On Domino Tilings of Rectangles
Aztec Diamonds in the Rough

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1 Introduction

What is the number of domino tilings of a 2-by-n rectangle? Viewing 2 as the height and $n \geq 2$ as the width, the right side of a 2-by-n rectangle must be tiled either with a single vertical domino or with two horizontal dominos. Thus the number of domino tilings of a 2-by-n rectangle is equal to the number of domino tilings of a 2-by-$(n-1)$ rectangle plus the number of domino tilings of a 2-by-$(n-2)$ rectangle. After a quick base case check, we find the number of domino tilings of the 2-by-n rectangle to be the nth Fibonacci number F_n, defined recursively by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

Such elegant and straightforward combinatorial arguments can easily be applied to determine the number of domino tilings of any family of rectangles of fixed height. The recurrences become exponentially more complicated as the height increases, but theoretically the number of domino tilings of an arbitrary rectangle can be computed using this very basic combinatorial trick. Combinatorists, in fact, know much more about the number of domino tilings of rectangles. In a seminal paper, Kasteleyn [9] proved that the number of domino tilings of the m-by-n rectangle is

$$\prod_{0 \leq i \leq m+1} \prod_{0 \leq j \leq n+1} \left(\cos^2 \frac{\pi i}{m+1} + \cos^2 \frac{\pi j}{n+1} \right)^{1/4}.$$

While it may seem that an explicit formula, if not an explicit method, entirely solves the problem of determining the number of domino tilings of rectangles, there are still many questions it does not answer immediately. For instance, is there a nice recurrence for the number of domino tilings of squares? We know there is a linear recurrence for the number of domino tilings of rectangles for each fixed height, but is there a nice recurrence for these recurrences? Are there more efficient ways of computing the number of domino tilings of arbitrary rectangles? What is the p-adic behavior of the number of domino tilings of rectangles?

Some parts of these questions have been answered using Kasteleyn's formula or algebraic methods, but finding combinatorial proofs is a high priority for combinatorists for the sake of fully understanding the underlying ideas. In this essay we explore the search for combinatorial proofs involved in the study of domino tilings of rectangles.

Because we will deal with elementary combinatorial proofs, very little general background is necessary. After presenting this background in Section 2, we will be able to quickly begin exploring recent results in tiling theory. In Section 3, we explore p-adic properties of the numbers of domino tilings of rectangles. We investigate the highest power of two that divides the number of domino tilings of squares and n-by-$2n$ rectangles. In Section 4, we describe another way of computing the number of domino tilings of rectangles using objects called Aztec diamonds. In this investigation, we chance upon a conjecture that generalizes one of the results in Section 3.

This essay is not meant to be a thorough exposition of any particular area or problem. Instead, it is intended to expose the reader to an array of problems concerning domino tilings of rectangles with elegant and elementary combinatorial proofs. The methods presented herein have undoubtedly not been used to their full potential in tiling theory, and it is the author's sincere hope that all methods and results presented will be generalized, rendering this essay essentially worthless, as soon as possible.

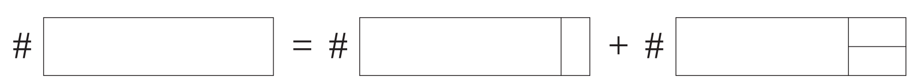
2 Background

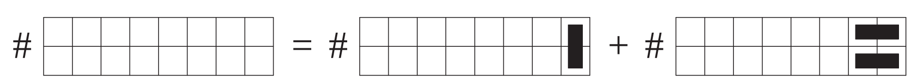
Given a collection of "small" closed plane figures and a single "large" closed plane figure, we may ask whether we can arrange the "small" figures to completely cover the "large" figure such that no two of the smaller figures overlap. If so, in how many ways can it be done? If we instead have a family of larger figures, in what ways does the number of arrangements depend upon the parameters of the family? What are the divisibility properties of the numbers of arrangements?

These types of questions are addressed in what is known as *tiling theory*. Given a set $\mathcal{S}$ of closed subsets of the plane, called *tiles*, and a single closed subset P of the plane, a *tiling of P using tiles from $\mathcal{S}$* is a nonoverlapping arrangement of tiles from $\mathcal{S}$ whose union is P. For example, the unit squares on a piece of graph paper tile the sheet of paper, assuming certain boundary conditions, and the hexagonal plane lattice constitutes a tiling of the plane. Many questions in tiling theory involve questions about the possibility of tiling an infinite region with some small set of geometric shapes, but in this essay we concentrate on finite regions and tiles for which questions of possibility are usually trivial and the enumeration of tilings becomes interesting.

A *domino* is a 1-by-2 rectangle of any orientation. In this essay we will be concerned with domino tilings of rectangles, and in particular the number of domino tilings of rectangles. If P is a planar region, we write $\#P$ for the number of domino tilings of P. Also, when we wish to refer to the number of domino tilings of a region after certain dominos have been fixed, we draw in the dominos we wish to fix and leave the rest of the region untiled.

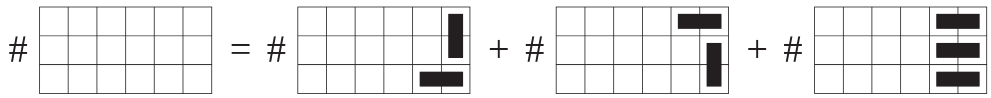
For instance, we began this essay by asking the number of domino tilings of the 2-by-n rectangle. Viewing 2 as the height and $n \geq 2$ as the width of our rectangle, the observation underlying our proof was that the right side of our rectangle must be tiled either with a single domino or with two horizontal dominos, and thus the number of domino tilings of a 2-by-n rectangle is equal to the number of domino tilings of a 2-by-$(n-1)$ rectangle plus the number of domino tilings of a 2-by-$(n-2)$ rectangle. Here we depict this observation for $n = 8$ in two slightly different ways, both of which may be beneficial in different situations.



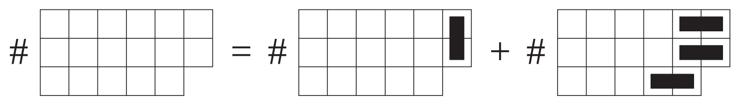


Notice that this is a combinatorial argument: if we let $\mathcal{T}_n$ be the set of domino tilings of the 2-by-n square, we essentially argue for the existence of a bijection between $\mathcal{T}_n$ and the disjoint union $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-2}$.

A similar bijective argument can be constructed for every family of rectangles of fixed height. Let us illustrate this here for rectangles of height 3. For $n \geq 2$, the right hand side of a 3-by-n rectangle must either be tiled with three horizontal dominos, a vertical domino at the top and a horizontal domino below it, or a vertical domino at the bottom and a horizontal domino above it. As in the 2-by-n case, the following picture illustrates our bijection of choice in the 3-by-6 case.



If we let T_n be the number of domino tilings of the 3-by-n rectangle and L_n be the number of domino tilings of the 3-by-n rectangle with a single corner unit square missing, our bijection gives $T_n = 2L_{n-1} + T_{n-2}$. Similarly, notice that $L_n = T_{n-1} + L_{n-2}$, as illustrated in the following picture.



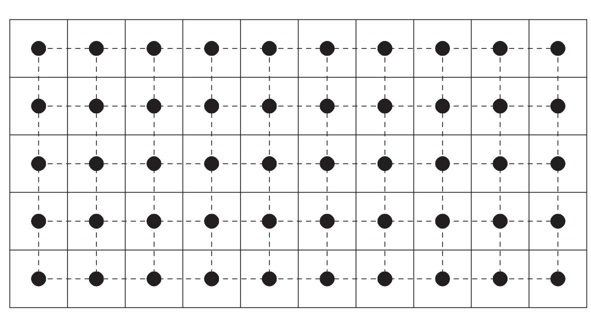
We now have a system of linear recurrences that, along with a few initial conditions, can be used to find the number of domino tilings of any 3-by-n rectangle or even find a single linear recurrence or generating function for the corresponding sequence.

2.1 Tilings and perfect matchings

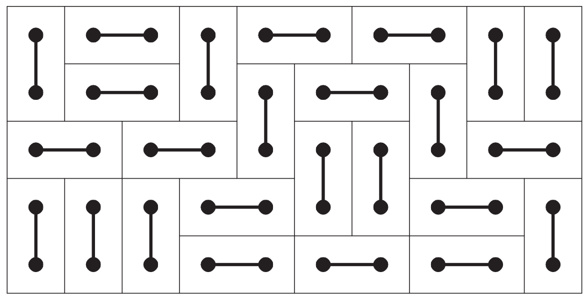
Let us depart from domino tilings for just a moment to consider a problem about graphs. A *perfect matching* of a graph is a subset of the edges of the graph that contains each vertex exactly once. Define the *m-by-n (rectangular) grid-graph* to be the graph with vertex set $\{(a, b) : 1 \leq a \leq m, 1 \leq b \leq n\}$ in which (a, b) and (c, d) are adjacent if and only if $|a - c| + |b - d| = 1$. One may ask how many perfect matchings the 2-by-n grid-graph has. Notice that, for $n \geq 2$, any perfect matching contains either the edge $\{(1, n), (2, n)\}$ or the two edges $\{(1, n), (1, n-1)\}$ and $\{(2, n), (2, n-1)\}$. The number of perfect matchings of the 2-by-n grid-graph is equal to the number of perfect matchings of the 2-by-$(n-1)$ grid-graph plus the number of perfect matchings of the 2-by-$(n-2)$ grid-graph, and therefore is given by the nth Fibonacci number.

Of course, it is no coincidence that this is equal to the number of domino tilings of the 2-by-n rectangle.

In general, there is an obvious bijection between the set of tilings of an m-by-n rectangle and the set of perfect matchings of an m-by-n grid-graph. If we draw the vertices of the m-by-n grid-graph as the centers of the unit squares that compose the m-by-n rectangle, two vertices are adjacent if and only if their corresponding unit squares share an edge. Below we illustrate this in the 5-by-10 case, with the graph's edges drawn as dotted line segments.



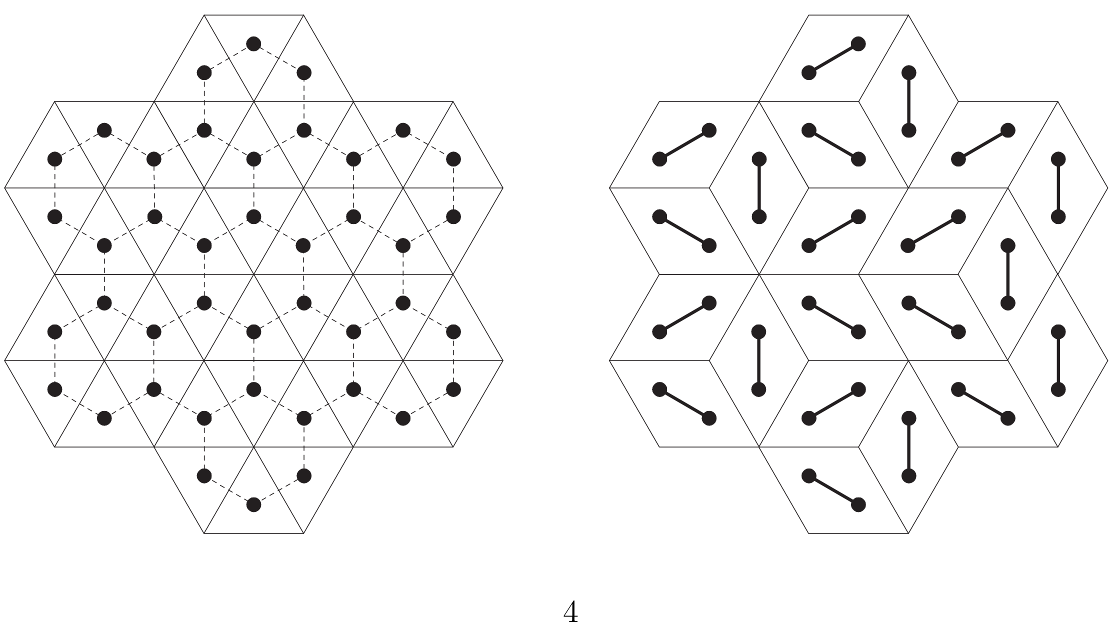
Given any domino tiling T of the m-by-n rectangle, we generate a perfect matching $f(T)$ of the m-by-n grid-graph by selecting each edge if and only if both of its endpoints are contained in a single domino in T. Clearly, f is a bijection from the set of domino tilings of the m-by-n rectangle to the set of perfect matchings of the m-by-n grid-graph for all natural numbers n and m. A 5-by-10 domino tiling and the corresponding perfect matching, drawn in bold, are illustrated below.



As we will see later, there are objects in graph theory other than perfect matchings, such as sets of nonintersecting paths, that are in bijective correspondence with domino tilings of rectangles. We will also make use of perfect matchings of various weighted graphs.

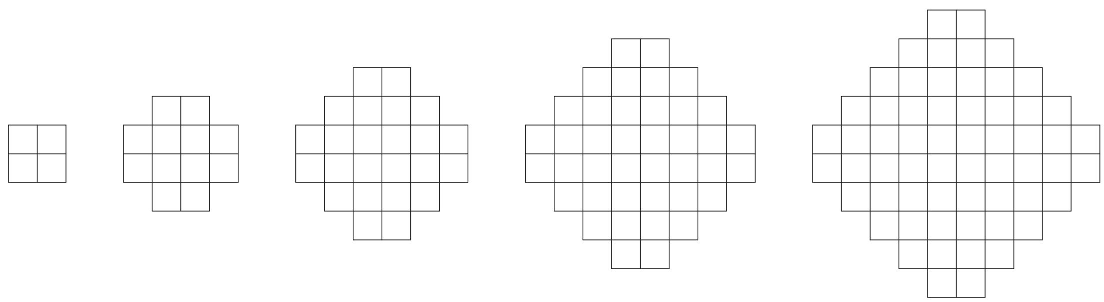
Before we move on, let us briefly discuss the connection between tilings and perfect matchings in slightly more generality. In this paper, the plane regions we tile are complexes built out of vertices, edges, and faces, and tiles correspond to pairs of faces that share an edge. In this framework, a collection of tiles is a tiling if each face of the plane region being tiled belongs to exactly one tile in the collection. Any tiling of a region can be represented as a perfect matching of its dual graph in this framework. For instance, in a rectangle, the faces are the unit squares (the infinite face is not considered a face in our context), and the dual graph is the rectangular grid-graph with one vertex in each face.

Another example where this framework is useful comes from rhombus tilings. Suppose a region is a union of equilateral triangles joined along edges and our tiles are rhombuses formed from the union of two of these equilateral triangles sharing a common edge. We wish to form the dual graph whose number of perfect matchings equals the number of rhombus tilings of the region. This graph can be formed by placing one vertex in each equilateral triangle and setting two vertices adjacent if their corresponding equilateral triangles share an edge.



2.2 Aztec diamonds

The main area of study in this paper is domino tilings of rectangles. In this pursuit, the study of domino tilings of another class of regions will prove quite useful. For $n \geq 1$, the *Aztec diamond of order* n is defined as the union of all unit squares whose corners are integer lattice points lying in the region $\{(x, y) : |x| + |y| \leq n + 1\}$. Below are depicted Aztec diamonds of orders 1 through 5.



The *Aztec diamond graph of order* n is the dual graph of the Aztec diamond of order n.

