PARIS SCHOOL OF ECONOMICS
ÉCOLE D'ÉCONOMIE DE PARIS

WORKING PAPER N° 2007 - 25

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JEL Codes: D84, C72, C62.
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Rationalizability in Games with a Continuum of Players*

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This version April 2008

Abstract

The concept of Rationalizability has been used in the last fifteen years to study stability of equilibria on models with a continuum of agents such as competitive markets, macroeconomic dynamics and currency attacks. However, Rationalizability has been formally defined in a general setting only for games with a finite number of players. We propose then a definition for Point-Rationalizable Strategies in the context of *Games with a Continuum of Players*. In a special class of these games, where the payoff of a player depends only on his own strategy and an aggregate value that represents the *state* of the game, state that is obtained from the actions of all the players, we define the sets of *Point-Rationalizable States* and *Rationalizable States*. These sets are characterized and some of their properties are explored. We study as well *standard* Rationalizability in a subclass of these games.

KEYWORDS: Rationalizable Strategies, Non-atomic Games, Expectational Coordination, Rational Expectations, Eductive Stability, Strong Rationality.

JEL CLASSIFICATION: D84, C72, C62.

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*This article is part of my dissertation, prepared at Paris School of Economics (PSE) under the supervision of Roger Guesnerie and Alejandro Jofré.

†I would like to thank Ali Khan, Thomas Boulogne, for helpful conversations; Bruno Strulovici and Nicolás Figueroa for pointing me out relevant references; and the audiences at the seminars at PSE, Department of Mathematical Engineering (DIM) of the University of Chile and Economics at Johns Hopkins University; as well as the audiences at Franco-Chilean Congress on Optimization, Chilean Economic Society Meeting and Chilean-Brazilean Workshop on Economic Theory, for their questions and remarks.

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1 Introduction

The concept of *Strong Rationality* was first introduced by Guesnerie (1992) in a model of a standard market with a continuum of producers. An equilibrium of the market is there said to be *Strongly Rational*, or *Eductively Stable*, if it is the only *Rationalizable Solution* of the economic system. Inspired in the work of Muth (1961), the purpose of such an exercise was to give a rationale for the Rational Expectations Hypothesis. Strong rationality has been studied as well in macroeconomic models in terms of stability of equilibria. See for instance Evans and Guesnerie (1993), where they study Eductive Stability in a general linear model of Rational Expectations or Evans and Guesnerie 2003; 2005 for dynamics in macroeconomics. More examples of applications of Strong Rationality can be found in the recent book by Chamley (2004) where he presents models of Stag Hunts in the context of coordination in games with strategic complementarities.

The *Rationalizable Solution* of the economic system assessed by Guesnerie in the definition of *Strong Rationality*, refers to the concept of *Rationalizable Strategies* as defined by Pearce (1984) in the context of games with a finite number of players and finite sets of strategies. *Rationalizable Strategies* were formally introduced by Bernheim (1984) and Pearce (1984) as the "adequate" solution concept under the premises that players are rational utility maximizers that take decisions independently and that rationality is common knowledge. Adequate because Rationalizable Strategy Profiles are outcomes of a game that cannot be discarded based only on agents' rationality and common knowledge. The work of Pearce focused mainly in refinements of equilibria of extensive form finite games, while Bernheim gave a definition and characterization in the context of general normal form games, along with comparison between the set of Nash Equilibria and the set of Rationalizable Strategy Profiles. In both papers and later treatments, however, the definition and characterization of rationalizable "solutions" were developed for games with a finite number of players.

On the other hand, each one of the works that are mentioned in the first paragraph of this introduction including the seminal work by Guesnerie (1992), feature intuitive and/or context-specific definitions of the concept of *Rationalizable Solution*, adapting the original definitions and characterizations of Rationalizable Strategies, based on the intuitions behind them, to models with a continuum of agents. It is this gap between the established theory and its' economic applications that motivate this work. Since there is no established definition for Rationalizable Strategies, or Rationalizability for what matters, in a general framework with a continuum of agents, in this paper then we link the game-theoretical concept of Rationalizability to its' applications in macroeconomics and economic models, proposing a general definition in the context of games with a continuum of players.

To motivate this presentation let us describe the model and illustrate how the Rationalizability concept is presented in Guesnerie's 1992 work.

Example 1. Consider that we have a group of farmers, represented by the $[0, 1] \equiv I$ interval, that participate in a market in which production decisions are taken one period before production is sold. Each farmer $i \in I$ has a cost function $c_i : \mathbb{R}_+ \to \mathbb{R}$. The price p at which the good is sold is obtained from the (given) inverse demand function $P : \mathbb{R}_+ \rightarrow [0, p_{\max}]$ evaluated in total aggregate production $p = P\big(\int q(i)\,\mathrm{di}\big)$ where $q(i)$ is farmer i's production. Since an individual change in production does not change the value of the price, the product is sold at price-taking behavior, so each farmer $i \in I$ maximizes his payoff function $u(i, \cdot, \cdot) : \mathbb{R}_+ \times [0, p_{\max}] \rightarrow \mathbb{R}$ defined by $u(i, q(i), p) \equiv pq(i) - c_i(q(i))$. An equilibrium of this system is a price p^* such that $p^* = P\big(\int q^*(i)\,\mathrm{di}\big)$ and $u(i, q^*(i), p^*) \geq u(i, q, p^*)$ $\forall$ $q \in \mathbb{R}$, $\forall$ $i \in I$.

At the moment of taking the production decision, farmers do not actually know the value of the price at which their production will be sold. Consequently they have to rely on forecasts of the price or of the production decision of the other farmers. The concept in scrutiny in our work is related to *how this (these) forecast(s) is (are) generated*.

Forecasts of farmers should be rational in the sense that no *unreasonable* price should be given positive probability of being achieved. It is in this setting that Guesnerie introduces the concept of *strong rationality* or *eductive stability* [1] as the uniqueness of *rationalizable prices* which are obtained from the elimination of the unreasonable forecasts of possible outcomes. To obtain these rationalizable prices, Guesnerie describes, in what he calls the *eductive procedure*, how the unreasonable prices can be eliminated using an iterative process of elimination of non-best-response strategies.

Now let us illustrate how the eductive process works in this setting. From the farmers problem we can obtain for each farmer his supply function $s(i, \cdot) : [0, p_{max}] \rightarrow \mathbb{R}_+$. The structure of the payoff function implies that for a given forecast μ of a farmer i over the value of the price, his optimal production is obtained evaluating his supply function in the expectation under μ of the price, $\mathbb{E}_\mu[p]$: $s(i, \mathbb{E}_\mu[p])$. Farmers know that a price higher than p_{max} gives no demand and so prices higher than those are unreasonable. Since all farmers can obtain this conclusion, all farmers know that the other farmers should not have forecasts that give positive weight to prices that are greater than p_{max}. The expectation of each of the farmers' forecasts then cannot be greater than p_{max} and so under necessary measurability hypothesis we can claim that aggregate supply can not be greater than $S(p_{max}) = \int_0^1 s(i, p_{max})\,\mathrm{di}$. Since all farmers know that aggregate supply can not be greater than $S(p_{max})$, they know then that the price, obtained through the inverse demand function, can not be smaller than $p^1_{min} = P(S(p_{max}))$. All farmers know then that forecasts are constrained by the interval $[p^1_{min}, p_{max}]$. They have discarded all the prices above p_{max} and below p^1_{min}. This same reasoning can be made now starting from this new interval.

In Figure 1 we can see the aggregate supply function depicted along with the demand function. We have seen that to eliminate unreasonable prices in this model we only need these two functions. The process, as described in the Figure, continues until the farmers eliminate all the prices except the unique equilibrium price p^*. We say then that this price is *(globally) eductively stable*. Note that the eductive process could "fail", in the sense

[1] An equilibrium of an economic system is said to be *strongly rational* or *eductively stable* if it is the only Rationalizable outcome of the system. We will refer equivalently to outcomes as begin strongly rational or eductively stable.

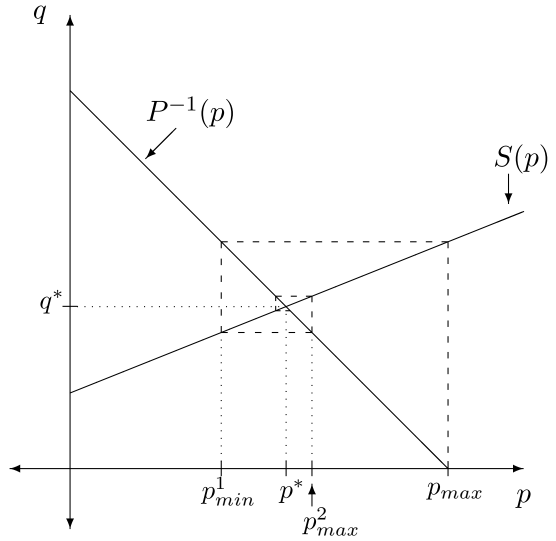


Figure 1: The *eductive process*

that it could give more than only the equilibrium point. This could happen for instance if $S(p_{max}) \geq P^{-1}(0)$. In this situation the *rationalizable set* would be the whole interval $[0, p_{max}]$, since farmers would not be able to eliminate prices belonging to this interval.

For more details on the example the reader is referred to the paper of Guesnerie (1992). The iterative process of elimination of unreasonable prices is inspired by the work of Pearce. However, Pearce's definition of Rationalizability is stressed in the particular framework of a game with a finite number of players where the sets of actions are finite. The approach followed by Pearce assesses rationality and common knowledge of rationality, by considering an iterative process of elimination of non-best-responses (or non-expected-utility-maximizers). This process is overtaken on the set of mixed strategies of the players. Starting with the whole set of mixed strategy profiles, players eliminate at each step of the process the mixed strategies that are not best response to some product probability measure over the set of remaining profiles of mixed strategies of the opponents. This process ends in a finite number of iterations and delivers a set that Pearce defines to be the set of Rationalizable Strategies.

Still, this argument may not be valid in more general contexts. Bernheim's approach to Rationalizable Strategies relies the formalization of the ideas of *system of beliefs* and *consistent system of beliefs*. A system of beliefs for a player represents the possible forecasts of the player concerning the forecasts over forecasts of his opponents, concerning what any player would do. These forecasts take the form of borel measurable subsets of the players' strategy sets. If a system of beliefs gives only singletons, then it is called a *system of point beliefs*. Rationality and common knowledge of rationality then imply that a system of beliefs must satisfy a consistency condition. A consistent system of beliefs simply emphasizes the idea that players should consider in their forecast that the opponents are rational and so are optimizing with respect to some forecasts of their own.

According to Bernheim, a strategy s_i is a *Rationalizable Strategy* for player i if there exists some consistent system of beliefs for this player and some subjective product probability measure over the set of strategy profiles of the opponents, that gives zero probability to actions of the opponents of i that are ruled out by this system of beliefs and such that the strategy s_i maximizes expected payoff with respect to this probability measure. In the particular case where the system is of point beliefs, Bernheim calls s_i a *Point-Rationalizable Strategy*.

In this context, the Rationalizable Set as defined by Bernheim may fail to be the result of the iterated elimination of non-best-responses as described by Pearce. Bernheim proves that in a game with a finite number of players, compact strategy sets and continuous payoff functions, the set of Rationalizable Strategy Profiles is in fact the result of the iterative elimination of strategies that are not best-replies to forecasts considering all of the remaining strategy profiles [2]. This result proves as well, as Bernheim and Pearce claim, that their definitions are indeed equivalent [3]. The characterizations of rationalizability presented by Bernheim are actually related to two properties that rationalizable sets should be asked to fulfill. This is, the rationalizable set must (i) be a subset (hopefully equal) of the set that results from the iterated elimination process, but above all it should (ii) be a fixed point of this process, or, at least, it should be contained in its image through the process [4].

Recent papers address the issue of the set obtained as the limit of processes of iterated elimination of non-best-response-strategies, not being a fixed point of the iterated process in general normal form games with finite number of players; and go beyond to explore more complex iterated processes of elimination of strategies (see for instance Dufwenberg and Stegeman (2002), Apt (2007), Chen et al. (2007) [5]). The problem rises then only if the assumptions on utility functions and strategy sets are relaxed (namely the cases for unbounded strategy sets and/or discontinuous utility functions).

The question surfaces on how should this process be defined in the context of a continuum of players? When can we claim that the result of the iterative elimination process gives a set that we may call *Rationalizable*?

Example 1 gives clear insight on how to face these questions. The particular structure of this example allows us to look at outcomes on the set of prices (or aggregate production), instead of the set of strategy profiles (production profiles), as is done in Pearce or Bernheim.

