ containing xy.*

Proof. Set $C = x_1y_1x_2y_2 \ldots x_sy_sx_1$, where x_iy_i are the red edges of C, $i = 1, 2, \ldots, s$. Then $d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \leq 2$, otherwise if $d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \geq 3$, then we have that the cycle $x_1y_1x_2y_2 \ldots x_ixyy_i \ldots x_sy_sx_1$ is the desired one. It follows that $\sum_i d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \leq 2\frac{|C|}{2} = |C|$, a contradiction to the hypothesis of the lemma. This completes the proof. □

Now we can prove the following theorem.

Theorem 6.1.2. *Let G^c be a 2-edge-colored multigraph on n vertices, $n \geq 4$. If $m \geq (n-1)(n-2)+n$, then G^c has a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise.*

Proof. Let red and blue be the two colors of G^c. The proof is by induction on n. The theorem is true for small values of n, say $n = 2, 3, 4$. Let us suppose that $n \geq 5$ and that the theorem is true until $n-1$. We will prove it for n. By a Theorem of [2], if for every vertex x we have that $d^r(x) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(x) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper hamiltonian cycle for n even and a proper cycle of length $n-1$ for n odd. Let us suppose therefore that for some vertex, say x, and for some color, say red, $d^r(x) \leq \lceil \frac{n+1}{2} \rceil - 1$. Notice now that $d^r(x) > 0$ and $d^b(x) > 0$, otherwise, for example if $d^r(x) = 0$, then $m \leq n(n-1) - (n-1) = (n-1)^2 < (n-1)(n-2) + n$, a contradiction. Similarly $d^r(x) + d^b(x) \geq 3$, otherwise, if $d^r(x) + d^b(x) \leq 2$, then $m \leq n(n-1) - 2n + 4 = n^2 - 3n + 4 < (n-1)(n-2) + n$, again a contradiction. Thus we may conclude that there are two distinct neighbors, say y and z, of x such that $c(xy) = r$ and $c(xz) = b$ in G^c. Replace now the vertices x, y, z by a new vertex s such that $N^b(s) = N^b_{G^c-\{x,y,z\}}(y)$ and $N^r(s) = N^r_{G^c-\{x,y,z\}}(z)$. The obtained graph, say G', has $n-2$ vertices and at least $(n-1)(n-2) + n - [(n-1) + \lceil \frac{n+1}{2} \rceil - 1 + 2(n-2)] = n^2 - \frac{11n}{2} + 8 > n^2 - 6n + 8$ edges, i.e., the number of edges needed for the inductive hypothesis in a graph on $n-1$ vertices. So, G' has a proper hamiltonian cycle for $n-2$ even and a proper cycle of length $n-3$ otherwise. If G' has a proper hamiltonian cycle then coming back to G^c we may easily find a proper hamiltonian cycle in G^c. Assume now that $n-2$ (and thus n) is odd. Let C be a proper cycle of length $n-3$ in G'. If s belongs to C, then as previously we may easily find a proper cycle of length $n-1$ in G^c. Assume therefore that s does not belong to C. By Lemma 6.1.1, if $d^b_C(x) + d^b_C(y) > |C|$ or $d^r_C(x) + d^r_C(z) > |C|$) then we may integrate the edge xy (respectively xz) in C in order to obtain a cycle of length $n-1$. Assume therefore that $d^b_C b(x) + d^b_C(y) \leq |C|$ and $d^r_C(x) + d^r_C(z) \leq |C|$. But then the number of edges of G^c is at most $n(n-1) - (n-3) - (n-3) = n^2 - 3n + 6 < (n-1)(n-2) + n$, again a contradiction. This completes the argument and the proof. □

Theorem 6.1.2 is the best possible for $n \geq 4$. Consider a rainbow complete 2-edge-colored multigraph on $n-1$ vertices (n even). Add one new vertex x. Then add all possible edges in one color, say red, between x and the complete graph. Clearly, the resulting graph has $m \geq (n-1)(n-2) + n - 1$ edges and it has no proper hamiltonian cycle since x has just color red incident to it.

Theorem 6.1.3. *Let G^c be a 2-edge-colored multigraph on n vertices, $n \geq 9$. Assume that for every vertex x of G^c, $rd(x) = 2$. If $m \geq (n-2)(n-3) + 2(n-2) + 4$, then G^c has a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise.*

Proof. The proof is by induction on n. For $n = 9, 10$, by inspection the result holds. Suppose now that $n \geq 11$. Observe that $E(\overline{G^c}) \leq 2n - 6$. By a Theorem of [2], if for every vertex $v \in G^c$ we have that $d^r(v) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(v) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise. Suppose then than there exists a vertex v such that $d^r(v) \leq \lceil \frac{n+1}{2} \rceil - 1$.

Clearly, v has two different neighbors u and w such that $c(vu) = b$ and $c(vw) = r$. Otherwise, if v has just one neighbor in both colors, then $E(\overline{G^c}) = 2n - 4 > 2n - 6$, a contradiction. We construct a new multigraph G'^c by replacing the vertices v, u and w with a new vertex z such that $N^r(z) = N^r_{G^c - \{v,u,w\}}(u)$ and $N^b(z) = N^r_{G^c - \{v,u,w\}}(w)$. Suppose first that $d^r(v) \leq 2$ and that in G'^c there exists a vertex x with $rd(x) < 2$. Therefore in G^c either $d^r(x) \leq 2$ or $d^b(x) \leq 2$. This vertex clearly cannont be v. Observe first that in both cases $E(\overline{G^c}) = 2n - 6$, and this happens when both v and x have their colored degrees exactly 2, otherwise we have a contradiction on the total number of edges. So G^c must have all possible edges but those edges already missing at v and x. We have the following cases now, depending on the vertex x.

- $x = u$, therefore $d^r(u) = 2$, u is adjacent in red to v and to w. Now, since $G^c - \{v, u\}$ is rainbow complete it has a proper hamiltonian cycle if n is even and a proper cycle of length $n-3$ otherwise. Take an blue edge x_1x_2 of that cycle. Now, since v and u have all possible blue edges, we just add the red edge vu to the cycle joining v with x_1 and u with x_2 in blue and removing the edge x_1x_2. Like this we obtain a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise.

- $x = w$, therefore $d^b(w) = 2$, w is adjacent in blue to v and to u. Again, since $G^c - \{v, w\}$ is rainbow complete it has a proper hamiltonian cycle if n is even and a proper cycle of length $n-3$ otherwise. For n odd, we can choose the proper cycle of length $n-3$ such that there is the second vertex z such that z is adjacent to v in red. Let y be the vertex in the cycle adjacent to z in color red. So, we add the blue edge vw to the cycle joining v to z, w to y both in red, and removing the edge zy. Observe that this is always possible since w is adjacent in red to every vertex. Then we obtain the desired proper hamiltonian cycle for n even, or the proper cycle of length $n-1$ for n odd.

- $x \neq u$ and $x \neq w$. Suppose first that $d^r(x) = 2$ then, x is adjacent to v and w in red. In this case, we proceed exactly as in the first case, adding the red edge vx to the proper hamiltonian cycle (or proper cycle of length $n-3$) that exists in $G^c - \{v, x\}$. Suppose now $d^b(x) = 2$ then, x is adjacent to v and u in blue. Here, we repeat the argument as in the second case, adding the blue edge vx to the proper hamiltonian cycle (or proper cycle of length $n-3$) that exists in $G^c - \{v, x\}$, choosing again the cycle that contains the second vertex z such that z is adjacent to v in red.

