Second-order inertial force on a sphere in a steady linear flow

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– Supplementary Material –

This supplementary material describes some additional details regarding the Fourier transforms we used to derive the inner solution (§ 3.2 and eqs. (3.37)-(3.38) in the manuscript).

1. Definitions

The three-dimensional Fourier transform used in the manuscript is defined as

$$\mathcal{F}(f(\boldsymbol{r})) = \hat{f}(\boldsymbol{k}) = \int_{\mathbb{R}^3} f(\boldsymbol{r})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}\,, \tag{1.1}$$

where $\boldsymbol{r} = x_i\boldsymbol{e}_i$ is the position vector. The inverse transform reads

$$\mathcal{F}^{-1}(\hat{f}(\boldsymbol{k})) = f(\boldsymbol{r}) = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \hat{f}(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k} \tag{1.2}$$

From these definitions, some basic properties can be directly inferred. For instance, if $g(k_1,\, k_2,\, k_3) = \mathcal{F}(f(x_1,\, x_2,\, x_3))$ denotes the Fourier transform of the function $f(x_1,\, x_2,\, x_3)$, then one has

$$\mathcal{F}(g(x_1,\, x_2,\, x_3)) = 8\pi^3\, f(-k_1,\, -k_2,\, -k_3)\,,$$

or equivalently

$$\frac{1}{8\pi^3}\mathcal{F}(g(-x_1,\, -x_2,\, -x_3)) = f(k_1,\, k_2,\, k_3)\,.$$

2. Fourier transforms of functions of $\boldsymbol{r}$ and $r = ||\boldsymbol{r}||$

The calculation of the inner solution makes repeated use of Fourier transforms of functions that have the form $f(\boldsymbol{r}) = x_1^{i_2}x_2^{i_3}x_3^{i_4}r^{i_1}$, where $(i_2,\, i_3,\, i_4) \in \mathbb{N}^3$, $i_1 \in \mathbb{Z}$ and $r = ||\boldsymbol{r}||$. Some transforms of such functions are given by Schwartz (1966) and are particularly useful for our purposes. We summarize them in the following.

• Functions in the form $1/r^\lambda$ with λ an even and strictly positive integer (i.e. $1/r^2$, $1/r^4$, $1/r^6$, ...), or with λ an odd and negative integer (i.e. r, r^3, r^5, ...), admit the Fourier transforms

$$\frac{1}{r^\lambda} \longrightarrow \frac{2^{3-\lambda}\pi^{3/2}\Gamma\left(\frac{3}{2}-\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)}k^{\lambda-3} \quad \text{with} \quad \lambda \neq 2h+3\,, \lambda \neq -2h\,, h = 0,1,2,\ldots\,, \tag{2.1}$$

with Γ the Gamma function. See examples with r, $1/r^2$ and $1/r^4$ in table 1.

$f(\boldsymbol{r})$	$\mathcal{F}(f)$	$f(\boldsymbol{r})$	$\mathcal{F}(f)$
$f(\boldsymbol{r})\cdot g(\boldsymbol{r})$	$\frac{1}{8\pi^3}\hat{f}(\boldsymbol{k})*\hat{g}(\boldsymbol{k})$	1	$8\pi^3\delta(\boldsymbol{k})$
$x_i\, f(\boldsymbol{r})$	$i\,\frac{\partial \hat{f}(\boldsymbol{k})}{\partial k_i}$	$\frac{1}{r}$	$\frac{4\pi}{k^2}$
$\delta(\boldsymbol{r})$	1	$\frac{1}{r^2}$	$\frac{2\pi^2}{k}$
r^2	$-8\pi^3\triangle_{\boldsymbol{k}}\delta(\boldsymbol{k})$	$\frac{1}{r^3}$	$4\pi\left(\ln\left(\frac{1}{k}\right)+1-\gamma\right)$
r	$-\frac{8\pi}{k^4}$	$\frac{1}{r^4}$	$-\pi^2 k$

TABLE 1. Fourier transforms of some functions of $\boldsymbol{r}$ and r (γ denotes the Euler constant).

