A Nasty Surprise in a Sequence

and Other Recent OEIS Stories

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Neil J. A. Sloane, Visiting Scholar, Math. Dept., Rutgers University;
and The OEIS Foundation, Highland Park, NJ
(njasloane … gmail.com)

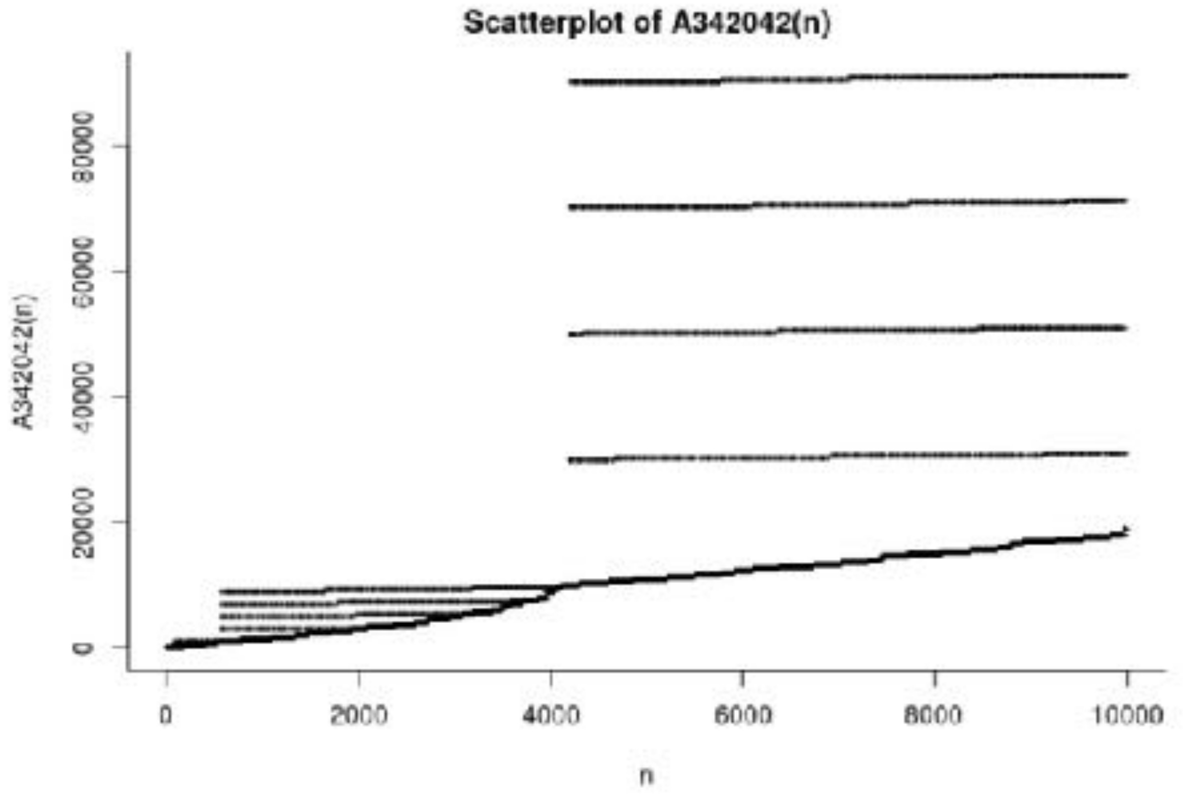
In Memoriam   Éric Angelini (Sep. 12 1951 - Sep. 27 2024)

Over 1500 sequences, brilliant, clever, surprising, witty.
One of my favorite contributors, and a friend for 20 years.

A121053 (2006): Lexicographically Earliest Sequence (LES)
that describes the positions of its prime terms:
2, 3, 5, 1, 7, 8, 11, 13, 10, 17, …

Many classics: Comma Sequence (2006), Sisyphus Sequence,
Choix de Bruxelles, “1995”, Sum and Erase, Triply Fractal, Look Left …,
Okapi, Same Game, Orphans, Palindromes, …

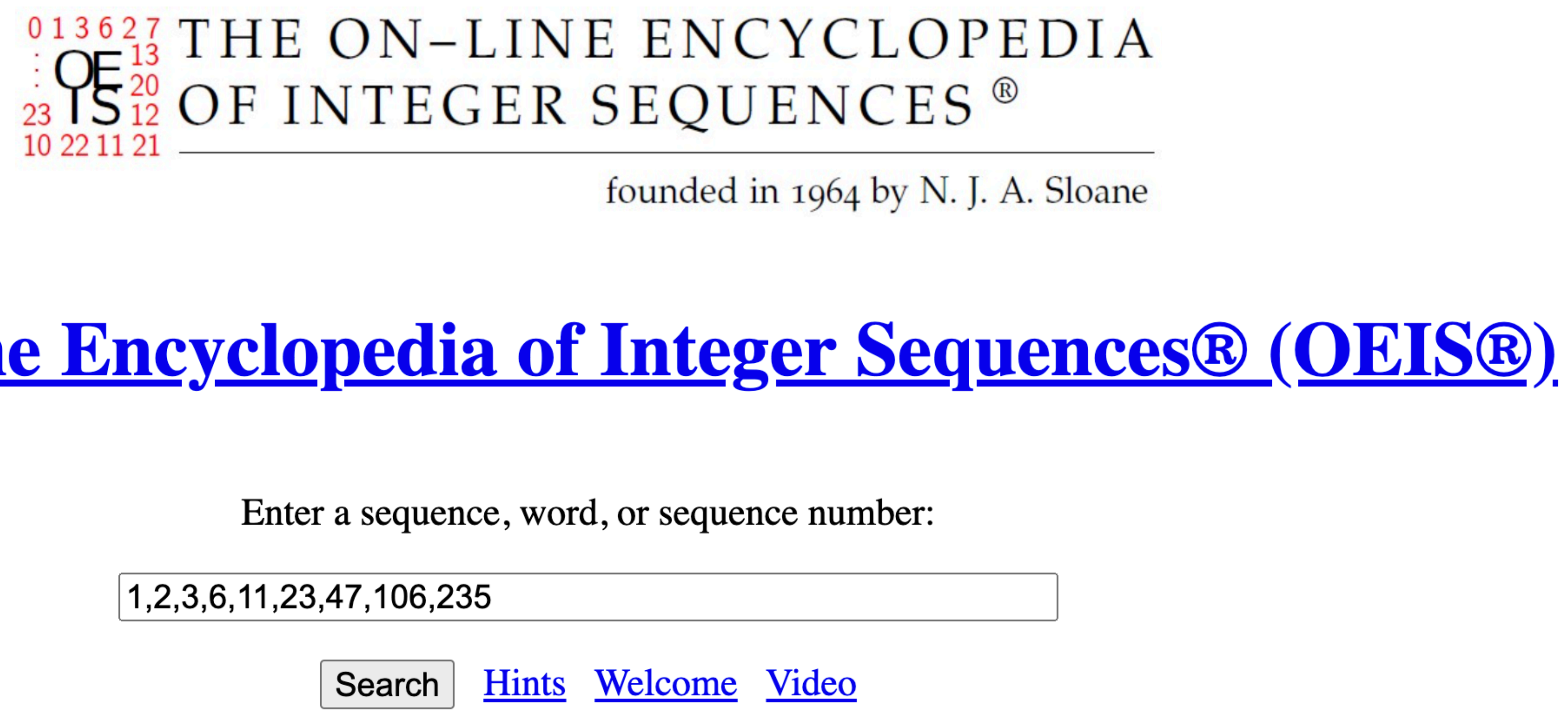
If a digit d is even, the next digit is > d:
A342042



Éric said that when he discovered the OEIS he thought
it was the eighth wonder of the world. He will be greatly missed

Outline

- 1. OEIS Foundation Seeks to Hire a Managing Editor



- 2. Covering a Sequence with Straight Lines

- 3. Dampening Down a Divergent Series

- 4. A Nasty Surprise in a Sequence (the Riesel and Sierpinski Problems)

1. The OEIS is Seeking to Hire a Managing Editor

Having raised the necessary endowment, the OEIS Foundation is looking to hire a full- or part-time managing editor to be in charge of processing submissions.

The OEIS is in its 60th year, contains 375,000 entries, has been cited over 11,000 times in the mathematical literature, and receives a million hits a day.

We have 170 volunteer editors, but they are finding it increasingly difficult to keep up with the nonstop flow of submissions of new sequences and updates.

The OEIS Foundation is therefore seeking to hire a full- or part-time managing editor who will be in charge of submissions.

OEIS Foundation seeks to hire a managing editor (2)

This person will probably have a PhD in mathematics, and could be an academic or someone working at a tech firm.
Since the OEIS Foundation is a United States-based 501(c)(3) Public Charity, it will be very difficult to pay someone who is not a U.S. resident

Candidates should be fluent in English, have a wide knowledge of mathematics and computer science, and be familiar with one or more of the computer languages widely used in the OEIS (Maple, Mathematica, PARI, Python, etc.).
A sympathetic personality and a good supply of tact are essential.

Further details of this job position and the application process will be posted soon on the OEIS Foundation website https://oeisf.org and the OEIS Wiki https://oeis.org/wiki/)).

Incidentally, the rest of this talk is based on sequences submitted to the OEIS in July-August 2024!

2. Covering a Sequence with Straight Lines

Let a(1), a(2), a(3), … be a sequence of numbers.