As we will see, domino tilings of Aztec diamonds are much simpler to handle than domino tilings of rectangles. Even weighted tilings, corresponding to weighted perfect matchings of Aztec diamond graphs, are quite well understood. We will later see that machinery constructed for weighted Aztec diamonds can be applied to the study of domino tilings of rectangles.

3 The p-adic behavior of the number of domino tilings of rectangles

There has been a lot of recent study of the p-adic behavior of the number of domino tilings of rectangles. For instance, Pachter [12] and Cohn [5] have proved very nice results about the p-adic behavior of the number of domino tilings of squares. We also prove a theorem about n-by-$2n$ rectangles, that the number of domino tilings of the n-by-$2n$ rectangle is congruent to 1 modulo 4. Many other strange behaviors conjectured by various combinatorists, some of which have been proven via Kasteleyn's formula (such as that the number of domino tilings of the $2n$-by-$2n$ square is divisible by 3 when n is congruent to 2 modulo 5), still seem to be itching for combinatorial proofs.

3.1 Domino tilings of squares

Kasteleyn's exact formula for the number of domino tilings of rectangles gives little intuition about properties of the number of domino tilings of squares. Even its specialization to $2n$-by-$2n$ squares,

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\left(4\cos^2\frac{\pi i}{2n+1}+4\cos^2\frac{\pi j}{2n+1}\right),$$

gives little insight. For instance, it is true though not at all obvious that this number is always a perfect square or twice a perfect square [11]. In this section we prove a generalization of this fact combinatorially.

The sequence whose nth term is the number of domino tilings of the $2n$-by-$2n$ rectangle begins

$$\begin{aligned}
2 &= 2^1\cdot 1^2,\\
36 &= 2^2\cdot 3^2,\\
6728 &= 2^3\cdot 29^2,\\
12988816 &= 2^4\cdot 901^2,\\
258584046368 &= 2^5\cdot 89893^2,\\
53060477521960000 &= 2^6\cdot 28793575^2,\\
112202208776036178000000 &= 2^7\cdot 29607089625^2,\\
2444888770250892795802079170816 &= 2^8\cdot 97725875584681^2,\\
548943583215388338077567813208427340288 &= 2^9\cdot 1035449388414303593^2,\\
126998401125623583424260275310229393429857624985\,6 &= 2^{10}\cdot 352167397836940296019 63^2.
\end{aligned}$$

As it turns out, the pattern continues.

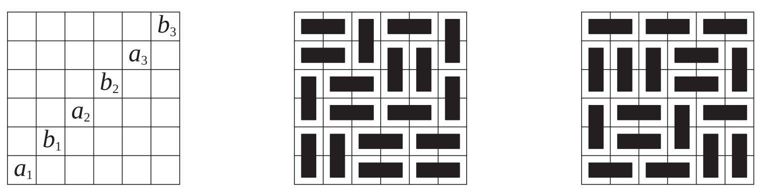
Theorem 1. *The number of domino tilings of the $2n$-by-$2n$ square grid is equal to 2^n times an odd square.*

This theorem has been proven from Kasteleyn's formula independently by many authors, including in [8]. Pachter [12] found the first combinatorial proof of this theorem.

Our proof of Theorem 1 roughly follows [12]. First, we prove that the number of domino tilings of a square is equal to a certain power of two times the number of pairs of tilings of certain subregions of the square. This step involves looking at the structure of domino tilings of the square. Then we prove that the certain subregions mentioned above have an odd number of domino tilings, completing the proof.

Given a domino and one of its two unit squares, the *direction* of the domino at that square is up, down, left, or right corresponding to the relative position of its other unit square, and the *orientation* of the domino at that square is *positive* if its direction is up or right and *negative* if its direction is down or left. Given any $2n$-by-$2n$ square grid, for the purposes of this section we label its diagonal squares $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ from lower left to upper right. The *d-type* and *o-type* of a domino tiling of such a grid are $(d_1, \ldots, d_n)$ and $(o_1, \ldots, o_n)$, respectively, where d_i is the direction and o_i the orientation of the domino at a_i. Call a d-type $(d_1, \ldots, d_n)$ and an o-type $(o_1, \ldots, o_n)$ *compatible* if, for all $i = 1, \ldots, n$, d_i is up or right if and only if o_i is positive, and call two d-types *compatible* if they are compatible with the same o-type.

Below, we depict a 6-by-6 square with the diagonal labelled as described above and two domino tilings with compatible d-types. The tiling on the left has d-type (up, left, down) and the tiling on the right has d-type (right, left, left); both have o-type (positive, negative, negative).



Lemma 2. *Let $d = (d_1, \ldots, d_n)$ and $d' = (d'_1, \ldots, d'_n)$ be two d-types. If d and d' are compatible, the number of domino tilings of d-type d of a $2n$-by-$2n$ square grid is equal to the number of domino tilings of d-type d'. In other words, the number of domino tilings of a $2n$-by-$2n$ square grid with dominos fixed at $a_1, \ldots, a_n$ depends only on the orientations of these dominos and not on their directions.*

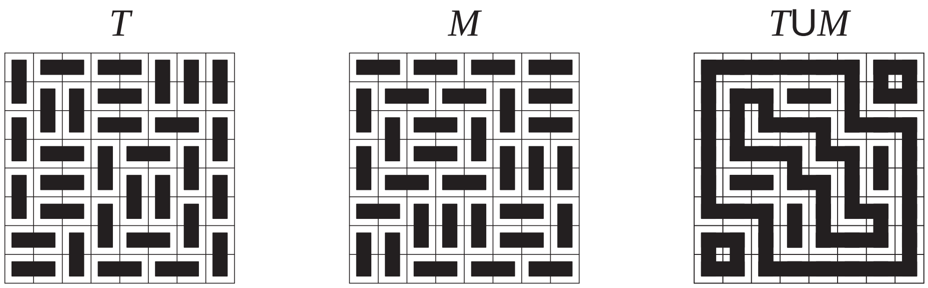
Proof. Let $\mathcal{T}_d$ (resp. $\mathcal{T}_{d'}$) be the set of domino tilings of the $2n$-by-$2n$ grid with d-type d (resp. d'), and set $I = \{i : d_i \neq d'_i\}$ to be the set of indices on which d and d' differ. We will provide an explicit bijection $f_I : \mathcal{T}_d \to \mathcal{T}_{d'}$.

Given a tiling $T \in \mathcal{T}_d$, let M be the tiling obtained by reflecting T in the diagonal. The dual graph of the union $D = T \cup M$, which is allowed to have multiple dominos, is a 2-factor and therefore a disjoint union of even cycles.

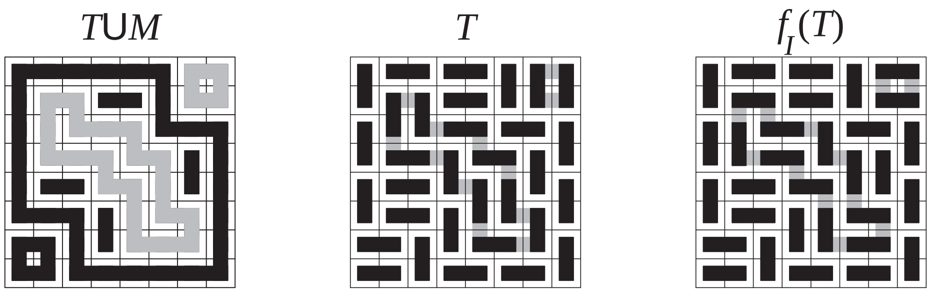
For each $i = 1, \ldots, n$, define C_i to be the cycle containing a_i. We first notice that the cycles C_i must be disjoint. Other than a_i, C_i contains at most one point on the diagonal; otherwise the fact that every vertex in D has degree two would be contradicted. Further, if this other point exists it must be of the form b_j; otherwise C_i would be odd. Finally, our observation is proven by noticing that no two of the cycles C_i may intersect at another point; otherwise again some vertex in D would have degree more than two.

Since $T \cap C_i$ is an alternating cycle in the tiling T for each C_i, we notice $M \cap C_i = C_i \smallsetminus (T \cap C_i)$ is the alternating cycle in D obtained by rotating C_i by one edge. Set $f_I(T) = T \cup \bigcup_{i \in I}(M \cap C_i)$, the tiling obtained from T by rotating each alternating cycle $T \cap C_i$ for which the directions of dominos on a_i in tilings with d-types d and d' differ. $\square$

An example can help illustrate this proof. Suppose $d =$ (right, left, right, up) and $d' =$ (right, left, up, right), so that $I = \{3, 4\}$. We illustrate the bijection $f_{\{3,4\}}$ by demonstrating its effect on the tiling T depicted below. First, we draw T, its mirror image M, and $T \cup M$, with multiple dominos represented simply by single dominos.

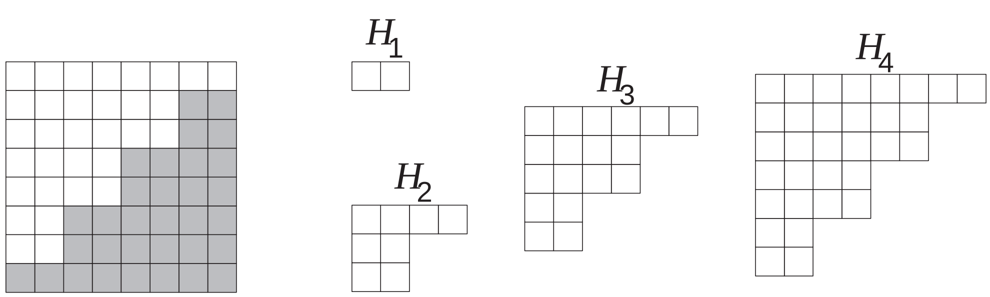


Now we depict $T \cup M$ with C_3 and C_4 highlighted, T with C_3 and C_4 watermarked, and $f_I(T)$ again with C_3 and C_4 watermarked.



Here, we can see that the only difference between T and $f_I(T)$ is that the alternating cycles $T \cap C_3$ and $T \cap C_4$ have been rotated.

Now that we have this preliminary result, we may move on. For a square grid S with unit square u, denote by $S(u)$ the subrectangle consisting of u and all unit squares to the right of and below u. For a $2n$-by-$2n$ square grid S, notice that $S_{\text{top}} = \bigcup_{i=1,\ldots,n} S(a_i)$ and $S_{\text{bottom}} = S \smallsetminus \bigcup_{i=1,\ldots,n} S(a_i)$ constitute a natural division of S into two congruent halves. For any integer $n \geq 1$, define H_n to be one half of the $2n$-by-$2n$ square grid as obtained by this method. Depicted below are an 8-by-8 grid dissected into halves as well as H_1, H_2, H_3, and H_4.



Lemma 3. *The number of domino tilings of a $2n$-by-$2n$ square grid is equal to 2^n times the square of the number of domino tilings of H_n.*

Proof. For a $2n$-by-$2n$ square S, any two domino tilings of S_{top} and S_{bottom} may be placed beside each other to form a tiling of S. In the resulting tiling, each domino at $a_1, \ldots, a_n$ is directed either right or down. Conversely, we may show that any tiling of S in which the dominos at $a_1, \ldots, a_n$ are directed right or down may be obtained in this way. If right- and down-directed dominos are placed at $a_1, \ldots, a_n$, the corners of S_{top} lying along the diagonal are all the same color in the standard chessboard 2-coloring of S, and thus in each tiling every domino must be contained wholly within either S_{top} or S_{bottom}. Thus the number of domino tilings of S in which the dominos at $a_1, \ldots, a_n$ are directed right or down is equal to the square of the number of domino tilings of H_n. We are done by Lemma 2. $\square$

To complete our proof of Theorem 1, it is sufficient to show that the number of domino tilings of H_n is odd for each $n \geq 1$.

Lemma 4. *For any integer $n \geq 1$, H_n is odd.*

Proof. For $n = 1$, the result is trivial. Assume $n \geq 2$ and proceed by induction. Viewing H_n as S_{top} for a $2n$-by-$2n$ square grid S, for $i = 2, \ldots, n$ denote by c'_{i-1} the unit square directly to the left of a_i and by c_{i-1} the unit square directly to the bottom left of a_i. Let $\mathcal{T}^n_{d_1, d'_1, d_2, d'_2, \ldots}$ be the set of domino tilings of H_n with the dominos at c_i in direction d_i and c'_i in direction d'_i for each $i = 1, \ldots, n-1$, where dominos are always listed in the prescribed order. For instance, $\mathcal{T}^n$ is the set of domino tilings of H_n, $\mathcal{T}^n_{\text{up}}$ is the set of domino tilings of H_n with the domino at c_1 directed up, and $\mathcal{T}^n_{\text{left, left}}$ is the set of tilings of H_n with the dominos at c_1 and c'_1 both directed left.

It is easy to see that

$$|\mathcal{T}^n| = |\mathcal{T}^n_{\text{left, left}}| + |\mathcal{T}^n_{\text{up}}| + |\mathcal{T}^n_{\text{left, up}}|.$$

Since $|\mathcal{T}^n_{\text{left, left}}| = |\mathcal{T}^n_{\text{up}}|$, we have $|\mathcal{T}^n| \equiv |\mathcal{T}^n_{\text{left, up}}| \pmod 2$. Continuing in this manner, we see that

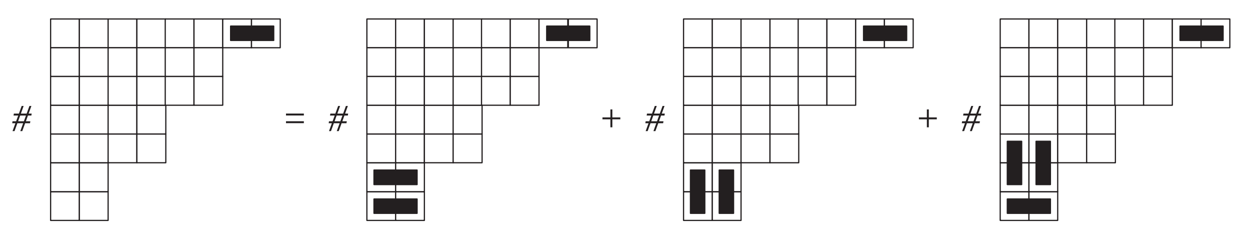
$$|\mathcal{T}^n_{\text{left, up}}| = |\mathcal{T}^n_{\text{left, up, left, left}}| + |\mathcal{T}^n_{\text{left, up, up}}| + |\mathcal{T}^n_{\text{left, up, left, up}}|.$$

Since $|\mathcal{T}^n_{\text{left, up, left, left}}| = |\mathcal{T}^n_{\text{left, up, up}}|$, we have $|\mathcal{T}^n| \equiv |\mathcal{T}^n_{\text{left, up}}| \equiv |\mathcal{T}^n_{\text{left, up, left, up}}| \pmod 2$. Continuing in this way, we see that

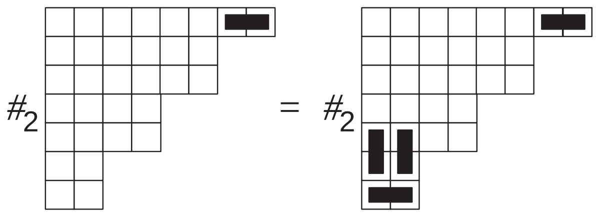
$$|\mathcal{T}^n| \equiv |\mathcal{T}^n_{\text{left, up, left, up, left, up, }\ldots}| \pmod 2.$$

Finally, notice that $|\mathcal{T}^n_{\text{left, up, left, up, left, up, }\ldots}| = |\mathcal{T}^{n-1}|$, so $|\mathcal{T}^n| \equiv |\mathcal{T}^{n-1}| \pmod 2$. $\square$

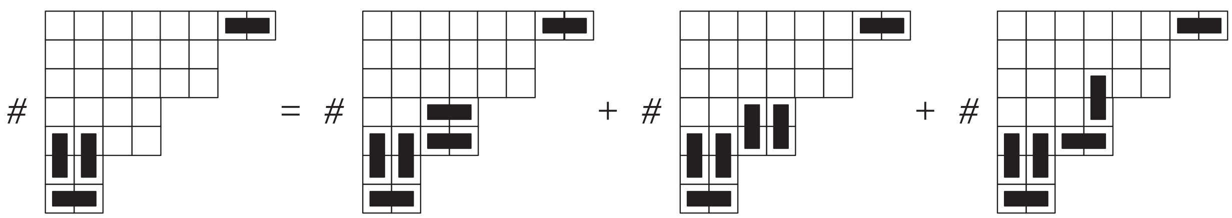
To illustrate this proof, we shall depict the inductive step for $n = 4$. In the following drawings, $\#R$ represents the number of tilings of the region R and $\#_2 R$ denotes the parity of the number of tilings of R. Also, forced dominos are drawn in.