• In the particular case where λ is odd and strictly greater than unity, that is $\lambda = 2h+3$, (i.e. $1/r^3$, $1/r^5$, $1/r^7$, ...), one has

$$\frac{1}{r^{2h+3}} \longrightarrow \frac{4\pi(-1)^h}{\Gamma(2+2h)}k^{2h}\left(\ln\left(\frac{1}{k}\right)+\ln(2)+\frac{1}{2}\left(\Psi\left(h+\frac{3}{2}\right)+\Psi(h+1)\right)\right), \qquad (2.2)$$

where

$$\Psi(x) = \frac{1}{\Gamma(x)}\frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x}\,.$$

Note that this result has been obtained after making use of the identity

$$\frac{8\pi^{3/2}}{2^{2h+2}h!\,\Gamma\left(h+\frac{3}{2}\right)} = \frac{4\pi}{\Gamma(2+2h)}\,.$$

See the example with $1/r^3$ in table 1.

• When λ is even and strictly negative (i.e. r^2, r^4, r^6, ...), one has

$$r^{2h} \longrightarrow 8\pi^3(-\triangle_{\boldsymbol{k}})^h\delta(\boldsymbol{k})\,, \quad h = 1,\,2,\,3,\ldots\,, \qquad (2.3)$$

with $\triangle_{\boldsymbol{k}}$ the Laplacian in Fourier space. See the example with r^2 in table 1.

3. Maple® algorithms

In this section we provide some Maple® algorithms we used to compute the Fourier transforms necessary to obtain the inner solution (§ 3.2 in the manuscript).

REFERENCES

Schwartz, L. 1966 Théorie des distribution, *Hermann*, pp. 257–258.

3.1 Script for the Fourier Transforms

$> \mathit{Fourier3D} := \mathbf{proc}(\arg)$

$\mathbf{local}\ i1 := 0:\ \mathbf{local}\ i2 := 0:\ \mathbf{local}\ i3 := 0:\ \mathbf{local}\ i4 := 0:\ \mathbf{local}\ i5 := 0:$
$\mathbf{local}\ f := \arg:\ \mathbf{local}\ fm1 := \arg^{-1}:\ \mathbf{local}\ F:\ \mathbf{local}\ h:\ \mathbf{local}\ case:$

$$\mathbf{if}\ \frac{subs(r=2,\arg)}{subs(r=1,\arg)} < 1\ \mathbf{then}\ case := 1:$$

$$\quad \mathbf{while}\ \mathit{diff}(fm1, r) \neq 0\ \mathbf{do}\ fm1 := \frac{fm1}{r}: i1 := i1+1:\ \mathbf{end\ do}:$$

$$\quad \mathbf{if}\ i1 = 1\ \mathbf{then}\ F := \frac{4\cdot \mathrm{Pi}}{k^2};$$

$$\quad \mathbf{elif}\ type(i1, odd)\ \mathbf{then}\ h := \frac{(i1-3)}{2}:$$

$$\quad F := \frac{4\pi(-1)^h}{\Gamma(2+2h)}\cdot k^{2\,h}\cdot(\ln(\frac{1}{k}) + \ln(2) + \frac{1}{2}\cdot(\mathrm{Psi}(h+\frac{3}{2}) + \mathrm{Psi}(h+1))):$$

$$\quad \mathbf{elif}\ type(i1, even)\ \mathbf{then}\ F := \frac{8\pi^{\frac{3}{2}}\Gamma(\frac{3}{2}-\frac{i1}{2})}{2^{i1}\Gamma(\frac{i1}{2})}\cdot k^{i1-3}:$$

$$\quad \mathbf{end\ if}:$$

$$\mathbf{elif}\ \frac{subs(r=2,\arg)}{subs(r=1,\arg)} = 1\ \mathbf{then}\ case := 2:$$

$$F := 8\cdot \mathrm{Pi}^3\cdot \delta(k1, k2, k3):$$

$$\mathbf{elif}\ \frac{subs(r=2,\arg)}{subs(r=1,\arg)} > 1\ \mathbf{then}\ case := 3:$$

$$\quad \mathbf{while}\ \mathit{diff}(f, r) \neq 0\ \mathbf{do}\ f := \frac{f}{r}: i1 := i1+1:\ \mathbf{end\ do}:$$