The last case that we have to consider is when in the new graph G'^c every vertex x has $rd(x) = 2$. We can see that this graph has at least $(n-2)(n-3) + 2(n-2) + 4 - (n-1) - (\lceil \frac{n+1}{2} \rceil - 1) - (n-3) - (n-3) - 2$ edges. This number, for $n \geq 11$, is greater or equal than $(n-4)(n-5) + 2(n-4) + 4$, i.e., the number of edges needed to have a proper hamiltonian cycle or a proper cycle of length $n-3$ (n odd) in G'^c. So by the inductive hypothesis we obtain such a cycle. For n even, it is easy to obtain a proper hamiltonian cycle for G^c, since we deleted the appropriate edges at u and w. For n odd, if the new vertex z is on the cycle of length $n-3$, it is exactly the same as for the even case to obtain a proper cycle of length $n-1$ for G^c. Now, if the vertex z is not on the proper cycle, let $x_1y_1x_2y_2 \ldots x_ky_kx_1$ with $2k = n-3$ be the proper cycle. Suppose without losing generality that the edges x_iy_i are red and the edges y_ix_{i+1} are blue. If we cannot add neither the blue edge vu nor the red edge vw to the cycle, then we have at most 2 red edges between the endpoints of the edge vu and the endpoints of the edges x_iy_i and at most 2 blue edges between the endpoints of the edge vw and the endpoints of the edges y_ix_{i+1}. So, since the length of the cycle is $n-3$, there are $\frac{n-3}{2} + \frac{n-3}{2} = 2n-6$ edges is $E(\overline{G^c})$. Therefore, we have all possible edges in G^c but those missing ones. In particular we have the red edge uv and the red edge uw. Now, if there is a blue edge from v to some vertex x_i (y_i) of the cycle, we extend the proper cycle to a proper cycle of length $n-1$ with the red edge uv adding the blue edge vx_i (vy_i), the blue edge uy_{i-1} (ux_{i+1}) and removing the blue edge x_iy_{i-1} (y_ix_{i+1}) from the cycle. The blue edge uy_{i-1} (ux_{i+1}) clearly exists since u has all possible blue incident edges. If there is no blue edge from v to some vertex x_i (y_i) of the cycle, we have that there exists a blue edge from w to some vertex x_i (y_i) of the cycle. Otherwise, we have that $d^b(v) = 2$ and $d^b(w) = 2$, and we have covered that case before. Finally, we extend the proper cycle to a proper one of length $n-1$ exactly as we did to add the red edge uv but now with the red edge uw.

The proof is now complete. □

Theorem 6.1.3 is the best possible for $n \geq 9$. Indeed, for n even, consider a complete blue graph, say A, on $n-2$ vertices. Add 2 new vertices v_1, v_2 and join them to a vertex v in A with blue edges. Finally, superpose the obtained graph with a complete red graph on n vertices. Although the resulting 2-edge-colored multigraph has $(n-2)(n-3)+2(n-2)+3$ edges, it has no proper hamiltonian cycle since there is not a perfect blue matching because v_1 and v_2 are only adjacent in blue to v. We remark that for $n = 8$, the graph $H^2_{k,k+2}$, $k = 3$, has $(n-2)(n-3)+2(n-2)+4$ edges but no proper hamiltonian cycle.

6.2 c-edge-colored multigraphs, $c \geq 3$

In this section we study the existence of proper hamiltonian cycles in 3-edge-colored multigraphs. We present two main results. The first one involes the number of edges and the second one, the rainbow degree and the number of edges. Both results are tight. Finally, we state a conjecture involving the rainbow degree, the number of edges and the connectivity.

First, we will present a similar result to that for proper hamiltonian paths, that allows us to consider just the case $c = 3$. We omit the proof since is exactly the same as for paths.

Lemma 6.2.1. *Let G^c be a c-edge-colored connected multigraph on n vertices, $c \geq 4$ and $m \geq c\ f(n) + 1$ edges. There exists one color c_j such that if we color its edges with another used color and we delete parallel edges with the same color, then the resulting $(c-1)$-edge-colored multigraph is connected and has $m' \geq (c-1)\ f(n) + 1$ edges, such that if G^{c-1} has a proper hamiltonian cycle then G^c has one too. Moreover, if $rd(G^c) = k$, then $rd(G^{c-1}) = k-1$ for $1 \leq k \leq c$.*

Theorem 6.2.2. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 4$ and $c \geq 3$. If $m \geq \frac{c(n-1)(n-2)}{2} + n$, then G^c has a proper hamiltonian cycle.*

Proof. By Lemma 6.2.1 it is enough to prove the theorem for $c = 3$. So we have that $m \geq \frac{3(n-1)(n-2)}{2} + n$. We prove the theorem by induction on n. For $n = 4$ the theorem is easily checked. Suppose then that $n \geq 5$. We take two vertices v and w such that they are adjacent in all colors r, b and g. It can be checked that two vertices like these always exists otherwise, if every pair of vertices have at most two edges between, the number of edges would be less than the hypothesis.

Observe first that $d(v), d(w) \leq 3n-5$. Otherwise, if $d(v) \geq 3n-4$ we can remove v from G^3 obtaining a graph with number of edges at least $\frac{3(n-1)(n-2)}{2} + n - 3(n-1) =$

$\frac{3(n-2)(n-3)}{2}+n-3 \geq \frac{3(n-2)(n-3)}{2}+1$ and therefore by Theorem 5.3.2 we have a proper hamiltonian path in G^3-v. Then, since $d(v) \geq 3n-4$ we can easily connect v to the endpoints of the path in an appropriate way in order to obtain a proper hamiltonian cycle. So, we can assume that $d(v), d(w) \leq 3n-5$.

Now we will contract v and w to a single vertex to apply the inductive hypothesis in this new graph on $n-1$ vertices. For that, we have to show that this graph has at least $\frac{3(n-2)(n-3)}{2}+n-1$. For this we can check that the difference between these two functions is $3n-5$ edges, that is, the maximum number of edges that we are allowed to delete in the contraction process. So, suppose that we contract v, w to a new vertex z such that $N^r(z)=N^r(v)-w, N^b(z)=N^b(w)-v$ and $N^g(z)=N^g(v)\cap N^g(w)$. Like this, since $d(v), d(w) \leq 3n-5$ we delete at most $3n-5$ edges as desired.

Finally, it is easy to see, once we obtain a proper hamiltonian cycle in the contracted graph, how to obtain a proper hamiltonian cycle in the original graph. □

Theorem 6.2.2 is the best possible for $n \geq 4$. Consider a rainbow complete c-edge-colored multigraph on $n-1$ vertices. Add one new vertex x. Then add all possible edges in one color, say red, between x and the complete graph. Clearly, the resulting graph has $m \geq \frac{c(n-1)(n-2)}{2}+n-1$ edges and it has no proper hamiltonian cycle since x has just color red incident to it.

Theorem 6.2.3. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 4$ and $c \geq 3$. Assume that for every vertex x of G^c, $rd(x)=c$. If $m \geq \frac{c(n-1)(n-2)}{2}+c+1$, then G^c has a proper hamiltonian cycle.*

Proof. By Lemma 6.2.1 it is enough to prove the theorem for $c=3$. So we have that $m \geq \frac{3(n-1)(n-2)}{2}+4$ and for every vertex x of G^3, $rd(x)=3$. We prove the theorem by induction on n. For $n=4,5$ the theorem is easily checked. Suppose then that $n \geq 7$. We take two vertices v and w such that they are adjacent in all colors r, b and g. It can be checked that two vertices like these always exists otherwise, if every pair of vertices have at most two edges between, the number of edges would be less than the hypothesis.