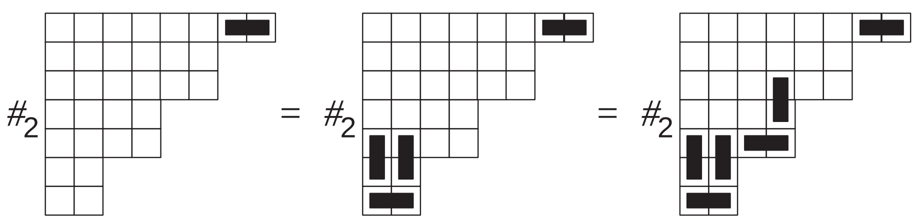


Since the first two terms on the right-hand side of this expression are equal, we have the following.

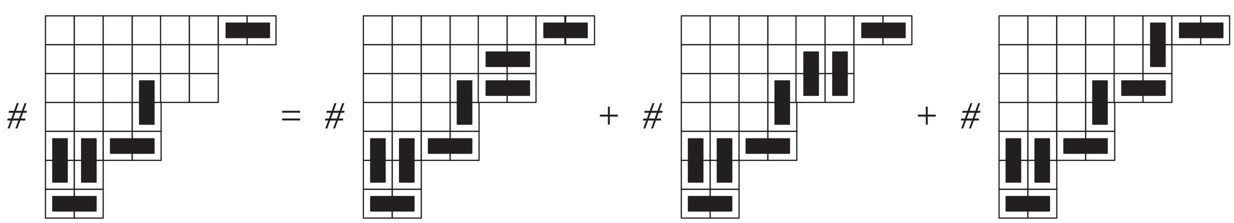


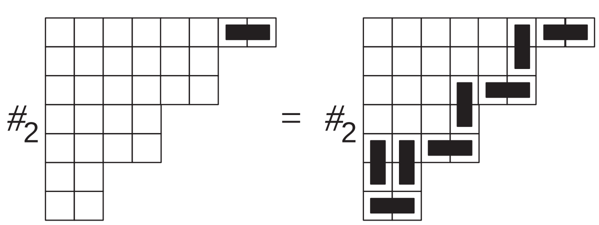
The next iteration is similar.





One more iteration finishes the process.





Our final observation is that the figure on the right of the last equation has the same number of domino tilings as H_3, as the untiled region is just H_3 with the forced domino inserted. Thus the number of domino tilings of H_4 is congruent modulo 2 to the number of domino tilings of H_3.

Together, Lemmas 3 and 4 prove Theorem 1.

The fact that there is an elegant combinatorial proof for the largest power of 2 dividing the number of domino tilings of squares suggests an investigation of the largest power of 2 dividing the number of domino tilings of all rectangles. In this vein, Pachter has posed the following problem: prove combinatorially that the number of domino tilings of the $2n$-by-$2m$ rectangular grid equals $\sqrt{2^{(2n+1,2m+1)-1}}(2r_1+1)$ and the number of domino tilings of the $2n+1$-by-$2m$ grid equals $\sqrt{2^{((n+1,2m+1)-1)(3+j)}}(2r_2+1)$, where (a,b) denotes the greatest common factor of integers a and b, j is defined by $n+1=2^j(2t+1)$, and r_1, r_2, and t are natural numbers that may vary for different values of n and m. The even-by-even case has been proven by algebraic methods, and the odd case has been empirically verified extensively with $n \leq 10$.

Pachter suggests another approach to determining the largest power of 2 that divides certain number of tilings, using tools similar to an observation of Lovász [11] that a graph G has an even number of perfect matchings if and only if there is a nonempty set $S \subseteq V(G)$ such that every vertex of G is adjacent to an even number of vertices in S. This result has been all but forgotten to those working in tiling theory, but has the potential to prove useful.

3.2 Domino tilings of n-by-$2n$ rectangles

The sequence whose nth term is the number of domino tilings of the n-by-$2n$ rectangle begins 1, 5, 41, 2245, 185921, 106912793, 51875781745329. All of these terms are odd, and the astute reader may have noticed that they are all also congruent to 1 (mod 4). As it turns out, this pattern continues for the entire sequence.

Theorem 5. *The number of domino tilings of the n-by-$2n$ rectangular grid is congruent to* 1 (mod 4).

This theorem was proven by the author [2]. Our proof begins by defining, for any n-by-$2n$ rectangle, a directed acyclic graph with the same number of routings as the rectangle has tilings. We then use a method based on dynamic programming as a computationally tractable way to calculate the number of routings modulo small numbers.

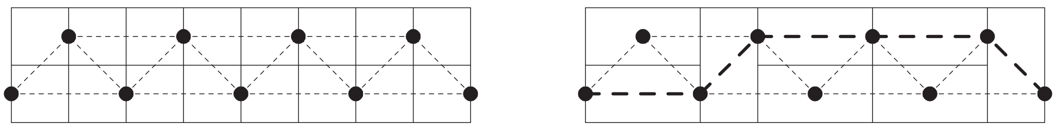
First, however, let us consider a simple example for the sake of motivation. In particular, given any 2-by-n rectangle, we will compute the number of domino tilings of the rectangle as the number of paths through a certain directed acyclic graph. We may think of our 2-by-n rectangle as oriented so that it has height 2 and width n. Label each square of the rectangle with the ordered pair consisting of the column followed by the row, where the topmost left square is $(1,1)$ and the bottommost right square is $(2,n)$, and we view the rectangle as part of a larger grid whose labelling scheme naturally extends the scheme above. Let $v_{a,b}=(a,b)_{\text{left}}$ denote the point that bisects the middle of the left-hand edge of unit square (a,b). We define G_n^2 to be the directed graph with vertex set

$$V(G_n^2)=\bigcup_{1\leq b\leq n+1}\{v_{a,b} : a=1 \text{ and } b \text{ is even}\}\cup\bigcup_{1\leq b\leq n+1}\{v_{a,b} : a=2 \text{ and } b \text{ is odd}\}$$

and such that $(v_{a,b}, v_{c,d})$ is an arc if and only if either (1) $c=a$ and $d=b+2$ or (2) $c=a\pm 1$ and $d=b+1$. In this graph let $s=v_{2,1}$ be the source and $t=v_{1,n+1}$ or $v_{2,n+1}$ (whichever is a vertex of the graph) be the sink.

We assert that the number of s–t paths is equal to the number of domino tilings of the 2-by-n grid. To prove this, we provide a bijection f between the set of tilings of the 2-by-n grid and the set of s–t paths in G_n^2. Given a tiling T of the grid, we form the path $f(T)$

recursively as follows: initialize $f(T) = \emptyset$. Add to $f(T)$ the only arc of G_n^2 emanating from s that lies entirely within a domino of T. Repeat this with s replaced by the end of the previously added arc, and continue the process until t is reached. It is easy to see that f is a bijection. Below, we superimpose the directed graph G_8^2 on the 2-by-8 rectangle, with arcs (directed to the right) indicated by dotted lines, and illustrate the bijection f by drawing a tiling T and darkening $f(T)$.



We can use *dynamic programming* to count the number of paths in any directed acyclic graph. Let G be any directed acyclic graph, and suppose without loss of generality that all arcs are directed from right to left and we wish to count paths from one of the leftmost vertices, s, to one of the rightmost vertices, t. Each vertex v in G is labelled, recursively from right to left, with the number of paths from v to t in G (t is labelled 1 and every other vertex in the same vertical line as t is labelled 0). In particular, s is labelled with the number of paths from s to t. At each step of the dynamic programming algorithm, the label of a vertex v in G is calculated by adding the labels of the heads of all arcs for which v is the tail. Since the vertices are labelled from right to left, these labels are already determined by the time the algorithm visits v.

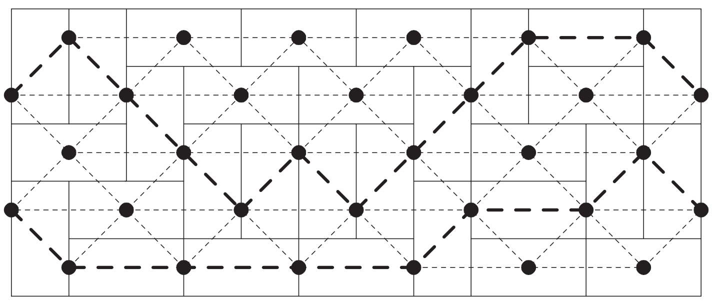
We are now ready to extend these observations to all rectangles; that is, we wish to compute the number of domino tilings of an m-by-n rectangle, for $m \geq 2$, as the number of paths through a certain directed acyclic graph. What follows is an extension of the above discussion for 2-by-n rectangles. We may think of an m-by-n rectangle as oriented so that it has height m and width n. Label each square of the rectangle with the ordered pair consisting of the column followed by the row, where the topmost left square is $(1,1)$ and the bottommost right square is (m,n), and we view the rectangle as part of a larger grid whose labelling scheme naturally extends the scheme above. Let $v_{a,b} = (a,b)_{\text{left}}$ denote the point that bisects the middle of the left-hand edge of unit square (a,b). We define G_n^m to be the directed graph with vertex set

$$V(G_n^m) = \bigcup_{1 \leq a \leq m} \bigcup_{1 \leq b \leq n+1} \{v_{a,b} : b \not\equiv a \pmod 2\}$$

and such that $(v_{a,b}, v_{c,d})$ is an arc if and only if either (1) $c = a$ and $d = b+2$ or (2) $c = a \pm 1$ and $d = b+1$. In this graph let $s_i = v_{2i,1}$ be the source vertices and $t_i = v_{2i-1,n+1}$ or $v_{2i,n+1}$ (whichever are vertices of the graph) be the sinks, for $i = 1, \ldots, k = \lfloor m/2 \rfloor$.

Two paths in a directed acyclic graph are said to *intersect* if they share any vertices. A k-*routing* from *sources* $\{s_1, \ldots, s_k\}$ to *sinks* $\{t_1, \ldots, t_k\}$ is a set of k pairwise nonintersecting paths from s_i to $t_{\sigma(i)}$ for $i = 1, \ldots, k$, where σ is any permutation of $\{1, \ldots, k\}$. It turns out that the number of routings from $\{s_1, \ldots, s_k\}$ to $\{t_1, \ldots, t_k\}$ is equal to the number of domino tilings of the m-by-n rectangle. We offer a slightly simplified version of the proof for 2-by-n rectangles here. Define a bijection f from the set of tilings to the set of routings as follows. Given a tiling T, define $f(T)$ to be the arcs that lie entirely within dominos in T. This map is clearly both injective and surjective. Notice, by the way, that all dominos in T that do not determine an arc in $f(T)$ must be horizontal (of width 2 and height 1), since for every possible vertical domino with lattice point corners there exists an arc lying completely inside it.

Below is drawn a 5-by-12 tiling with the corresponding routing superimposed.



3.2.1 Enumerating routings

Counting routings is not quite as easy as counting paths. However, it is still quite tractable and, as we will see, particularly computationally tractable modulo small numbers.

Before investigating the general case, we count 2-routings. In a directed acyclic graph G, let s_1, s_2, t_1, and t_2 be vertices such that any s_1–t_2 path intersects any s_2–t_1 path. Our goal is to determine the number of routings from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, in other words the number of nonintersecting pairs of s_1–t_1 and s_2–t_2 paths.

For $i = 1, 2$ and $j = 1, 2$, let n_{ij} be the number of paths from s_i to t_j. The total number of pairs of s_1–t_1 and s_2–t_2 paths is $n_{11}n_{22}$.

Next, we assert that the number of pairs of s_1–t_1 and s_2–t_2 paths that *do* intersect is $n_{21}n_{12}$. We find a bijection between intersecting pairs of s_1–t_1 and s_2–t_2 paths and paths of s_1–t_2 and s_2–t_1 paths. Given an s_1–t_1 path P_1 and an s_2–t_2 path P_2 that intersect, let p be the leftmost point of intersection. Exchanging the partial paths connecting s_1 to p and s_2 to p yields one s_1–t_2 path and one s_2–t_1 path. Conversely, any pair of s_1–t_2 and s_2–t_1 paths must intersect, so let q be the leftmost point of intersection. Exchanging the partial paths connecting s_1 to q and s_2 to q yields one s_1–t_1 path and one s_2–t_2 path.

Therefore, the number of 2-routings from $\{s_1, t_1\}$ to $\{s_2, t_2\}$ is $n_{11}n_{22} - n_{12}n_{21}$. This is known as the *Exchange Principle*.

The astute reader may have noticed that the formula in the Exchange Principle seems to suggest that the number of k-routings can be calculated as a determinant.

Let G be a directed acyclic graph and $s_1, \ldots, s_k; t_1, \ldots, t_k$ be vertices in G such that the existence of a nonintersecting pair of s_i–t_j and $s_{i'}$–$t_{j'}$ paths implies that $(i' - i)(j' - j)$ is positive (that is, either $i' > i$ and $j' > j$ or $i' < i$ and $j' < j$). Clearly, this implies that if σ is a nonidentity permutation on $\{1, \ldots, k\}$, there is no set of nonintersecting s_i–$t_{\sigma(i)}$ paths for $i = 1, \ldots, k$. For all $i, j = 1, \ldots, k$, let n_{ij} be the number of s_i–t_j paths. The following lemma was originally formulated for graphs with a certain planarity condition, but the above constraints work just as well for our purposes. It was also originally proven in slightly more generality and for weighted graphs.