[2] See Propositions 3.1 and 3.2 in Bernheim (1984). Proposition 3.2 states that the set of Rationalizable Strategies is as well the largest set that satisfies being a fixed point of the process of elimination of strategies. Proposition 3.1 gives an analogous characterization for Point-Rationalizable Strategies considering in the definition of the process of elimination of non-best-response-strategies only the Dirac measures over the remaining sets, instead of all the measures.

[3] Proposition 3.2 in Bernheim (1984). Then, what Pearce defines as the rationalizable set, is named by Bernheim the set of *rationalizable mixed strategies*.

[4] This pertains to some type of *best response property* that the rationalizable set must satisfy.

[5] Dufwenberg and Stegeman (2002) and Chen et al. (2007) put emphasis in Reny's 1999 better-reply secure games.

This allows for a special characterization of the *rationalizable set* as the limit of an iterative process of elimination of unreasonable prices, and not necessarily production profiles. The eductive procedure consists in eliminating the prices that do not emerge as a consequence of farmers taking productions decisions that are best responses to the remaining strategy profiles, or equivalently, remaining values of aggregate production or prices [6].

There are three main issues to take into account when we pass from the finite to the continuous player sets. The first one is how to address forecasts. In the finite player case it is direct to use product measures as forecasts and take expectation over payoff functions to make decisions. This is not evident in the continuum case. The second issue stems from the first one and is related to the space in which one should seek the *rationalizable set*. The set of strategy profiles may not be appealing in contexts where the set of players is a continuum. The third one relates to give conditions to have a well defined process of iterated elimination of *outcomes*. As we have already said, Guesnerie's approach is Pearce's approach in a situation with a continuum of players. This approach is a reasonable and natural way to overtake the rationalizability argument. Nevertheless, and in the light of Bernheim's Proposition 3.2, we see that care is needed to claim that the limit of the process of iterated elimination is in fact a set that we could call of *rationalizable outcomes*. Moreover, the process itself could well be undefined without proper assumptions. Of course, as we prove below in Theorems 3.6 and 4.5, this is not an issue in Guesnerie's setting.

We make the emphasis then in two features of this example [7]: (i) there is a continuum of producers that interact and (ii) payoffs of producers depend on an aggregate value that cannot be affected unilaterally by any agent, this aggregate variable has all the relevant information that producers need to take a decision. We are interested in defining *Rationalizability* in a general setting considering these features. We will adapt the concept of Rationalizable Strategy from the finite game-theoretical world to the context of a class of non-atomic non-cooperative games with a continuum of players. One part of the task then is to find a suitable model of game with a continuum of players, in which one could be able to define and characterize *Rationalizable Outcomes*.

In what follows, we will present a framework of a general class of non-cooperative games with a continuum of players, in which we explore the ideas of rationalizability. We will begin by loosely defining the concept of *Point-Rationalizable Strategies* in a general setting. Then we will turn to the special case where payoffs depend on players' own actions and the average of the actions taken by all the players. We will call this average the *state* of the game, and we will define the sets of *Point-Rationalizable States* and *Rationalizable*

[6] A second characteristic of this setting is that the eductive procedure can be done by simply eliminating prices that are beyond the upper and lower bounds that are obtained in each iteration. However, this comes from the monotonicity properties of the aggregation operator (the integral) and the supply function of the farmers. It is not always the case that the eductive process works this way. This second feature of example 1 is more related to the ordered structure of the games studied in Milgrom and Roberts (1990) and so is left for a further treatment Guesnerie and Jara-Moroni (2007).

[7] Similar features and structure can be found as well in Evans and Guesnerie (1993), in Chapter 11 of Chamley (2004), in Stag Hunt models (see also Morris and Shin (1998)), Chatterji and Ghosal (2004) and in Guesnerie (2005), among others.

States. This last approach is not evident nor a generalization of finite player games, since in "small" games, and as opposed to what we do here, players can actually affect directly and unilaterally the payoff of other players. Our main results are Theorems 3.6 and 4.5 where we characterize these sets as the results of iterated elimination of states. More precisely, we extend Propositions 3.1 and 3.2 in Bernheim (1984) to (Point-)Rationalizable States in the context of games with compact strategy sets, continuous utility functions and a continuum of players. The need for these two Theorems comes from the proof of Proposition 3.2, where a convergent subsequence extraction argument is used, argument that is no longer valid in the context of a continuum of players. A different limit concept is needed to conclude. Moreover, certain measurability properties must be required to have a well defined process of iterated elimination. Consequently, we will get a setting with a continuum of players in which it is possible to study rationalizability and general properties of (locally) strongly rational equilibria as in the economic applications.

The remainder of the paper is as follows: in section 2 we introduce *games with a continuum of players* and some notation; in section 3 we define Point-Rationalizable Strategies in the context of these games and, for the particular class of games with an aggregate state, we define as well Point-Rationalizable States. The main result of this section is the study of the set of Point-Rationalizable States, for which we give a characterization and show its' convexity and compactness. In 3.4 we introduce the concept of *Strongly Point Rational Equilibrium* and explore the relation between Point-Rationalizable Strategies and States. We argue in favor of the use of this last approach, states instead of strategies, in the context of these games. In section 4 we define and characterize Rationalizable States. Before concluding, we explore the concept of Rationalizability in terms of strategy profiles, in the particular setting in which (pure) strategies are chosen from finite sets and payoffs depend on the integral of the profile of mixed strategies Schmeidler (1973). We close the presentation with comments and conclusions in section 5.

2 Games With a Continuum of Players

Since the concept of Strong Rationality introduced by Guesnerie in his paper, relies on a concept that comes from the game-theory literature, our interest is to look at the setting described in the example as a strategic interaction situation. This idea of strategic interaction is then: payoffs of agents depend on the actions of other agents. This interaction would occur through the aggregation of the production and the evaluation in the price function. The payoff of a single farmer depends on the production of all the farmers through P and $u(i, \cdot, \cdot)$, as follows: each farmer $i \in I$ chooses a production $q(i)$ in the positive interval. The price is determined by evaluating the price function in the value of total production, that is on the integral of the production profile. Each agent $i \in I$ obtains payoff $u(i, q(i), p)$. The second feature we need in the mathematical formulation, is that it allows to model the inability of single agents to influence the state of the system, in this case the price, or for what matters, total production, which calls for a mathematical formulation where the weight of single agent is small compared to the whole set or the remaining agents. These two features are captured in the mathematical model presented below.

We consider then games with a continuum of players. Schmeidler (1973) introduced a concept of equilibrium and gave existence results in games where a strategy profile is an equivalence class of measurable functions from the set of players into a strategy set, and the payoff function of a player depends on his own strategy and the strategy profile played. A different approach was presented later by Mas-Colell (1984) [8] and more general frameworks can be found in Khan and Papageorgiou (1987) and Khan et al. (1997) as well. For a comprehensive review of games with many players see Khan and Sun (2002). We will focus mainly in Schmeidler's general setting and specially in games where payoffs depend on an "average" of the actions taken by all the players Rath (1992).

In a Non-Atomic Game the set of players is a non-atomic measure space $(I, \mathcal{I}, \lambda)$ where I is the set of interacting agents $i \in I$ and λ is a non atomic measure on $\mathcal{I}$. This is, $\forall E \in \mathcal{I}$ such that $\lambda(E) > 0$, $\exists F \in \mathcal{I}$ such that $0 < \lambda(F) < \lambda(E)$. We will consider the set of players I as the unit interval in $\mathbb{R}$ and the non-atomic measure λ to be the Lebesgue measure.

Given a set $X \subseteq \mathbb{R}^n$ we will denote the set of equivalence classes of measurable functions from I to X as X^I. We identify then, for a general set X of available actions, X^I with the set of strategy profiles. So a strategy profile is a measurable function from I to X, the set of strategies. By doing this we are assuming that all players have the same strategy set. We will denote S the set of strategies and we will not make a difference a priori between pure or mixed strategies. However, since we assume that S is in $\mathbb{R}^n$ it is better to think of this set as a set of pure strategies. We will come back to this issue on section 4.

For each player $i \in I$, we will denote by $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$ the general payoff functions of a game, that depend on the action of each player as an element of the set S and the profile of strategies as an element of the set S^I described as above. To specify how the functions $\pi(i, \cdot, \cdot)$ depend on these variables, we will use auxiliary functions that depend on the action taken by the player in his strategy set S and some vector taken from a set $X \subset \mathbb{R}^K$, that is obtained from the strategy profile $\mathbf{s}$. The functions $\pi(i, \cdot, \cdot)$ will be obtained then by an operation between these auxiliary functions and some other mathematical objects[9].

2.1 Payoff Functions that Depend on the Integral of the Strategy Profile

Our aim is to capture the relevant features of a wide variety of models that are similar to the one described in Example 1, in the Introduction. Consider then the class of models

[8] In Mas-Colell (1984) what matters is not strategy profiles but a distribution on the product set of payoff functions and strategies.

[9] See equations 2.1 and 4.5.

where there is a set $\mathcal{A} \subseteq \mathbb{R}^K$ and a variable $a \in \mathcal{A}$ that represent, respectively, the *set of states* and the *state* of an economic system. For each agent $i \in I$, the payoff function is now defined on the product of S and $\mathcal{A}$, $u(i, \cdot, \cdot) : S \times \mathcal{A} \to \mathbb{R}$ and depends on his own action $s(i) \in S$ and the state of the system $a \in \mathcal{A}$. Finally, we have an aggregation operator: $A : S^I \to \mathcal{A}$ that gives the state of the system $a = A(\mathbf{s})$ when agents take the action profile $\mathbf{s}$.

In the example, the state of the system could have been identified with aggregate production or the price, and the aggregation operator would have been the integral of the production profile or the evaluation of the price function on such a quantity (respectively).

Agents' impossibility of affecting unilaterally the state of the system is formalized by the following property of A:

$$A(\mathbf{s}) = A(\mathbf{s}') \ \ \forall\, \mathbf{s}, \mathbf{s}' \in \operatorname{dom} A \text{ such that } \lambda(\{l \in I \ : \ s(l) \neq s'(l)\}) = 0$$

That is, since A is defined on S^I, for all strategy profiles that are in the same equivalence class of S^I, the value of the mapping A is the same.

To capture this setting, let S be now a compact subset of $\mathbb{R}^n$. The aggregation operator is chosen for convenience to be the integral with respect to the Lebesgue measure:

$$A(\mathbf{s}) \equiv \int_I \mathbf{s}(i)\,\mathrm{di}$$

so that S^I, the set of measurable functions from I to S, is contained in $\operatorname{dom} A$, the set of integrable functions from I to $\mathbb{R}^n$, and the set $\mathcal{A}$ is $\mathcal{A} \equiv \operatorname{co}\{S\}$ [10].

The payoff functions $\pi(i, \cdot, \cdot)$ mentioned above in the description of a game are calculated by composing the functions $u(i, \cdot, \cdot)$ and A of the economic system, that is

$$\begin{aligned} \pi(i, s(i)\,, \mathbf{s}) &:= u(i, s(i)\,, A(\mathbf{s})) \\ &\equiv u\Big(i, s(i)\,, \int_I s(i) \ \mathrm{di}\Big)\,. \end{aligned} \tag{2.1}$$

In this way we are in Rath's extension of Schmeidler's formulation of games with a continuum of players, where, in a particular class of these games, agents' utility functions depend on their own actions, that are elements of a general compact set, and an "average" of all agents' actions. The description of a game will be given then by a mapping that associates each player $i \in I$ with a real valued continuous function $u(i, \cdot, \cdot)$ defined on $S \times \mathcal{A}$.

[10] The aggregation operator can as well be the integral of the strategy profile with respect to any measure that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\Big(\int_I s(i) \ \mathrm{d}\bar{\lambda}(i)\Big)$$

where $\bar{\lambda}$ is absolutely continuous with respect to the lebesgue measure and $G : \operatorname{co}\{S\} \to \mathcal{A}$ is a continuous function; the results in this work could well be extended to this setting. For instance Theorem 3.6 holds and if G is affine, Corollary 3.7 holds.

We denote the set of real valued bounded continuous functions defined on a space X by $C_{\mathrm{b}}(X)$. Let $\mathcal{U}_{S\times\mathcal{A}} := C_{\mathrm{b}}(S\times\mathcal{A})$ denote the set of real valued continuous functions defined on $S\times\mathcal{A}$ endowed with the sup norm topology.