Observe first that $d(v), d(w) \leq 3n-5$. Otherwise, if $d(v) \geq 3n-4$ we can remove v from G^3 obtaining a graph with at least $\frac{3(n-1)(n-2)}{2}+4-3(n-1)=\frac{3(n-2)(n-3)}{2}+1$ edges. This is exactly the number of edges needed by Theorem 5.3.2 to have a proper hamiltonian path in G^3-v. Therefore, since $d(v) \geq 3n-4$ we can easily connect v to the endpoints of the path in an appropriate way in order to obtain a proper hamiltonian cycle. So, we can assume that $d(v), d(w) \leq 3n-5$.

Now we will contract v and w to a single vertex to apply the inductive hypothesis in this new graph on $n-1$ vertices. For that, we have to show that this graph has at least $\frac{3(n-2)(n-3)}{2}+4$ edges and for every vertex x, $rd(x)=3$. For this we can check that the difference between these two functions is $3n-6$ edges, that is, the maximum number of edges that we are allowed to delete in the contraction process. So, suppose that we contract v,w to a new vertex z in a trivial way such that $N^r(z)=N^r(v)-w, N^b(z)=N^b(w)-v$ and $N^g(z)=N^g(v)\cap N^g(w)$. Like this, since $d(v),d(w)\leq 3n-5$ we delete at most $3n-5$ edges. So we need to delete one edge less to apply the inductive hyphotesis. Clearly, if either $d(v)\leq 3n-6$ or $d(w)\leq 3n-6$, we can choose the colors to delete in order to remove $3n-6$ edges as desired. Suppose then that $d(v)=d(w)=3n-5$. Remove now v from the graph. As before we have a proper hamiltonian path $P=v_1v_2\dots v_{n-1}$ in G^3-v where $w=v_i$ for some i. Now we extend this path with v to a proper hamiltonian cycle. If we cannot trivially extend it adding v properly to the endpoints, we have that v (without losing generality) has the three colors to v_1 and just one color to v_{n-1} where this color is the same as in the edge $v_{n-2}v_{n-1}$ in P (say r). Therefore since $d(v)=3n-5$, v is fully connected to all vertices but v_{n-1}. We can suppose that $w\neq v_1$, otherwise it is easy to see how to extend P with v to a cycle using the fact that $d(w)=3n-5$. Now, suppose that v_{n-1} has an edge to $w=v_i$ with color b (similar with g). If wv_{i+1} is b we have the proper hamiltonian cycle $v_{n-1}wv_{i-1}\dots v_1vv_{i+1}\dots v_{n-1}$. Otherwise, if $v_{i-1}w$ is b we have the following one: $v_{n-1}wv_1\dots v_{i-1}vv_{i+1}\dots v_{n-1}$. We can conclude that w is adjacent to v_{n-1} only in r and fully connected to the other vertices. Finally, since $rd(v_{n-1})=3$, there exists a vertex v_j adjacent to v_{n-1} with some color different than r. As we have just done, we can always find a proper hamiltonian cycle in this situation considering different cases when $j<i$ and $i<j$. Finally, either $d(v)\leq 3n-6$ or $d(w)\leq 3n-6$. So we can therefore contract v,w to a vertex z removing at most $3n-6$ in order to obtain a graph on $n-1$ with at least $\frac{3(n-2)(n-3)}{2}+4$. The problem now is that in this new graph, the rainbow degree condition may be not respected anymore since we can have vertices (at most 2) with just one color incident to them. Suppose then, that there exists a vertex $x\in G^3$ and a color, say r, such that $d^r(x)=1$. We show this case independently on the rest of the proof. Let z be the only neighbor of x with color r and let $y\neq z$ be another neighbor of x in another color, say b. Clearly y must exist otherwise the number of edges on the graph would be less than the hypothesis. Now we contract the vertices x,y and z to a new vertex w in order to apply Theorem 6.2.2 to a graph on $n-2$ vertices. For this, we can check that we can delete at most $5n-9$ edges. If we contract them in a trivial way such that $N^r(w)=N^r_{G^3-\{x,y,z\}}(y), N^b(w)=N^b_{G^3-\{x,y,z\}}(z)$

and $N^g(w) = N^g_{G^3-\{x,y,z\}}(z) \cap N^g_{G^3-\{x,y,z\}}(y)$ we remove at most $5n-7$ edges, so we need to delete two edges less to apply Theorem 6.2.2. Observe that if we do not have colors b and g between x and z we obtain these two edges and therefore we remove $5n-9$ edges as desired. So, we have at least two colors between x and z, r and, say b. Now consider the graph $G^3 - \{x, z\}$. We can check that by Theorem 5.3.2, the graph has a proper hamiltonian path $P = v_1v_2 \dots v_{n-2}$. We can suppose also that $y = v_1$ (or $y = v_{n-2}$), since otherwise if x is not adjacent to any of v_1, v_{n-2} in colors b or g we find the two edges less to delete for the contraction. We have the following cases now. Suppose first that there are exactly colors r and b between x and z. So, we need to find one more edge in order to arrive to $5n-9$. Therefore, x should be adjacent to both $y = v_1$ and v_{n-2} in colors b and g, otherwise we obtain the last edge to get $5n-9$. Now, if z is adjacent to v_1 in both colors b and g, we easily obtain a proper hamiltonian cycle otherwise we obtain the edge less to delete. Suppose last that there are the three colors r, b and g between x and z. Therefore, x should be adjacent without losing generality to v_1 in at least color b and to v_{n-2} in colors b and g, otherwise we obtain two edges less to get $5n-9$. If x is adjacent to v_1 just in color b, as before, if z is adjacent to v_1 in both colors b and g we are done since either we have a proper hamiltonian cycle or we obtain the $5n-9$ edges. If x is adjacent to v_1 in colors b and g, depending on the color of v_1v_2, if z is adjacent to v_1 in two colors we obtain again a proper hamiltonian cycle. Otherwise, two edges are missing between z and v_1. As we covered all cases, we can then contract x, y and z as we described above.

Finally, it is easy to see once we obtain a proper hamiltonian cycle in the contracted graph (when we contract 2 or 3 vertices) how to obtain a proper hamiltonian cycle in the whole graph. □

Theorem 6.2.3 is the best possible for $n \geq 4$. Consider a rainbow complete c-edge-colored multigraph on $n-1$ vertices. Add one new vertex x. Then, add all possible edges in all colors between x and one vertex of the complete graph. Clearly, the resulting graph has $m \geq \frac{c(n-1)(n-2)}{2} + c$ edges, every vertex has rainbow degree c but it has no proper hamiltonian cycle since it is not 2-connected.

Finally, we state a conjecture for the existence of proper hamiltonian cycles involving not also the rainbow degree and the number of edges, but also the connectivity.

Conjecture 6.2.4. *Let G^c be a 2-connected c-edge-colored multigraph on n vertices, $n \geq 10$ and $c \geq 3$. Assume that for every vertex x of G^c, $rd(x) = c$. If $m \geq \frac{c(n-2)(n-3)}{2} + 4c + 1$, then G^c has a proper hamiltonian cycle.*

If true, Conjecture 6.2.4 is the best possible for $n \geq 10$. For this, consider a rainbow complete c-edge-colored multigraph on $n-2$ vertices. Add two new vertices x_1 and x_2. Consider two vertices of the complete graph y_1 and y_2. Finally, add all possible edges in all colors between x_1 and y_1, x_1 and y_2, x_2 and y_1, and x_2 and y_2. The resulting graph has $m \geq \frac{c(n-2)(n-3)}{2} + 4c$ edges, it is 2-connected and every vertex has rainbow degree c. However, this graph has no proper hamiltonian cycle since one of the vertices x_1 or x_2 cannot be together in a proper hamiltonian cycle. If $n = 7, 9$, the graphs $H^c_{k,k+1}$, $k = 3, 4$, are exceptions for Conjecture 6.2.4.