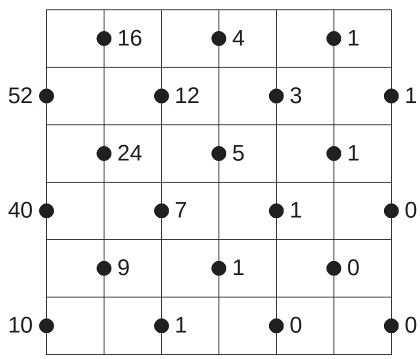
Lemma 6 (Lindström's Lemma). *The number of routings from $\{s_1, \ldots, s_k\}$ to $\{t_1, \ldots, t_k\}$ is equal to the determinant of the matrix $N = (n_{ij})_{1 \le i,j \le k}$.*

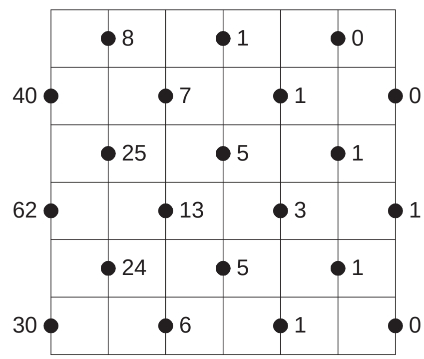
Proof. The determinant of N is

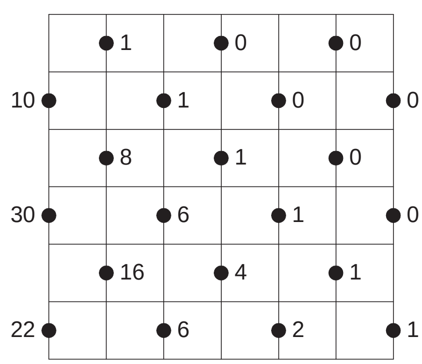
$$\det N = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) n_{1,\sigma(1)} n_{2,\sigma(2)} \cdots n_{k,\sigma(k)},$$

where S_k is the symmetric group of order k. The first term in this sum, $n_{11}n_{22}\cdots n_{kk}$, is equal to the number of k-tuples of s_i–t_i paths for $i = 1, \ldots, k$. Let $\mathcal{P} = (P_1, \ldots, P_k)$ be such a path family in which paths P_j and P_{j+1} intersect exactly once and no other paths intersect. The set of edges covered by $\mathcal{P}$ is also covered by the k-tuple $(P_1, \ldots, P_{j-1}, P'_j, P'_{j+1}, P_{j+2}, \ldots, P_k\}$ obtained by swapping paths P_j and P_{j+1} before their intersection. This second k-tuple is counted in the term $n_{1,1}\cdots n_{j-1,j-1}n_{j+1,j}n_{j,j+1}n_{j+2,j+2}\cdots n_{k,k}$ with sign -1 because the corresponding permutation is a transposition (of j and $j+1$). Together, these two k-tuples contribute zero to the determinant. Similarly, a k-tuple with m intersections is counted 2^m times in the determinant, 2^{m-1} times with sign $+1$ and 2^{m-1} times with sign $+1$. The only k-tuples that are counted exactly once are those in which no two paths intersect, and these have sign $+1$. $\square$

This result is quite straightforward to apply in enumerating domino tilings, but it rarely hurts to try an example. Here we compute the number of domino tilings of the 6-by-6 square, or equivalently the number of routings in G_6^6 from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$.







$$\begin{array}{lll} n_{11} = 52 & n_{12} = 40 & n_{13} = 10 \\ n_{21} = 40 & n_{22} = 62 & n_{23} = 30 \\ n_{31} = 10 & n_{32} = 30 & n_{33} = 22 \end{array} \quad \Rightarrow \quad N = \begin{bmatrix} 52 & 40 & 10 \\ 40 & 62 & 30 \\ 10 & 30 & 22 \end{bmatrix}$$

Thus $\det N = 6728 = 2^3 \cdot 29^2$, verifying one case of Pachter's Theorem.

3.2.2 The number of domino tilings of n-by-$2n$ rectangles modulo 2

Armed with new gear in our toolbox, we are now able to more directly approach the proof of Theorem 5. The goal of the current section is to prove that the number of domino tilings of an n-by-$2n$ rectangle is congruent to 1 modulo 2. We will be able to apply the same machinery for the full proof of Theorem 5 in the next section.

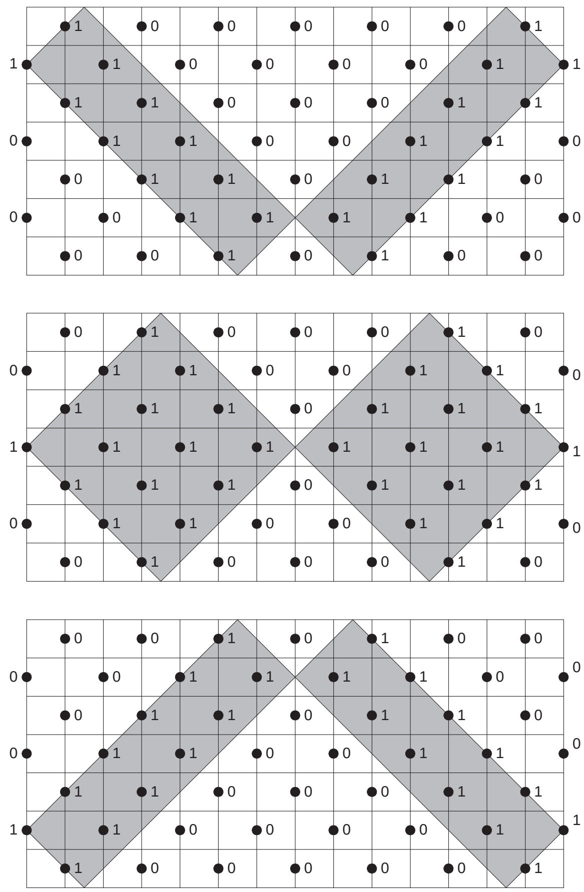
Let R_n be the n-by-$2n$ rectangle and G_n (called G_{2n}^n in our previous notation) be the n-by-$2n$ rectangular grid-graph as superimposed on R_n as before. Label the sources and sinks of G_n, from top to bottom, $s_{n,1}, \ldots, s_{n,k}$ and $t_{n,1}, \ldots, t_{n,k}$, respectively, where $k = \lfloor n/2 \rfloor$.

Suppose a particle begins at sink $t_{n,i}$ and moves directly northwest, reflecting when it comes into contact with an edge of R_n (reflecting from edges at the same angle as a physical object would reflect from walls), until it returns to $t_{n,i}$. Let $P_{n,i}$ be the path this particle traces out. Notice that $P_{n,i}$ is a closed (though not simple) plane figure consisting of the union of two congruent rectangles joined at a common vertex. Let $G_{n,i}$ be the subgraph consisting of all vertices of G_n that lie inside $P_{n,i}$ or on its boundaries. Notice that $t_{n,i}$ is the only sink of G_n contained in $G_{n,i}$ and that $s_{n,i}$ is the only source of G_n contained in $G_{n,i}$.

Proposition 7. *For any $n \geq 1$ and $1 \leq i \leq k$, $s_{n,i}$ is the only source of G_n that lies in $G_{n,i}$.*

Observe that the vertices that have odd numbers of paths to $t_{n,i}$ are precisely the vertices that lie in $G_{n,i}$. Formally, let $w_{n,i}(v)$ be the label of vertex v assigned by our dynamic programming algorithm for counting paths when the label of each sink $t_{n,j}$ is initialized to 1 if $i = j$ and 0 otherwise. In other words, $w_{n,i}(v)$ is the number of v–$t_{n,i}$ paths in G_n. Let $w_{n,i}(v)_k$ be the remainder when $w_{n,i}(v)$ is divided by k.

Below we write the numbers $w_{7,i}(v)_2$ superimposed on R_7, indicating $P_{7,i}$ by shading the interior.



The following proposition formalizes one pattern evident in the diagram.

Proposition 8. *For any $n \geq 1$, $1 \leq i \leq k$, and $v \in G_n$, $w_{n,i}(v)_2 = 1$ if and only if $v \in V(G_{n,i})$.*

Proof. This Lemma can easily be proven by induction. The induction step can be broken up into six very straightforward cases (for what a given column will look like in terms of the two columns to the right of it), three corresponding to even n and three to odd n. □

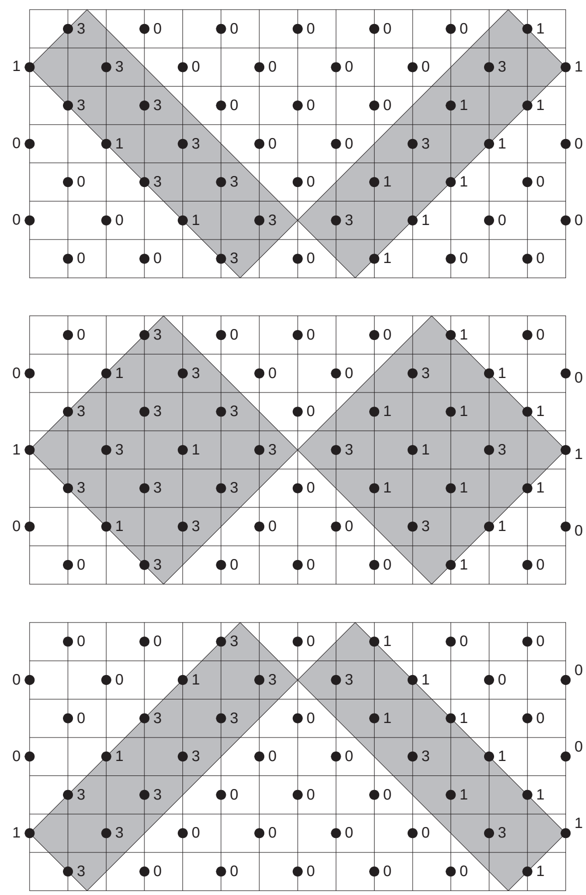
Let $N_{n,i,j}$ be the number of paths in G_n from $s_{n,i}$ to $t_{n,j}$.

Theorem 9. *The matrix $(N_{n,i,j})_{1\leq i,j\leq k}$ is equivalent modulo 2 to the k-by-k identity matrix. Thus the number of domino tilings of the n-by-$2n$ rectangle is odd.*

Proof. The proof of the first sentence is immediate from Propositions 7 and 8. The second sentence follows from the first sentence by Lindström's Lemma (since the determinant of the identity matrix is 1) and our bijection between routings and domino tilings. □

3.2.3 The number of domino tilings of n-by-$2n$ rectangles modulo 4

The goal of this section is to prove that the number of domino tilings of the n-by-$2n$ rectangle is congruent to 1 modulo 4. In light of the results of the last section, the following proposition is sufficient to prove this. Recall that we already know that $w_{n,i}(s_{n,i})_4 = 1$ or 3.



Proposition 10. *For any $n \geq 1$, $1 \leq i \leq k$, and $v \in G_n$, $w_{n,i}(s_{n,i})_4 = 1$.*

Proof. This can easily be proved by a slight generalization of the induction used to prove Proposition 8. The only difference is that cases taking numbers modulo 4 rather than just modulo 2 must be considered. □

This proposition immediately yields the following result, which includes Theorem 5.

Theorem 11. *The matrix $(N_{n,i,j})_{1\leq i,j\leq k}$ is equivalent modulo 4 to the k-by-k identity matrix, so the number of domino tilings of the n-by-2n rectangle is congruent to 1 modulo 4.*

3.2.4 The number of domino tilings of n-by-$2rn$ rectangles

Notice that the results of these sections extend naturally to the study of n-by-$2rn$ rectangles for every $n \geq 1$ and $r \geq 1$. The following theorem can be proven simply by pasting copies of n-by-$2n$ rectangles together horizontally and redefining the paths traced out by particles to reflect off the outside edges of the entire rectangle (meaning, not the edges of each n-by-$2n$ rectangle). The pattern that emerges has exactly the same inductive proof as for n-by-$2n$ rectangles, as *no new cases need be considered* in the induction step.

Theorem 12. *For any positive integers n and r, the number of domino tilings of the n-by-$2rn$ rectangle is congruent to 1 modulo 4.*

4 Domino tilings of rectangles and Aztec diamonds

The rest of this essay is concerned with applications of a simple method due to the current author [4], based on a paper of Kuo [10], for studying domino tilings of rectangles by viewing the rectangles as subregions of Aztec diamonds.

4.1 Dodgson condensation and lambda determinants

In order to fully understand the context in which Kuo's method arose, we need to begin by exploring an observation due to Charles Dodgson [6], better known as the author Lewis Carroll of *Alice in Wonderland*, about a very elegant method for taking determinants of matrices. We can then describe a beautiful extension of the determinant, based on Dodgson's method, formulated by Robbins and Rumsey [13].

4.1.1 Dodgson condensation ...

Charles Dodgson developed a now largely obscure method for computing matrix determinants known as *Dodgson condensation* [6]. Dodgson condensation begins with a n-by-n square matrix $A = A_n$ for $n \geq 1$ and recursively computes k-by-k matrices A_k with k decreasing from $n-1$ to 1. It is helpful to let A_{n+1} be the $n+1$-by-$n+1$ matrix each of whose entries is 1, and to think of the matrices $A_{n+1}, A_n, \ldots, A_1$ as stacked to form a pyramid, with A_{n+1} as the base and A_1 on the top. To determine an element in the kth level A_k, take the determinant of the 2-by-2 submatrix that sits just below it in the $(k+1)$st level A_{k+1} and divide it by the entry that sits directly below it in the $(k+2)$nd level A_{k+2}. Then the sole entry of A_1 is the determinant of A.

Formally, for a matrix A let $A(i,j)$ denote its (i,j) entry. The matrix A_k is then defined recursively in terms of A_{k+1} and A_{k+2} by

$$A_k(i,j)A_{k+2}(i+1,j+1) = A_{k+1}(i,j)A_{k+1}(i+1,j+1) - A_{k+1}(i+1,j)A_{k+1}(i,j+1),$$

and $\det A = A_1(1,1)$.

Let us try an example of this method with the 4-by-4 matrix $A = A_4$ whose (i,j)th entry is the minimum of i and j.

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_1 = [1]$$

Thus $\det A = 1$.

The astute reader may have noticed one problem with this method. It is certainly possible that division by zero may occur, and the method does not account for what to do if it does happen. As it turns out, this is not a major problem. Charles Dodgson himself developed an elaborate element-shuffling process to circumvent the problem, and since his time several application-specific methods have been developed. For some applications, for instance, it is appropriate to set all zeros to an infinitesimal and take limits.

The rule that Dodgson condensation depends upon, however, is completely unambiguous. Here we present a beautiful combinatorial proof of the rule due to Zeilberger [16]. In the original paper, the inimitable Zeilberger uses the apt analogy of two-timing men and women, making his original proof as amusing as it is elegant.

For $n \geq 2$ and A an n-by-n matrix, let $A^{i,j}_{k,l}$ be the $(k-i)$-by-$(l-j)$ submatrix of A whose upper left and lower right entries are the (i,j)th and (k,l)th entries of A, respectively. For example, $A^{1,1}_{n,n} = A$ and $A^{i,j}_{i,j}$ is the 1-by-1 matrix whose sole entry is $A(i,j)$.

Theorem 13 (Dodgson's rule). *If $n \geq 2$ and A is an n-by-n matrix,*

$$\det A \det A^{2,2}_{n-1,n-1} = \det A^{1,1}_{n-1,n-1} \det A^{2,2}_{n,n} - \det A^{1,2}_{n-1,n} \det A^{2,1}_{n,n-1}.$$

Proof. We may regard any permutation σ from $\{i,\ldots,j\}$ to $\{k',\ldots,l'\}$ as a perfect matching from $\{i,\ldots,j\}$ to $\{k',\ldots,l'\}$. Let $\mathcal{S}^{i,j}_{k',l'}$ be the set of perfect matchings from $\{i,\ldots,j\}$ to $\{k',\ldots,l'\}$, and let the *weight* of a permutation $\sigma \in \mathcal{S}^{i,j}_{k',l'}$ be

$$w(\sigma) = w^{i,j}_{k',l'}(\sigma) = \operatorname{sign}(\sigma) \prod_{m=i}^{j} A(m, \sigma(m)).$$

Define

$$\mathcal{A} = \mathcal{S}^{1,n}_{1',n'} \times \mathcal{S}^{2,n-1}_{2',(n-1)'}, \quad \mathcal{B} = \mathcal{S}^{1,n-1}_{1,(n-1)'} \times \mathcal{S}^{2,n}_{2',n'}, \quad \text{and } \mathcal{C} = \mathcal{S}^{1,n-1}_{2',n'} \times \mathcal{S}^{2,n}_{1',\ldots,(n-1)'}.$$

Set the *weight* of $(\sigma,\pi) \in \mathcal{A} \cup \mathcal{B}$ to $w(\sigma,\pi) = w(\sigma)w(\pi)$ and the *weight* of $(\sigma,\pi) \in \mathcal{C}$ to $w(\sigma,\pi) = -w(\sigma)w(\pi)$. The left side of Dodgson's rule is the sum of the weights of all the elements of $\mathcal{A}$ and the right side is the sum of the weights of all the elements of $\mathcal{B} \cup \mathcal{C}$.