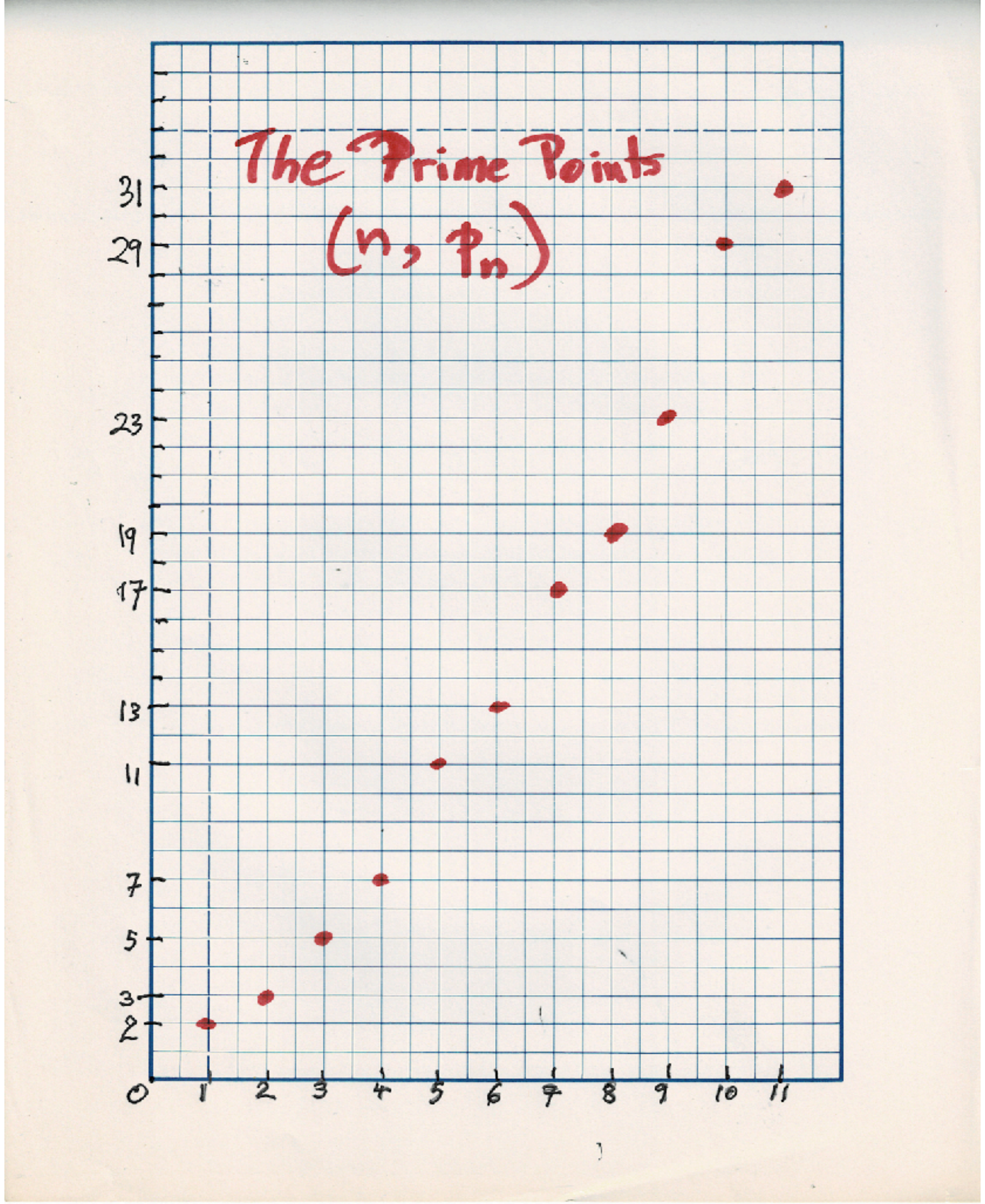
Draw points (1,a(1)), (2,a(2)), (3,a(3)), … on graph paper.

Let b(n) = minimal number of straight lines needed to intersect all the points (1,a(1)), …, (n,a(n)).

Example 1: The points are the “prime-points” (k, prime(k)):

(1,2), (2,3), (3,5), (4,7), (5,11), … A373813 gives number of lines needed

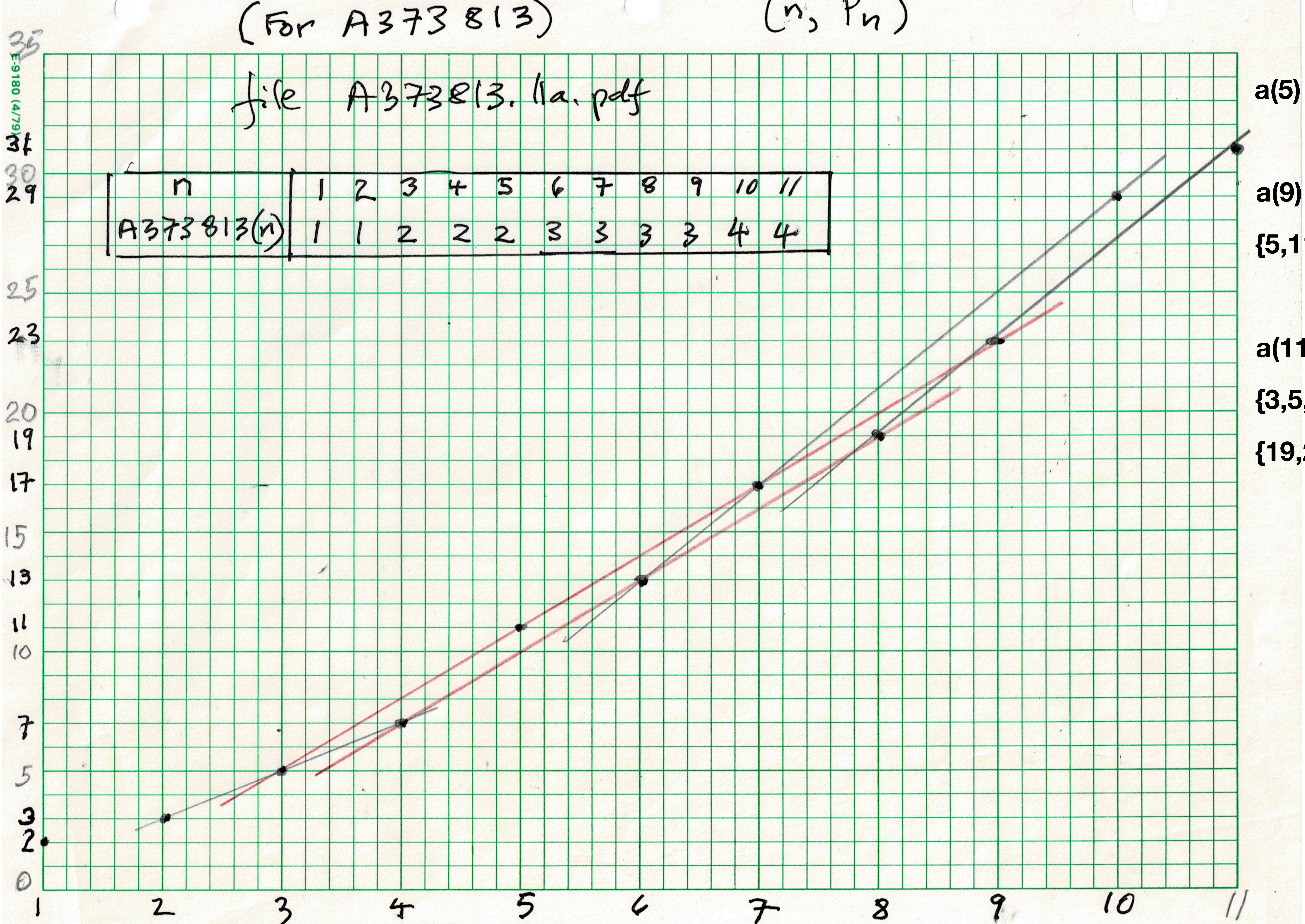
(The lines need not be disjoint.)



The Prime Points
(n, prime_n)

(1,2), (2,3), (3,5), (4,7), (5,11), (6,13), ...

Covering the primes with lines (2)



a(5) = 2: {2,11}, {3,5,7}

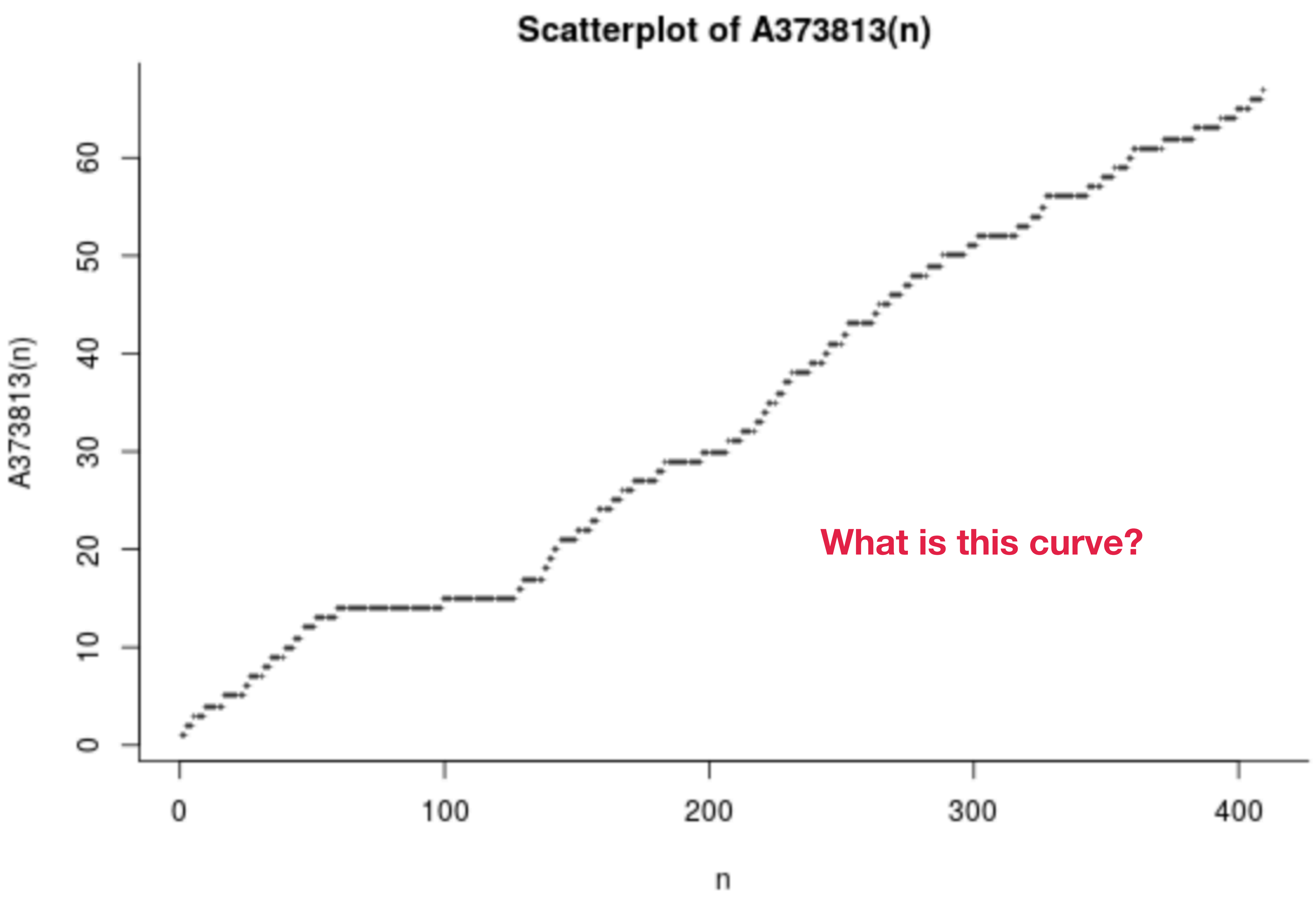
a(9) = 3: {2,3}, {5,11,17,23}, {7,13,19}