To denote *games with a continuum of players* that have an *aggregate state* as above, we will use the notation $\mathbf{u}$. Throughout the document when we refer to such games, we will be using the assumption that the function $\mathbf{u}: I \to \mathcal{U}_{S\times\mathcal{A}}$ that associates players with their payoff functions is measurable Rath (1992).

This is in opposition to when we refer to more general games related to the function π that to each player $i \in I$ associates a payoff function $\pi(i): S\times S^I \to \mathbb{R}$ over which we make no general assumptions. We will note then equivalently $\mathbf{u}(i)$ and $u(i,\cdot,\cdot)$. Since the set S is compact, so is $\mathcal{A}$ and so the payoff functions $\mathbf{u}(i)$ are as well bounded. We will call *states* the elements of the set $\mathcal{A}$. Under this description of the game, the fact that payoffs depend on the strategy profiles is given by the rules of the game, and not the payoff function, i.e. the fact that the state of the game is calculated with the integral of the strategy profile.

A *Nash Equilibrium* of a game π is a strategy profile $\mathbf{s}^* \in S^I$ such that λ-almost-everywhere in I:

$$\pi(i, s^*(i), \mathbf{s}^*) \geq \pi(i, y, \mathbf{s}^*) \qquad \forall\, y \in S,$$

This is simply re-stated for a game $\mathbf{u}$ as a strategy profile $\mathbf{s}^* \in S^I$ such that λ-almost-everywhere in I:

$$u\left(i, s^*(i), \int_I \mathbf{s}^*\ \mathrm{di}\right) \geq u\left(i, y, \int_I \mathbf{s}^*\ \mathrm{di}\right) \qquad \forall\, y \in S,$$

In this framework Rath shows that for every game there exists a Nash Equilibrium.

Theorem 2.1 (Rath, 1992)**.** *Every game* $\mathbf{u}$ *has a (pure strategy) Nash Equilibrium.*

We present a proof for this Theorem in the Appendix. The proof in Rath's paper uses Kakutani's fixed point theorem on the mapping Γ that maps a state $a \in \mathcal{A}$ into all the possible states that rise as the consequence of agents taking best response actions to this state. This mapping goes from the convex and compact set $\mathcal{A} \subset \mathbb{R}^n$ into itself and is proved to have a closed graph with non-empty, convex values. The only step where one should be careful is on the proof for non-emptiness of $\Gamma(a)$ in which a measurable selection argument is needed. This is a consequence of the assumption on measurability of the mapping that defines the game. The proof presented herein makes use of Lemma 3.2 stated below. As Rath mentions in his paper, the assumptions on continuity and measurability of the payoff functions are both hidden in the definition of the function $\mathbf{u}$ that represents a game.

3 Point-Rationalizability

Recall that we are interested in situations where players act in ignorance of the actions taken by their opponents. Thus, they must rely on forecasts or subjective priors over the possible outcomes. We assume that agents are rational not only in the sense that they act by maximizing their payoff, but also considering that the subjective priors that they form do not contradict any information that they may have.

The two main assumptions on player's behavior that justify *Rationalizable Strategies* as a solution concept can be summed up to two basic principles: rationality of agents and common knowledge (structural and of rationality of agents) Pearce (1984); Bernheim (1984); Tan and da Costa Werlang (1988). The implications of these assumptions can be exhausted, as is done in Pearce (1984) and Guesnerie (1992), by considering sequential and independent reasoning by the agents, where they rule out certain outcomes of the system as impossible.

Since agents are rational, they only use strategies that are best responses to some forecast over the possible strategy profiles that can actually be played by the others. Hence, the assumption of rationality implies that strategies that are not best responses will never be played. Following the assumption of common knowledge, each agent knows that all other agents are rational. They can then reach the same conclusion: that only best responses can be played; and taking that into account, each agent may discover that some of his (remaining) strategies are no longer best responses and so he will eliminate them. Then rationality implies that forecasts will be restricted to strategy profiles that are not eliminated. Since all agents are rational and know this second conclusion, they can continue this process of elimination of strategies. This generates a sequence of elimination of non-best-responses that under suitable hypothesis will converge in a sense to be formalized to some (hopefully strict) subset of the original strategy profile set. Guesnerie names this procedure the *eductive process* and we will use this terminology.

Following the terminology of Bernheim we will make a difference between *Rationalizability*, understood as forecasts being general probability measures on the sets of outcomes, and *Point-Rationalizability*, understood as forecasts being points or dirac probability measures on the sets of outcomes. We will continue now by giving a formal definition of the concept of Point-Rationalizability for the case of games with a continuum of agents. Further-on we will address the issue of standard Rationalizability.

3.1 Point-Rationalizable Strategies

The first and natural attempt is to go directly from the finite player case into the continuous case. In this approach, players have forecasts over the set of strategies of each of their opponents. These forecasts are in the form of points in these sets and are so represented by functions from I into S.

Consider the following line of reasoning. Given the strategy profile set S^I, all players know that each player will only play a strategy that is a best response to some strategy profile $\mathbf{s} \in S^I$. For each player then we may define the best response mapping $\mathrm{Br}(i, \cdot)$:

$S^I \rightrightarrows S$:

$$\mathrm{Br}(i, \mathbf{s}) := \operatorname{argmax}\{\pi(i, y, \mathbf{s}) \ : \ y \in S\}. \tag{3.1}$$

The mapping $\mathrm{Br}(i, \cdot)$ gives the optimal set for player $i \in I$ facing a strategy profile $\mathbf{s}$. We use the function $\pi(i, \cdot, \cdot)$ that associates strategy profiles to payoffs in a general way. As we said before, rationality of players implies that they will only use strategies that are optimal to some forecast. So players can discard for each player $i \in I$ strategies that are outside the sets

$$\mathrm{Br}\big(i, S^I\big) \equiv \bigcup_{\mathbf{s} \in S^I} \mathrm{Br}(i, \mathbf{s}),$$

so strategy profiles can be actually secluded into the set:

$$S_1^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of the correspondence} \\ i \rightrightarrows \mathrm{Br}\big(i, S^I\big) \end{array} \right\}.$$

That is, players will not play a strategy that is not a best response to some strategy profile. This is captured by selections of the mapping $i \rightrightarrows \mathrm{Br}\big(i, S^I\big)$. Taking this into account, agents can deduce, at a step t of this process, that strategy profiles must actually be in the set S_t^I,

$$S_t^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of the correspondence} \\ i \rightrightarrows \mathrm{Br}\big(i, S_{t-1}^I\big) \end{array} \right\}.$$

This exercise motivates the definition of a recursive process of elimination of non best responses. For this, denoting by $\mathcal{P}(X)$ the set of subsets of a certain set X, we define the mapping $Pr : \mathcal{P}\big(S^I\big) \to \mathcal{P}\big(S^I\big)$ that to each subset $H \subseteq S^I$ associates the set $Pr(H)$ defined by:

$$Pr(H) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of the correspondence} \\ i \rightrightarrows \mathrm{Br}(i, H) \end{array} \right\}. \tag{3.2}$$

This definition is analogous to the one given by Pearce and by Bernheim [11]. In the context of a continuum of players, however, the set $Pr(H)$ could well be empty if we do not make appropriate assumptions about the payoff function π. A sufficient condition for non-emptiness of $Pr(H)$ is non-emptyness of the sets $\mathrm{Br}(i, H)$ λ-almost-everywhere in I along with measurability of the correspondence $i \rightrightarrows \mathrm{Br}(i, H)$. The mapping Pr represents strategy profiles that are obtained as the reactions of players to strategy profiles contained

[11] See Definition 1 in Pearce (1984) and Section 3(b) in Bernheim (1984).

in the set $H \subseteq S^I$. It has to be kept in mind that strategies of different players in a strategy profile in $Pr(H)$ can be the reactions to (possibly) different strategy profiles in H.

The line of reasoning developed above implies that a strategy profile that is point rationalizable should never be eliminated during the process generated by the iterations of Pr. Let us note $Pr^t(S^I) \equiv Pr(Pr^{t-1}(S^I))$ and $Pr^0(S^I) \equiv S^I$. The set $Pr^t(S^I)$ is the one obtained in the t^{th} step of the process of elimination of non-best-response strategy profiles. It is direct to see that $Pr^1(S^I) \equiv S_1^I$ and $Pr^t(S^I) \equiv S_t^I$. Note that the process $\{Pr^t(S^I)\}_{t=0}^{+\infty}$ gives a nested family of subsets of S^I and so a point that is never eliminated should be in the intersection of all of them. This means that the set of point-rationalizable strategies, from now on denoted $\mathbb{P}_S$, must satisfy:

$$\mathbb{P}_S \subseteq \bigcap_{t=0}^{+\infty} Pr^t(S^I)\,. \tag{3.3}$$

However, it is not enough to ask for this property, since rationality of players implies that a strategy should only be played if it is justified by a rationalizable strategy profile. The point-rationalizable set must have the *best response property*: each strategy that participates in a strategy profile in $\mathbb{P}_S$ must be a best response to some (possibly different) strategy profile in $\mathbb{P}_S$.We capture this second feature by asking condition (3.4),

$$\mathbb{P}_S \subseteq Pr(\mathbb{P}_S)\,. \tag{3.4}$$

Note that condition (3.4) implies (3.3), since a set that satisfies (3.4) would never be eliminated. The ideal situation would be that the result of the eductive process gave the set of point-rationalizable strategies. This would be the case only if $Pr\big(\bigcap_{t=0}^{+\infty} Pr^t(S^I)\big) = \bigcap_{t=0}^{+\infty} Pr^t(S^I)$, which as we mentioned in the introduction is not necessarily true in all generality, we give an example in the next subsection.

Nevertheless, with the concepts displayed so far, we are able to give a definition for the Point-Rationalizable Strategy Profiles set.

Definition 3.1. The set of *Point-Rationalizable Strategy Profiles* is the maximal subset $H \subseteq S^I$ that satisfies condition (3.4) and we note it $\mathbb{P}_S$.
For each player, $i \in I$, there will be a set of Point-Rationalizable Strategies, namely the union, over all the Point-Rationalizable Strategy Profiles in $\mathbb{P}_S$, of the best response set of the considered player. That is, the set of Point-Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{P}_S(i) := \bigcup_{\mathbf{s} \in \mathbb{P}_S} \mathrm{Br}(i, \mathbf{s})$$

A well known result for the case of games with a finite number of players is that all Nash Equilibria of the game are elements of the Point-Rationalizable Strategies set Bernheim (1984). The same is true for our definition, since if $\mathbf{s}^*$ is a Nash Equilibrium, then it is a selection taken from $i \rightrightarrows \mathrm{Br}(i, \mathbf{s}^*)$ and so it satisfies $\{\mathbf{s}^*\} \subseteq Pr(\{\mathbf{s}^*\})$ which implies the property.

We now turn to a different approach to Rationalizability. In the context that interests us, players form expectations not on the space of strategy profiles, but on the set of states of the game. Thus Rationalizability should also be stated in terms of forecasts on this set of states. This is what we present in the next subsection.

3.2 Point-Rationalizable States

We turn to the particular class of games with a continuum of players where payoffs depend explicitly on the average of the actions of all the players, which we call the state of the game. In this framework it is natural to model agents as having forecasts on the set of states, rather than on the set of strategy profiles, since the relevant information that agents need to take a decision is the value of the state a [12].

In what follows, we will define Point-Rationalizability on the set of states. So now instead of using the correspondence $\mathrm{Br}(i,\,\cdot\,)$ defined in (3.1), we use the mapping $B(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S$ that gives the optimal strategy set given a state of the system,

$$B(i,a) := \operatorname{argmax}\left\{u(i,y,a)\ :\ y \in S\right\}.$$

There are two main differences between this approach an the one presented in the previous subsection. First, here we use the specific function $\mathbf{u}$ that defines a game with an aggregate state instead of the general function π as in (3.1), and second, the mapping $B(i,\,\cdot\,)$ goes from $\mathcal{A} \subset \mathbb{R}^n$, instead of S^I, to $S \subset \mathbb{R}^n$. It is direct to see, however, that for a given strategy profile $\mathbf{s}$, in the context of a game $\mathbf{u}$, $\mathrm{Br}(i,\mathbf{s}) \equiv B\big(i, \int \mathbf{s}\big)$. For each $i \in I$ and a set $X \subseteq \mathcal{A}$, consider the image through $B(i,\,\cdot\,)$ of the set X

$$B(i,X) := \bigcup_{a\in X} B(i,a)\,.$$

Let us now look at the process of elimination of non reachable or non generated states. Suppose that initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$. Then, in a first order basis, an agent can assume that any of the states $a \in X$ can be the actual state, but point expectations are actually constrained by X, so the possible actions of a player $i \in I$ are constrained to the set $B(i,X)$. Since all players know this, each one of them can discard all strategy profiles $\mathbf{s} \in S^I$ that are not selections of the set valued mapping $i \rightrightarrows B(i,X)$. Then, if the players know that forecasts are restricted to $X \subseteq \mathcal{A}$, they will know that the actual outcome has to be a state associated through the aggregation operator to some measurable selection of that mapping.