If $\sigma(a) = b$, we may say that $\{a,b\} = \{b,a\}$ is an edge in the perfect matching corresponding to σ. If E and F are matchings in the complete bipartite graph with bipartition $\{i,\ldots,j\} \cup \{k',\ldots,l'\}$ such that E matches a if and only if F matches a and E matches a' if and only if F matches a', we represent by $\sigma - E + F$ the permutation obtained by subtracting the edges in E and adding the edges in F.

Define a mapping $f : \mathcal{A} \to \mathcal{B} \cup \mathcal{C}$ as follows. Given $(\sigma,\pi) \in \mathcal{A}$, define an alternating sequence $s_1, s_1', s_2, s_2', \ldots, s_r, s_r'$, where $s_1 = n$ and s_r' is $1'$ or n', such that $\sigma(s_i) = s_i'$ and $\pi^{-1}(s_i) = s_{i+1}'$ for all i. This sequence must terminate at either $s_r' = 1'$ or $s_r' = n'$, since $\pi^{-1}(1')$ and $\pi^{-1}(n')$ are undefined. Let $E = \{\{s_1,s_1'\},\ldots,\{s_r,s_r'\}\}$ be a subset of the edges in the perfect matching corresponding to σ and $E' = \{\{s_1',s_2\},\ldots,\{s_{r-1}',s_r\}\}$ be a subset of the edges in the perfect matching corresponding to π. Then $f(\sigma,\pi) = (\sigma - E + E', \pi - E' + E)$. If $s_r = n'$ then $f(\sigma,\pi) \in \mathcal{B}$ and if $s_r = 1'$ then $f(\sigma,\pi) \in \mathcal{C}$.

The mapping f is weight preserving and one-to-one. Unfortunately f is not onto (for then we would be done immediately), but something close enough is true. Call a member of $\mathcal{B} \cup \mathcal{C}$ *bad* if it is not in $f(\mathcal{A})$. We claim that the sum of all the bad members of $\mathcal{B} \cup \mathcal{C}$ is zero. This follows from the fact that there is a natural bijection g between the bad members of $\mathcal{B}$ and those of $\mathcal{C}$ such that $w(S(\sigma,\pi)) = -w(\sigma,\pi)$ for every member $(\sigma,\pi) \in \mathcal{B}$. $\square$

4.1.2 ... and lambda determinants

The recurrence used in Dodgson condensation,

$$A_k(i,j)A_{k+2}(i+1,j+1) = A_{k+1}(i,j)A_{k+1}(i+1,j+1) - A_{k+1}(i+1,j)A_{k+1}(i,j+1),$$

suggests a certain variation. At the very least, a certain generalization occurred to Robbins and Rumsey [13]. If A_n is an n-by-n matrix and A_{n+1} is an $(n+1)$-by-$(n+1)$ matrix, define A_k recursively in terms of A_{k+1} and A_{k+2} by

$$A_k(i,j)A_{k+2}(i+1,j+1) = A_{k+1}(i,j)A_{k+1}(i+1,j+1) + \lambda A_{k+1}(i+1,j)A_{k+1}(i,j+1),$$

setting $A_k(i,j) = 0$ if $A_{k+2}(i+1,j+1) = 0$, and write $A_n \to_\lambda A_{n-1} \to_\lambda \cdots \to_\lambda A_1$ (or just $A_n \to A_{n-1} \to \cdots \to A_1$ in the case $\lambda = 1$). The *λ-determinant of the pair* (A_{n+1}, A_n) is the sole entry of A_1. Define the *λ-determinant of the matrix* A_n to be the λ-determinant of the pair (C_{n+1}, A_n), where C_{n+1} is the $(n+1)$-by-$(n+1)$ matrix each of whose entries is 1.

Note that in this definition, which is a slight variation of the original definition of Robbins and Rumsey, we have dealt with the aforementioned division by zero problem. While the way we have handled it is not appropriate in all situations, it suffices for our purposes.

Also notice that the (-1)-determinant of a matrix is just its determinant, assuming there is no discrepancy caused by our zero division handling. While it is tempting to assume that the 1-determinant of a matrix is its permanent, this is not the case. For instance, C_3 has 1-determinant 8 but permanent 9.

For the purposes of this paper we will be almost solely interested in 1-determinants. Let $\Lambda(A)$ denote the 1-determinant of the matrix A. Here we compute the 1-determinant of the matrix whose (i,j)th entry is the minimum of i and j as above to be 465. We henceforth adopt the common convention that a 1-by-1 matrix may be written as its sole entry.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 4 & 4 \\ 4 & 10 & 12 \\ 4 & 12 & 21 \end{bmatrix} \to \begin{bmatrix} 23 & 44 \\ 44 & 118 \end{bmatrix} \to 465$$

4.2 Kuo condensation

The sequence whose nth term is the number of domino tilings of the order n Aztec diamond begins $2, 8, 64, 1024, 32768$. The following theorem of Elkies, Kuperberg, Larsen, and Propp [7] asserts that this pattern continues indefinitely.

Theorem 14 (The Aztec diamond theorem). *The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.*

Here we will prove this theorem using a technique due to Kuo [10]. His method easily extends to a result on weighted Aztec diamonds crucial to our study of domino tilings of rectangles. It also very closely resembles Dodgson condensation, and for this reason is called *graphical condensation* by Kuo and *Kuo condensation* by everyone else.

The Aztec diamond theorem follows almost immediately from the following lemma.

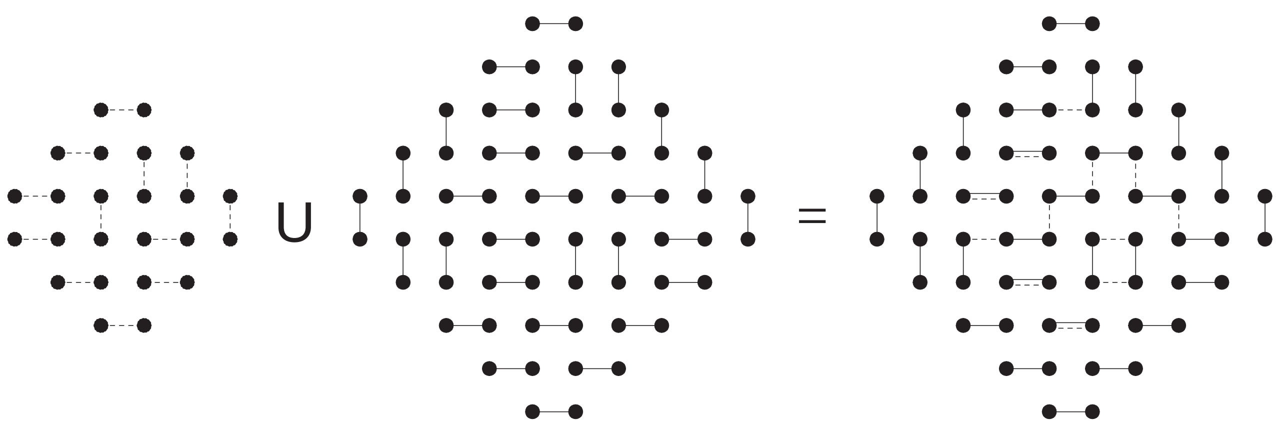
Lemma 15. *Let T_n be the number of domino tilings of the Aztec diamond of order n. Then $T_n T_{n-2} = 2T_{n-1}^2$ for $n \geq 3$.*

Proof. Let $\mathcal{M}_n$ be the set of perfect matchings of the Aztec diamond graph of order n, so that $T_n = |\mathcal{M}_n|$. We will attempt to show that $|\mathcal{M}_n \times \mathcal{M}_{n-2}| = 2|\mathcal{M}_{n-1} \times \mathcal{M}_{n-1}|$ for $n \geq 3$.

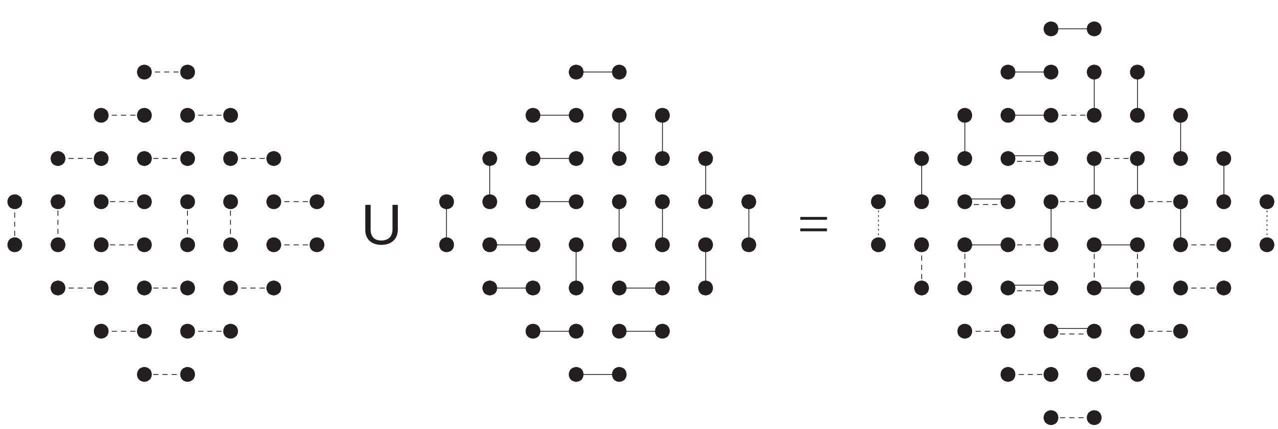
A *doubled Aztec diamond graph of order* n is any multigraph whose vertices are the vertices of the Aztec diamond graph of order n such that the (*inner*) vertices that form an order $n-2$ Aztec diamond graph centered in the order n Aztec diamond graph have order 2 and all other (*outer*) vertices have degree 1.

Every doubled Aztec diamond graph can potentially be partitioned into two perfect matchings of regular Aztec diamond graphs in three ways. First, it can potentially be partitioned into an order n perfect matching and an order $n-2$ perfect matching in a natural way. Second, it can potentially be partitioned into two order $n-1$ perfect matchings by superimposing two $n-1$ Aztec diamond graphs side by side plus the topmost and bottommost edges. Third, it can potentially be partitioned into two order $n-1$ perfect matchings by superimposing two $n-1$ Aztec diamond graphs top to bottom plus the leftmost and rightmost edges.

Below we show one way in which an order 5 doubled Aztec diamond graph can be partitioned into order 3 and order 5 perfect matchings plus two additional edges.



Next we show one way in which the same order 5 doubled Aztec diamond graph can be partitioned into two order 4 perfect matchings.



We claim that every doubled Aztec diamond graph can be partitioned the first way and either the second or third way, but not both, in the same number of ways. Suppose then without loss of generality that a given doubled Aztec diamond graph can be partitioned the second but not the third way. Since an order n and order $n-2$ perfect matchings can be superimposed only one way to form a doubled Aztec diamond graph, but two order $n-1$ perfect matchings can be superimposed in two ways (left and right or right and left), by the claim we have $|\mathcal{M}_n \times \mathcal{M}_{n-2}| = 2|\mathcal{M}_{n-1} \times \mathcal{M}_{n-1}|$, proving the theorem.

To prove our remaining claim, we will show that the number of partitions of a doubled Aztec diamond graph G into two perfect matchings of order n and $n-2$ is equal to the number of partitions of G into two order $n-1$ perfect matchings (and two line segments). In particular, we demonstrate that this common value is 2^k, where k is the number of cycles in G.

The edges of G form cycles, lattice paths of length greater than one whose ends are outer vertices, and single and double edges. Since G is bipartite, all cycles have even length, and all cycles are contained in the inner vertices. Each cycle can be partitioned so that alternating edges are put in the same part of the partition, so that there are two ways to decide which half of the cycle goes in which perfect matching. All double edges in G are split between the subgraphs. Single edges are also placed in an obvious way. It remains to show that the edges lying in paths of length greater than one must be partitioned uniquely.

In any doubled Aztec diamond graph, one vertex in the row next to each (diagonal) side of the diamond is matched to a vertex not on that side. Let x, y, z, and w be these four distinguished vertices, arranged clockwise with x nearest the upper left side of the doubled Aztec diamond graph. Either there are paths from x to y and from z to w or there are paths from x to w and z to y (if there were paths from x to w and y to z, these paths would intersect at one of the inner vertices of G, causing it to have degree more than two).

Notice that these are the only two paths of length more than one in G (in fact, one or both of them may simply be edges). Since $\{x, z\}$ and $\{y, w\}$ must be subsets of different parts of any bipartition of G, any x–y, z–w, x–w, or z–y path is odd, so the segments from both ends of any path in G must belong to the same part in any partition of G into perfect matchings.

When G is partitioned into perfect matchings of orders n and $n-2$, the ending segments of the paths must be placed in the order n perfect matching before the rest of the partition is determined. From our above arguments, such a partition always exists.

Next we show that G can be partitioned into two perfect matchings of order $n-1$ along with two additional side edges. As stated before there are two ways in which this partition can occur: the centers of the perfect matchings can either lie above and below each other or to the right and left of each other. If there are paths in G from x to y and from z to w, the perfect matchings must lie above and below each other, and if there are paths from x to w and y to z, the perfect matchings must lie to the right and left of each other. Thus any given doubled Aztec diamond graph may be partitioned into order $n-1$ perfect matchings in exactly one of these two ways, since the partition of the paths is uniquely determined. From our above arguments, such a partition always exists. □

Using the fact that the number of domino tilings of the order 1 and 2 Aztec diamonds are 2 and 8, respectively, as initial conditions in conjunction with the previous lemma, Theorem 14 can easily be proven by induction.

4.2.1 Weighted Aztec diamonds

Kuo [10] extends the same idea used in his proof of the Aztec diamond theorem to *weighted Aztec diamonds*, diamonds in which each possible domino (each pair of squares that share an edge) has a real number associated with it. The dual graph of a weighted Aztec diamond is just an edge weighted graph, where the edge that lies in the union of two squares inherits the weight of that pair of squares.

In a weighted multigraph G, the *weight* of a subgraph H is the product of the weights of the edges in H. The *weighted sum* $W(G)$ of G is the sum of the weights of all perfect matchings of G. This standard terminology from weighted graphs carries over to weighted Aztec diamonds. In a weighted Aztec diamond D, the weight of collection of dominos is the product of the weights of the dominos, and the weighted sum of D is the sum of the weights of all domino tilings of D.