a(11) = 4: {2,11}, {3,5,7}, {13,17,29}, {19,23,31}

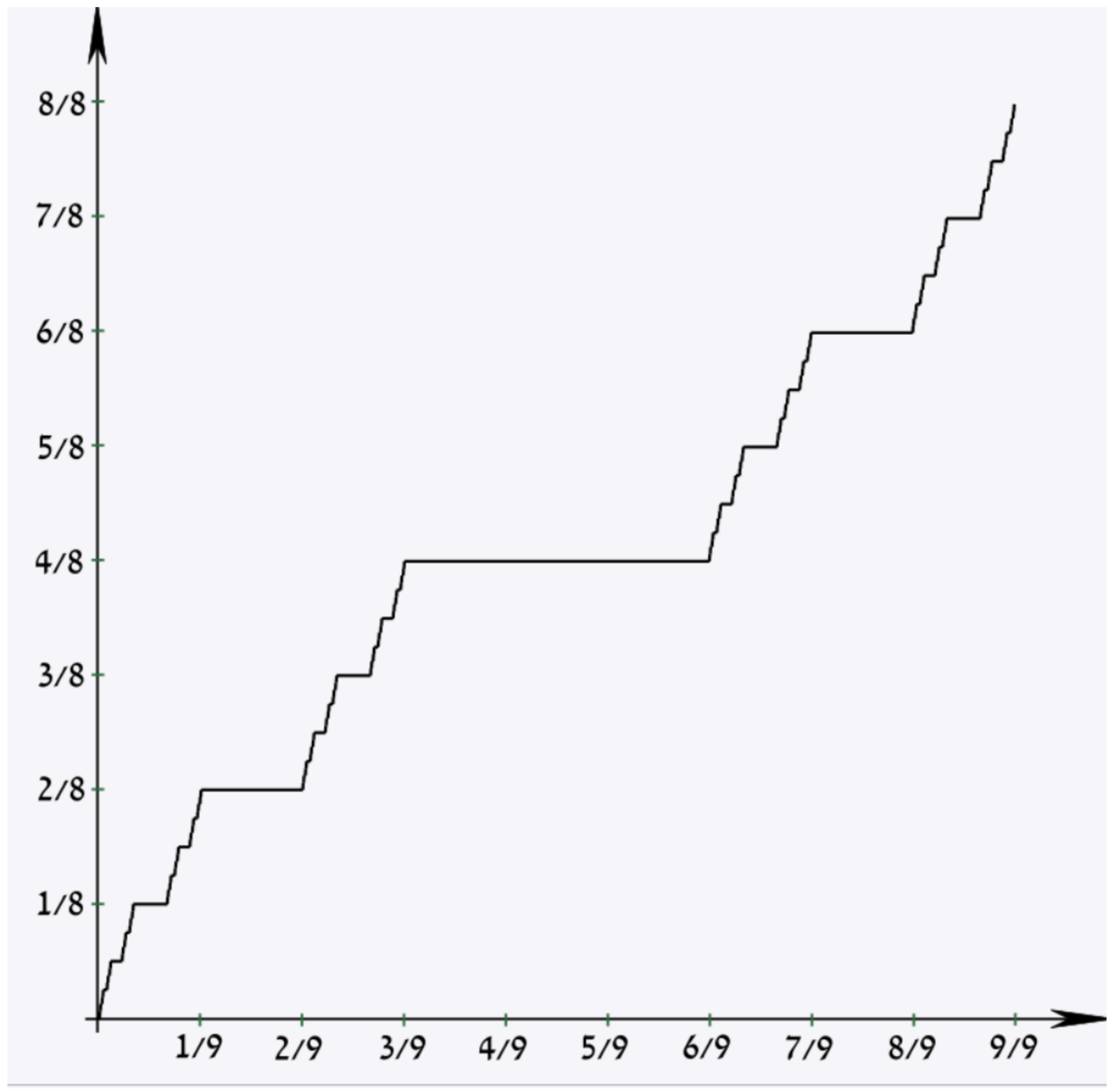
A373814 gives lengths of runs: 2, 3, 4, 7, 8, 2, ...

Covering the Primes with Straight Lines (continued)

Using SET COVER , Max Alekseyev found 410 terms of A373813:



Dan Asimov: Is this related to the Cantor function (or Devil's Staircase)?



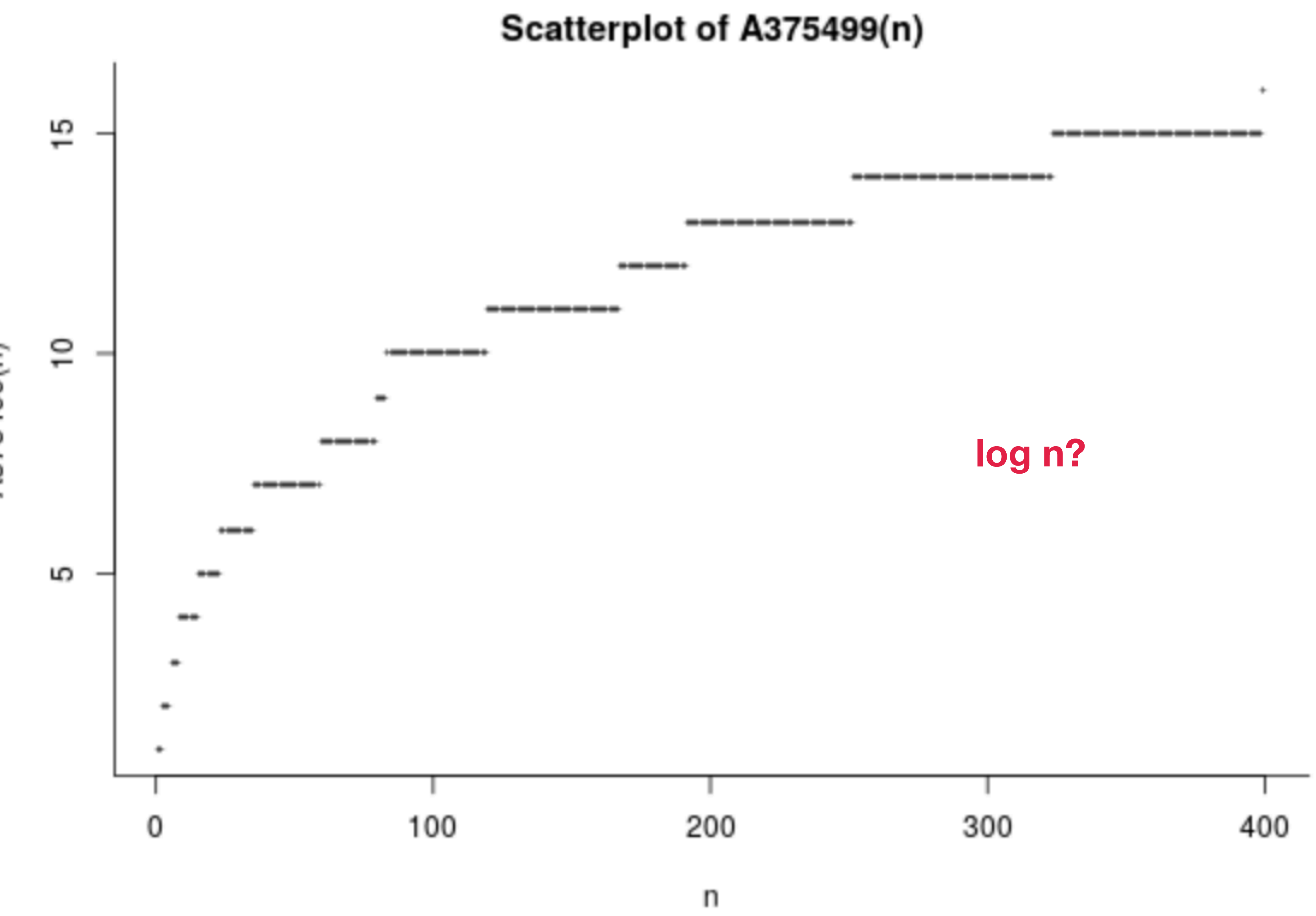
An early stage in the construction of the Devil's Staircase

Covering a sequence with straight lines (4)

What is relation between growth of seq. and number of lines needed to cover it?