Chapter 7

Bicliques and Graphs Without False-Twins Vertices

This chapter is organized as follows: In Section 7.1 we present known results about the convergence and divergence of the biclique operator, along with the $O(n^4)$ algorithm. In Section 7.2 We prove several results that imply a linear time algorithm for deciding the behavior of a graph under the biclique operator. Finally, in Section 7.3 we study structural properties of bicliques in false-twin free graphs. We assume, unless we clarify, that all graphs in this chapter are connected.

7.1 Preliminary results

We start with this easy observation.

Observation 7.1.1 ([53]). *If G is and induced subgraph of H, then $KB(G)$ is a subgraph (not necessarily induced) of $KB(H)$.*

The following proposition is central for characterize convergent and divergent graphs under the biclique operator.

Proposition 7.1.2 ([53]). *Let G be a graph that contains K_n as a subgraph, for some $n \geq 4$. Then, $K_{2n-4} \subseteq KB(G)$ or $K_{(n-2)(n-3)} \subseteq KB^2(G)$.*

As in [53], consider all maximal sets of false-twin vertices $Z_1, ...Z_k$ and let $\{z_1, z_2, ..., z_k\}$ be the set of *respresentative vertices* such that $z_i \in Z_i$. The graph obtained by the deletion of all vertices of $Z_i \setminus \{z_i\}$, for $i = 1...k$ is denoted *Tw(G)*. Observe that $Tw(G)$ has no false-twin vertices.

Next result remarks that we can delete false-twin vertices of the graph, since if two vertices are false-twins, they belong exactly to the same bicliques. Therefore, the deletion of one of them does not change neither the number of the bicliques of the graph nor the structure of its biclique graph.

Proposition 7.1.3 ([53]). *For any graph G, we have $KB(G) = KB(Tw(G))$.*

Next theorem characterize the behavior of a graph under the biclique operator.

Theorem 7.1.4 ([53]). *If $KB(G)$ contains either K_5 or the gem or the rocket as an induced subgraph, then G is divergent. Otherwise, G converges to K_1 or K_3 in at most 3 steps.*

Notice that, differently than the clique operator, a graph is never periodic under the biclique operator (with period bigger than 1). We remark the importance of the graph K_5 to decide the behavior of a graph under the biclique operator, since we have that $KB(gem) = K_5$ and $K_5 \subseteq KB(house)$.

As a corollary of Theorem 7.1.4 the next useful result was obtained.

Corollary 7.1.5 ([53]). *A graph G is convergent if and only if $Tw(KB(G))$ has at most four vertices. Moreover, $Tw(KB(G)) = K_n$ for $n = 1, ..., 4$.*

Note that if some vertex lies in five bicliques, then $KB(G)$ contains a K_5 and then G diverges. Therefore, Corollary 7.1.5 gives a polynomial time algorithm to test convergence of G: if some vertex lies in five bicliques, answer that G is divergent. Else, the computation of $KB(G)$ and $Tw(KB(G))$ is polynomial (we remark however, that the number of bicliques of a graph can be exponential [98]). If $Tw(KB(G))$ has at most four vertices, answer that G is convergent, otherwise, answer that G is divergent.

Constructing $KB(G)$ takes $O(n^4)$ time, since for the case that is done, the input graph G has at most $2n$ bicliques and generating each biclique is $O(n^3)$ [38, 39]. To build $Tw(KB(G))$ can be done in $O(n^2)$ time. Therefore, the algorithm runs in $O(n^4)$ time.

7.2 Linear time algorithm

In this section we give a linear time algorithm for deciding whether a given graph is divergent or convergent under the biclique operator.

Motivated by Theorem 7.1.4 and Corollary 7.1.5, we study the structure of biclique graphs with false-twin vertices, in order to find conditions to have K_5 as a subgraph that will guarantee the divergence of the graph.

We obtain the following two lemmas.

Lemma 7.2.1. *Let $G = KB(H)$ for some graph H. Let b_1, b_2 be false-twin vertices of G and B_1, B_2 their associated bicliques in H. Suppose that there are no edges between vertices of B_1 and vertices of B_2. Then there exists a vertex $v \in H$ such that v is adjacent to every vertex of B_1 and B_2. Furthermore, G contains a K_5 as induced subgraph.*

Proof. Let b_1, b_2 be false-twin vertices of G and B_1, B_2 their associated bicliques in H, such that there are no edges between vertices of B_1 and vertices of B_2. Since G is connected, take the shortest path from some vertex of B_1 to B_2. Let w be the first vertex in the path such that $w \notin B_1$. Clearly, $w \notin B_2$. Let $v \in B_1$ be a vertex adjacent to w. Suppose that there exists a vertex $x \in B_1$ such that x is not adjacent to w. Then consider the following alternatives:

Case 1: $xv \in E(H)$. Then, $\{x, v, w\}$ is contained in some biclique B, $B \neq B_1$ and $B \neq B_2$, that does not intersect B_2 since there is no edge between B_1 and B_2. A contradiction, because b_1 and b_2 are false-twin vertices.

Case 2: $xv \notin E(H)$. Then there exists a vertex $y \in B_1$ adjacent to v and x. If y is adjacent to w, similar to Case 1, we have that $\{x, y, w\}$ is contained in a biclique B, $B \neq B_1$ and $B \neq B_2$, that does not intersect B_2, a contradiction. Now, if y is not adjacent to w, $\{y, v, w\}$ is contained in a biclique B, $B \neq B_1$ and $B \neq B_2$, that does not intersect B_2, a contradiction.

Then for all $x \in B_1$, x is adjacent to w.

Suppose now that there is no vertex z in B_2 adjacent to w. Then there is a P_3 starting at v that is contained in a biclique B, $B \neq B_1$ and $B \neq B_2$, that does not intersect B_2 leading to a contradiction. Then, let $z \in B_2$ be a vertex adjacent to w. Now, the same argument as in case $v \in B_1$ holds for $z \in B_2$. Then for all $z \in B_2$, z is adjacent to w.

Finally, let v, v' be adjacent vertices in B_1 and let z, z' be adjacent vertices in B_2. Since v, v', z, z' are adjacent to w, then $\{v, w, z\}$, $\{v', w, z\}$, $\{v, w, z'\}$ and $\{v', w, z'\}$ are contained in four bicliques B_3, B_4, B_5 and B_6 such that $B_i \neq B_j$, for $1 \leq i \neq j \leq 6$. So, as $B_i \cap B_j \neq \emptyset$, for $1 \leq i \neq j \leq 5$, K_5 is an induced subgraph of G. □

Lemma 7.2.2. *Let $G = KB(H)$ for some graph H. Let b_1, b_2, b_3 be false-twin vertices of G and let B_1, B_2, B_3 be their associated bicliques in H. Suppose that for any pair of bicliques B_i, B_j, $1 \leq i \neq j \leq 3$, there is an edge between some vertex of B_i and some vertex of B_j. Then, K_5 is an induced subgraph of G.*

Proof. Let b_1, b_2, b_3 be the false-twin vertices of G and B_1, B_2, B_3 their associated bicliques in H such that for any pair of bicliques B_i, B_j, $1 \leq i \neq j \leq 3$, there is an edge between some vertex of B_i and some vertex of B_j. We have the following cases:

Case 1: There is a K_3 with one vertex in each biclique. Let $u \in B_1$, $v \in B_2$, $w \in B_3$ be the K_3. Now, uv, uw and uw are contained in 3 different bicliques of H. Since the biclique that contains uv has to intersect B_3 there exists a vertex $z \in B_3$ adjacent to v and not adjacent to u. Consider the cases:

Case 1.1: $zw \notin E(H)$. Since the biclique that contains uw has to intersect B_2 there exists a vertex $x \in B_2$ adjacent to w and not adjacent to u. Now we have more cases:

Case 1.1.1: $xv \in E(H)$, $xz \in E(H)$. Then we have that H contains the gem $\{u, v, z, x, w\}$ as induced subgraph and therefore there is a K_5 in G.