$$\quad \mathbf{if}\ i1 = 1\ \mathbf{then}\ F := -\frac{8\cdot \mathrm{Pi}}{k^4};$$

$$\quad \mathbf{elif}\ type(i1, odd)\ \mathbf{then}\ F := \frac{8\pi^{\frac{3}{2}}\Gamma(\frac{3}{2}-\frac{-i1}{2})}{2^{-i1}\Gamma(\frac{-i1}{2})}\cdot k^{-i1-3}:$$

$$\quad \mathbf{elif}\ type(i1, even)\ \mathbf{then} h := \frac{i1}{2}:\ F := 8\cdot \mathrm{Pi}^3\cdot(-\Delta)^h\cdot\delta(k1, k2, k3):$$

$$\quad \mathbf{end\ if}:$$

$$\mathbf{end\ if}:$$

$$\mathbf{while}\ \mathit{diff}(f, x1) \neq 0\ \mathbf{do}\ f := \frac{f}{x1}: i2 := i2+1: \mathbf{end\ do}:$$

$$\mathbf{while}\ \mathit{diff}(f, x2) \neq 0\ \mathbf{do}\ f := \frac{f}{x2}: i3 := i3+1:\ \mathbf{end\ do}:$$

$$\mathbf{while}\ \mathit{diff}(f, x3) \neq 0\ \mathbf{do}\ f := \frac{f}{x3}: i4 := i4+1:\ \mathbf{end\ do}:$$

$$\mathbf{for}\ i5\ \mathbf{from}\ 1\ \mathbf{to}\ i2\ \mathbf{do}\ F := I\cdot(\mathit{diff}(F, k1) + \frac{k1}{k}\cdot \mathit{diff}(F, k))\ \mathbf{end\ do};$$

$$\mathbf{for}\ i5\ \mathbf{from}\ 1\ \mathbf{to}\ i3\ \mathbf{do}\ F := I\cdot(\mathit{diff}(F, k2) + \frac{k2}{k}\cdot \mathit{diff}(F, k))\ \mathbf{end\ do};$$

$$\mathbf{for}\ i5\ \mathbf{from}\ 1\ \mathbf{to}\ i4\ \mathbf{do}\ F := I\cdot(\mathit{diff}(F, k3) + \frac{k3}{k}\cdot \mathit{diff}(F, k))\ \mathbf{end\ do};$$

$$\mathbf{if}\ case = 1\ \mathbf{then}\ expand(\frac{r^{i1}\cdot\arg}{x1^{i2}\cdot x2^{i3}\cdot x3^{i4}}\cdot F);\ \mathbf{elif}\ case = 2\ \mathbf{then}\ expand(\frac{\arg}{x1^{i2}\cdot x2^{i3}\cdot x3^{i4}}\cdot F)\ \mathbf{elif}\ case = 3\ \mathbf{then}\ expand(\frac{\arg}{r^{i1}\cdot x1^{i2}\cdot x2^{i3}\cdot x3^{i4}}\cdot F)\mathbf{end\ if}:$$

$$\mathbf{end\ proc}:$$

3.2 Script for the Inverse Fourier Transforms

> $InvFourier3D := \textbf{proc}(\arg)$

$\textbf{local}\, i1 := 0: \textbf{local}\, i2 := 0: \textbf{local}\, i3 := 0: \textbf{local}\, i4 := 0: \textbf{local}\, i5 := 0:$
$\textbf{local}\, f := \arg: \textbf{local}\, fm1 := \arg^{-1}: \textbf{local}\, F: \textbf{local}\, h: \textbf{local}\, case:$

$\textbf{if}\ \frac{subs(k=2,\arg)}{subs(k=1,\arg)} < 1\ \textbf{then}\ case := 1:$

$\quad \textbf{while}\ diff(fm1, k) \neq 0\ \textbf{do}\ fm1 := \frac{fm1}{k}: i1 := i1 + 1: \textbf{end do}:$

$\quad \textbf{if}\ i1 = 1\ \textbf{then}\ F := \frac{1}{8\cdot \mathrm{Pi}^3}\cdot\frac{4\cdot \mathrm{Pi}}{r^2};$