Points are (n, d(n))

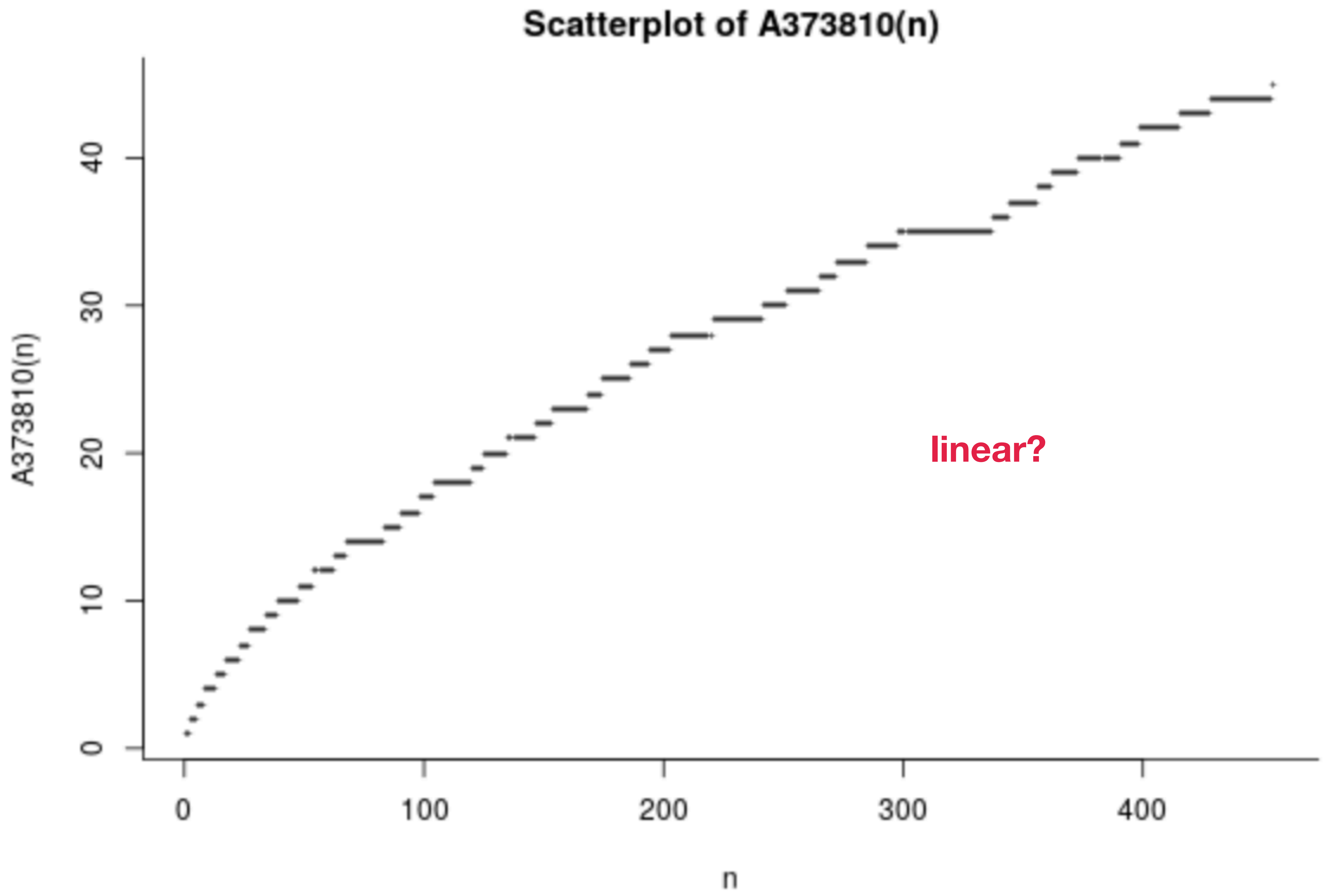
no. of divisors



A375499

Points are (n, phi(n))

Euler phi(n)



A373810

[Rémy Sigrist, Max Alekseyev, NJAS]

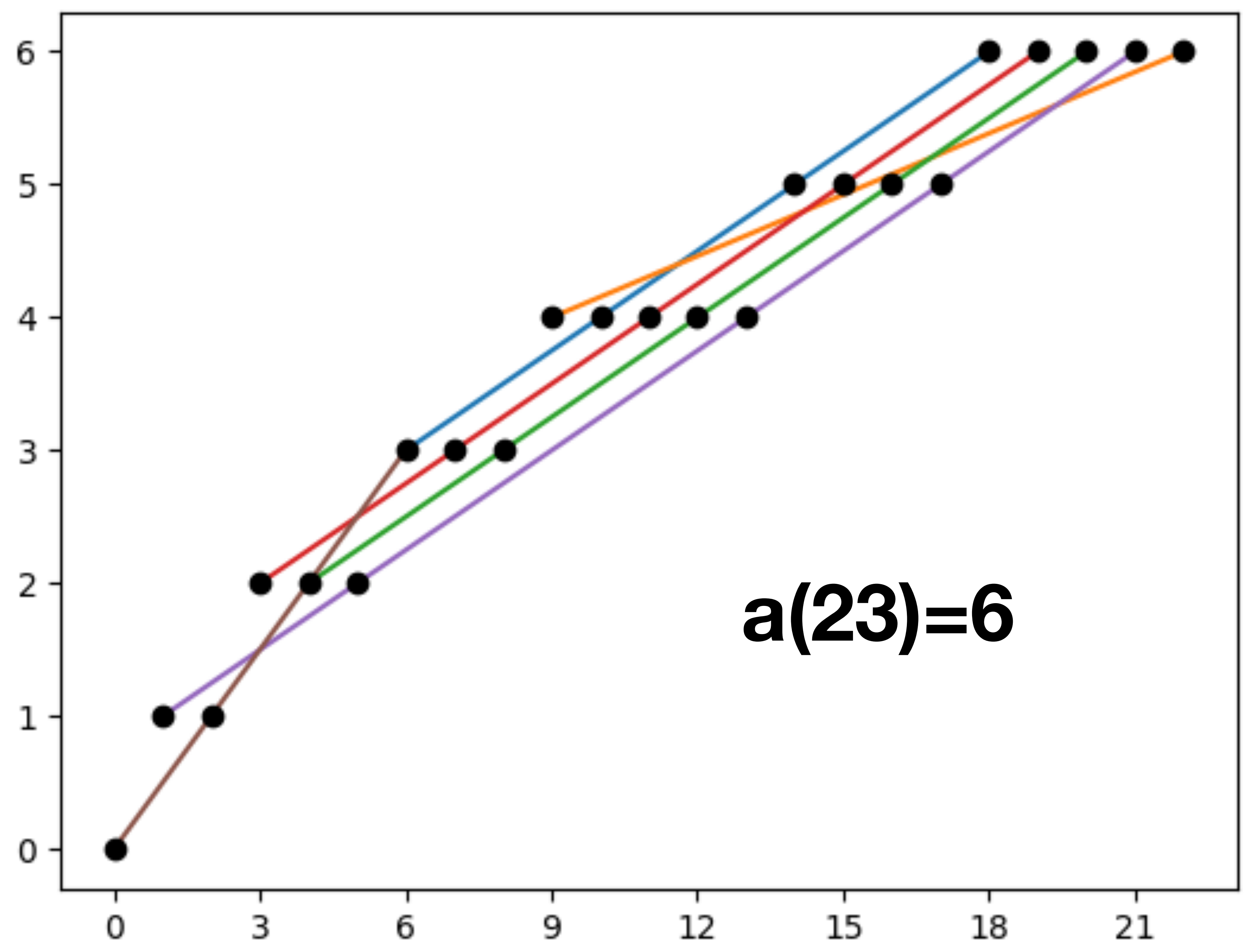
Covering a sequence with straight lines (5)

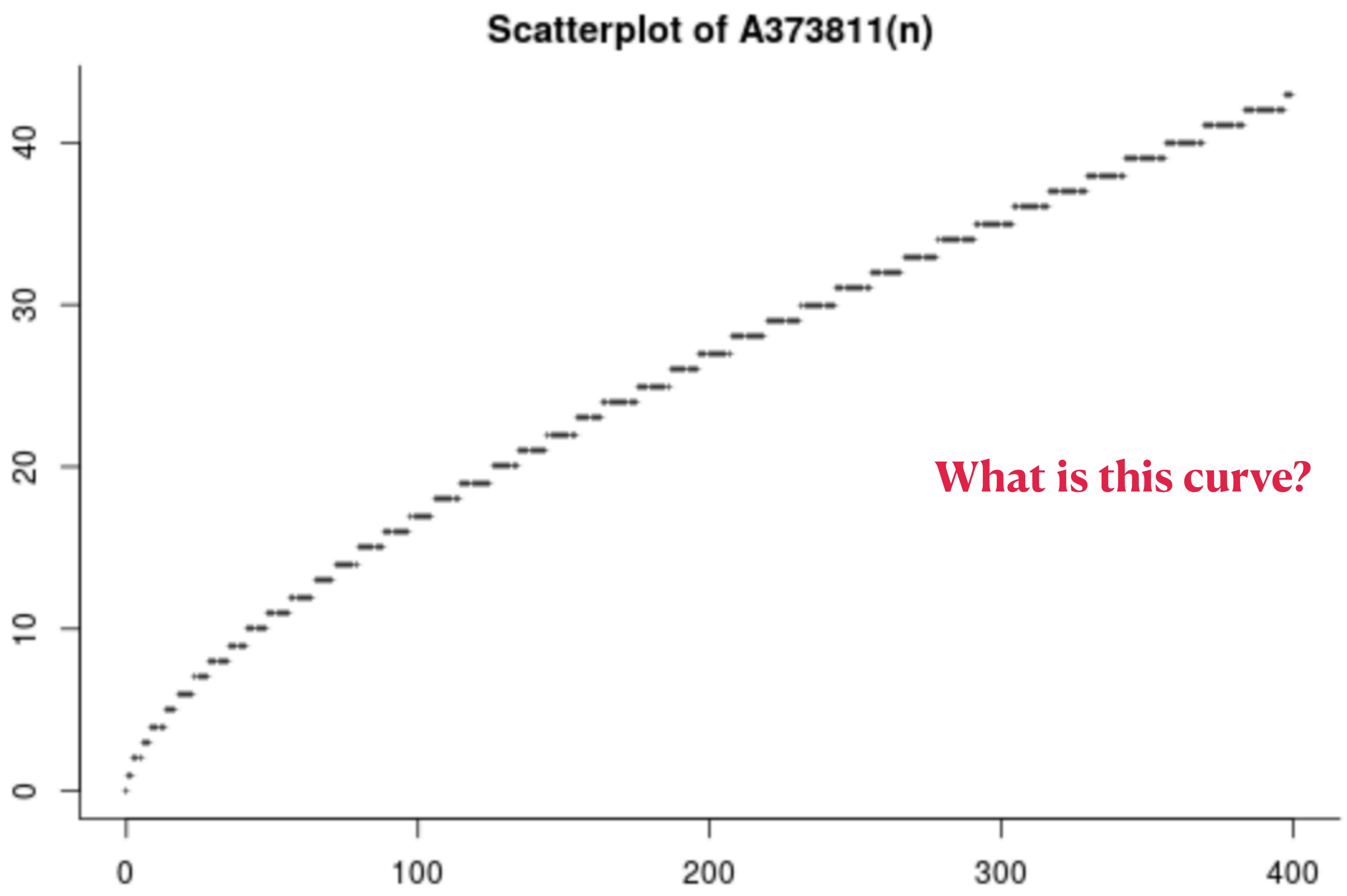
Points are (n, a(n)) !   0, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, ...

a(0)=0; for n>0, a(n) = minimal number of straight lines needed to intersect all points (k, a(k)) for k < n.