Therefore, given $X \subseteq \mathcal{A}$ consider the set of all the measurable selections taken from the correspondence $i \rightrightarrows B(i,X)$ that to each agent $i \in I$ associates the set $B(i,X)$. Then, take all the possible images through the aggregation mapping of such functions. We

[12]See as well Guesnerie (2002) for a discussion on this matter.

define then the mapping $\tilde{Pr} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ that to each set $X \subseteq \mathcal{A}$ associates the set $\tilde{Pr}(X) \subseteq \mathcal{A}$ defined by:

$$\tilde{Pr}(X) := \left\{ a \in \mathcal{A} : \ a = A(\mathbf{s}), \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the correspondence} \\ i \rightrightarrows B(i, X) \end{array} \right\}. \tag{3.5}$$

Our assumptions on the aggregation operator A allow us to re-write definition (3.5) as the integral of a set valued mapping [13]:

$$\tilde{Pr}(X) \equiv \int_I B(i, X) \,\mathrm{di}.$$

Before continuing, we state a relevant property associated to the mapping B.

Lemma 3.2. *In a game $\boldsymbol{u}$, for a non-empty closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$ is measurable and has non-empty compact values.*

The proof is relegated to the Appendix. Lemma 3.2 above and Theorem 2 in Aumann (1965) assure that $\tilde{Pr}(X)$ is non empty and closed whenever X is non empty and closed. With this set to set mapping we can define a set of point rationalizable states.

As we did in the previous subsection, consider the process given by iterations of $\tilde{Pr}$. That is,

$$\begin{aligned} \tilde{Pr}^{0}(\mathcal{A}) &:= \mathcal{A} \\ \tilde{Pr}^{t+1}(\mathcal{A}) &:= \tilde{Pr}\Big(\tilde{Pr}^{t}(\mathcal{A})\Big) \quad \text{for } t \geq 1. \end{aligned}$$

Observe that $\tilde{Pr}^{t+1}(\mathcal{A}) \subseteq \tilde{Pr}^{t}(\mathcal{A})$, this is not necessarily true for any subset $X \subseteq \mathcal{A}$. The set of Point-Rationalizable States, $\mathbb{P}_{\mathcal{A}}$, must then satisfy:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t=0}^{\infty} \tilde{Pr}^{t}(\mathcal{A}). \tag{3.6}$$

The right hand side of (3.6) represents the iterative elimination of non reachable states. At each step of this process, players only keep in mind the states that could be reached following rational actions based on point expectations given by the set of the previous step. If a state is not reached by actions following forecasts constrained at a certain step of the process, then it is not rationalizable. Since the family of sets $\left\{\tilde{Pr}^{t}(\mathcal{A})\right\}_{t=0}^{+\infty}$ is a

[13] The integral of a correspondence $F : I \rightrightarrows \mathbb{R}^n$ is calculated, following Aumann (1965), as the set of integrals of all the integrable selections of F. This is,

$$\int_I F(i) \,\mathrm{di} \equiv \left\{ \int_I f(i) \,\mathrm{di} \ : \ f \text{ is an integrable selection of } F \right\}$$

where $\int f \mathrm{di} := \left(\int f_1(i) \,\mathrm{di}, \ldots, \int f_n(i) \,\mathrm{di} \right)$.

nested (decreasing) family of closed subsets of $\mathbb{R}^n$, the infinite intersection in expression (3.6) turns out to be the exact *Painlevé-Kuratowski* limit of the sequence of sets.

The second condition that the set of Point-Rationalizable States must satisfy is:

$$\mathbb{P}_{\mathcal{A}} \subseteq \tilde{Pr}(\mathbb{P}_{\mathcal{A}})\,. \tag{3.7}$$

Condition (3.7) stands for the fact that Point-Rationalizable States should be justified by Point-Rationalizable States. This means that if a state is Point-Rationalizable, it should rise as the consequences of players taking actions as reactions to point forecasts in the set of Point-Rationalizable States. Analogously to the case where point forecasts are taken over strategy profiles, it is direct to see that condition (3.7) implies (3.6). That is, if a set $X \subseteq \mathcal{A}$ satisfies condition (3.7) then $X \subseteq \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A})$. So we define the set of Point-Rationalizable States as follows:

Definition 3.3. The set of Point-Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies condition (3.7) and we note it $\mathbb{P}_{\mathcal{A}}$.

Remark 3.4. Note that for the case of forecasts over the set of states, defining player-specific rationalizable states set makes no sense. This approach calls for different mathematical tools since now we are dealing with a set in a finite dimensional space as opposed to Definition 3.1. Moreover, the exercise of obtaining Point-Rationalizable States gives clear insights on properties of the Point-Rationalizable Strategy Profiles set, particularly for strongly rational equilibria, as can be seen in Proposition 3.11 below.

Remark 3.5. Conditions (3.4) and (3.7) are related to the definition of *Tight Sets Closed Under Rational Behavior* (Tight CURB Sets) given in Basu and Weibull (1991). Indeed Basu and Weibull make the observation that the set of rationalizable strategy profiles in a finite game with compact strategy sets and continuous payoff functions, is in fact the maximal tight curb set, which is analogous to our definitions of Point-Rationalizability.

We know give an answer to the question of whether we can obtain the same conclusion as in Bernheim's Proposition 3.2 in our context. Our main result, Theorem 3.6, states that under the hypothesis of Rath's setting we have that the set of Point-Rationalizable States, is actually the one obtained from the eductive process, and so we obtain a first characterization of this set.

Theorem 3.6. *Let us write $\mathbb{P}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A})$. The set of Point-Rationalizable States of a game $\boldsymbol{u}$ can be calculated as*

$$\begin{aligned}\mathbb{P}_{\mathcal{A}} &\equiv \mathbb{P}'_{\mathcal{A}}\\ &\equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A})\end{aligned}$$

Proof.

We will show that:

$$\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}}$$

Let us begin by showing that $\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \subseteq \mathbb{P}'_{\mathcal{A}}$. Indeed, if $a \in \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})$ then, by the definition of $\tilde{Pr}$, there exists a measurable selection $\mathbf{s} : I \to S$ of $i \rightrightarrows B(i, \mathbb{P}'_{\mathcal{A}})$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{Pr}^t(\mathcal{A})\ \forall\ t \geq 0$, we have that $B(i, \mathbb{P}'_{\mathcal{A}}) \subseteq B\Big(i, \tilde{Pr}^t(\mathcal{A})\Big)\ \forall\ t \geq 0\ \forall\ i \in I$. So $\mathbf{s}$ is a selection of $i \rightrightarrows B\Big(i, \tilde{Pr}^t(\mathcal{A})\Big)$ and then $a \in \tilde{Pr}^{t+1}(\mathcal{A})\ \forall\ t \geq 0$, which means that $a \in \mathbb{P}'_{\mathcal{A}}$.

Now we show that $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})$. For this inclusion, consider the following sequence $F^t : I \rightrightarrows S$, $t \geq 0$, of set valued mappings:

$$\begin{aligned} & F^0(i) := S && \forall\ i \in I \\ \forall\ i \in I \quad & F^t(i) := B\Big(i, \tilde{Pr}^{t-1}(\mathcal{A})\Big) && t \geq 1 \end{aligned}$$

As we said before, we have that

$$\tilde{Pr}^t(\mathcal{A}) \equiv \int_I F^t(i)\ \mathrm{di}.$$

Since $\mathbf{u}(i) \in \mathcal{U}_{S \times \mathcal{A}}$, then $\forall\ i \in I$ the mapping $B(i, \cdot\,) : \mathcal{A} \rightrightarrows S$ is u.s.c. and, as a consequence, the set $B(i, X)$ is compact for any compact subset $X \subseteq \mathcal{A}$ Berge (1997). Since $\mathcal{A} \equiv \int_I F^0$, Aumann (1965) gives that $\mathcal{A}$ is non empty and compact[14]. From Lemma 3.2 we get that F^1 is measurable and compact valued and by induction over t, we get that for all $t \geq 1$, $\tilde{Pr}^{t-1}(\mathcal{A})$ is non empty, convex and compact, and F^t is measurable and compact valued.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t, noted p-lim $\sup_t F^t$, obtained as:

$$F(i) := \big(\text{p-lim}\sup\nolimits_t F^t\big)(i) \equiv \limsup_t F^t(i)$$

where the right hand side is the set of all cluster points of sequences $\{y^t\}_{t \in \mathbb{N}}$ such that $y^t \in F^t(i)$. From Rockafellar and Wets[15] we get that F is measurable and compact valued.

So now let us take a point $a \in \mathbb{P}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t \in \mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976) we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t \in \mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int \mathbf{s}^t$.

Now it suffices to check that $F(i) \subseteq B(i, \mathbb{P}'_{\mathcal{A}})$, since then we would have

$$a \in \int_I F \mathrm{di} \subseteq \int_I B(i, \mathbb{P}'_{\mathcal{A}})\, \mathrm{di} \equiv \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})\,.$$

This comes from the upper semi continuity of $B(i, \cdot\,)$. Take $y \in F(i)$. From the definition of F, y is a cluster point of a sequence $\{y^t\}_{t \in \mathbb{N}}$ such that $y^t \in F^t(i)$. That is, there

[14] We have already noted that in fact $\mathcal{A} \equiv \mathrm{co}\,\{S\}$ and so in particular it is also convex, which is of no relevance in this proof, but is the key property in Corollary 3.7.

[15] See Rockafellar and Wets (1998) Ch. 4 and 5 and Theorem 14.20 .

is a sequence of elements of $\mathcal{A}$, $\{a^k\}_{k\in\mathbb{N}}$ such that $a^k \in \tilde{Pr}^{t_k-1}(\mathcal{A})$, $y^{t_k} \in B(i, a^k)$ and $\lim_k y^{t_k} = y$. Through a subsequence of $\{a^k\}_{k\in\mathbb{N}}$ we get that the limit of $\{y^{t_k}\}_{k\in\mathbb{N}}$ must belong to $B(i, \mathbb{P}'_{\mathcal{A}})$, since all cluster points of $\{a^k\}_{k\in\mathbb{N}}$ are in $\mathbb{P}'_{\mathcal{A}}$, being the intersection of a nested family of compact sets.

■

The previous theorem gives a characterization of Point-Rationalizable States that is analogous to the one given for Point-Rationalizable Strategies in Bernheim, in the case of finite player games with compact strategy sets and continuous utility functions. The difficulty of Theorem 3.6 is to identify the adequate convergence concept for the eductive process. In the case of finite player games there is no such a question, since in that case the technique is simply to take a convergent subsequence of points (in the finite dimensional strategy profile set) from the sequence of sets that participate in the eductive procedure and using (semi) continuity arguments of the best response mappings obtain the result [16]. However, in our setting these arguments fail to prevail. From the proof of the Theorem, we see that the set of Point-Rationalizable States is obtained as the integral of the point-wise upper limit of a sequence of set valued mappings. So the relevant improvement in the proof (besides measurability requirements) is to give the adequate limit concept.

To see that the Theorem is not vacuous consider the following example.

Example 2. Consider the game where $I \equiv [0,1]$, $S \equiv [0,1]$ and $\mathbf{u}(i) \equiv u : [0,1]^2 \to \mathbb{R}$ for all $i \in I$, such that it generates the following best reply correspondence, depicted in Figure 2a:

$$B(a) = \begin{cases} a^* & \text{if } a \leq \bar{a}, \\ \{0, \bar{a}(1-\alpha) + a\alpha\} & \text{if } a > \bar{a}, \end{cases}$$

where a^*, $\bar{a}$ and α are in $]0,1[$ and $a^* < \bar{a}$. It is clear that this game does not satisfy the hypothesis of Theorem 3.6 since no continuous utility function may give rise to such a best response correspondence.

The only equilibrium of the game is a^*.