Case 1.1.2: $xv \in E(H)$, $xz \notin E(H)$. As B_1 is a biclique, then there exists a vertex $u' \in B_1$ adjacent to u. Now, it is easy to see that if we add any number of edges from u' to the other vertices we form 4 bicliques mutually intersecting and taking B_1, B_2 or B_3 we obtain a K_5 in G.

Case 1.1.3: $xv \notin E(H)$, $xz \notin E(H)$. Then we have that $\{u, v, z\}$, $\{w, v, z\}$, $\{v, w, x\}$ and $\{u, w, x\}$ are contained in 4 different intersecting bicliques and therefore, counting one of the bicliques B_1, B_2 or B_3 we have that K_5 is in G.

Case 1.1.4: $xv \notin E(H)$, $xz \in E(H)$. As B_2 is a biclique there exists a vertex $x' \in B_2$ adjacent to x and v such that x' has to be also adjacent to w or z or both since $B_2 \cap B_3 = \emptyset$. If x' is adjacent to w we have two cases. First, if x' is not adjacent to u then H contains the gem $\{u, w, v, x, x'\}$ as induced subgraph and therefore G contains a K_5. Second, if x' is adjacent to u, the same set of vertices induces a house in H and so K_5 is present in G. Now, if x' is adjacent to z and not adjacent to w we have this four sets contained in different bicliques mutually intersecting, $\{v, x, x', w\}$, $\{z, x, w, v\}$, $\{v, z, u\}$ and $\{u, w, x\}$. Finally, taking one of B_1, B_2 or B_3, G contains a K_5.

Case 1.2: $zw \in E(H)$. Since B_1 is a biclique there exists a vertex $u' \in B_1$ adjacent to u. It is easy to see that if we add any number of edges from u' to the other vertices we form 4 bicliques mutually intersecting and taking B_1, B_2 or B_3 we obtain a K_5 in G.

We covered all the cases when a K_3 is in H.

Case 2: There is an induced C_4 in H with two vertices in B_1, one in B_2 and one in B_3. Let $u, x \in B_1$, $v \in B_2$, $w \in B_3$ be the C_4, that is $ux, uv, vw, xw \in E(H)$. As B_2 is a biclique there exists a vertex $v' \in B_2$ adjacent to v, such that v' does not extend the C_4. Then, v' cannot be adjacent to x and not adjacent to u and w. We have the following cases:

Case 2.1: if v' is adjacent to x and w or adjacent to u and w we have a triangle and we already covered that case.

Case 2.2: v' is only adjacent to u and x. Now as B_3 is a biclique there exists a vertex $w' \in B_3$ adjacent to w such that w' does not extend the C_4. Then, it is easy to see that if we add any number of edges from w' to the other vertices we obtain either four mutually intersecting bicliques or a triangle. Therefore a K_5 is present in G.

Case 2.3: v' is only adjacent to w. As B_1 is a biclique there exists a vertex $x' \in B_1$ adjacent to x such that x' does not extend the C_4. As in the previous case, we can see that adding any number of edges from x' to the other vertices we obtain either four mutually intersecting bicliques or a triangle. So, a K_5 in G.

Case 2.4: v' is not adjacent to any one of the other vertices (only to v). Now, as B_3 is a biclique there exists a vertex $w' \in B_3$ adjacent to w such that does not extend the C_4. Then w' cannot be adjacent to any vertex different of v' because if that happens we are in one of the cases **2.1**, **2.2** or **2.3**. So, if w' is adjacent to v' we have that the sets $\{u, x, v, w\}$, $\{v, v', w, w'\}$, $\{u, v, v', w\}$ and $\{x, w, w', v\}$ are contained in four different bicliques and they are mutually intersecting, so G contains a K_5. Now, w' is only adjacent to w. As B_1 is a biclique there exists a vertex $x' \in B_1$ adjacent to x such that does not extend the C_4. As we saw before, x' can be only adjacent to v' and w'. In each one of the cases we have that the sets $\{u, x, v, w\}$, $\{v, v', w, u\}$, $\{w, w', x, v\}$ and $\{x, x', w, u\}$ are contained in four different bicliques and they are mutually intersecting, and therefore K_5 is a subgraph of G.

We covered all the cases when a C_4 is in H with all of the vertices in the bicliques B_1, B_2 and B_3. Now we have the last case.

Case 3: There is an induced C_k, $k \geq 5$ in H with vertice in the bicliques B_1, B_2 and B_3. This case is easy since C_k has k bicliques an each of them has to intersect the three bicliques B_1, B_2 and B_3. So K_5 is a subgraph of G.

Since we covered all cases the proof is done.

□

Next, we present the main theorem of this section. This theorem shows that almost every graph is divergent under the biclique operator. Also, it helps us two obtain the result that will imply the linear time algorithm.

Theorem 7.2.3. *Let G be a graph. If G has at least 7 bicliques, then G diverges under the biclique operator.*

Proof. By way of contradiction, suppose that G has at least 7 bicliques and G converges under the biclique operator. By Corollary 7.1.5, $Tw(KB(G)) = K_n$ for $n = 1, ..., 4$.

Consider the following cases.

Case $n = 1$. Then $KB(G) = K_1$ is a contradiction since G has at least 7 bicliques.

Case $n = 2$. Then $KB(G) = K_2$ or $KB(G)$ is bipartite with more than two vertices. In the first case, G has only 2 bicliques and therefore a contradiction. If $KB(G)$ is bipartite with more than two vertices $KB(G)$ is not a biclique graph [54] and that leads to a contradiction.

Case $n = 3$. Since G has at least 7 bicliques, it follows that in $KB(G)$ there exists a set of false-twin vertices of size at least three. Consider the bicliques B_1, B_2, B_3 of G associated to the three false-twin vertices. If there is a pair of bicliques B_i, B_j such that there is no edge between any vertex of B_i and any vertex of B_j, by Lemma 7.2.1, it follows that K_5 is an induced subgraph of $KB(G)$. Otherwise, for every two pair of bicliques B_i, B_j there is an edge between some vertex of B_i and some vertex of B_j and by Lemma 7.2.2, $KB(G)$ contains K_5 as an induced subgraph. In any case, by Theorem 7.1.4, G diverges under the biclique operator, a contradiction.

Case $n = 4$. There are two alternatives. Suppose that $KB(G)$ has a set of false-twin vertices of size at least three. Then, following the proof of the case $n = 3$, we arrive to a contradiction. Otherwise, there are only two possible graphs isomorphic to $KB(G)$ ($KB(G)$ has 7 or 8 vertices, and it has no set of three false-twin vertices). By inspection, using the characterization given in [54], we prove that these two graphs are not biclique graphs. We conclude that this case can not occur.