(Zach DeStefano, Arthur O'Dwyer, Daniel Mondot, Max Alekseyev, NJAS)

Dominic McCarty,   August 13 2024,   A373811





Covering the Primes with Straight Lines (cont.) The points are the “prime-points” (k, prime(k).

What is the earliest line to contain exactly n prime-points?

(“earliest” means minimize max prime)

3 points: [3,5,7], slope 2

4 points: [5,11,17,23], slope 3

5 points: [19,23,31,43,47], slope 4

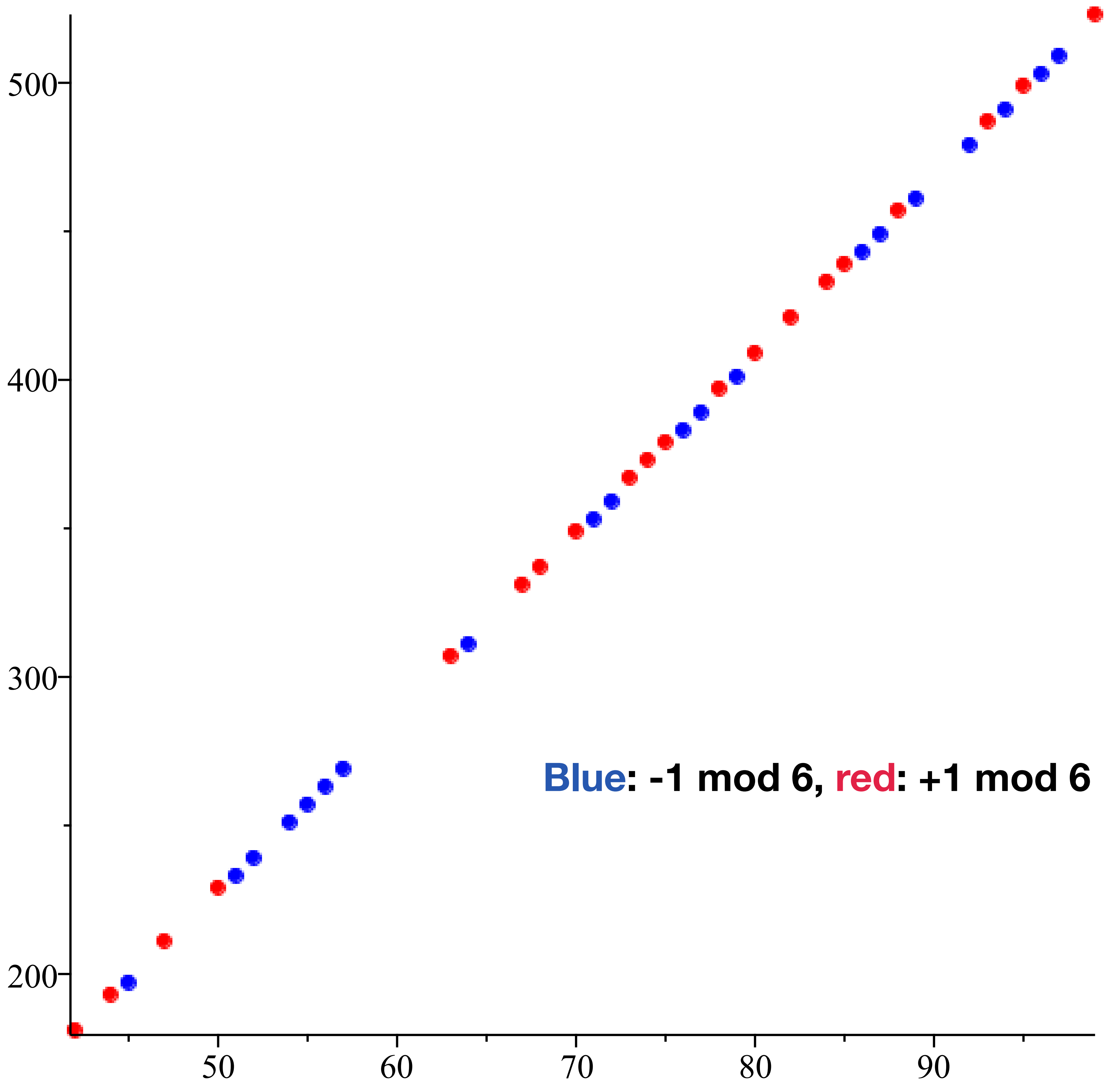
7 points: [7,11,59,67,71,79,83], slope 4

...

A376187:

2, 3, 7, 23, 47, 181, 83, 73, 1069, 521, 701, 1627, 691, ...

Surprise: There are two parallel lines of slope 6 each with 20 points: so a(20) <= 509.



Contrast this with finding 20 primes in an A.P.: have to out to 5729450393512 (A005115), and these aren't on a straight line anyway!

[Edwin Clark]

Don Reble, Oct 2 2024: Found 5 lines of 54 primes & slope 12
also a 79-prime line of slope 12

3. Dampening Down a Divergent Series

(with Rémy Sigrist)

Chernobyl 1996 (no dampening!)

Dampening down a divergent series (1)

The harmonic series diverges:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \log(n) + \text{gamma} + O(1/n).$$

Rémy Sigrist suggested inserting dampening coefficients a_i so that the partial sums are (just) less than 1:

S_n = 1/(a_1 x 1) + 1/(a_2 x 2) + ... 1/(a_n x n) < 1,

and choose the lexicographically earliest positive integers a_i.

n = 1: S_1 = 1/a_1, so a_1 = 2, S_1 = 1/2, difference d_1 = 1 - S_1 = 1/2.
n = 2: d_1 > 1/(a_2 x 2), so a_2 = 2, S_2 = 1/2 + 1/4 = 3/4, d_2 = 1/4.
n = 3: d_2 > 1/(a_3 x 3), so a_3 = 2, S_3 = 3/4 + 1/6 = 11/12, d_3 = 1/12.

The "Subharmonic" series (continued)

$$S_n = \sum_{k=1}^{n} \frac{1}{a_k k} < 1$$

n	1	2	3	4	5	6	7	8
a_n	2	2	2	4	10	201	34458	1212060151
S_n	1/2	3/4	11/12	47/48	1199/1200	241199/241200	...	...
d_n	1/2	1/4	1/12	1/48	1/1200	1/241200	...	...
e_n	2	4	12	48	1200	241200	8696481200	$1.17 \cdot 10^{19}$

What are these numbers?
How fast do they grow?
Sigrist: S_n = (c-1)/c for c an integer for n <= 36

a_n = A374663, S_n = A374983/A375516

Surely someone has already studied this?

The "Subharmonic" series (continued)

The Recurrence:

<u>Theorem 1</u> a_n and d_n are given by

$a_1 = 2$, $d_1 = \frac{1}{2}$, and for $n \geq 2$

$$a_n = \left\lfloor \frac{1}{n\, d_{n-1}} \right\rfloor + 1 \quad (1)$$

$$d_n = d_{n-1} - \frac{1}{a_n n} \quad (2)$$

Proof. Want smallest a_n such that

$$S_{n-1} + \frac{1}{a_n n} < 1$$

$$\frac{1}{a_n n} < 1 - S_{n-1} = d_{n-1}$$

$$a_n n > \frac{1}{d_{n-1}}$$

$$a_n > \frac{1}{n\, d_{n-1}} \quad \text{gives (1)}$$

$$d_n = 1 - S_n = 1 - \left(S_{n-1} + \frac{1}{a_n n}\right)$$

$$= d_{n-1} - \frac{1}{a_n n} \quad \text{gives (2)}$$

□

The "Subharmonic" series (continued)

a_n, e_n = denominator of S_n are doubly exponential

n	1	2	3	4	5	6	7	8	9	10	11	...
length(e_n)	1	1	2	2	4	6	10	20	38	74	147	...

Roughly $e_n \doteq 10^{2^{n-3}}$

Why? $S_4 = \frac{47}{48}$ $\quad \frac{1}{48} > \frac{1}{a_5 \cdot 5}$ so $a_5 = 10$

$S_5 = \frac{47}{48} + \frac{1}{5 \cdot 10} = \frac{2398}{2400} = \frac{1199}{1200}$

$S_{n+1} = \frac{\text{close to } e_n^2}{e_n(\text{close to } e_n)}$

so $e_{n+1} \approx e_n^2$

In fact, a_{n+1} is very close to $\frac{n}{n+1} a(n)^2$ for $n = 6, 8, 9, 10, 12$.

Rémy Sigrist's Theorem

$$S_n = \sum_{i=1}^{n} \frac{1}{a_i i} < 1, \quad d_n = 1 - S_n \quad \text{(as above)}$$

$$= \frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{48}, \frac{1}{1200},$$

Theorem 2

(i) $d_n = \dfrac{1}{\text{pos. integer}} = \dfrac{1}{e_n}$ (say)

(ii) e_n is divisible by all integers $\leq n$

(iii) $e_{n-1} \mid e_n$

Proof Induction on n, plus Eq. (1) □

Corollary: S_n = (e_n - 1) / e_n

Other Examples:

The general divergent series problem

Dampening Down Divergent Series

Given nonnegative real numbers $b_1, b_2, \ldots$ such that $\sum_{i=1}^{\infty} b_i$ diverges find lex. earliest positive integers $a_1, a_2, \ldots$ such that $S_n = \sum_{i=1}^{n} \frac{b_i}{a_i} < 1$.