Since all the players have the same best reply correspondence, the process of elimination of non-generated-states is obtained by:

$$\tilde{Pr}(X) \equiv \text{co}\{B(X)\}$$

The image through the best reply correspondence of the state set is

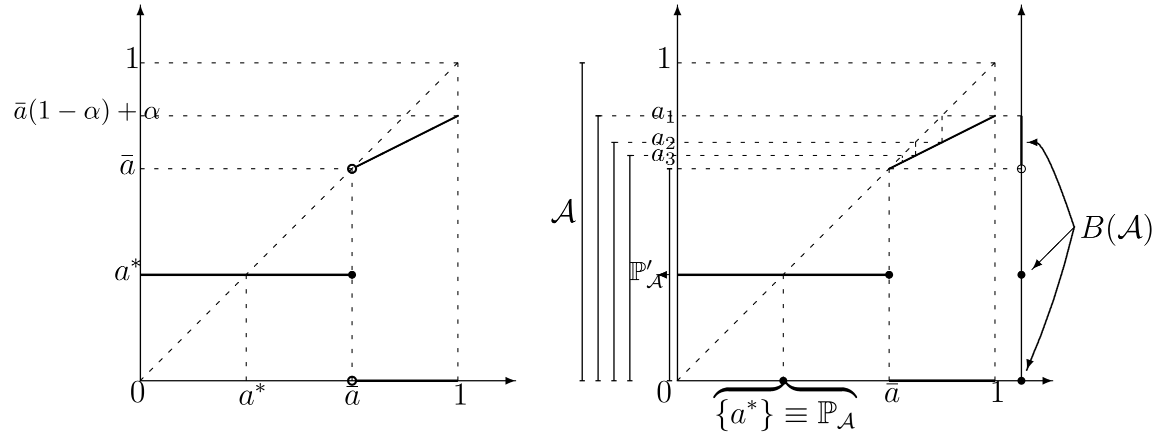
$$B(\mathcal{A}) \equiv \{0, a^*\} \cup]\bar{a}, \bar{a}(1-\alpha) + \alpha],$$

then $\tilde{Pr}(\mathcal{A}) \equiv \text{co}\{B(\mathcal{A})\} \equiv [0, \bar{a}(1-\alpha) + \alpha]$.

Then the second iteration is obtained by

$$\begin{aligned} \tilde{Pr}^2(\mathcal{A}) \equiv \text{co}\left\{B\left(\tilde{Pr}(\mathcal{A})\right)\right\} &\equiv \text{co}\{B([0, \bar{a}(1-\alpha) + \alpha])\} \\ &\equiv \text{co}\{\{0, a^*\} \cup]\bar{a}, \bar{a}(1-\alpha) + \alpha(\bar{a}(1-\alpha) + \alpha)]\} \\ &\equiv \left[0, \bar{a}\left(1-\alpha^2\right) + \alpha^2\right]. \end{aligned}$$

[16] See the proof of Proposition 3.1 in Bernheim (1984).



(a) The best reply correspondence $B : [0,1] \rightrightarrows [0,1]$.

(b) The elimination of non-generated-states with $\tilde{Pr}(X) \equiv \text{co}\,\{B(X)\}$ and the Point-Rationalizable State set.

Figure 2: Example 2: The set of Point-Rationalizable States is not the set $\mathbb{P}'_{\mathcal{A}}$.

We see from the form of the best reply correspondence that on each iteration of $\tilde{Pr}$ we get an interval of the form $[0, a_t]$ where the sequence $\{a_t\}_{t=0}^{+\infty}$ satisfies:

$$a_{t+1} = \bar{a}(1-\alpha) + a_t\alpha,$$

which gives, for $t \geq 1$,

$$a_t = \bar{a}\big(1-\alpha^t\big) + \alpha^t,$$

with $a_0 = 1$, and so the sequence is decreasing and converges to $\bar{a}$ (see Figure 2b). This allows for us to see that $\mathbb{P}'_{\mathcal{A}} \equiv [0, \bar{a}]$. However,

$$\begin{aligned}\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) &\equiv \text{co}\,\{B(\mathbb{P}'_{\mathcal{A}})\} \equiv \text{co}\,\{B([0,\bar{a}])\} \equiv \text{co}\,\{\{a^*\}\} \equiv \{a^*\} \\ &\subsetneq \mathbb{P}'_{\mathcal{A}}.\end{aligned}$$

and so $\mathbb{P}'_{\mathcal{A}}$ is not equal to $\mathbb{P}_{\mathcal{A}}$, which is in fact equal to the set of equilibria: the singleton $\{a^*\}$.

Theorem 2.1 implies that if the set of Point-Rationalizable Strategies (or States for what it matters) is well defined, then it is not empty, since as we already said, all the equilibria belong to this set. From Theorem 3.6 we get as a Corollary that in the games that we are considering, the set of Point-Rationalizable States is well behaved.

Corollary 3.7. *The set of Point-Rationalizable States of a game* $\boldsymbol{u}$ *is well defined, non-empty, convex and compact.*

Proof.

From Theorem 3.6, $\mathbb{P}_{\mathcal{A}}$ is the intersection of a nested family of non-empty compact convex sets. From Theorem 2.1 we get that there is point $a^* \in \mathcal{A}$ such that $a^* \in \tilde{Pr}^t(\mathcal{A})$ $\forall\, t$. These two facts lead us to conclude that this intersection is compact, convex and non empty.

■

The properties stated in this Corollary are not trivial. In games with finite number of players we can find examples where the outcome of the iterative elimination of non-best-replies is an empty set. The same can be true in our context when we withdraw the continuity hypothesis of utility functions, we present below an example of a game with non-continuous payoffs:

Example 3 (Based on Dufwenberg and Stegeman (2002)). Consider the game where $I \equiv [0,1]$, $S \equiv [0,1]$ and $\mathbf{u}(i) \equiv u : [0,1]^2 \to \mathbb{R}$ for all $i \in I$, such that:

$$u(y,a) = \begin{cases} 1-y & \text{if } 0 < \frac{a}{2} \leq y, \\ y & \text{if not.} \end{cases}$$

Then, the best reply correspondence is the same for all players:

$$B(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

The mapping Γ turns out to be equal to the best reply correspondence:

$$\Gamma(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

This mapping has no fixed point and so in this game there is no equilibrium.

Let us study the Point-Rationalizable States set. The image through the best reply correspondence of the state set is $B(\mathcal{A}) \equiv \{1\} \cup]0,1/2]$, then

$$\tilde{Pr}(\mathcal{A}) \equiv \text{co}\,\{B(\mathcal{A})\} \equiv]0,1]\,,$$

the second iteration gives

$$\tilde{Pr}^2(\mathcal{A}) \equiv \text{co}\left\{B\Big(\tilde{Pr}(\mathcal{A})\Big)\right\} \equiv \text{co}\,\{B(]0,1])\} \equiv]0,1/2]$$

and the third

$$\tilde{Pr}^3(\mathcal{A}) \equiv \text{co}\left\{B\Big(\tilde{Pr}^2(\mathcal{A})\Big)\right\} \equiv \text{co}\,\{B(]0,1/2])\} \equiv]0,1/4]\,,\ \ldots,\ \text{etc..}$$

Then the set $\mathbb{P}'_{\mathcal{A}} \equiv \emptyset$ and so $\mathbb{P}_{\mathcal{A}} \equiv \emptyset$, that is, in this example there is no set $X \subseteq \mathcal{A}$ that satisfies $X \subseteq \tilde{Pr}(X)$.

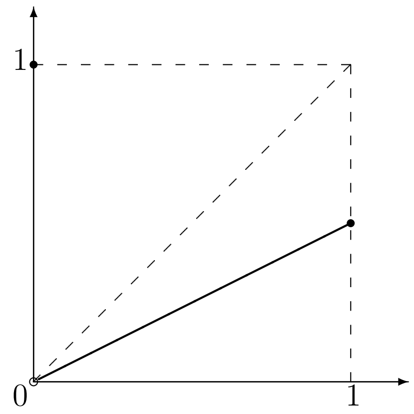


Figure 3: The best reply correspondence $B : [0,1] \rightrightarrows [0,1]$.

Although Theorem 3.6 and Corollary 3.7 assure that the eductive procedure achieves a non-empty Point-Rationalizable set of states, we see from Examples 2 and 3 that even in cases where the eductive procedure fails, in the sense that it does not deliver the Point-Rationalizable set, we may still identify a set as the *correct* Point-Rationalizable State set following our Definition 3.3 (in Example 2 the set $\mathbb{P}_{\mathcal{A}}$ is the singleton that contains the equilibrium and in Example 3 it is the empty set). Even more, the eductive procedure can (obviously) help to locate such set even in the case of failure and emptiness. The original motivation to introduce rationalizability in economic contexts is the plausibility of the Rational Expectations Hypothesis. In consequence we allow an empty (Point-)Rationalizable set, under the definition of rationalizability, interpreting such as a pessimistic answer to the possibility of the coordination of expectations.

Another property stated in Corollary 3.7 and an important consequence of Theorem 3.6, is the convexity of the Point-Rationalizable States set, since in the case where we have multiple equilibria in the set of states, we know that the convex hull of this set of equilibria is also contained in the set of Point-Rationalizable States. This inclusion may be strict since if we have multiple equilibria in the set of strategies, even with uniqueness in the set of states, we may have multiple Point-Rationalizable States [17]. Convexity of the Point-Rationalizable States set is a relevant property since it has not been obtained (to our knowledge) for any other concept related to Rationalizability.

3.3 Point-Rationalizable Strategies vs. Point-Rationalizable States

It is straightforward to ask how the concepts that we just defined are related. To address this issue, note that for the class of non-atomic games that we are considering,

[17] See Proposition 3.11.

the iterations of the strategy-elimination and state-elimination mappings (Pr and $\tilde{Pr}$) are equivalent in the following sense. Consider the set to set mappings $\bar{A}$ and $\bar{B}$ defined below. Let $\bar{A} : \mathcal{P}(S^I) \to \mathcal{P}(\mathcal{A})$ be defined by:

$$\begin{aligned}\bar{A}(H) &:= \left\{ a \in \mathcal{A} : \begin{array}{l} a = \int_I s(i)\,\mathrm{di} \text{ and } \mathbf{s} \text{ is a mea-} \\ \text{surable function in } H \end{array} \right\} \\ &\equiv A(H)\end{aligned}$$

and let $\bar{B} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(S^I)$ be:

$$\bar{B}(X) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of} \\ i \rightrightarrows B(i, X) \end{array} \right\}.$$

These mappings satisfy

$$\begin{aligned} X_1 \subseteq X_2 \subseteq \mathcal{A} \quad &\Longrightarrow \quad \bar{B}(X_1) \subseteq \bar{B}(X_2) \\ H_1 \subseteq H_2 \subseteq S^I \quad &\Longrightarrow \quad \bar{A}(H_1) \subseteq \bar{A}(H_2)\,. \end{aligned} \tag{3.8}$$

Then, in the context of a game $\mathbf{u}$, the mappings Pr and $\tilde{Pr}$ satisfy:

$$\begin{aligned} Pr(H) &\equiv \bar{B}\big(\bar{A}(H)\big) \\ \tilde{Pr}(X) &\equiv \bar{A}\big(\bar{B}(X)\big) \end{aligned} \tag{3.9}$$

In particular, we get,

$$\tilde{Pr}^0(\mathcal{A}) \equiv \mathcal{A} \equiv \mathrm{co}\,\{S\} \equiv \bar{A}\big(S^I\big)$$

which implies by induction that:

$$\begin{aligned} \tilde{Pr}^t(\mathcal{A}) &\equiv \bar{A}\big(Pr^t\big(S^I\big)\big) \\ Pr^t\big(S^I\big) &\equiv \bar{B}\Big(\tilde{Pr}^{t-1}(\mathcal{A})\Big) \end{aligned}$$

Theorem 3.8. *In a game $\mathbf{u}$ we have:*

$$\mathbb{P}_S \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \quad \textit{and} \quad \mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_S)\,.$$

Proof.
Note that from (3.9) we have

$$\begin{aligned} Pr\big(\bar{B}(\mathbb{P}_{\mathcal{A}})\big) &\equiv \bar{B}\Big(\tilde{Pr}(\mathbb{P}_{\mathcal{A}})\Big) \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \\ \tilde{Pr}\big(\bar{A}(\mathbb{P}_S)\big) &\equiv \bar{A}(Pr(\mathbb{P}_S)) \;\equiv \bar{A}(\mathbb{P}_S) \end{aligned}$$

That is, the sets $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq S^I$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathcal{A}$ satisfy conditions (3.4) and (4.1) respectively, which implies that $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathbb{P}_{\mathcal{A}}$. Then

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S) \equiv \bar{B}\big(\bar{A}(\mathbb{P}_S)\big) \subseteq \bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S.$$

The second equality comes from (3.9) while the first inclusion comes from (3.8) and the previous observation. The proof for the second statement is analogous.