Since we covered all cases, G diverges under the biclique operator and the proof is finished. □

Based on last theorem and in the fact that $KB(G) = KB(Tw(G))$, it is interesting to know when a graph without false-twin vertices has at least 7 bicliques. Next theorem answers this question and, moreover, it gives us the linear time algorithm for the recognition of divergent and convergent graphs under the biclique operator.

Theorem 7.2.4. *Let G be a graph with no false-twin vertices. If G has at least 13 vertices then G has at least 7 bicliques.*

Proof. We prove the result by induction on n. For $n = 13$, by inspection of all graphs without false-twin vertices, the result holds. Suppose now that $n \geq 14$. By a Theorem in [30], there is a vertex v such that $G - v$ has no false-twin vertices. Consider the graph $G' = G - v$. If G' is connected, since it has at least 13 vertices, by inductive hypothesis, it has at least 7 bicliques. Now, as G' is an induced subgraph of G, we conclude that G also has at least 7 bicliques. Suppose now that G' is not connected. Let $G_1, G_2, \dots, G_s$ be the connected components of G' on $n_1, n_2, \ldots, n_s$ vertices respectively. Since G has no false-twin vertices, it can be at most one G_i such that $n_i = 1$. If there is one component with at least 13 vertices, then by inductive hypothesis, this component has at least 7 bicliques and so does G. So, every component has at most 12 vertices. Now, by inspection we can verify that every component G_i (but maybe one with just 1 vertex) has at least $\lceil \frac{n_i}{2} \rceil$ bicliques. Also, since G' is disconnected, v along with at least one vertex of each of the s components is a biclique in G isomorphic to $K_{1.r}$ that is lost in G'. Summing up and assuming the worst case, this is, there exists one $n_i = 1$ (suppose $i = s$) we obtain that the number of bicliques of G is at least

$$\left(\sum_{i=1}^{s-1} \left\lceil \frac{n_i}{2} \right\rceil \right) + 1 \geq \left\lceil \frac{11}{2} \right\rceil + 1 = 7$$

as we wanted to prove. Now the proof is complete. □

Theorem 7.2.4 implies that the number of convergent graphs without false-twin vertices is finite since convergent graphs without false-twin vertices have at most 12 vertices. This fact leads to the following linear time algorithm.

Algorithm: Given a graph G, build $H = Tw(G)$. If H has at least 13 vertices, answer "G diverges" and STOP. Otherwise, build $Tw(KB(H))$. If $Tw(KB(H))$ has at most 4 vertices answer "G converges" and STOP. Otherwise, answer "G diverges" and STOP.

The algorithm has $O(n + m)$ time complexity. For this, observe that H can be built in $O(n + m)$ time by the known modular decomposition and if H has at most 12 vertices any further operation takes $O(1)$ time complexity.

7.3 Bicliques in false-twin free graphs

To finish this chapter and motivated by the fact that the amount of bicliques in graphs without false-twin vertices is lower bounded, we will study this class of graphs more extensively.

We start with the case when the graph is also K_3-free.

Lemma 7.3.1. *Let G be a K_3-free graph without false-twin vertices. Then every vertex is contained in a biclique isomorphic to $K_{1,r}$ for some $r \geq 1$.*

Proof. Let v be a vertex. If $|N(v)| = 1$ then the result clearly follows. Suppose now that $|N(v)| > 1$. Now, since $N(v)$ is an independent set, $\{v\} \cup N(v)$ is contained in one biclique. If there is no vertex w such that $N(v) \subseteq N(w)$ then $\{v\} \cup N(v)$ is a biclique isomorphic to $K_{1,r}$, $r = |N(v)|$. Otherwise, let u be the vertex with maximum degree among all vertices in $N(v)$. Clearly, since G is K_3-free and without false-twin vertices, if there are two vertices of same maximum degree, they must have some different neighbors. Therefore, $\{u\} \cup N(u)$ is a biclique isomorphic to $K_{1,r}$, $r = |N(u)|$ and contains the vertex v as desired. □

Based on last lemma, we obtain this immediat result.

Corollary 7.3.2. *Let G be a graph without false-twin vertices. Let v be a vertex such that $d(v) = \Delta(G)$ and v does not belong to a K_3. Then $\{v\} \cup N(v)$ is a biclique.*

Now, we obtain the following important result that will help us to prove the main theorem of the section.

Lemma 7.3.3. *Let G be a K_3-free graph without false-twin vertices. Let v be a vertex such that $d(v) = k$. Then v belongs to at least k different bicliques.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the neighbors of v. Clearly they are an independent set. Let $x_1, x_2, \ldots, x_\ell$ be the set of vertices adjacent to the vertices $v_1, v_2, \ldots, v_k$. Let G' be the subgraph induced by $\{v\} \cup \{v_1, v_2, \ldots, v_k\} \cup \{x_1, x_2, \ldots, x_\ell\}$. It is easy to see that $v_1, v_2, \ldots, v_k$ are not false-twins in G' and since G is K_3-free, v is not adjacent to any x_j, $1 \leq j \leq \ell$. Now, for each $1 \leq i \leq k$, let $S_{v_i} = \{N_{G'}(x_j) : v_i \in N_{G'}(x_j), 1 \leq j \leq \ell\} \cup N(v)$. Let $Cl(S_{v_i}) = \bigcap_{S \in S_{v_i}} S$. Observe first that $S_{v_i} \neq \emptyset$ for all i since $N(v)$ belongs to all of them. Observe then that $Cl(S_{v_i}) \cup (\{x_j : N_{G'}(x_j) \in S_{v_i}, 1 \leq j \leq \ell\} \cup \{v\})$ is a biclique in G' and therefore a biclique in G. We show now that, for all $i \neq j$ we have that $Cl(S_{v_i}) \neq Cl(S_{v_j})$, i.e., v belongs to k different bicliques in G. Suppose by contrary, that $Cl(S_{v_i}) = Cl(S_{v_j})$. Now, since $v_i \in Cl(S_{v_i})$, we have that $v_i \in Cl(S_{v_j})$. Similarly, $v_j \in Cl(S_{v_i})$. So, for all $S \in S_{v_i}$, we have that $v_j \in S$. Also, for all $S \in S_{v_j}$, we have that $v_i \in S$. This is $N(v_i) = N(v_j)$, a contradiction since G has no false-twin vertices. Now, the result follows. □

As a corollary, we obtain the following.

Corollary 7.3.4. *Let G be a K_3-free graph without false-twin vertices. Suppose that there is a vertex v such that the graph $G - v$ has k sets of false-twin vertices. Then $G - v$ has at least k bicliques less than G.*

Proof. Observe first that, since G has no false-twin vertices, every set of false-twin vertices in $G - v$ has size exactly 2. Let v_i, w_i for $1 \leq i \leq k$ the k sets of false-twin vertices, such that v is adjacent to v_i. Observe now that, since v_i and w_i are false-twins, they belong to exactly the same bicliques but those bicliques containing the edge vv_i. Consider now the subgraph induced by the vertices $\{v\} \cup \{v_1, v_2, \ldots, v_k\} \cup N(v_1) \cup \cdots \cup N(v_k)$. Clearly, $v_1, v_2, \ldots, v_k$ are not false-twins in this graph. Now, by Lemma 7.3.3, v belongs to k different bicliques. Those bicliques are either bicliques or are contained in bigger bicliques in G, but they do not contain any of the vertices w_i. Now, after removing v, these k bicliques are lost in $G - v$ since any other biclique containing any v_i contains also w_i. □

Combining last three results, we obtain the main theorem of the section. It gives a tight lower bound for the number bicliques of a K_3-free graph without false-twin vertices.