Given an weighted Aztec diamond D of order n, define D_{top}, D_{bottom}, D_{left}, and D_{right} to be the upper, lower, left, and right order $n-1$ weighted Aztec sub-diamonds in D, respectively, and let D_{middle} be the centered order $n-2$ weighted Aztec sub-diamond. Let t, b, l, and r be the weights of the topmost, bottommost, leftmost, and rightmost possible dominos in D, respectively.

Theorem 16 (The weighted Aztec diamond theorem). *For any weighted Aztec diamond D of order at least three,*

$$W(D) \cdot W(D_{\text{middle}}) = l \cdot r \cdot W(D_{\text{top}}) \cdot W(D_{\text{bottom}}) + t \cdot b \cdot W(D_{\text{left}}) \cdot W(D_{\text{right}}).$$

Proof. Recall that a doubled Aztec diamond graph G of order n can be decomposed into subgraphs in two of three ways. First, it can be partitioned into an order n perfect matching and an order $n-2$ perfect matching in a natural way. Second, it can potentially be partitioned into two order $n-1$ perfect matchings by superimposing two $n-1$ Aztec diamond graphs side by side plus the topmost and bottommost edges. Third, it can potentially be partitioned into two order $n-1$ perfect matchings by superimposing two $n-1$ Aztec diamond graphs top to bottom plus the leftmost and rightmost edges. As we know, G can be decomposed in the first way and exactly one of the second or third ways. The number of possible decompositions in either way is 2^k, where $k(G)$ is the number of cycles in G. Since each edge in G becomes part of exactly one of the subgraphs, the product of the weights of the subgraphs equals the weight of G.

Notice that

$$W(D) \cdot W(D_{\text{middle}}) = \sum_G 2^{k(G)} w(G),$$

where G ranges over all doubled Aztec diamond graphs of order n whose edge weights agree with the domino weights of D. Each term in the sum is the weight of G times the number of ways to partition G via the first method above, and each partition is accounted for in the left-hand side. Similarly,

$$l \cdot r \cdot W(D_{\text{top}}) \cdot W(D_{\text{bottom}}) + t \cdot b \cdot W(D_{\text{left}}) \cdot W(D_{\text{right}}) = \sum_G 2^{k(G)} w(G),$$

a common quantity. ∎

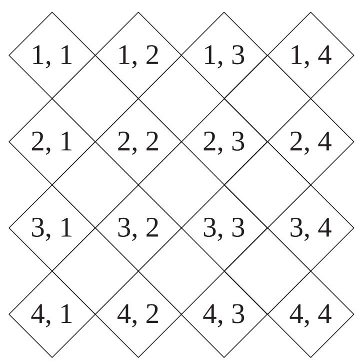
4.3 Kuo condensation and domino tilings of rectangles

Henceforth we refer to a weighted Aztec diamond graph with weights from the set S as an *S-WAD*. We will be using $\{0,1\}$-WADs almost exclusively. It is also useful for us to consider WADs as being tilted 45 degrees. In this context, for a WAD D let D_{ne}, D_{nw}, D_{se}, and D_{sw} be the northeast, northwest, southeast, and southwest order $n-1$ weighted Aztec sub-diamonds of D, respectively, and let D_{middle} be the inner order $n-2$ weighted Aztec sub-diamond. Let w_{ne}, w_{nw}, w_{se}, and w_{sw} be the weights of the northeast, northwest, southeast, and southwest edges in D, respectively. We can restate Kuo's weighted Aztec diamond theorem in this context as

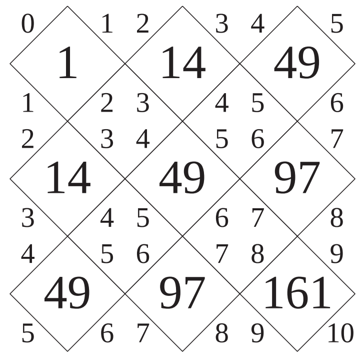
$$W(D) \cdot W(D_{\text{middle}}) = w_{\text{sw}} \cdot w_{\text{ne}} \cdot W(D_{\text{nw}}) \cdot W(D_{\text{se}}) + w_{\text{nw}} \cdot w_{\text{se}} \cdot W(D_{\text{sw}}) \cdot W(D_{\text{ne}}).$$

The *faces* of a WAD are the bounded faces of the WAD viewed as a graph. Given a face F of a WAD, denote by F_{ne}, F_{nw}, F_{se}, and F_{sw} the weights of the northeast, northwest, southeast, and southwest edges bordering F, respectively. The *edge factor of F* is defined to be $F_{\text{ne}} \cdot F_{\text{sw}} + F_{\text{nw}} \cdot F_{\text{se}}$.

Call two faces F, G of a WAD D *diagonal* (in which case we write $F|G$) if they share a common vertex but not an edge, and *diagonally equivalent* if either $F|G$ or there is a finite sequence $F_1, F_2, \ldots, F_n$ of faces in D such that $F_i|F_{i+1}$ for all i, $F|F_1$, and $F_n|G$. Call a face a *major face* if it is vertex connected to the northwest face. The centers of the major faces of D form a finite square lattice (actually, an n-by-n square lattice), so we may coordinatize the major faces in a natural way. Denote by $D(i,j)$ the face in row i, column j, where rows are labelled 1 through n from top to bottom and columns are labelled 1 through n from left to right. Here we depict this labelling scheme for an order 4 Aztec diamond graph.



The *edge factor matrix* of D, denoted $M(D)$, is the n-by-n square matrix whose (i,j)th entry is equal to the edge factor of $D(i,j)$. Below we depict an order 3 WAD with edge factors written in a larger font than edge weightings.



The edge factor matrix corresponding to this WAD is

$$\begin{bmatrix} 1 & 14 & 49 \\ 14 & 49 & 97 \\ 49 & 97 & 161 \end{bmatrix}.$$

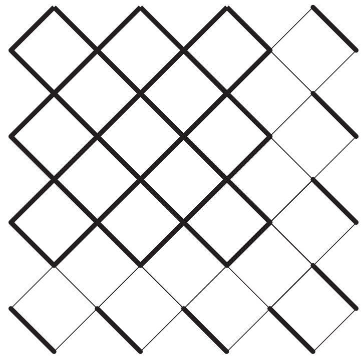
There is a striking similarity between Kuo condensation and the lambda determinant recurrence, and now we have a way of encoding information about a WAD (whose number of tilings we compute using Kuo condensation) in a square matrix (whose λ-determinant we also know how to compute for every λ). In fact, the formula for edge factors very closely resembles the recurrence used to compute 1-determinants. We will see that there is in fact a fundamental connection between the weighted Aztec diamond theorem and 1-determinants. After all, we did go to all the trouble of tilting the Aztec diamond graph 45 degrees and reformulating our notation.

Before we continue, let us try a couple of experiments. Consider the order 4 WAD with all edges weighted 1. Notice that $W(D)$ is just the number of perfect matchings of the order 4 Aztec diamond graph, which is 1024. The edge factor matrix of our WAD is the 3-by-3 matrix each of whose entries is 2 ($= 1 \cdot 1 + 1 \cdot 1$). The 1-determinant of this matrix is 1024, since

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix} \to 1024.$$

It is not hard to see that this trick will work for Aztec diamonds of every order.

Now consider the order 4 WAD weighted according to the following scheme: each edge the northwest order 3 sub-diamond is weighted 1, each edge in the only perfect matching of the rest of the graph is weighted 1, and the remaining edges are weighted 0. We depict this Aztec diamond by drawing the edges weighted 1 in bold.

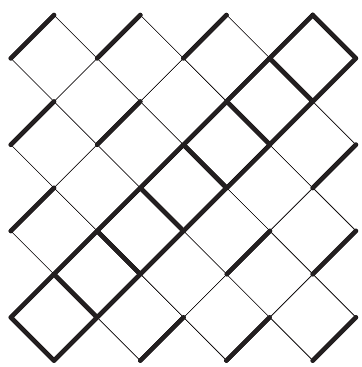


Notice that the number of perfect matchings of the subgraph consisting of the edges whose weights are equal to 1 is equal to the number of perfect matchings of the order 3 Aztec diamond graph, which is 64. Lo and behold, the edge factor matrix of this WAD has 1-determinant 64, since

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 8 & 8 & 0 \\ 8 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 32 & 0 \\ 0 & 16 \end{bmatrix} \to 64.$$

This trick also works for Aztec diamonds of any size.

Let us try one more experiment before continuing. Consider the order 4 WAD weighted according to the following scheme: each edge in the 2-by-8 rectangular grid-graph stretching from the southwest to northeast corner is weighted 1, each edge in the only perfect matching of the rest of the graph is weighted 1, and the remaining edges are weighted 0. Again we depict this Aztec diamond by drawing the edges weighted 1 in bold.



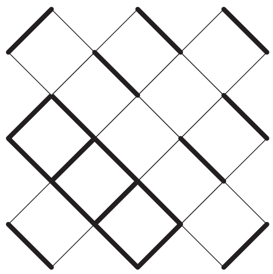
As above, the number of perfect matchings of the subgraph consisting of the edges whose weights are equal to 1 is equal to the number of perfect matchings of the 2-by-8 rectangular grid-graph, which is 34 (recall that the sequence whose nth term is the number of domino tilings of the 2-by-n rectangle, or equivalently the nth Fibonacci number F_n, begins 1, 2, 3, 5, 8, 13, 21, 34). The 1-determinant of our edge factor matrix matches again.

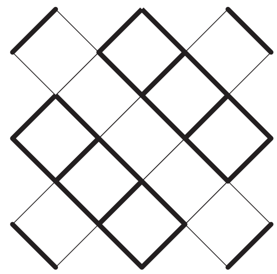
$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 13 \\ 13 & 1 \end{bmatrix} \to 34.$$

This generalizes to larger Aztec diamonds.

(On a very tangential side note, the fact that this generalization works provides a very quick proof that, for all $n \geq 5$, $F_n \cdot F_{n-4} = F_{n-2}^2 + 1$.)

These experiments seem at first to indicate that for a $\{0,1\}$-WAD D, the number of perfect matchings of the subgraph of D formed from all edges of weight 1 is equal to the 1-determinant of the edge factor matrix of D. While this is on the right track, and something close to it is true, it is not entirely accurate. The problem arises partially from the inherent limitation of the lambda-determinant recurrence from handling zeros, but that is not the only issue. Here we depict two examples in which the problem arises. In the one on the left, the 1-determinant is 0 where the number of perfect matchings is 5. In the one on the right, the 1-determinant is 45 where the number of perfect matchings is 25 ($= 5 \cdot 5$).





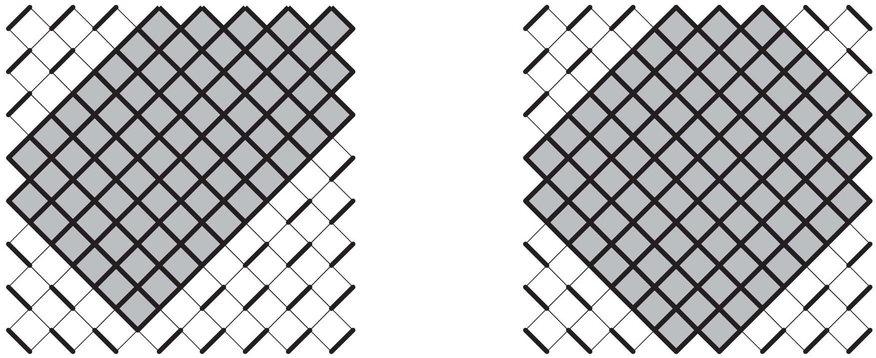
While the precise classification of weightings for which the trick works has not been completed, the present author [4] has found a relatively general class of weightings for which it does work, in the form of a general class of graphs whose number of perfect matchings can be determined in this way.

If a $\{0,1\}$-WAD is weighted all 1 within some subgraph and in a *unique* 1-0 brickwork pattern outside this subgraph as above, we say that the subgraph has been *embedded* in the WAD, and we refer to the resulting WAD as the *embedding*. We define an *Aztec octagon graph* to be a grid-graph obtained from an Aztec diamond graph after removing zero or more corners by removing all vertices above or below lines with slope ± 1.

Theorem 17. *The number of perfect matchings of an Aztec octagon graph equals the 1-determinant of the edge factor matrix of any embedding of the Aztec octagon graph into an Aztec diamond graph.*

This comes almost immediately from the weighted Aztec diamond theorem.

Two sample Aztec octagon graphs are shown below embedded in the Aztec diamond graphs from which they were obtained. For ease of viewing, the faces of the Aztec octagon graphs are shaded. Aztec octagons derive their name from the fact that they can have up to eight "sides," as does the one shown here on the right.

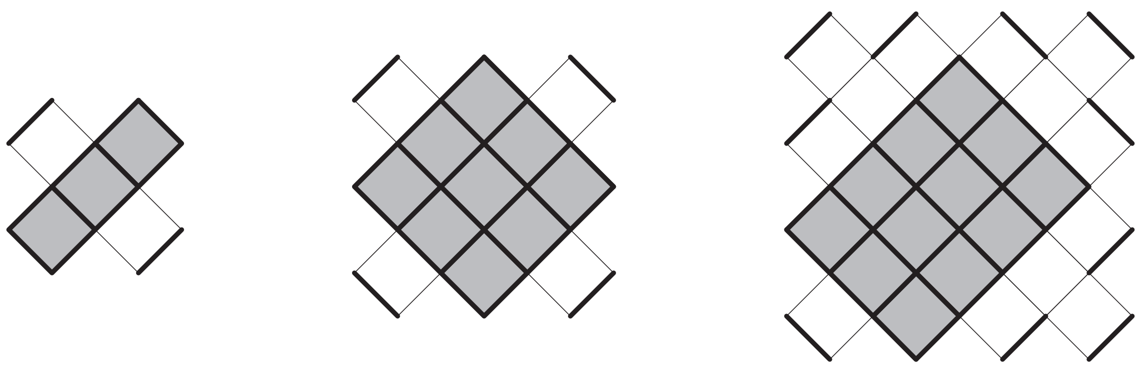


4.3.1 The part about domino tilings of rectangles

At this point, there is one trivial observation that will help us immensely: *every rectangular grid-graph is an Aztec octagon graph.* Therefore, we can compute the number of domino tilings of a rectangle as the 1-determinant of a matrix. In particular,

Proposition 18. *The number of domino tilings of an m-by-n rectangle is equal to the 1-determinant of an N-by-N matrix, where $N = \lfloor (n+m-1)/2 \rfloor$.*

To illustrate, we will quickly give three examples. Say we want to compute the numbers of domino tilings of the 2-by-4, 4-by-4, and 4-by-5 rectangles. To do this, we first embed their dual graphs in weighted Aztec diamond graphs.



We then compute the 1-determinants of the edge factor matrices of the embeddings.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \to 5$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \to 36$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 6 & 8 & 4 \\ 6 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 16 & 12 \\ 42 & 16 \end{bmatrix} \to 95$$

The numbers of domino tilings of 2-by-4, 4-by-4, and 4-by-5 rectangles are 5, 36, and 95, respectively.