■

We see from Theorem 3.8 that in the context that we are considering, the set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. This implies that in the models that interest us, it is equivalent to study Point-Rationalizability in terms of strategies or states, an intuition claimed by Guesnerie and, of course, present in Example 1.

Theorems 3.8 and 3.6 together imply that the set of Point-Rationalizable Strategies $\mathbb{P}_S$ can be actually computed, in this setting, as the limit of the strategy elimination process governed by Pr (see condition (3.3)) answering a question that remained unanswered. In consequence, we have that in a game $\mathbf{u}$ we can obtain the sets of Point-Rationalizable States and Strategies through the eductive process in the respective set ($\mathcal{A}$ or S^I).

Corollary 3.9. *Let us write $\mathbb{P}'_S := \bigcap_{t=0}^{\infty} Pr^t\big(S^I\big)$. The set of Point-Rationalizable Strategy Profiles of a game $\boldsymbol{u}$ can be calculated as*

$$\begin{aligned}\mathbb{P}_S &\equiv \mathbb{P}'_S \\ &\equiv \bigcap_{t=0}^{\infty} Pr^t\big(S^I\big)\end{aligned}$$

3.4 Strongly Point Rational Equilibrium

As we have already said, Guesnerie defines the concept of (local) Strongly Rational Equilibrium as an equilibrium that is the only Rationalizable State of an economic system. A particular feature of the work therein developed is that although the definition of Rationalizability is stressed in terms of strategy profiles, that is, on the profile of individual production quantities, the study of the (local) stability of the (unique) equilibrium can be developed in terms of aggregate production or even prices (see note 12). In our context, Strong Rationality would be defined in terms of the aggregate variable $a \in \mathcal{A}$.

Our purpose in this section then is to explore the relation between strategy profiles and aggregate states when we are interested in Strong Rationality and Point-Rationalizability.

Definition 3.10. An equilibrium state $a^* \in \mathcal{A}$ is a *Strongly Point Rational Equilibrium* if $\mathbb{P}_{\mathcal{A}} = \{a^*\}$.

Note that if a state a^* satisfies $\mathbb{P}_{\mathcal{A}} = \{a^*\}$, then it is the unique equilibrium of the system since (i) all equilibria are in $\mathbb{P}_{\mathcal{A}}$ and (ii) $\tilde{Pr}(\{a^*\}) \equiv \{a^*\}$ implies that a^* is the unique value obtained from the integral of the best response mapping $i \rightrightarrows B(i, a^*)$ and so is an equilibrium. Analogously if $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$, then $\mathbf{s}^*$ is the unique Nash Equilibrium of the game, since all Nash Equilibria are in $\mathbb{P}_S$ and $Pr(\{\mathbf{s}^*\}) \equiv \{\mathbf{s}^*\}$ implies that $\mathbf{s}^*$ is the (unique) measurable selection of $i \rightrightarrows \mathrm{Br}(i, \mathbf{s}^*)$ and so it is a Nash Equilibrium. In particular this says that $\mathrm{Br}(i, \mathbf{s}^*)$ is λ-a.e. single valued and hence can be associated to the concept of strict Nash Equilibrium.[18]

Proposition 3.11. *If $\boldsymbol{s}^*$ is a Nash Equilibrium of $\boldsymbol{u}$ and $\int_I \boldsymbol{s}^* = a^*$, then:*

$$a^* \text{ is Strongly Point Rational} \implies \mathbb{P}_S \equiv \left\{ \boldsymbol{s} \in S^I : \begin{array}{l} s \text{ is a measur-} \\ \text{able selection of} \\ i \rightrightarrows B(i, a^*) \end{array} \right\}$$

$$\mathbb{P}_S \equiv \{\boldsymbol{s}^*\} \implies a^* \text{ is Strongly Point Rational}$$

In particular, if $B(i, \cdot)$ is single valued at a^* λ-a.e. on I, then,

$$a^* \text{ is Strongly Point Rational} \iff \mathbb{P}_S \equiv \{\boldsymbol{s}^*\}$$

Proof.

By the definition of $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{P}_S$ and the property of $\mathbb{P}_{\mathcal{A}}$ stated in Theorem 3.8 we have,

$$\bar{B}(\mathbb{P}_{\mathcal{A}}) \equiv \mathbb{P}_S \tag{3.10}$$

Suppose that $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$. Then, a^* is an equilibrium, so it satisfies $a^* \in \mathbb{P}_{\mathcal{A}}$ which in turn implies that $\mathbb{P}_{\mathcal{A}} \neq \emptyset$. Property (3.10) gives

$$\mathbb{P}_S \equiv \{\mathbf{s}^*\} \implies \tilde{Pr}(\mathbb{P}_{\mathcal{A}}) \equiv \{a^*\}$$

and by the definition of $\mathbb{P}_{\mathcal{A}}$,

$$\mathbb{P}_{\mathcal{A}} \equiv \tilde{Pr}(\mathbb{P}_{\mathcal{A}}) \,.$$

For the proof in the opposite sense, analogously we get:

$$\bar{A}(\mathbb{P}_S) \equiv \mathbb{P}_{\mathcal{A}} \tag{3.11}$$

Since $\mathbf{s}^*$ is a Nash Equilibrium, $\mathbb{P}_S \neq \emptyset$ and then from (3.11) we get:

$$\mathbb{P}_{\mathcal{A}} = \{a^*\} \implies Pr(\mathbb{P}_S) \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measur-} \\ \text{able selection of} \\ i \rightrightarrows B(i, a^*) \end{array} \right\}$$

And from the definition of $\mathbb{P}_S$,

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S)$$

■

[18] A situation in which any unilateral deviation incurs in a loss.

Proposition 3.11 shows that it is possible to study Eductive Stability of models with continuum of agents that fit our framework using the set of states as well as the set of strategies. Moreover, it can be even desirable to use the former approach since (local) uniqueness and stability are more pertinent in terms of the state of the system rather than in terms of strategy profiles, as is discussed in the previous section. For instance, the study of Strategic Complementarities in coordination games or Strategic Substituability in general models as well, can be undertaken by looking at states of the system rather than strategy profiles (see the books by Cooper (1999) and Chamley (2004). See as well Guesnerie (2005) and Guesnerie and Jara-Moroni (2007)).

4 Rationalizability vs Point-Rationalizability

Rationalizability differs from Point-Rationalizability on the way we address forecasts. For Rationalizability, forecasts are no longer points in the corresponding sets but probability distributions whose supports are contained in these sets. Then, when a player has a subjective probability forecast over what may occur with the rest of the economic system, he maximizes his expected utility with respect to such a probability distribution to make a decision. Rationality implies that players should not give positive weight in their forecasts to strategies that are not best response to some rationally generated forecast.

Rationalizable Strategy Profiles, for instance, should be obtained from a similar exercise as done in Subsection 3.1, but considering forecasts as probability measures over the set of strategies of the opponents. Loosely speaking, each player should consider a profile of probability measures (his subjective forecasts over each of his opponents' play) and maximize some expected utility, expectation taken over an induced probability measure over the set of strategy profiles.

A difficulty in a context with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the player's name. There is no straightforward solution in any case. However, in our framework it is possible to bypass this difficulty. Using the intuition just described for the strategy profiles case, we will reformulate the processes of elimination of strategies and states described by equations (3.2) and (3.5) by considering procedures where players eliminate strategies that are not best response to any possible (subjective probability) forecast (profile) over a given set of states or strategy profiles.

We present first, in the next Subsection, the concept of Rationalizable States, where forecasts and the process of elimination are taken over the set of states $\mathcal{A}$. Then, in Subsection 4.4 we will make use of Schmeidler's original framework of games with continuum of players, to formalize the idea of Rationalizable Strategy Profiles in that context.

4.1 Forecasts over the set of states

Before we enter into context we need to introduce some concepts and some notation. For a Borel set $X \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}(X)$ the set of probability measures on the Borel subsets of X. Equivalently this is the set of probability measures on the Borel sets of $\mathbb{R}^n$ whose support is in X. We will endow the set $\mathcal{P}(X)$ with the weak* topology $w^* = \sigma(\mathcal{P}(X), C_{\mathrm{b}}(X))$ [19]. For a Borel subset Z of X in $\mathbb{R}^n$, $\mathcal{P}(Z)$ can be considered to be a subset of $\mathcal{P}(X)$ and the weak* topology on $\mathcal{P}(Z)$ is the relativization of the weak* topology on $\mathcal{P}(X)$ to $\mathcal{P}(Z)$. The set X can be topologically identified with a subspace of $\mathcal{P}(X)$ by the embedding $x \mapsto \delta_x$. An important property that we will use is that X is compact if and only if $\mathcal{P}(X)$ is compact (and metrizable, since we use the norm in $\mathbb{R}^n$) [20].

As we said before, in the setting of Rath there is a simple way to get through the inconvenience of defining an induced probability measure over the set of strategy profiles, using the presence of the state variable of the game over which players have an infinitesimal influence.

We consider then games with an aggregate state. In this setting, we again consider players as having forecasts over the set of states rather than over each of the individual strategy sets. That is, forecasts are probability measures over the set of states rather than profiles of probabilities over the set of strategies. We define then for each player the set valued mapping that gives the actions that maximize expected utility given a probability measure μ over the set of states $\mathcal{A}$, $\mu \in \mathcal{P}(\mathcal{A})$, $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$:

$$\begin{aligned}\mathbb{B}(i, \mu) :&= \operatorname{argmax}_{y \in S} \mathbb{E}_\mu \left[u(i, y, a)\right] \\ &\equiv \operatorname{argmax}_{y \in S} \int_{\mathcal{A}} u(i, y, a)\, \mathrm{d}\mu(a)\,.\end{aligned}$$

As it has been along all this document, we can describe then, using $\mathbb{B}(i, \cdot)$, the process of elimination of unreasonable states, considering that players could now use probability forecasts over the set of states. If it is common knowledge that the actual state is restricted to a subset $X \subseteq \mathcal{A}$ then players will use strategies only in the set $\mathbb{B}(i, \mathcal{P}(X)) := \cup_{\mu \in \mathcal{P}(X)} \mathbb{B}(i, \mu)$. This is, each player $i \in I$ will behave optimally with respect to some subjective belief about the outcome of the game, whose support is contained in X. This means that rational strategy profiles have to be selections of the correspondence $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$ that maps the set of players on their set of optimal responses with respect to any possible forecast over X. The state of the game will then be the integral of one of these selections. This process is described with the mapping $\tilde{R} : \mathcal{B}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$:

$$\begin{aligned}\tilde{R}(X) :&= \left\{ \int_I s(i)\, \mathrm{di} : \begin{array}{l} \mathbf{s} \in S^I, \mathbf{s} \text{ is a measurable selec-} \\ \text{tion of } i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X)) \end{array} \right\}. \\ &\equiv \int_I \mathbb{B}(i, \mathcal{P}(X))\, \mathrm{di}\end{aligned}$$

The set $\tilde{R}(X)$ gives the set of states that are obtained as consequence of optimal behavior when common knowledge about the outcome of the game is represented by X. As we

[19] Recall that $C_{\mathrm{b}}(X)$ is the space of real valued bounded continuous functions on X.

[20] See Aliprantis and Border (1999) for detailed treatments of these and other results.

said before, the difference between $\tilde{Pr}$ and $\tilde{R}$ is that the second process considers expected utility maximizers. For a given Borel set $X \subseteq \mathcal{A}$, X can be embedded into $\mathcal{P}(X)$. This means that $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$ and so we have that $\tilde{Pr}(X) \subseteq \tilde{R}(X)$.

Proposition 4.1. *In a game* $\mathbf{u}$*, if* $X \subseteq \mathcal{A}$ *is nonempty and closed, then* $\tilde{R}(X)$ *is nonempty, convex and closed.*

For the proof we will make use of the following Lemma whose proof is relegated to the appendix.

Lemma 4.2. *Let* Y *and* X *be compact subsets of* $\mathbb{R}^n$*. Given a function* $u \in C_{\mathrm{b}}(Y \times X)$*, the function* $U : Y \times \mathcal{P}(X) \to \mathbb{R}$*, which to each* $(y, \mu) \in Y \times \mathcal{P}(X)$ *associates the expectation*

$$U(y, \mu) \equiv \int_X u(y, x)\, \mathrm{d}\mu(x)\,,$$

is continuous when $\mathcal{P}(X)$ *is endowed with the weak* topology.*

Proof of Proposition 4.1.