Theorem 7.3.5. *Let G be a K_3-free graph of order $n \geq 4$ without false-twin vertices. Then G has at least $\lceil \frac{n}{2} \rceil$ bicliques.*

Proof. The proof is by induction on n. For $n = 4$ the result trivially holds. Suppose $n \geq 5$. Now, by Lemma 7.3.1 there is a vertex v contained in a biclique isomorphic to $K_{1,r}$. Without losing of generality, we can suppose that v is the center, otherwise we take its unique adjacent vertex in the biclique. Consider the graph $G' = G - v$. We consider the following two cases.

- G' is disconnected. Let $G_1, G_2, \ldots, G_s$ be the connected components of G' on $n_1, n_2, \ldots, n_s$ vertices respectively. Since G has no false-twin vertices, it can be at most one G_i such that $n_i = 1$. Suppose that there are ℓ components, $G_{i_1}, G_{i_2}, \ldots .G_{i_\ell}$ with $k_{i_1}, k_{i_2}, \ldots, k_{i_\ell}$ pairs of false-twin vertices. So, by Lemma 7.3.4 we have that G' has $k_{i_1} + k_{i_2} + \cdots + k_{i_\ell}$ bicliques less than G. Also, since G' is disconnected, v along with at least one vertex of each of the s components is a biclique in G isomorphic to $K_{1.r}$ that is lost in G', and clearly different than the ones we have just counted. Consider now for each G_{i_j} the graph $Tw(G_{i_j})$. Each of these graphs have $n_{i_j} - k_{i_j}$ vertices and no false-twin vertices. If $n_{i_j} - k_{i_j} = 2$ then $Tw(G_{i_j}) = K_2$, and therefore it has 1 biclique, i.e., at least $\left\lceil \frac{n_{i_j} - k_{i_j}}{2} \right\rceil$ bicliques. If $n_{i_j} - k_{i_j} \geq 4$, by inductive hypothesis, $Tw(G_{i_j})$ has also at least $\left\lceil \frac{n_{i_j} - k_{i_j}}{2} \right\rceil$ bicliques. Now, for all other G_i without false-twin vertices, if $n_i = 2$, G_i has, as before, $1 = \lceil \frac{n_i}{2} \rceil$ biclique and for $n_i \geq 4$, by inductive hypothesis, G_i has at least $\lceil \frac{n_i}{2} \rceil$. If we sum up everything (and suppose the worst case, this is, there exists one G_i, say G_s, such that $n_i = 1$) we have that the number of bicliques of G is at least

$$\left(\sum_{j=1}^{\ell}\left\lceil\frac{n_{i_j}-k_{i_j}}{2}\right\rceil + k_{i_j}\right) + \left(\sum_{i=1, i\neq i_j}^{s-1}\left\lceil\frac{n_i}{2}\right\rceil\right) + 1 \geq$$

$$\left(\sum_{j=1}^{\ell}\left\lceil\frac{n_{i_j}}{2}\right\rceil\right) + \left(\sum_{i=1, i\neq i_j}^{s-1}\left\lceil\frac{n_i}{2}\right\rceil\right) + 1 \geq \left(\sum_{i=1}^{s-1}\left\lceil\frac{n_i}{2}\right\rceil\right) + 1 \geq \left\lceil\frac{n}{2}\right\rceil$$

as desired.

- Suppose now that G' is connected. Suppose that in G' there are k pairs of false-twin vertices. As before, by Lemma 7.3.4, G' has k bicliques less than G. Consider now the graph $Tw(G')$. This graph has $n-k-1 \geq 4$ vertices and no false-twin vertices, therefore we can apply the inductive hypothesis. So, we have that $Tw(G)$ has at least $\left\lceil\frac{n-k-1}{2}\right\rceil$ bicliques. Therefore G has at least $\left\lceil\frac{n-k-1}{2}\right\rceil + k \geq \lceil\frac{n}{2}\rceil$ bicliques as desired. Now suppose that G' has no false-twin vertices. We have by inductive hypothesis that it has at least $\lceil\frac{n-1}{2}\rceil$ bicliques. Finally, since the biclique isomorphic to $K_{1,r}$ with center at v is lost in G' we conclude that G has at least $\lceil\frac{n-1}{2}\rceil + 1 \geq \lceil\frac{n}{2}\rceil$ bicliques.

Since we covered all cases the proof is now complete. □

Notice that this result directly implies Theorem 7.2.4 for K_3-free graphs. Also, we obtain this direct corollary.

Corollary 7.3.6. *Let T be a tree of order $n \geq 4$ without false-twin vertices. Then T has at least $\lceil\frac{n}{2}\rceil$ bicliques.*

Following Corollary 7.3.6 we obtain this immediat result.

Corollary 7.3.7. *For $\lceil\frac{n}{2}\rceil \leq k \leq n-2$ and $n \geq 4$ there exists a tree T without false-twin vertices of order n and k bicliques.*

Now, we state the following conjecture that generalizes Theorem 7.3.5.

Conjecture 7.3.8. *Let G be a graph without false-twin vertices of orden n. Then G has at least $\lceil\frac{n}{2}\rceil$ bicliques.*

We remark that we verified this conjecture for all graphs without false-twin vertices up to 13 vertices. Note that this is exactly the basic step for the induction in Theorem 7.2.4. We could prove the conjecture if similar results as we have for K_3-free graphs are valid for general graphs. In particular, if the following conjectures are true, we have a proof of Conjecture 7.3.8

Conjecture 7.3.9. *Let G be a graph without false-twin vertices. Then there exists at least one biclique isomorphic either to $K_{1,r}$, $r \geq 1$, or to $K_{2,p}$, $p \geq 2$.*

Conjecture 7.3.10. *Let G be a graph without false-twin vertices. Let v be a vertex such that $d(v) = k$. Then, if it has a neighbor of degree 1, v belongs to at least $k-2$ different bicliques, otherwise, to at least $k-1$.*

Conjecture 7.3.11. *Let G be a graph without false-twin vertices. Suppose that there is a vertex v such that the graph $G-v$ has k sets of false-twin vertices. Then $G-v$ has at least k bicliques less than G.*

Our main goal to prove Conjecture 7.3.8 was to prove that we always have a biclique isomorphic either to $K_{1,r}$, $r \geq 1$, or to $K_{2,p}$, $p \geq 2$. Having this result, we can always remove one or two vertices in order to obtain a graph with less bicliques than the original and then obtain the result by induction as in Theorem 7.3.5.

Because of this we have the following results that guarantee the existence of a biclique isomorphic either to $K_{1,r}$, $r \geq 1$, or to $K_{2,p}$, $p \geq 2$ for special classes of graphs.