4.3.2 Edge weights versus face weights

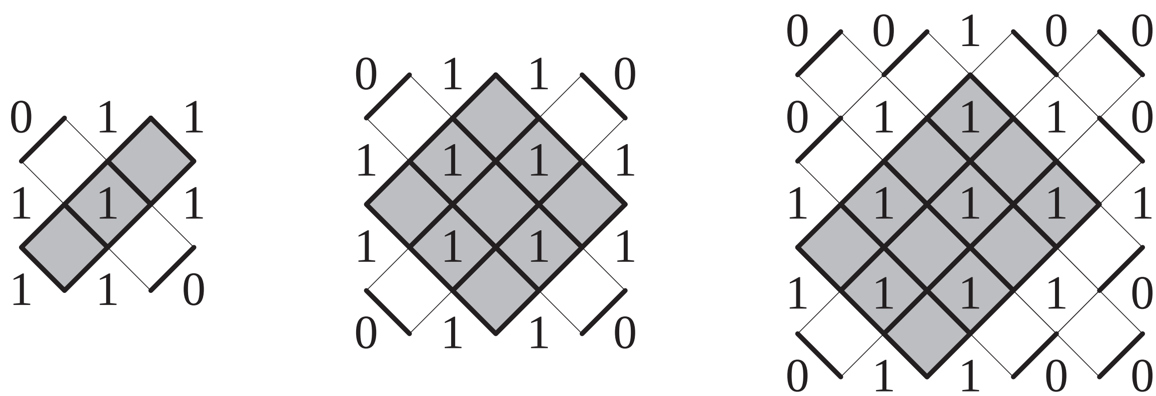
There is an alternate, but just slightly different, way to compute the number of perfect matchings of an Aztec octagon graph, also from [4]. Until this point we have considered edge factors of major faces. Given an order n WAD D, the *minor faces* of D are the (bounded) faces of D that are not major faces. Like with the major faces, we may coordinatize the minor faces as $D'(i,j)$, where i and j each range from 1 to $n-1$. It is helpful to add some additional faces, called the *outer minor faces* of D, that continue the lattice of minor faces one face in each direction. The *total minor faces*, which consist of both the minor and outer minor faces, can be coordinatized as $D'(i,j)$, where i and j each range from 0 to n, naturally respecting the existing labelling of the minor faces.

The *face weight* of a total minor face F is 1 if no edges in the border of F appear in any perfect matching of D with nonzero weight, 0 if the number of edges in the border of F that appear in a perfect matching of D is always 1 (that is, even if the edges that appear are different, their number is the same), and 1 otherwise. The *face weight matrix* of D is the $(n+1)$-by-$(n+1)$ matrix whose (i,j)th entry is the face weight of $F(i-1,j-1)$.

Theorem 19. *The number of perfect matchings of an Aztec octagon graph equals the 1-determinant of the face weight matrix of any embedding of the Aztec octagon graph into an Aztec diamond graph.*

The proof of this theorem is immediate upon noticing that the intermediate n-by-n matrix used in taking the 1-determinant of an $(n+1)$-by-$(n+1)$ edge factor matrix of a WAD is precisely the edge factor matrix of the WAD (and noticing that division by zero will never occur in the case of Aztec octagons).

To illustrate, we compute the edge factor matrices of the WADs illustrated above. Here we redraw them with face weights superimposed.



Now we begin finding the 1-determinants of the associated face weight matrices.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \to \cdots$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \to \cdots$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \to \cdots$$

We realize the matrices obtained from the first step of the 1-determinant recurrence are the edge factor matrices of the WADs, as promised in our one-line proof of Theorem 19.

Alternately, we could view the face weight matrix as a step *backward* in the 1-determinant recurrence. This point of view will come in very useful in the next section, especially if the reader wishes to ignore the details of face weightings and stick to edge factors, as is recommended. The important detail that makes this viewpoint feasible is that, except for in regions of 0s we encounter in the corners of matrices, *there is a unique way to take a backward step in the 1-determinant recurrence* in all the matrices we will henceforth encounter. Try it! The results in the following section were originally obtained solely through the edge weighting picture in this way.

4.4 Two related conjectures

In this section we present a pair of conjectures of the present author [3]. We will consider domino tilings of even-by-even squares once again. In particular, we will be interested in what happens if we embed a $2n$-by-$2n$ square grid-graph in the center of weighted Aztec diamond graphs of orders $2n-1$ and $2n$. Let $E_{n,k}$ and $F_{n,k}$ be the edge factor and face weight matrices, respectively, of the embedding of the $2n$-by-$2n$ square grid-graph in the center of the order k Aztec diamond. By the results of the last section, $\Lambda(E_{n,2n-1}) = \Lambda(E_{n,2n}) = \Lambda(F_{n,2n-1})$. Also notice $F_{n,2n-1}$ and $E_{n,2n}$ are both $2n$-by-$2n$ matrices. Further, $F_{n,2n}$ can be obtained from $E_{n,2n-1}$ by going backward one step back in the 1-determinant recursion. Thus we can determine $E_{n,2n-1}$, $E_{n,2n}$, and $F_{n,2n-1}$ solely from the edge factor picture.

Let us examine a few small cases for the sake of motivation.

$$F_{1,1} = E_{1,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_{2,3} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad E_{2,4} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$F_{3,5} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{3,6} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$F_{4,7} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{4,8} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For $n = 2, 3$, and 4, these pairs of matrices are equal except in certain entries that are each 2 instead of 1. This suggests that we set those entries to a parameter and investigate the behavior of the resulting matrix when we vary the parameter beyond simply 1 and 2.

We define $M_{n,n}(a)$ to be the result of replacing the entries for which $F_{n,2n-1}$ and $E_{n,2n}$ differ by the parameter a. We will make this definition precise below. Let us first list a few small examples.

$$M_{1,1}(a) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad M_{2,2}(a) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & a & a & 1 \\ 1 & a & a & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M_{3,3}(a) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & a & a & 1 & 0 \\ 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & 1 \\ 0 & 1 & a & a & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad M_{4,4}(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & a & 1 & 0 & 0 \\ 0 & 1 & a & a & a & a & 1 & 0 \\ 1 & a & a & a & a & a & a & 1 \\ 1 & a & a & a & a & a & a & 1 \\ 0 & 1 & a & a & a & a & 1 & 0 \\ 0 & 0 & a & a & a & a & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Remember throughout that $M_{n,n}(a)$ is defined so that $M_{n,n}(1) = F_{n,2n-1}$ and $M_{n,n}(2) = E_{n,2n}$ for any $n \geq 1$, so that $\Lambda(M_{n,n}(1) = \Lambda(M_{n,n}(2))$ because these quantities both represent the number of domino tilings of the $2n$-by-$2n$ square.

Is $\Lambda(M_{n,n}(a))$ constant for all real a? One can easily see this is not true for $n \geq 2$, but is there some other well-known function of a to which it is equal? If not, can we at least determine some of its properties? After formalizing the definition of $M_{n,n}(a)$, we conjecture a functional equation for $\Lambda(M_{n,n}(a))$.

4.4.1 Stating the conjectures

To be precise, $M_{n,n}(a)$ is the $2n$-by-$2n$ square matrix defined as follows. Divide a $2n$-by-$2n$ matrix into equal-size quarters by slicing it vertically (with horizontal line H) and horizontally (with vertical line V). Define the bottom left quarter submatrix as the upper triangular matrix such that each diagonal entry is 1 and each entry above the diagonal is a. Rotate this quarter matrix by 90, 180, and 270 degrees about the point of intersection of H and V to define the three other quarter submatrices.

Let $m_{n,n}(a) = \Lambda(M_{n,n}(a))$.

Conjecture 20. *For all positive integers n, $m_{n,n}(a) = (2/a)^n p_n^2(a)$, where $p_n(a)$ is an integer polynomial of degree n such that $p_n(1)$ is odd.*

Notice that this would strengthen Pachter's theorem. In our new notation, Pachter's theorem states simply that $m_{n,n}(1)/2^n$ is an odd square.

We can extend the definitions for $M_{n,n}(a)$ and $m_{n,n}(a)$ in a very natural way to $M_{k,n}(a)$ and $m_{k,n}(a)$ for any $2k$-by-$2n$ rectangle.

Conjecture 21. *For all positive integers k and n, we have the functional equation $m_{k,n}(a) = m_{k,n}(2/a)$.*

These two conjectures have been tested extensively. However, a general method for solving the conjectures has so far proven elusive. In the next two sections, we prove some special cases of the conjectures. In particular, we prove Conjecture 21 for all rectangles of width 2 or 4.

For rectangles of fixed width, notice that the conjecture is a one-dimensional problem. For rectangles of width $2k$, in the matrix $M_{k,n}(a)$ all entries equal to a lie along there are k adjacent diagonals. One-dimensional problems are much more tractable in combinatorics than two-dimensional problems.

4.4.2 The case of 2-by-even rectangles

In this section we attempt to prove Conjecture 21 for the case of 2-by-even rectangles. We wish to show that for all positive integers n, the functional equation $m_{1,n}(a) = m_{1,n}(2/a)$ holds. We will do this by finding the generating function in terms of n of $m_{1,n}(a)$ and demonstrating that the generating function obeys the same functional equation. We find the generating function in a few steps. First, we find a recurrence in terms of n for $m_{1,n}(a)$ (dependent on other recurrences). We then convert this into a linear recurrence (dependent on other linear recurrences), which immediately allows us to find the generating function for $m_{1,n}(a)$.

For the purposes of this section, let $m_n(a) = m_{1,n}(a)$, so that $m_n(1) = m_n(2)$ is the number of domino tilings of the 2-by-$2n$ rectangle, and let $M_n(a) = M_{1,n}(a)$.

$$M_1(a) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad M_2(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_3(a) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & a & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & a & 1 \\ 0 & 1 & a & 1 & 0 \\ 1 & a & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad M_5(a) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & a & 1 \\ 0 & 0 & 1 & a & 1 & 0 \\ 0 & 1 & a & 1 & 0 & 0 \\ 1 & a & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first few values of $m_n(a)$ are

$$m_1(a) = 2, \quad m_2(a) = a + 2 + \frac{2}{a}, \quad \text{and} \quad m_3(a) = a^2 + 2a + 2 + \frac{4}{a} + \frac{4}{a^2}.$$

Notice that

$$m_1(1) = m_1(2) = 2, \quad m_2(1) = m_2(2) = 5, \quad \text{and} \quad m_3(1) = m_3(2) = 13,$$

agreeing with the numbers of tilings of 2-by-2, 2-by-4, and 2-by-6 rectangles, respectively, and also that $m_i(a) = m_i(2/a)$ for $i = 1, 2, 3$. Our values were obtained by implementing the 1-determinant recurrence, as demonstrated below for the sake of clarity.

$$M_1(a) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to 2$$

$$M_2(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & a+1 \\ a+1 & 1 \end{bmatrix} \to a + 2 + \frac{2}{a}$$

$$\begin{aligned} M_3(a) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & a & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} &\to \begin{bmatrix} 0 & 1 & a+1 \\ 1 & a^2+1 & 1 \\ a+1 & 1 & 0 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & a^2+a+1+\frac{2}{a} \\ a^2+a+1+\frac{2}{a} & 1 \end{bmatrix} \\ &\to a^2 + 2a + 2 + \frac{4}{a} + \frac{4}{a^2} \end{aligned}$$

To help find a recurrence for $M_n(a)$, we define a few other types of matrices. Let $S_n(a)$ be the middle n-by-n submatrix of $M_{n+2}(a)$, and let $s_n = s_n(a) = \Lambda(S_n(a))$. In other words, $S_n(a)$ is the matrix obtained from $M_n(a)$ by replacing its upper-right and lower-left entries with a. Let $T_n(a)$ be the upper right n-by-n submatrix of $M_{n+1}(a)$, and let $t_n = t_n(a) = \Lambda(T_n(a))$. In other words, $T_n(a)$ is the matrix obtained from $M_n(a)$ by replacing its lower-left entry with a. Note that if $T'_n(a)$ is lower left n-by-n submatrix of $M_{n+1}(a)$, then $\Lambda(T'_n(a)) = \Lambda(T_n(a)) = t_n(a)$ by symmetry.

Now we may find simultaneous recurrences for the sequences s_n, t_n, and $m_n, n \geq 1$.

Lemma 22. *The sequences s_n, t_n, and $m_n, n \geq 1$ satisfy*

$$\begin{aligned} &(1) \qquad m_n s_{n-2} = t_{n-1}^2 + 1, \\ &(2) \qquad s_n s_{n-2} = s_{n-1}^2 + 1, \text{ and} \\ &(3) \qquad t_n s_{n-2} = t_{n-1} s_{n-1} + 1 \end{aligned}$$

with initial conditions

$$\begin{array}{rcl} m_0 = 1, & & m_1 = 2, \\ s_0 = a, & & s_1 = a^2 + 1, \\ t_0 = 1, & \text{and} & t_1 = a + 1. \end{array}$$

Proof. The proof is immediate from the definitions of the sequences upon noting that any upper (or lower) triangular matrix with each diagonal element 1 has 1-determinant 1.

The perspective in the following alternate proof may be useful as well. We have $M_1(a) = 1 = m_1, M_2(a) \to 2 = m_2$,

$$M_3(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_2 \\ t_2 & 1 \end{bmatrix} \to \frac{t_2^2 + 1}{s_1} = s_3,$$

$$M_4(a) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & a & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & t_2 \\ 1 & s_2 & 1 \\ t_2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_3 \\ t_3 & 1 \end{bmatrix} \to \frac{t_3^2 + 1}{s_2} = t_4,$$

and so forth, demonstrating Equation (1). Similarly, $S_1(a) = a = s_1, S_2(a) \to a^2 + 1 = s_2$,

$$S_3(a) = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & s_2 \\ s_2 & 1 \end{bmatrix} \to \frac{s_2^2 + 1}{s_1} = s_3,$$

and so forth, demonstrating Equation (2). Finally, $T_1(a) = 1 = t_1, T_2(a) \to a + 1 = t_2$,

$$T_3(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_2 \\ s_2 & 1 \end{bmatrix} \to \frac{s_2 t_2 + 1}{s_1} = t_3,$$

and so forth, demonstrating Equation (3). □

Now that we have recurrences for the sequences s_n, t_n, and $m_n, n \geq 1$, we can turn these into linear recurrences by an elegant observation due to the present author's brother [1].

Lemma 23. *The sequences $s_n, t_n, m_n, n \geq 1$ satisfy the three linear recurrence relations*

$$\begin{aligned} s_n - \left(a + \frac{2}{a}\right) s_{n-1} + s_{n-2} &= 0, \\ s_n - t_n - (a-1)s_{n-1} &= 0, \text{ and} \\ m_n - t_n + (a-1)t_{n-1} &= 0. \end{aligned}$$

Proof. Taking equation (2) minus equation (3) gives $s_{n-2}(s_n - t_n) = s_{n-1}(s_{n-1} - t_{n-1})$, so the quantity $\frac{s_n - t_n}{s_{n-1}}$ has the same value for all n; this value is determined to be $a - 1$ by plugging in initial conditions. Taking equation (3) minus equation (1) gives $\frac{t_n - m_n}{t_{n-1}} = \frac{s_{n-1} - t_{n-1}}{s_{n-2}} = a - 1$ for all n as well. □

Alternatively, if we assume s_n obeys a linear recurrence relation of order two, we can find it (for approaching things this way, finding equations (1), (2), and (3) is unnecessary). Setting $s_3 = cs_2 + ds_1, s_4 = cs_3 + ds_2$ and solving the system, we find $c = a + \frac{2}{a}, d = -1$. The sequence s_n can easily be shown to obey this order-two linear recurrence relation via straightforward induction or by viewing s_n as a linear combination $a\alpha^n + b\beta^n$ of exponentials and straightforwardly finding constraints on α and β. While the method of proof above is far nicer, this method generalizes more easily.