From our assumptions, $\mathbf{u}(i)$ belongs to $C_{\mathrm{b}}(S \times \mathcal{A})$, and S and $\mathcal{A}$ are compact sets in $\mathbb{R}^n$, so Lemma 4.2, along with Berges' Theorem, imply that for each $i \in I$ the correspondence $\mathbb{B}(i, \cdot\,) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ is upper semi continuous. Since X is closed, it is compact, which gives that $\mathcal{P}(X)$ is a compact subset of $\mathcal{P}(\mathcal{A})$ and so the set $\mathbb{B}(i, \mathcal{P}(X))$ is closed for every $i \in I$. From Theorem 4 in Aumann (1965) we get that $\int_I \mathbb{B}(i, \mathcal{P}(X))\,\mathrm{di}$ is closed.

On the other hand, Lemma 3.2 states that the correspondence $i \rightrightarrows B(i, X)$ is measurable, which means that it has a measurable selection $\mathbf{s}$. Since $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$, $\mathbf{s}$ is also a selection of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$. This implies that $\int_I \mathbb{B}(i, \mathcal{P}(X))\,\mathrm{di}$ is nonempty.

Convexity comes from the fact that $\tilde{R}(X)$ is obtained as an integral of a set valued mapping.

■

The previous Proposition allows us to define the Eductive Process in this case. As usual then we consider the iterative elimination of non generated states, but now allowing for probability forecasts of players. The iterative process begins with the whole set of outcomes, in this case $\mathcal{A}$.

$$\tilde{R}^0(\mathcal{A}) := \mathcal{A}$$

Then, on each iteration, the states that are not reached by the process $\tilde{R}$ are eliminated:

$$\tilde{R}^{t+1}(\mathcal{A}) := \tilde{R}\Big(\tilde{R}^t(\mathcal{A})\Big)\,.$$

Recall that since we start the process at $\mathcal{A}$, what we get is a nested family of sets that, following Proposition 4.1, are nonempty convex and closed. The Eductive Process gives then the set,

$$\mathbb{R}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A})\,.$$

Theorem 4.3. *In a game $\boldsymbol{u}$, the set $\mathbb{R}'_{\mathcal{A}}$ is non empty, convex and closed.*

Proof.
$\mathbb{R}'_{\mathcal{A}}$ is the intersections of closed convex sets, so it is convex and closed. Theorem 2.1 assures that $\mathbb{R}'_{\mathcal{A}}$ is nonempty, since equilibria belong to every set $\tilde{R}^t(\mathcal{A})$.

■

As it was the case before, the assumptions of rationality and common knowledge of rationality imply that players must take into account that all their opponents construct their subjective forecasts rationally. This is formalized by asking that the set of Rationalizable States must be a subset of $\mathbb{R}'_{\mathcal{A}}$ (analogously to (3.6)), in the sense that states that are eliminated can not rise with positive probability and hence are not rationalizable. On the other hand, if a state is rationalizable then it must be an outcome associated to optimal reactions to forecasts with support in the set of Rationalizable States, this means that the set of Rationalizable States must satisfy an analogous condition to (3.7). This is, the set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$ must satisfy

$$\mathbb{R}_{\mathcal{A}} \subseteq \tilde{R}(\mathbb{R}_{\mathcal{A}})\,. \tag{4.1}$$

Note that if a set satisfies condition (4.1), then it is a subset of the set $\mathbb{R}'_{\mathcal{A}}$.

Definition 4.4. The set of Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies:

$$X \subseteq \tilde{R}(X)$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

As we have already said, from the definition of the set of Rationalizable States, we have $\mathbb{R}_{\mathcal{A}} \subseteq \mathbb{R}'_{\mathcal{A}}$. The following Theorem, analogous of Theorem 3.6, shows that these two sets are actually the same.

Theorem 4.5. *The set of Rationalizable States of a game $\boldsymbol{u}$ can be calculated as*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

Although there are some aspects in which attention must be put. The proof of Theorem 4.5 follows the proof of Theorem 3.6 and is therefore relegated to the appendix.

We get directly that,

Proposition 4.6. *In a game $\boldsymbol{u}$ we have:*

$$\mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

In Proposition 4.6 it is not possible to obtain the equality in a general context. We give a sufficient condition in our setting to have it.

Proposition 4.7. *If in a game* $\boldsymbol{u}$*, we have* $\forall\ \mu \in \mathcal{P}(\mathcal{A})$*:*

$$\mathbb{E}_\mu\left[u(i,y,a)\right] \equiv u(i,y,\mathbb{E}_\mu\left[a\right])$$

then

$$\mathbb{P}_\mathcal{A} \equiv \mathbb{R}_\mathcal{A}$$

Proposition 4.7 says that if the utility functions are affine in the state variable, then we have that the Point-Rationalizable States set is equal to the set of Rationalizable States. Below we will see that we do have a general setting where the set of Rationalizable Strategies is well defined and in which we can get a result of equivalence between Point and standard Rationalizability in the strategy sets, improving the statement of Proposition 4.6. We address this issue in Subsection 4.4.

Proof.
If $\mathbb{E}_\mu\left[u(i,y,a)\right] \equiv u(i,y,\mathbb{E}_\mu\left[a\right])$ then

$$\mathbb{B}(i,\mu) \equiv B(i,\mathbb{E}_\mu\left[a\right])\,,$$

which implies that

$$\mathbb{B}(i,\mathcal{P}(X)) \equiv \bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu\left[a\right])\,.$$

For a convex set $X \subseteq \mathcal{A}$ we have $\mathbb{E}_\mu\left[a\right] \in X$, $\forall\ \mu \in \mathcal{P}(X)$.

This implies that under the hypothesis of the Proposition if X is convex,

$$\bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu\left[a\right]) \subseteq \bigcup_{a\in X} B(i,a)$$

and consequently

$$B(i,X) \subseteq \mathbb{B}(i,\mathcal{P}(X)) \equiv \bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu\left[a\right]) \subseteq \bigcup_{a\in X} B(i,a) \equiv B(i,X)\,.$$

This is, $\mathbb{B}(i,\mathcal{P}(X)) \equiv B(i,X)$ and so $\tilde{R}(X) \equiv \tilde{Pr}(X)$.

Noting that $\mathcal{A}$ is convex, we get $\tilde{R}(\mathcal{A}) \equiv \tilde{Pr}(\mathcal{A})$ which are as well convex. By induction we get that $\tilde{Pr}^t(\mathcal{A}) \equiv \tilde{R}^t(\mathcal{A})\ \forall\ t$ which gives (using the previous notation)

$$\mathbb{R}'_\mathcal{A} \equiv \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A}) \equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A}) \equiv \mathbb{P}'_\mathcal{A} \tag{4.2}$$

where these intersections give closed convex sets.

Finally we get,

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \tilde{R}(\mathbb{P}'_{\mathcal{A}}) \equiv \tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

which implies that $\mathbb{R}'_{\mathcal{A}} \equiv \mathbb{R}_{\mathcal{A}}$. The first inequality comes from (4.2), the second one is true because $\mathbb{P}'_{\mathcal{A}}$ is convex, the third one comes from Theorem 3.6 which states that $\mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$ and the last one comes again from (4.2).

We conclude that $\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}} \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$.

■

4.2 More of Example 1

Let us now illustrate our results and definitions with the example that motivates this presentation. In this example the strategy set was $\mathbb{R}_+$. Without loss of generality we can assume it to be a compact interval $S \equiv [0, q_{\max}]$, where $q_{\max}$ could be the quantity that makes the price equal to 0 : $q_{\max} := \inf\{P^{-1}(0)\}$.

Now we identify the state set. As we have already said, we could choose the state set to be the set of aggregate production quantities or the set of prices. This depends on the aggregation operator that we are considering.

- Let us consider first A to be the operator that gives aggregate production quantities. This is, $A : S^I \to \mathbb{R}_+$

  $$A(\mathbf{s}) \equiv \int_I s(i)\,\mathrm{di}.$$

  The set $\mathcal{A}$ is the interval

  $$\begin{aligned} \mathcal{A} &\equiv \left\{ q \in \mathbb{R} \ : \ \exists\, \mathbf{s} \in S^I,\ q = \int_I s(i)\,\mathrm{di} \right\} \\ &\equiv \mathrm{co}\,\{[0, q_{\max}]\} \\ &\equiv [0, q_{\max}]\,. \end{aligned}$$

  The payoff function $u(i, \cdot, \cdot) : [0, q_{\max}] \times [0, q_{\max}] \to \mathbb{R}$ is then:

  $$u(i, s(i)\,, Q) \equiv P(Q)\, s(i) - c_i(s(i))\,.$$

  If we assume P and c_i to be continuous and that the measurability requirement over the function $i \to c_i(\cdot)$ is met (for instance if all the producers have the same cost function), Theorem 3.6 holds and so we can compute the Point-Rationalizable State set using the eductive procedure described by the right hand side of (3.6). We get as well the result of Corollary 3.7 and we know then that this set is a compact interval.

- Now suppose that we use a variation, as in footnote 10, of the aggregation operator. In the same setting we will consider the state set to be the set of prices. This is, $A : S^I \to \mathbb{R}_+$

  $$A(\mathbf{s}) \equiv P\left(\int_I s(i)\,\mathrm{di}\right).$$

  This is not the aggregation operator for which we obtained the results. However, we will see below that they still hold. The set of states $\mathcal{A}$ will be identified with the set:

  $$\begin{aligned}\mathcal{A} &\equiv \{p \in \mathbb{R} \ : \ \exists\, q \in [0, q_{\max}]\,,\ p = P(q)\} \\ &\equiv P([0, q_{\max}]) \\ &\equiv [0, p_{\max}]\,.\end{aligned}$$

  We see that since P is a continuous function that goes from one-dimensional aggregate-production set $\mathbb{R}_+$ to the set of prices $[0, p_{\max}] \subset \mathbb{R}$, this set turns out to be convex.

  The utility function is now $u(i, \cdot\,, \cdot) : [0, q_{\max}] \times [0, p_{\max}] \to \mathbb{R}$:

  $$u(i, s(i)\,, p) \equiv ps(i) - c_i(s(i))\,.$$

  Continuity of P implies that Theorem 3.6 still holds and so we can compute the Point-Rationalizable State set using the eductive procedure described by the right hand side of (3.6) using A instead of the integral. Since the strategy and state sets are unidimensional[21] we get the result of Corollary 3.7 and we know then that this set is a compact interval.

  Furthermore, in this second approach to the example we have that the payoff function is affine in the state variable and so Proposition 4.7 holds and we then see that what we are actually calculating is in fact the set of Rationalizable States (Rationalizable Prices).

The equivalence between Rationalizable and Point-Rationalizable sets for the case of prices is obtained directly from Proposition 4.7. The question remains on whether this holds for the first approach. Clearly the payoff function is not necessarily affine on production quantities and so we can not apply this Proposition directly. However, without making any further assumptions we do have that the set of Rationalizable and Point-Rationalizable States in the " aggregate production quantities" approach are the same. For this, note that we have already argued that the set of Point-Rationalizable States is an interval in $\mathbb{R}$. Then, since the model presents strategic substitutes the limits of this interval are actually the largest and smallest Rationalizable States Guesnerie and Jara-Moroni (2007). This implies that the whole interval is the set of Rationalizable States.

[21]Otherwise we would have it if we had used only the integral and so aggregate production instead of prices.

4.3 Forecasts over the set of strategies

When we consider forecasts over the sets of strategies, players should have a prior over each of the other player's individual actions. A forecasts in this case then would be a profile of probability measures with support in the set of strategies S. The question is not trivial since what we would have in this case is a continuum of random variables indexed by the set of players and it is not clear how payoff should depend on this profile of probability measures. However, in the setting of the original paper by Schmeidler (1973) this technical issue can be overtaken since in this setting payoffs depend on the profiles of probability measures that represent mixed strategies, which are the same mathematical objects as forecasts. We will give first a loose description of how the eductive process should work when agents use forecasts over the set of strategies, to continue with a formal description for the case where S is a finite pure strategy set.

Given a set of strategy profiles $H \subseteq S^I$, consider the set of strategies that a player $i \in I$ may use in strategy profiles in H and denote it $H(i)$:

$$H(i) := \{ \ y \in S \ : \ y = s(i) \, , \text{and } \mathbf{s} \in H \} \, .$$

Ideally we would like to have forecasts on this set, and use the set $\mathcal{P}(H(i))$. Since we do not know whether $H(i)$ is a Borel set, we may use for instance the closure of $H(i)$, $\text{cl}\{H(i)\}$ and consider then $\mathcal{P}(\text{cl}\{H(i)\})$.