Lemma 7.3.12. *Let G be a graph without false-twin vertices. Let $v_1v_2v_3v_4$ be an induced C_4. If no vertex of the C_4 belongs to a K_3 then are there two bicliques isomorphic to K_{1,r_1} and K_{1,r_2}, for some $r_1, r_2 \geq 1$.*

Proof. Since G has no false-twin vertices There exist without losing of generality, one vertex u_1 adjacent to v_1 not adjacent to the others v_i and another vertex u_2 adjacent to v_2 not adjacent to all others v_i. Clearly $u_1 \neq u_2$ because any of v_1, v_2, v_3 and v_4 belongs to a K_3. Now, if $\{v_1\} \cup \{u_1, v_2, v_4\}$ and $\{v_2\} \cup \{u_2, v_1, v_3\}$ are in bicliques isomorphic to K_{1,r_1} and K_{1,r_2}, we are done. Otherwise there are vertices $w_1 \neq w_2$ that extend them respectively. This is w_1 is not adjacent to v_1 but adjacent to u_1, v_2 and v_4, and w_2 is not adjacent to v_2 but adjacent to u_2, v_1 and v_3. We have then that, if $\{v_1\} \cup \{u_1, v_2, v_4, w_2\}$ and $\{v_2\} \cup \{u_2, v_1, v_3, w_2\}$ are in bicliques isomorphic to K_{1,r_1} and K_{1,r_2}, we are done. Otherwise, as before, we have two vertices $z_1 \neq z_2$ that extend them. But then we have that $\{v_1\} \cup \{u_1, v_2, v_4, w_2, z_2\}$ and $\{v_2\} \cup \{u_2, v_1, v_3, w_2, z_1\}$ are contained in two different bicliques. If they are not isomorphic to K_{1,r_1} and K_{1,r_2} we have again two different vertices that extend them. Finally, since G is finite, at some point we will have no more vertices to extend them and therefore the result holds. □

Lemma 7.3.13. *Let G be a graph without false-twin vertices. Let v be a vertex such that v does not belong to an induced C_4. Then v belongs to a biclique isomorphic to $K_{1,r}$, $r \geq 1$.*

Lemma 7.3.14. *Let G be a graph without false-twin vertices. Let v be an universal vertex. Then v belongs to a biclique isomorphic to $K_{1,r}$, $r \geq 1$.*

Notice that last lemma is also true when the graph has false-twin vertices.

Lemma 7.3.15. *Let G be a graph without false-twin vertices. Let v be a vertex such that $N(v) = K_p$, $p \geq 1$ then v belongs to a biclique isomorphic to $K_{1,r}$, $r \geq 1$.*

Lemma 7.3.16. *Let G be a graph without false-twin vertices. If G has a K_p, $p \geq 1$, as a cut-set, then there exists a biclique isomorphic to $K_{1,r}$, $r \geq 1$.*

Lemma 7.3.17. *Let G be a graph without false-twin vertices. Let v, w be two non-adjacent vertices such that $G - \{v, w\}$ is disconnected. Then, there exists a biclique isomorphic either to $K_{1,r}$, $r \geq 1$, or to $K_{2,p}$, $p \geq 2$*

To finish this chapter, we present the following results about the structure of graphs without false-twin vertices.

Lemma 7.3.18. *Let G be a graph without false-twin vertices. If G has a K_3 as a subgraph then there is no vertex that belongs to all bicliques.*

Lemma 7.3.19. *Let G be a graph without false-twin vertices. There are at most two vertices v, w that belong to all bicliques and they must be adjacent. Moreover, for every other vertex u, u is adjacent to v if and only if u is not adjacent to w.*

Lemma 7.3.20. *Let G be a graph without false-twin vertices. For every biclique B there exists at most one vertex that belongs only to B.*

Lemma 7.3.21. *Let G be a graph without false-twin vertices. Let v a vertex such that $d(v) \geq 2$. Then v belongs to at least 2 different bicliques.*

From last lemma, we obtain this immediat result.

Corollary 7.3.22. *Let G be a graph without false-twin vertices. There are at least $\lceil \frac{n}{2} \rceil$ vertices that belong to at least 2 bicliques.*

Lemma 7.3.23. *Let G be a graph without false-twin vertices, $G \neq K_2$. If there are two vertices of degree one, they belong to different bicliques.*

Chapter 8

Conclusions and Perspectives

In the present thesis, we studied different problems in edge-colorings and in edge-colored multigraphs. In particular, we studied the proper connection number of graphs, strong edge-colorings in k-degenerate and outerplanar graphs, and proper hamiltonian paths and cycles in edge-colored multigraphs. Finally, we studied properties of bicliques of graphs and, in particular, we gave a linear time algorithm to recognize convergent and divergent graphs under the biclique operator

8.1 Contribution summary

In Chapter 3 we studied proper connection in graphs. We proved several upper bounds for $pc_k(G)$. We stated some conjectures for general and bipartite graphs, Conjectures 3.2.6 and 3.1.1 respectively, and we proved them for the case when $k = 1$. In particular, we proved a variety of conditions on G which imply $pc(G) = 2$. We can remark that, from Theorem 3.3.6, it is clear that if G is 2-connected and $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$. We believe that this degree condition can be greatly improved in the 2-connected case. In particular, we propose the following conjecture.

Conjecture 8.1.1. *If $\kappa(G) = 2$ and $\delta(G) \geq 3$, then $pc(G) = 2$.*

By the proof of Theorem 3.2.2 and the standard ear decomposition of a 2-connected graph, it is easy to produce a linear-time algorithm to 3-color any 2-connected graph to be proper connected with the strong property. Also since there is an $O(n+m)$ algorithm for finding a block decomposition of a graph G with $\kappa(G) = 1$ on n vertices with m edges, we can find an $O(n+m)$ algorithm to produce a proper connected coloring of such graphs. Therefore, in practice, these colorings are not difficult to find.

In Chapter 4, we showed that the strong chromatic index is linear in the maximum degree for any k-degenerate graph where k is fixed. This is an extension of the results due to Chang and Narayanan [29] where they prove the same for 2-degenerate graphs. As a corollary, our result leaded to considerable improvement of the constants and also gave an easier and more efficient algorithm. We also gave a sketch of the algorithm.

Further, we considered outerplanar graphs. We gave a formula to find exact strong chromatic index for bipartite outerplanar graphs. We also improved the upper bound for the general outerplanar graphs from the $3\Delta - 3$ stated in [60].

A recent work [66] gives an algorithm to find the strong chromatic index of any maximal outerplanar graph, but notice that when you extend the graph to maximal outerplanar, the maximum degree and the index can increase. We provided an algorithm to color any outerplanar graph with number of colors close to optimum and bipartite outerplanar graphs with optimum colors.

In some special cases of the general outerplanar graph, (where we use η extra colors), we were not able to show the optimality of the bounds. We believe that it is very close to the exact bound within an additive factor of a small constant. It would be interesting to prove if our bounds are optimal and if not, to find a way to close the gap.

In Chapters 5 and 6, we studied the existence of proper hamiltonian paths and proper hamiltonian cycles, respectively, in edge-colored multigraphs depending on the number of edges, the rainbow degree and the connectivity. Here, the notable fact is that the proofs were sometimes long and tedious despite the lower bounds for the edges in the considered multigraphs were really high. Finally, we stated Conjecture 6.2.4 that guarantees the existence of a proper hamiltonian cycle in a 2-connected edge-colored multigraph with bounded number of edges and fixed rainbow degree.

It should be also interesting to study similar conditions for other patterns such as trees, etc.

In Chapter 7 we studied different structural properties of bicliques in graphs without false-twin vertices and then, we applied them to the study of the iterated biclique operator. There exists an $O(n^4)$ time algorithm to decide if a given graph converges or diverges under the biclique operator and the possible behaviors have been characterized. We proved that graphs with at least 7 bicliques are divergent under the biclique operator. Furthermore, we proved that this sufficient condition implies that graphs with no false-twin vertices that are convergent, have at most 12 vertices, and therefore, there is a finite number of them. We also proposed a linear time algorithm to decide the behavior of a graph under the biclique operator. It is worth mentioning that no polynomial time algorithm is known for deciding the behavior of a graph under the clique operator. Finally, we proved a lower bound of the number of bicliques in K_3-free graphs without false-twin vertices and we stated Conjecture 7.3.8 for a similiar result for the general case, i.e., the condition of the graph being K_3-free is dropped. We proposed several results and conjectures that might help to solve the general one.

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