Now define generating functions

$$\begin{aligned} S(a,x) &= s_1(a) + s_2(a)x + s_3(a)x^2 + \cdots = \sum_{n=0}^{\infty} s_{n+1}(a)x^n, \\ T(a,x) &= \sum_{n=0}^{\infty} t_{n+1}(a)x^n, \text{ and} \\ M(a,x) &= \sum_{n=0}^{\infty} m_{n+1}(a)x^n. \end{aligned}$$

By the above linear recurrences and initial conditions,

$$\begin{aligned} S(x) - \left(a + \frac{2}{a}\right) xS(x) + x^2 S(x) &= a - x, \\ S(x) - T(x) - (a-1)xS(x) &= a - 1, \text{ and} \\ M(x) - T(x) + (a-1)xT(x) &= 0. \end{aligned}$$

Solving these equations for the generating functions yields

$$M(a,x) = \frac{(a - 2a + ax)(1 - x - ax)}{a - 2x - a^2x + ax^2},$$

which obeys the functional equation $M(a,x) = M(2/a, x)$. This completes the proof of Conjecture 21 for 2-by-even rectangles.

4.4.3 The case of 4-by-even rectangles

Here we prove Conjecture 21 for 4-by-even rectangles. For the purposes of this section, let $M_n(a) = M_{2,n}(a)$ and $m_n(a) = m_{2,n}(a)$, since we are examining 4-by-$2n$ rectangles.

If we assume m_n obeys a linear recurrence of order at most 10 (arbitrary and much too large), we can solve for this recurrence in the same way we solved for the recurrence of s_n in the previous section. The result can be proved inductively. The sequence m_n can thus be found to obey a linear recurrence relation of order 5, and thus we can solve for its generating function,

$$\frac{(1-x)(a^2-(6a+2a^2+3a^3)x+(4+4a+14a^2+2a^3+a^4)x^2-(6a+2a^2+3a^3)x^3+a^2x^4)}{x^2(36a^2-(94a+10a^2+47a^3)x+(40+22a+86a^2+11a^3+10a^4)x^2-(8+26a+10a^2+13a^3+2a^4)x^3+(2a+2a^2+a^3)x^4)}.$$

This obeys the desired functional equation, completing the proof of Conjecture 21 for 4-by-even rectangles.

However, we can also prove the 4-by-even case in the same way we proved the 2-by-even case. Let us first examine what $M_n(a)$ looks like.

$$M_1(a) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad M_2(a) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & a & a & 1 \\ 1 & a & a & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad M_3(a) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & a & a & 1 \\ 1 & a & a & a & 1 \\ 1 & a & a & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & a & a & 1 \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ 1 & a & a & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad M_5(a) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & a & a & 1 \\ 0 & 0 & 1 & a & a & a & 1 \\ 0 & 1 & a & a & a & 1 & 0 \\ 1 & a & a & a & 1 & 0 & 0 \\ 1 & a & a & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We proceed similarly as in the last section by defining some other sequences of matrices that will help us find a recurrence for $M_n(a)$. Let $S_n(a)$ be the inner n-by-n submatrix of $M_{n+2}(a)$, $C_n(a)$ the bottom left n-by-n submatrix of $M_{n+1}(a)$, $Q_n(a)$ the bottom right n-by-n submatrix of $M_{n+1}(a)$, $B_n(a)$ the bottom right n-by-n submatrix of $S_{n+1}(a)$, and $D_n(a)$ the bottom right n-by-n matrix of $C_{n+1}(a)$. It is easy to see that these matrices will be sufficient by symmetry of the 1-determinant.

$$S_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & a & a \\ 0 & 0 & 1 & a & a & a \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ a & a & a & 1 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 \end{bmatrix} \qquad C_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & a & a \\ 0 & 0 & 1 & a & a & a \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ 1 & a & a & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & 1 \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ a & a & a & 1 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & a \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ a & a & a & 1 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & a \\ 0 & 1 & a & a & a & 1 \\ 1 & a & a & a & 1 & 0 \\ a & a & a & 1 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As before let $s_n(a) = \Lambda(S_n(a))$, $c_n(a) = \Lambda(C_n(a))$, $q_n(a) = \Lambda(Q_n(a))$, $b_n(a) = \Lambda(B_n(a))$, and $d_n(a) = \Lambda(D_n(a))$. The following lemma is immediate from these definitions and the fact that the 1-determinant of a matrix is 1 if it is upper triangular with each entry on the diagonal equal to 1.

Lemma 24. *The sequences m_n, s_n, t_n, r_n, c_n, and $d_n, n \geq 1$ satisfy*

$$\begin{aligned}
m_n s_{n-2} &= q_{n-1}^2 + c_{n-1}^2, \\
s_n s_{n-2} &= b_{n-1}^2 + s_{n-1}^2, \\
c_n s_{n-2} &= d_{n-1}^2 + c_{n-1} s_{n-1}, \\
q_n b_{n-2} &= d_{n-1}^2 + s_{n-1}, \\
b_n b_{n-2} &= b_{n-1}^2 + s_{n-1}, \text{ and} \\
d_n b_{n-2} &= b_{n-1} d_{n-1} + s_{n-1}
\end{aligned}$$

with initial conditions

$$\begin{aligned}
m_1 &= \frac{a^2+2a+2}{a}, & m_2 &= 36, \\
s_1 &= 5a^3 + 2a^2 + a, & s_2 &= 13a^4 + 12a^3 + 12a^2 + 8a + 5, \\
c_1 &= 6a, & c_2 &= 17a^2 + 14a + 11, \\
q_1 &= 6a, & q_2 &= 4a^2 = 17a + 1, \\
b_1 &= a^3 + 2a^2 + 3a, & b_2 &= a^4 + 3a^3 + 7a^2 + 10a + 1, \\
d_1 &= 2a^2 + 4a, \quad \text{and} & d_2 &= a^4 + 3a^3 + 7a^2 + 10a + 1.
\end{aligned}$$

These recurrences and initial conditions can be used to prove linear recurrences as in the 2-by-even case.

4.4.4 The case of 6-by-even rectangles

The proof of Conjecture 21 for 6-by-even rectangles can also been checked as above. The purpose of this section, however, is to write out recurrences for the 6-by-even case to give the readers more data to investigate. For the purposes of this section, let $M_n(a) = M_{3,n}(a)$ and $m_n(a) = m_{3,n}(a)$, since we are examining 6-by-$2n$ rectangles.

$$M_1(a) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad M_2(a) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & a & a & 1 & 0 \\ 1 & a & a & a & 1 \\ 0 & 1 & a & a & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M_3(a) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & a & a & 1 & 0 \\ 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & 1 \\ 0 & 1 & a & a & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad M_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ 1 & a & a & a & a & 1 & 0 \\ 0 & 1 & a & a & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_5(a) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 0 & 1 & a & a & a & a & 1 \\ 0 & 1 & a & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 & 0 \\ 1 & a & a & a & a & 1 & 0 & 0 \\ 0 & 1 & a & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_6(a) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 0 & 0 & 1 & a & a & a & a & 1 \\ 0 & 0 & 1 & a & a & a & a & a & 1 \\ 0 & 1 & a & a & a & a & a & 1 & 0 \\ 1 & a & a & a & a & a & 1 & 0 & 0 \\ 1 & a & a & a & a & 1 & 0 & 0 & 0 \\ 0 & 1 & a & a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we define some other sequences of matrices. Let $S_n(a)$ be the inner n-by-n submatrix of $M_{n+2}(a)$, $T_n(a)$ the upper right n-by-n submatrix of $M_{n+1}(a)$, $R_n(a)$ the lower right n-by-n submatrix of $M_{n+1}(a)$, $P_n(a)$ the inner n-by-n submatrix of $S_{n+2}(a)$, $Q_n(a)$ the upper right n-by-n submatrix of $S_{n+1}(a)$, $A_n(a)$ the lower right n-by-n submatrix of $S_{n+1}(a)$, $B_n(a)$ the upper right n-by-n submatrix of $T_{n+1}(a)$, $C_n(a)$ the lower right n-by-n submatrix of $T_{n+1}(a)$, $D_n(a)$ the lower right n-by-n submatrix of $R_{n+1}(a)$, $F_n(a)$ the lower right n-by-n submatrix of $A_{n+1}(a)$, and $G_n(a)$ the lower right n-by-n submatrix of $C_{n+1}(a)$.

$$S_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & a & a & 1 \\ 0 & 0 & 1 & a & a & a & a \\ 0 & 1 & a & a & a & a & a \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ 1 & a & a & 1 & 0 & 0 & 0 \end{bmatrix} \qquad T_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ 1 & a & a & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$R_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ 1 & a & a & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & a & a & a \\ 0 & 0 & 1 & a & a & a & a \\ 0 & 1 & a & a & a & a & a \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & a & a & 1 \\ 0 & 0 & 1 & a & a & a & a \\ 0 & 1 & a & a & a & a & a \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & a & a \\ 0 & 1 & a & a & a & a & a \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_4(a) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \end{bmatrix} \qquad C_4(a) = \begin{bmatrix} 0 & 0 & 1 & a & a & 1 & 0 \\ 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_4(a) = \begin{bmatrix} 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_4(a) = \begin{bmatrix} 0 & 1 & a & a & a & a & a \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4(a) = \begin{bmatrix} 0 & 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & a & 1 \\ a & a & a & a & a & 1 & 0 \\ a & a & a & a & 1 & 0 & 0 \\ a & a & a & 1 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $s_n(a) = \Lambda(S_n(a))$, $t_n(a) = \Lambda(T_n(a))$, $r_n(a) = \Lambda(R_n(a))$, $p_n(a) = \Lambda(P_n(a))$, $q_n(a) = \Lambda(Q_n(a))$, $a_n(a) = \Lambda(A_n(a))$, $b_n(a) = \Lambda(B_n(a))$, $c_n(a) = \Lambda(C_n(a))$, $d_n(a) = \Lambda(D_n(a))$, $f_n(a) = \Lambda(F_n(a))$, and $g_n(a) = \Lambda(G_n(a))$.

Lemma 25. *The sequences m_n, s_n, t_n, r_n, p_n, q_n, a_n, b_n, c_n, d_n, f_n, and $g_n, n \geq 1$ satisfy*

$$\begin{aligned} m_n s_{n-2} &= t_{n-1}^2 + r_{n-1}^2, \\ s_n p_{n-2} &= q_{n-1}^2 + a_{n-1}^2, \\ t_n q_{n-2} &= b_{n-1} s_{n-1} + c_{n-1}^2, \\ r_n a_{n-2} &= c_{n-1}^2 + d_{n-1} s_{n-1}, \\ p_n p_{n-2} &= a_{n-1}^2 + p_{n-1}^2, \\ q_n p_{n-2} &= q_{n-1} p_{n-1} + a_{n-1}^2, \\ a_n a_{n-2} &= f_{n-1} p_{n-1} + a_{n-1}^2, \\ b_n q_{n-2} &= b_{n-1} q_{n-1} + c_{n-1}^2, \\ c_n a_{n-2} &= a_{n-1} c_{n-1} + q_{n-1} g_{n-1}, \\ d_n f_{n-2} &= f_{n-1} a_{n-1} + g_{n-1}^2, \\ f_n f_{n-2} &= f_{n-1}^2 + a_{n-1}, \text{ and} \\ g_n f_{n-2} &= g_{n-1} f_{n-1} + a_{n-1} \end{aligned}$$

with initial conditions

$$\begin{aligned}
m_1 &= a^2+2a+2+\frac{4}{a}+\frac{4}{a^2}, & m_2 &= \frac{61a^2+98a+122}{a},\\
s_1 &= 36a^4+24a^3+4a^2, & s_2 &= 281a^5+320a^4+198a^3+80a^2+25a,\\
t_1 &= 4a^2+17a+1, & t_2 &= 48a^3+220a^2+164a+32,\\
r_1 &= 17a^2+14a+11, & r_2 &= 48a^3+220a^2+164a+32,\\
p_1 &= 50a^4+12a^3+2a^2, & p_2 &= 373a^5+260a^4+170a^3+76a^2+25a,\\
q_1 &= 42a^4+20a^3+2a^2, & q_2 &= 281a^5+320a^4+198a^3+80a^2+25a,\\
a_1 &= 14a^2+22a^4+20a^3, & a_2 &= \frac{255a^6+485a^5+618a^4+468a^3+171a^2+11a}{3a+1},\\
b_1 &= 4a^2+18a, & b_2 &= 34a^3+156a^2+50a+8,\\
c_1 &= 6a^3+14a^2+4a, & c_2 &= 36a^4+108a^3+158a^2+42a,\\
d_1 &= 4a^2+18a, & d_2 &= 4a^3+16a^2+70a,\\
f_1 &= a^4+3a^3+7a^2+11a, & f_2 &= a^5+4a^4+12a^3+28a^2+45a,\\
g_1 &= 2a^3+6a^2+14a, \quad \text{and} & g_2 &= 2a^4+8a^3+24a^2+56a.
\end{aligned}$$

4.4.5 The future

It is very likely that any arbitrary $2k$-by-even case, for fixed k, can be solved using the same technique. However, doing so by hand becomes intractable very quickly, so automation is highly recommended for continuing the investigation in this way. If Conjecture 21 is in fact true (and it has been tested extensively), then for each fixed k the generating function $M_k(a,x)$ in terms of n for $m_{k,n}(a)$ obeys $M_k(a,x) = M_k(2/a,x)$, and since the problem for each fixed k is a one-dimensional combinatorial problem and thus obeys a linear recurrence (and thus a rational generating function), we can prove that it is *possible* to generate an infinitely long proof. But the actual generation of the proof won't happen any time soon.

But perhaps there is a way to show that enough things are *possible* to prove unequivocally that it is possible to generate a proof for Conjecture 21, for then the conjecture must be true. For instance, a natural question to ask in extending the results of the last two sections is: if you have a finite number N of equations of the form

$$a_n b_{n-2} = c_{n-1}d_{n-1} + e_{n-1}f_{n-1},$$

where a, b, c, d, e, and f are not necessarily distinct, do the involved sequences satisfy linear recurrences of order $\leq C(N)$, where C is some bounded function? The answer to this question is No, for as we have seen the recurrence $a_n a_{n-2} = a_{n-1}a_{n-1} + a_{n-1}a_{n-1}$ with appropriate initial conditions has $a_n = 2^{n(n+1)/2}$ as a solution, and this grows too quickly to have a linear recurrence. However, there is probably something along these lines that is true. This has the potential to suffice for a proof of Conjecture 21.

Alternatively, we can first show that a rational generating function $M_k(a,x)$ exists for each family of $2k$-by-$2n$ rectangles, holding k fixed. Conjecture 20 probably follows easily from here. To prove Conjecture 21 it is clearly sufficient to show that these generating functions obey $M_k(a,x) = M_k(2/a,x)$ for all $k \geq 1$. One possible way to do this is to obtain the generating function $G(a, x, y) = M_0(a, x) + M_1(a, x)y + M_2(a, x)y^2 + \cdots$ and show $G(a, x, y) = G(2/a, x, y)$, but G is certainly not rational (it grows too quickly) and is unlikely to have a nice closed form. This is because the problem of finding a closed form for G is a direct generalization of the following problem (i.e. a solution to the former would immediately give a solution to the latter, by setting $a = 1$ or 2): if $q_{n,k}$ is the number of domino tilings of the $2n$-by-$2k$ rectangle, and $f_n(x) = q_{n,0} + q_{n,1}x + q_{n,2}x^2 + \cdots$ for each n, find a closed form for $F(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots$. The simplest known formula for domino tilings of rectangles is Kasteleyn's, which involves a large double product of trigonometric functions.

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