Examples

b_n	a_n	denominator of S_n
$1/n$	A374663	A375516
$1/\text{prime}(n)$	A375781	A375522*
$1/\text{lucky}(n)$	A375527	A375528
1	A000058	A007018
n	A295391	A275611
prime(n)	A375529	A375530
2^n	A059917	A059723
$n!$	A375531	A375532
$a(n-1)$	A376043	A376044

plus many others

* The only case where numerator of $S_n \neq$ denominator $- 1$ for all n

Dampening down a divergent series (7)

$b\_n = 1 / \text{prime}(n)$

ANALOG FOR DIVERGING SEQUENCE $\sum_{i=1}^{n} \frac{1}{\text{prime}(i)}$

(SIGRIST)

$$S_n = \frac{1}{a_1 2} + \frac{1}{a_2 3} + \frac{1}{a_3 5} + \cdots + \frac{1}{a_n p_n} < 1$$

n	1	2	3	4	5	6	7	8	
p_n	2	3	5	7	11	13	17	19	
a_n	1	1	2	3	5	89	39304	$\sim 4.10^{10}$	A375781
S_n	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{14}{15}$	$\frac{103}{105}$	$\frac{1154}{1155}$	$\frac{1336333}{1336335}$			A375521 / A375522

A375781

A375521

A375522

Write $S_n = \frac{e_n - \epsilon_n}{e_n}$. The values of ϵ_n are

ϵ_n	1	1	1	2	1	2	1

. In fact, $\epsilon_n = 1$ for $n = 7 \ldots 34$

Conjecture $\epsilon_n = 1$ except for $n = 4$ and 6.

The Egyptian fraction:

$$1 = \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{5.11} + \frac{1}{89.13}$$

$$+ \frac{1}{39304 \cdot 17} + \frac{1}{46994541278 \cdot 19}$$

$$+ \frac{1}{1733182118440905147145 6 \cdot 23} + \cdots$$

The Positive Integers Case

Theorem 3: Given nonneg. ints. b_1, b_2, b_3, ..., the lex. earliest pos. ints. a_1, a_2, a_3, ... such that S_n = Sum_{i=1..n} b_i/a_i are given by:

S_n = (e_n - 1)/e_n, n >= 1, e_n positive integers satisfying

e_{n+1} = b_{n+1} * e_n^2 + e_n, n>1; e_1 = b_1 + 1,

a_{n+1} = b_{n+1} * e_n + 1, n >= 0

The General Case

Conjecture 4: In the general divergent series problem there is a constant c and integers e_i such that

S_n = (e_n - c)/e_n

for all sufficiently large n.

The Positive Integers Case, Four Important Examples

Ex. 1. Dampening coefficients for b_i = 1, the divergent series 1+1+1+1+…

n	0	1	2	3	4	5		
b_n	–	1	1	1	1	1		
a_n	–	2	3	7	43	1807	Sylvester A58	
s_n	0	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{41}{42}$	$\frac{1805}{1806}$			A000058
d_n	1	1/2	1/6	1/42	1/1806			
e_n	1	2	6	42	1806	3263442	A7018	A007018

$$a_{n+1} = a_n^2 - a_n + 1, \quad a_1 = 2$$

Arises from Greedy Egyptian Fraction expansion of 1:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \dots$$

Ex. 2. Dampening coefficients for b_n = 2^(n-1), the divergent series 1+2+4+8+16+…

n	1	2	3	4	5	6
b_n	1	2	4	8	16	32
a_n	2	5	41	3281	21523361	A059917
$s_n = (e_n - 1)/e_n$						
e_n	2	10	410	1345210		A059723

A059917

A059723

$$a_n = \frac{3^{2^n} + 1}{2} \qquad e_n = \frac{3^{2^n} - 1}{2^{n+1}}$$

Generalized Fermat numbers.

a_n is prime for n = 1, 2, 3, 5, 6, 7:

2, 5, 41, ~~3281~~, 21523361, 926510094425921, 1716841910146256242328924544641

Ex. 3. Dampening coefficients for the recursive case: b_n = a_{n-1}

Defn. $S_n = \sum_{i=1}^{n} \frac{a_{i-1}}{a_i} < 1, \quad a_0 = 1$

Implies by Th. 3

$$a_n = e_{n-2}\, a_{n-1} + 1, \quad a_0 = 1$$

$$e_n = a_n\, e_{n-1}^2 + e_{n-1}, \quad e_0 = 1$$

$$S_n = \frac{e_n - 1}{e_n}$$

n	0	1	2	3	4	5
b_n	–	1	2	5	51	26011
a_n	1	2	5	51	26011	
S_n	0	$\frac{1}{2}$	$\frac{9}{10}$	$\frac{509}{510}$		
e_n	1	2	10	510	13265610	

b_n, a_n: A376043

e_n: A376044

A376043

A376044

Solution:

$$a_n = a_{n-1} \cdot \prod_{i=0}^{n-1} a_i + 1, \quad a_0 = 1$$

$$e_n = \prod_{i=0}^{n} a_i .$$

Ex. 4. Examples where S_n = (e_n - c) / e_n with c different from 1 (Sigrist)

$$S_n = \frac{e_n - c}{e_n} \quad \text{with } c \neq 1$$

Ex. 4a. $b_1 = \frac{5}{4}, \; b_{2k} = \frac{3}{2}, \; b_{2k+1} = \frac{6}{5} \; (k > 0)$

$$\Rightarrow S_n = \frac{e_n - 3}{e_n} \quad \forall n$$

A376184, A376186

Ex. 4b. $b_1 = \frac{7}{6}, \; b_k = \frac{5}{4} \; (k > 1)$

$$\Rightarrow S_n = \frac{e_n - 5}{e_n} \quad \forall n$$

A376062, A376185

Think outside the box: change the base!

Interpret 1/n in base 11

(Bob Lyons and others)

Although Sum 1/n diverges, let U(n) mean write n in base 10, but read it in base 11, then Sum 1/U(n) converges

If n = Sum c_i 10^i, then U(n) = Sum c_i 11^i

If n = 25,
U(n) = 2 x 11 + 5 = 27

Proof is easy, but what is value of Sum_{n=1..oo} 1/U(n) ?
Hard (impossible?) by brute force.
But sum is also Sum 1/m, where m runs through all numbers which are missing the digit A (“ten”) in base 11.
This converges by a 1914 theorem of Kempner, and can be evaluated using method of Baillie (1979).

Gareth McCaughan found (see A375805):
26.2833282048814207699401516874442229241887980925...

Dampening down a divergent series (14)

Open Problem

But what is Sum_{n=1..oo} 1/U(prime(n)) ?
Not yet in OEIS, may be very hard

Hans Havermann, 10^9 terms: 2.89, 66 x 10^9 terms: 2.91, estimate: 10^12 terms: 2.94, … ?? terms: 3 or higher? …

Not a single digit is known!

This is the limit of partial sums A375533/A375534:

1/2, 5/6, 31/30, 247/210, 5891/4830, 175669/140070, 6639823/5182590, ...

4. A Nasty Surprise in a Sequence

A nasty surprise in a sandwich,
A drawing pin caught in your sock,
The limpest of shakes from a hand which
You'd thought would be firm as a rock.
……


Drawing pins

from "God: A Poem", by James Fenton

1. An innocent looking sequence and a surprise
2. What is going on? The Riesel problem
3. Lucas Brown to the rescue
4. The Sierpinski problem

1. In July 2024 Bill McEachen submitted an innocent-looking sequence:

$a(1) = 1$; $a(n) = 2 \times a(n-1) + 1$ if $a(n-1)$ is not a prime,
otherwise $a(n) = \text{prime}(n) - 1$.

The sequence

n	1	2	3	4	5	6	7	8	9
a(n)	1	3	4	9	19	12	25	51	103
			p_3-1			p_6-1			

A374965 — **A374965**

n	10	11	12	13	14	15	...
a(n)	28	57	115	231	463	46	...
	$p_{10}-1$					$p_{15}-1$	

The primes

1	2	3	4	5	6	...
3	19	103	463	751	283	...

A375028 (primes) — **A375028**
A373799 (where) — **A373799**
A373804 (sorted) — **A373804**

A Nasty Surprise in a Sequence (3)

Harvey Dale computed the first 289 primes. Some large primes appear early on.
At n = 3612, the 203rd prime appeared, 134851. So a(3613) = 33748.
Then no prime for 2225 steps. At n = 5837, 204th prime is about 10^674:

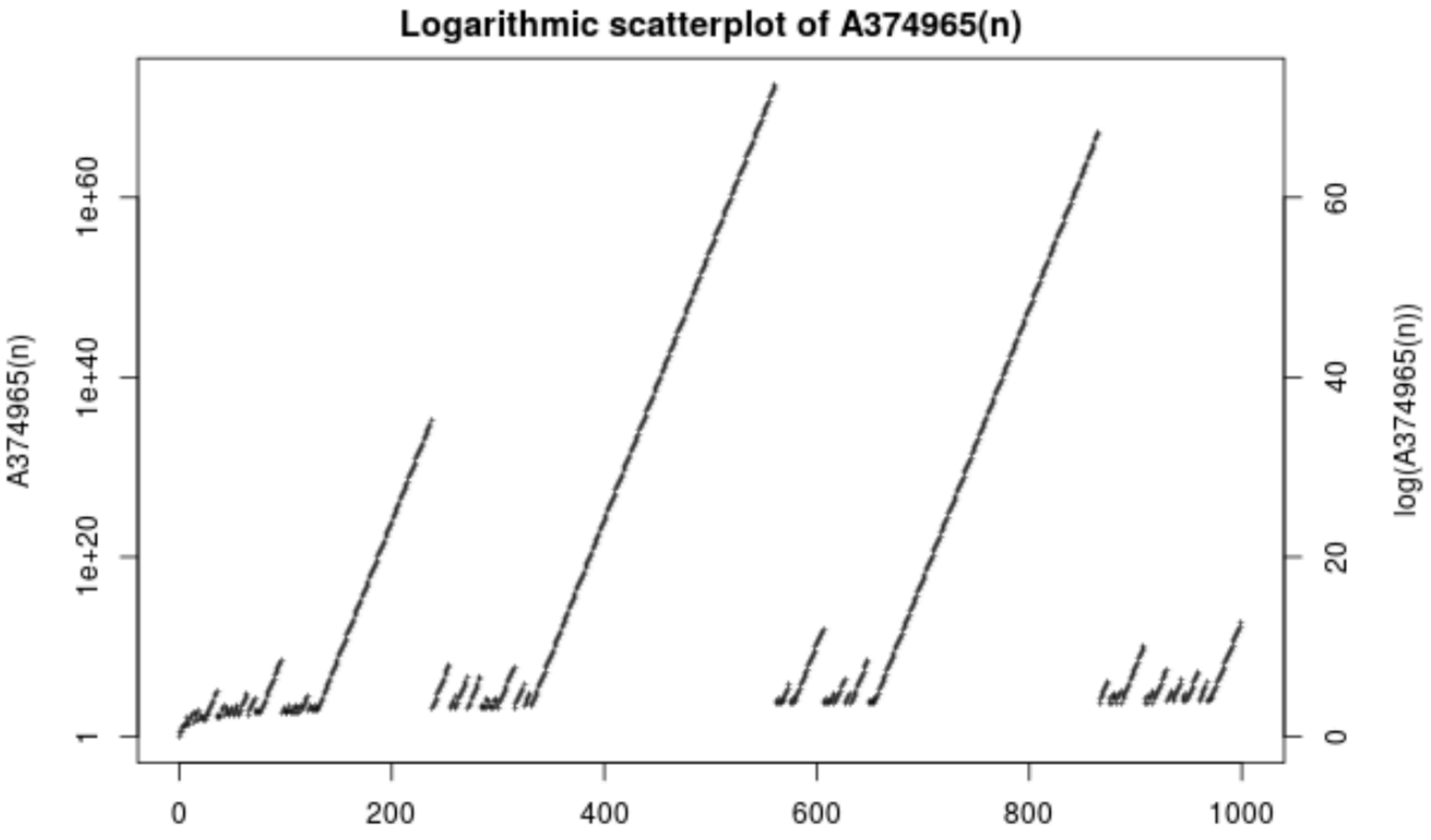
```
203 134851
204
104465109947123485908937460665577942483040301930458165787150790499888569218419468979484743891586
570785659823074952818919216807478911712598351248252713631779069062164935086502031495456675149237
025035569814921747718411026423303388190073168515943977512877312238680598233923367465378465453191
749707065543973702588750459454791489761772060955285836134729382889814015475551577460856584616777
099079247121361629309831838079454544405892019599615209991870935285609006981091019933367569942609
802807523179612151407912382028172327718740674605049445482940340613238762508353798534822887988233
670317263711583710198517963983474429770469504441601056786954267723041908525896676646476838846070
783
205 230563
```