$\quad \textbf{elif}\ type(i1, odd)\ \textbf{then}\ h := \frac{(i1-3)}{2}: F := \frac{1}{8\cdot \mathrm{Pi}^3}\cdot\frac{4\pi(-1)^h}{\Gamma(2+2h)}\cdot r^{2h}\cdot(\ln(\frac{1}{r})$
$\quad + \ln(2) + \frac{1}{2}\cdot(\mathrm{Psi}(h+\frac{3}{2}) + \mathrm{Psi}(h+1))):$

$\quad \textbf{elif}\ type(i1, even)\ \textbf{then}\ F := \frac{1}{8\cdot \mathrm{Pi}^3}\cdot\frac{8\pi^{\frac{3}{2}}\Gamma(\frac{3}{2}-\frac{i1}{2})}{2^{i1}\Gamma(\frac{i1}{2})}\cdot r^{i1-3}:$

$\quad \textbf{end if}:$

$\textbf{elif}\ \frac{subs(k=2,\arg)}{subs(k=1,\arg)} = 1\ \textbf{then}\ case := 2: F := \delta(x1,\ x2,\ x3):$

$\textbf{elif}\ \frac{subs(k=2,\arg)}{subs(k=1,\arg)} > 1\ \textbf{then}\ case := 3:$

$\quad \textbf{while}\ diff(f, k) \neq 0\ \textbf{do}\ f := \frac{f}{k}: i1 := i1 + 1: \textbf{end do}:$

$\quad \textbf{if}\ i1 = 1\ \textbf{then}\ F := -\frac{1}{8\cdot \mathrm{Pi}^3}\cdot\frac{8\cdot \mathrm{Pi}}{r^4}:$

$\quad \textbf{elif}\ type(i1, odd)\ \textbf{then}\ F := \frac{1}{8\cdot \mathrm{Pi}^3}\cdot\frac{8\pi^{\frac{3}{2}}\Gamma(\frac{3}{2}-\frac{-i1}{2})}{2^{-i1}\Gamma(\frac{-i1}{2})}\cdot r^{-i1-3}:$

$\quad \textbf{elif}\ type(i1, even)\ \textbf{then}\ h := \frac{i1}{2}: F := (-\Delta)^h\cdot\delta(x,y,z):$

$\quad \textbf{end if}:$

$\textbf{end if}:$

$\textbf{while}\ diff(f, k1) \neq 0\ \textbf{do}\ f := \frac{f}{k1}: i2 := i2 + 1: \textbf{end do}:$

$\textbf{while}\ diff(f, k2) \neq 0\ \textbf{do}\ f := \frac{f}{k2}: i3 := i3 + 1: \textbf{end do}:$

$\textbf{while}\ diff(f, k3) \neq 0\ \textbf{do}\ f := \frac{f}{k3}: i4 := i4 + 1: \textbf{end do}:$

$\textbf{for}\ i5\ \textbf{from}\ 1\ \textbf{to}\ i2\ \textbf{do}\ F := -I\cdot(diff(F, x1) + \frac{x1}{r}\cdot diff(F, r))\ \textbf{end do};$

$\textbf{for}\ i5\ \textbf{from}\ 1\ \textbf{to}\ i3\ \textbf{do}\ F := -I\cdot(diff(F, x2) + \frac{x2}{r}\cdot diff(F, r))\ \textbf{end do};$

$\textbf{for}\ i5\ \textbf{from}\ 1\ \textbf{to}\ i4\ \textbf{do}\ F := -I\cdot(diff(F, x3) + \frac{x3}{r}\cdot diff(F, r))\ \textbf{end do};$

$\textbf{if}\ case = 1\ \textbf{then}\ expand(\frac{k^{i1}\cdot\arg}{k1^{i2}\cdot k2^{i3}\cdot k3^{i4}}\cdot F)\ \textbf{elif}\ case = 2\ \textbf{then}\ expand(\frac{\arg}{k1^{i2}\cdot k2^{i3}\cdot k3^{i4}}\cdot$
$F)\ \textbf{elif}\ case = 3\ \textbf{then}\ expand(\frac{\arg}{k^{i1}\cdot k1^{i2}\cdot k2^{i3}\cdot k3^{i4}}\cdot F)\textbf{end if}:$

$\textbf{end proc}:$