Dale stopped at the 289th prime, a(10016) = 838951. Next term is
a(10017) = prime(10017) - 1 = 104916. Now repeat: double and add 1,
until reaching a prime, or double-and-add-1 forever???

I ran a program for 24 hours: it just kept growing.
What is going on?

A Nasty Surprise in a Sequence (4)

A log plot of 1000 terms of the sequence (A374965)



2. What is going on? The Riesel problem from 1956.

The Riesel problem: Double and add 1, repeat: how long to reach a prime?

4 - 9 - 19 - 39 - 79 - ..., or equivalently $5 \times 2^i - 1$, $i \geq 0$.

Double and add 1	Sequence	A052333 First prime	A050412 Steps
1 (3) 7 15 ...	$2 \cdot 2^i - 1$	3	1
2 (5) 11 23 ...	$3 \cdot 2^i - 1$	5	1
3 (7) 15 31 ...	$4 \cdot 2^i - 1$	7	1
4 9 (19) 39 ...	$5 \cdot 2^i - 1$	19	2
... ...	...	...	...
28 57 115 231 (463)	$29 \cdot 2^i - 1$	463	4
... ...	...		
33748 67497 ... ($\approx 10^{674}$)	$33749 \cdot 2^i - 1$	$\approx 10^{674}$	2224
... ...	...		
104916 209833 ... ??	$104917 \cdot 2^i - 1$	$\approx 10^{102409}$	340181

See below!

But do we always reach a prime?

This whole subject was started by the article:
Hans Riesel, "Some Large Prime Numbers" [Swedish], Elementa 39 (1956),258-260.

(Was not on the Web, but is now, thanks to Lars Blomberg, who located a copy and translated it into English. See **A076337**)

Theorem (Riesel): There are infinitely many positive integers k such that the sequence $k \times 2^i - 1$, $i > 0$, contains no primes.

For proof, see the Appendix to this talk (the last two slides).

The smallest example he found was k = 509203.

No one has found a smaller example.

Since 1998 there has been a vast project underway to prove that 509203 is indeed the smallest example.

To prove it, for each of the 254601 odd numbers k < 509203, the project tries to find a prime in the sequence $k \times 2^i - 1$, $i > 0$.

This has now been done for all except 42 values of k. See:

Ray Ballinger and Wilfrid Keller, [The Riesel Problem: Definition and Status](http://www.prothsearch.com/rieselprob.html) [http://www.prothsearch.com/rieselprob.html].

Shortest sequence in OEIS, A076337: Riesel numbers: odd k such that $k \times 2^i - 1$ is composite for $i>0$: 509203

A Nasty Surprise in a Sequence (7)

Since 104917 is less than 509203, it had been investigated by the Riesel team - see

Ray Ballinger and Wilfrid Keller, The Riesel Problem: Definition and Status [http://www.prothsearch.com/rieselprob.html]

and a prime had been found, namely
104917 x 2^340181 - 1
but it was not known if 340181 was the smallest exponent that gives a prime.

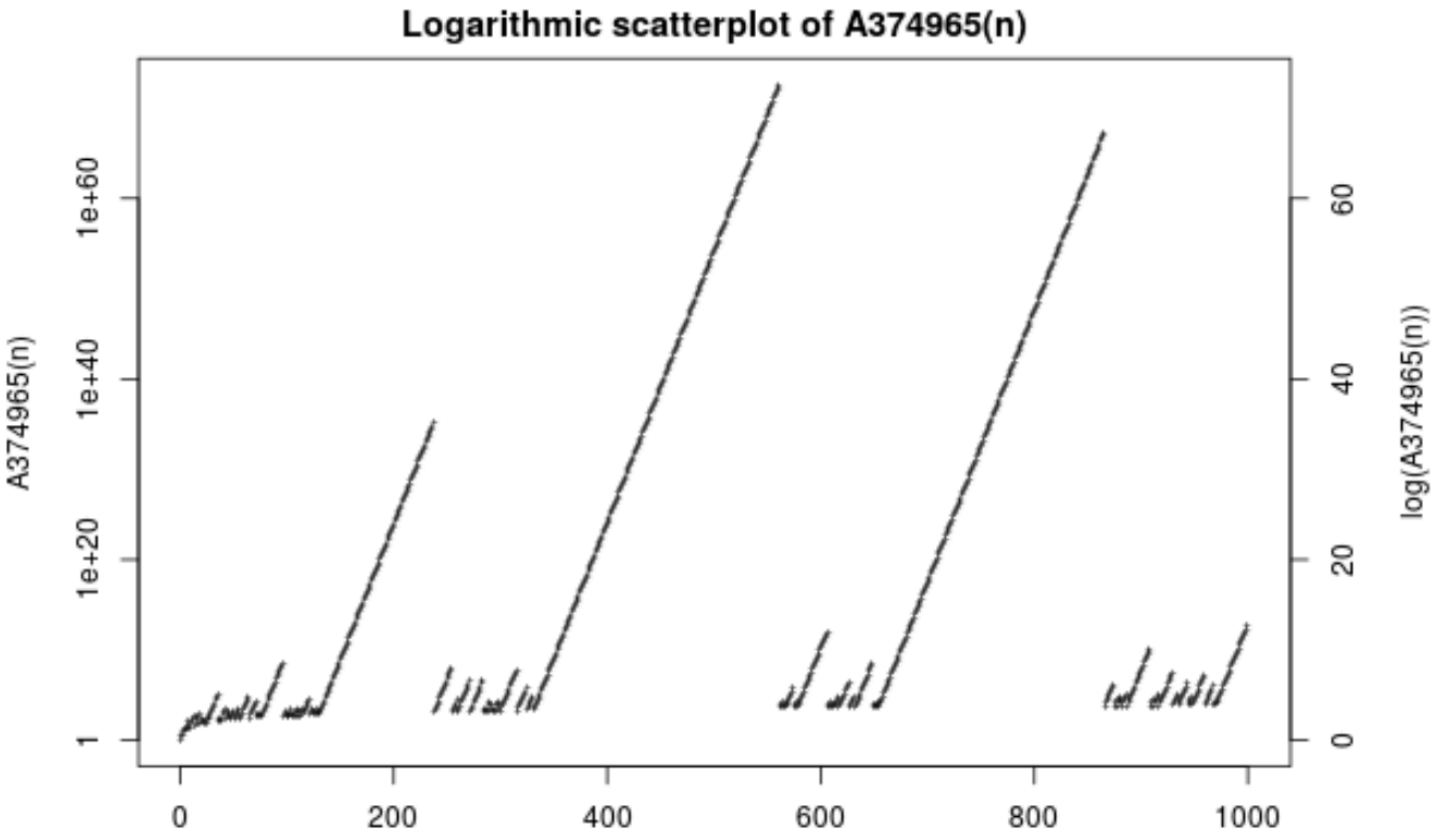
3. Lucas Brown to the rescue.

Theorem (Lucas Brown): The smallest prime of the form 104917 x 2^i - 1, i > 0, is indeed 104917 x 2^340181 - 1.

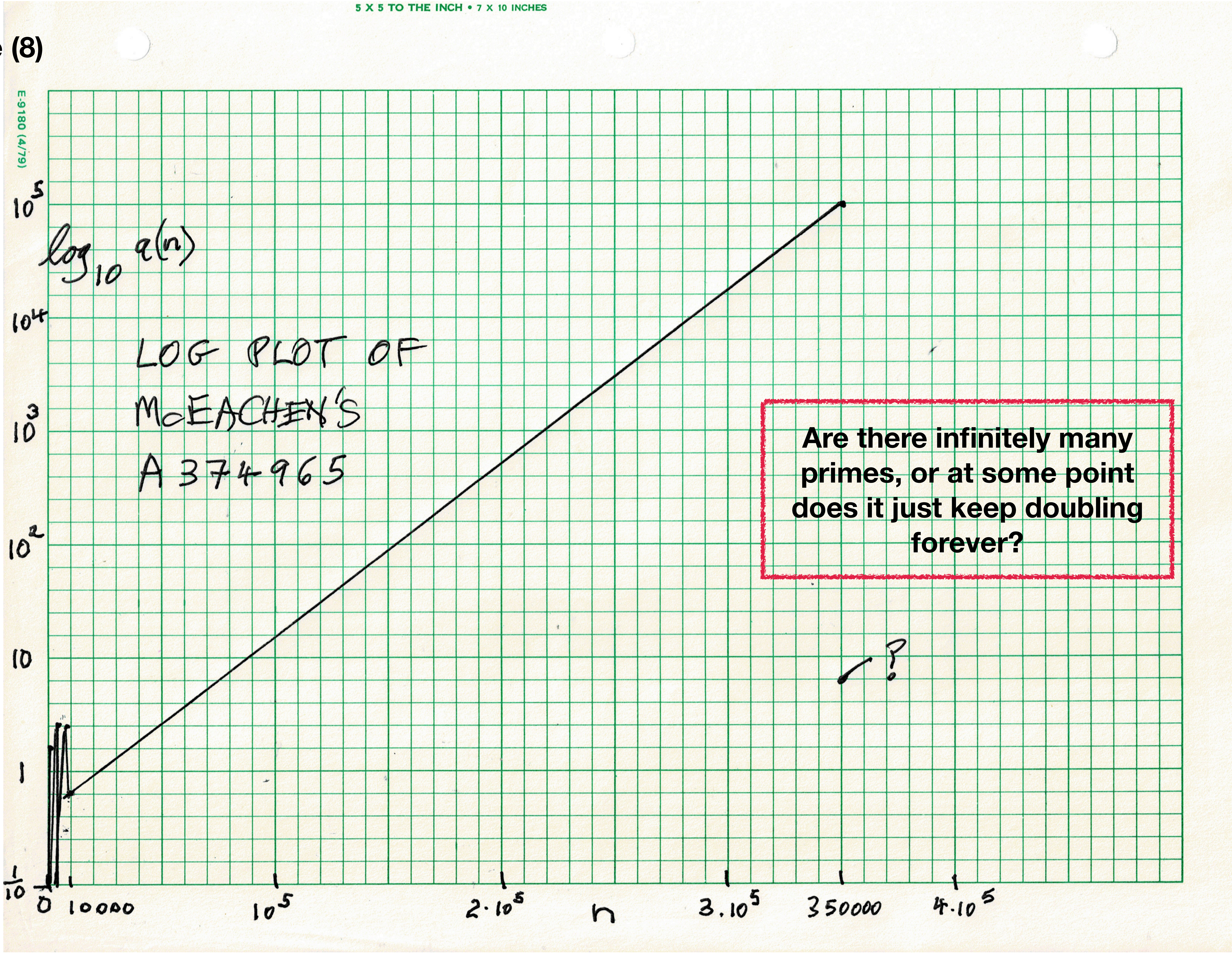
Proof by computer, using a Python program, with gmpy2 to handle the arithmetic and BPSW for the primality testing.
The program ran July 30 - 31, 2024. It took 15 hours of wall-clock time, and used 24 threads running in parallel.
The Python program is given in A050412.

A Nasty Surprise in a Sequence (4)

A log plot of 1000 terms of the sequence (A374965)



A Nasty Surprise in a Sequence (8)



Are there infinitely many primes, or at some point does it just keep doubling forever?

4. The Sierpinski Problem from 1960

Riesel looks for primes k x 2^i - 1 ("Mersenne"), Sierpinski for primes k x 2^i + 1 ("Fermat")

The Sierpinski problem: Double and subtract 1, repeat: how long to reach a prime?

4 - 7 - 13 - 25 - 49 - ..., or equivalently 3 x 2^i + 1, i >= 0.

Given k, A078683 = first prime reached,
A078680 = number of doubling steps needed,
or -1, if no prime ever reached

Theorem (W. Sierpinski, 1960): There are infinitely many positive integers k such that the sequence k x 2^i + 1, i > 0, contains no primes.

It is conjectured that the smallest k is 78557 - see A076336 for references.

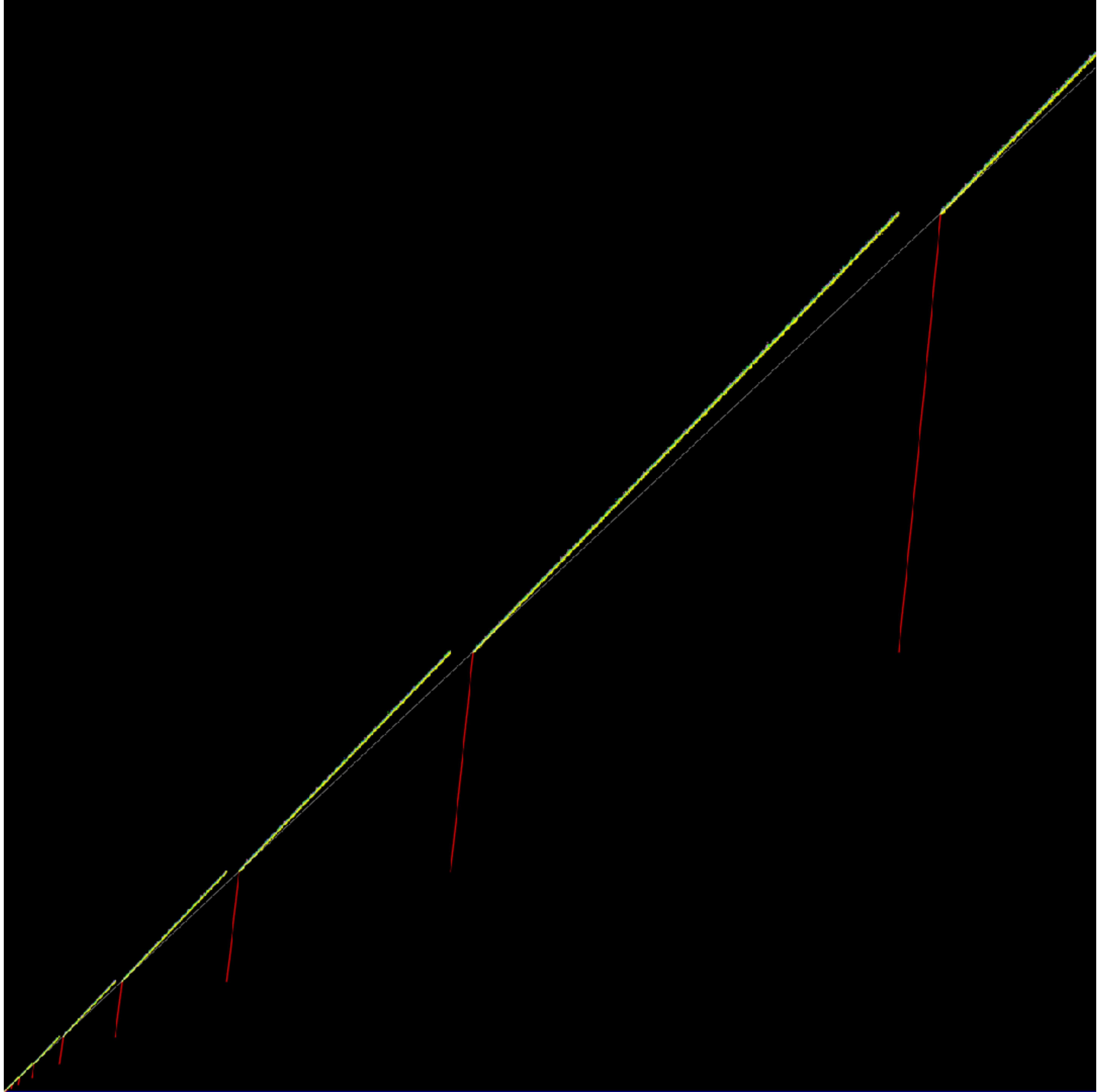
See A373801 for an analog of McEachen's sequence.

With thanks to:

Contributors to this talk: Max Alekseyev, Dan Asimov, Lars Blomberg, Lucas Brown, Harvey Dale, Robert Gerbicz, Alois Heinz, Marc LeBrun, Bob Lyons, Dominic McCarty, Gareth McCaughan, Bill McEachen, Alex Meiburg, Ed Pegg, Kevin Ryde, Rémy Sigrist, and others.

Thanks also to all the Editors and Trustees who keep the OEIS running, especially Harvey Dale, Sean Irvine, Bob Price, and above all, our President Russ Cox, who has made great improvements to the software, the lookup process, the editing mechanism, and the graphical interface.

Many other sequences from this summer, such as Scott Shannon's A375564:



Appendix: Riesel's Proof of his Theorem

Theorem: There are infinitely many numbers h such that
N = h x 2^e - 1 is composite for all e >= 0

[Example: h = 509203 + k x 5592405 for k = 0, 1, 2, …]

Proof: Useful facts about Fermat numbers
t_n = 2^(2^n) + 1, n >=0.
Note 2^(2^n) == -1 (mod p) if p divides t_n.
Also t_0, t_1, t_2, t_3, t_4 are primes, but t_5 = 2^32 + 1 = 641 x 700417

We deduce 7 further facts:
2^2 == 1 (mod 3), 2^4 == 1 (mod 5), 2^8 == 1 (17), 2^16 == 1 (257),
2^32 == 1 (65537), 2^64 == 1 (641), 2^64 == 1 (6700417)

Just as every number e is either even or congruent to 1 or 3 mod 4,
so every number e is one of 7 types:
1+2k, 2+4k, 4+8k, 8+16k, 16+32k, 32+64k, or 64k.

For each of the 7 types of e, we will show N = h x 2^e - 1 is divisible by
the corresponding prime mentioned above.

Appendix: Riesel's Proof of his Theorem (2)

Case 1: If e = 1+2k, assume h satisfies h x 2 == 1 mod 3.
Then N-1 = h.2^e = (2h)(2^2)^k == 1.1^k (mod 3) = 1 mod 3, so N == 0 mod 3

Case 2: If e = 2+4k, assume h satisfies h x 4 == 1 mod 5.
Then N-1 = h.2^e = (4h)(2^4)^k == 1.1^k (mod 5) = 1 mod 5, so N == 0 mod 5

Case 3: e = 4+8k, h x 2^4 == 1 mod 17, N == 0 mod 17

Case 4: e = 8+16k, h x 2^8 == 1 mod 257, N == 0 mod 257

Case 5: e = 16+32k, h x 2^16 == 1 mod 65537, N == 0 mod 65537

Case 6: e = 32+64k, h x 2^32 == 1 mod 641, N == 0 mod 641

Case 7: e = 64k, h x 2^64 == 1 mod 6700417, N == 0 mod 6700417

All possibilities for e are covered, and in each case N is composite. Also h must satisfy the 7 underlined congruences, the solution to which is
h = 2935363327246958234 (mod 2^64 - 1). QED

A similar argument, using a different set of congruences, has solution
h = 509203 (mod 5592405).