We will say that a (measurable) mapping $\mathbf{m} : I \to \mathcal{P}(S)$ is a *forecast profile over H* if $m(i)$ has support on $\text{cl}\{H(i)\}$ λ-a.e.. The question is, how should rational forecasts be generated?

If originally any strategy can be used, when thinking about possible actions taken by their rivals players should consider any possible forecast profile in the set of actions. This should generate for each player a set of best-replies-to-forecasts. The issue is still on how players use their forecasts to generate this set, but suppose we have this. Once all the players have done this exercise, we will have a correspondence that maps players to the set of strategies that represents all the possible strategy profiles that could be played reacting optimally to some forecast (where different players could be using different forecasts, recall that this is an issue of forecasting the forecasts of the others). This correspondence would be the result of a first iteration and should be the point of depart for the second iteration. Forecasts now should be profiles of probability measures where the support of each probability should be in the closure of the set associated to each player by this correspondence.

We define the mapping of best-reply-to-forecasts as the set of strategies that maximize some "expected" utility given a forecast profile on S, $\mathbb{B}\text{r}(i, \cdot\,) : \mathcal{P}(S)^I \rightrightarrows S$:

$$\mathbb{B}\text{r}(i, \mathbf{m}) := \text{argmax}_{y \in S} \, \mathbb{E}_{\mathbb{P}(\mathbf{m})} \left[\pi(i, y, \mathbf{s}) \right] .$$

We use the notation $\mathbb{E}_{\mathbb{P}(\mathbf{m})}$ to represent that the payoff that is being maximized is an expected payoff where the expectation comes from the fact that players are using non-degenerate forecasts over the set of strategies. We note $\mathbb{P}(\mathbf{m})$ to indicate that the profile

$\mathbf{m}$ induces in some sense a probability measure over the set of strategy profiles. We do not give an answer here on to how this is done.

Given a measurable set valued mapping $F : I \rightrightarrows S$, we can obtain, for each agent $i \in I$, the set of best-replies to forecasts over this mapping as:

$$\mathbb{B}\mathrm{r}(i, F) := \left\{ y \in S : \begin{array}{l} y \in \mathbb{B}\mathrm{r}(i, \mathbf{m}), \mathbf{m} \text{ is a fore-} \\ \text{cast profile over } F \end{array} \right\}. \quad (4.3)$$

Finally, we can define the process of elimination of non best-reply-to-forecasts, described in the previous paragraphs, with the mapping R that takes a set valued mapping $F : I \rightrightarrows S$ and returns a subset $R(F) \subseteq S^I$,

$$R(F) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measur-} \\ \text{able selection of} \\ i \rightrightarrows \mathbb{B}\mathrm{r}(i, F) \end{array} \right\}. \quad (4.4)$$

The process[22] described by equation (4.4) considers that strategy profiles that are "kept" are those that can be constructed from best replies of agents taking decisions considering any of the possible forecast profiles over F. Of course, as was the case before, on a same strategy profile $\mathbf{s}$ of $R(F)$ the strategies of two different agents can be best-responses to two different forecast profiles over F.

Definition 4.8. The set of Rationalizable Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$H \subseteq R(H)$$

and we note it $\mathbb{R}_S$.
For each player, $i \in I$, there will be a set of Rationalizable Strategies, namely the union, over all the rationalizable strategy profiles in $\mathbb{R}_S$, of the best response set of the considered player. That is, the set of Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{R}_S(i) := \mathbb{B}\mathrm{r}(i, \mathbb{R}_S)$$

So now that we have presented the main ideas of rationalizability in terms of strategies, let's turn to a setting where the best-reply-to-forecast mapping has a concrete sense.

4.4 Games with a continuum of players and finite strategy set

In Schmeidler's formulation of a game with a continuum of players, payoff functions $\pi(i, \cdot, \cdot)$ are defined on the product set $\Delta \times \Delta^I$, where $\{e^1, \cdots, e^L\} \subseteq \mathbb{R}^L$ is a finite set of pure strategies that we identify with the canonical base of $\mathbb{R}^L$ and $\Delta \equiv \mathrm{co}\{\{e^1, \cdots, e^L\}\}$

[22] The set $R(F)$ can itself be regarded as a set valued mapping from I to S. This has to be taken in account to have the set of Rationalizable Strategies well defined.

is the set of mixed strategies, the convex hull of $\{e^1, \cdots, e^L\}$ and the simplex in $\mathbb{R}^L$. The functions $\pi(i, \cdot, \cdot) : \Delta \times \Delta^I \to \mathbb{R}$ take in this setting the form:

$$\pi(i, y, \mathbf{m}) := y \cdot h(i)(\mathbf{m}) \tag{4.5}$$

where $h(i) : \Delta^I \to \mathbb{R}^L$ is an auxiliary vector utility function whose coordinate l gives the utility of player i when he chooses action e^l, $\mathbf{m}$ is a (mixed) strategy profile and $y \in \Delta$ is a (mixed) strategy of player i. If payoffs of players depend on the integral of the (mixed) strategy profile $\mathbf{m}$, $\int_I \mathbf{m}$, then we can say that Schmeidler's setting is ours with $S \equiv \mathcal{A} \equiv \Delta$. In this case the functions $h(i)$ can be regarded as depending only on the values of the integrals (as in our setting)[23]. As quoted by Schmeidler himself, the central result of his paper is the existence and purification theorem, Theorem 4.9 below, where the main assumption is precisely this last one. We state this theorem in the context of our framework.

Theorem 4.9 (Schmeidler, 1973)**.** *If the following conditions are satisfied:*

1. *The functions $h(i)$ depend only on the integral of the mixed strategy profile $\boldsymbol{m}$,*
2. *The functions $\hat{h}(i) : \Delta \to \mathbb{R}^L$ such that $h(i)(\boldsymbol{m}) \equiv \hat{h}(i)\big(\int \boldsymbol{m}\big)$, are continuous,*
3. *For all $\boldsymbol{m} \in \Delta^I$ and all $l, k \in \{1, \ldots, L\}$ the set*

$$\left\{ i \in I \ : \ h(i)^l(\boldsymbol{m}) > h(i)^k(\boldsymbol{m}) \right\}$$

is measurable,

then there exists a Pure Strategy *Nash Equilibrium of the game π.*

This theorem motivates that we look at Schmeidler's formulation from a slightly different point of view. As we have already said, one possibility is to consider this setting in the context of a game $\mathbf{u}$ where the set of strategies, S, is $S \equiv \Delta$. The implicit properties imposed on payoff functions in this definition imply the hypothesis of Theorem 4.9 and so we know that the results stated so far are true for the set of mixed strategies. However, if we keep focusing on the set of pure strategies, we can benefit from the structure of Schmeidler's formulation to have a well defined best-reply-to-forecast mapping. The payoff functions depend on the integral of the mixed strategy profile, in particular when we consider only pure strategies they also depend on the integral of pure strategy profiles. We can make then a difference between (Point-)Rationalizability in pure or mixed strategies. At this point we have to introduce some more notation: we will represent by a subscript on the corresponding set or operator whether we are considering pure or mixed strategies [24]. In Schmeidler's formulation we have $S_p \equiv \{e^1, \cdots, e^L\}$ and $S_m \equiv \Delta$. We continue to

[23]See Rath (1992) for a discussion on this matter.

[24]For instance, the pointwise eductive procedure Pr can be defined on pure strategies:

$$Pr_p(H) \equiv \left\{ \mathbf{s} \in S_p^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows \mathrm{Br}_p(i, H) \end{array} \right\}$$

consider the game where $S \equiv \mathcal{A} \equiv \Delta$ and so $\mathcal{A}$ is the same as the mixed strategy set Δ which in turn is equal to the set of probability measures over the set of pure strategies $\mathcal{P}(S_p)$, and so we get:

$$S \equiv S_m \equiv \mathcal{A} \equiv \Delta \equiv \mathcal{P}\big(\{e^1, \cdots, e^L\}\big) \equiv \mathcal{P}(S_p)\,.$$

If S_p is a finite pure strategy set, then any forecast in the form of a probability distribution over a subset $Y \subseteq S_p$ can be considered as a point in $\mathcal{P}(Y) \equiv \mathrm{co}\,\{Y\} \subseteq \Delta \equiv \mathrm{co}\,\{S_p\}$. So a forecast profile would be a function $\mathbf{m} : I \to \mathrm{co}\,\{Y\}$. Since mixed strategies are the same mathematical objects as probability forecasts over the set of actions of each player, in the current setting we get to identify the expected utility $\mathbb{E}_{\mathbb{P}(\mathbf{m})}\left[\pi(i, y, \mathbf{s})\right]$ mentioned above, with the following expression:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}(\mathbf{m})}\left[\pi(i, y, \mathbf{s})\right] &\equiv y \cdot h(i)(\mathbf{m}) \\ &\equiv u\left(i, y, \int \mathbf{m}\right) \end{aligned} \tag{4.6}$$

Where now $\mathbb{E}_{\mathbb{P}(\,\cdot\,)}\left[\pi(i,\,\cdot\,, \mathbf{s})\right] : S_p \times \Delta^I \to \mathbb{R}$ is a function that depends on the *pure* strategy $y \in S_p$ and we interpret the profile of probability distributions $\mathbf{m} \in \Delta^I \equiv \mathcal{P}(S)^I$ as a forecast, under the hypothesis that $h(i)$ depends on $\mathbf{m}$ through the integral.

In this setting then we get that the set of forecast profiles on a mapping $F : I \rightrightarrows S_p$ can be described by the mapping $\mathrm{co}\,\{F\} : I \rightrightarrows \Delta$ defined as $\mathrm{co}\,\{F\}(i) := \mathrm{co}\,\{F(i)\}$. With this, we can now consider six different rationalizable sets:

1. The set of Point-Rationalizable Pure Strategies $\mathbb{P}_{S_p}$
2. The set of Point-Rationalizable Mixed Strategies $\mathbb{P}_{S_m}$
3. The set of Rationalizable Pure Strategies $\mathbb{R}_{S_p}$
4. The set of Rationalizable Mixed Strategies $\mathbb{R}_{S_m}$
5. The set of Point-Rationalizable States $\mathbb{P}_{\mathcal{A}}$
6. The set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$

Where the first three have been proved to be well defined and can be obtained from the eductive processes defined by optimal strategies and forecasts on the corresponding sets. The last two have been discussed with more detail in Subsections 3.2 and 4.1.

Theorem 4.10. *In a game $\boldsymbol{u}$ where $S \equiv \Delta$ we have,*

$$\forall\ t, \quad Pr_p^t\big(S_p^I\big) \equiv R_p^t\big(S_p^I\big)\,.$$

---

where $H \subseteq S_p^I$ and $\mathrm{Br}_p(i,\,\cdot\,) : S_p^I \rightrightarrows S_p$ is defined by:

$$\mathrm{Br}_p(i, \mathbf{s}) \equiv \mathrm{argmax}\,\{\pi(i, y, \mathbf{s}) \;:\; y \in S_p\}$$

Proof.

From the relation in (4.6) we can see that to look at all forecast profiles is equivalent to look at all the integrals of such profiles. Moreover, we see that in equation (4.3), and considering F to be the constant mapping $F(i) \equiv S_p$, the set of forecast profiles over F is the set valued mapping $\text{co}\{F\}$ as defined above. That is, to obtain $\text{Br}_p\big(i, S_p^I\big)$ we are interested in the integral of F while the calculus of $\mathbb{B}\text{r}_p(i, F)$ considers the integral of $\text{co}\{F\}$. From Aumann (1965) the integral of the convex hull mapping is equal to the integral of the mapping itself and so we have:

$$\int_I F(i)\,\text{di} \equiv \int_I \text{co}\{F\}(i)\,\text{di}.$$

So no matter whether we are considering point or standard forecasts we obtain the same set of states. In consequence we get that the set of maximizers is the same:

$$\text{Br}_p\big(i, S_p^I\big) \equiv \mathbb{B}\text{r}_p(i, F)$$

Thus we take measurable selections from the same mapping and so,

$$Pr_p\big(S_p^I\big) \equiv R_p\big(S_p^I\big)\,.$$

By induction over t we get that

$$Pr_p^t\big(S_p^I\big) \equiv R_p^t\big(S_p^I\big)\,.$$

■

Corollary 4.11. *In a game* $\boldsymbol{u}$ *where* $S \equiv \Delta$ *we have,*

$$\mathbb{R}_{S_p} \equiv \mathbb{P}_{S_p}$$

Corollary 4.11 says that in the setting where payoff functions depend on the integral of the mixed strategy profile, and we consider S to be the set of mixed strategies associated to a finite pure strategy set, then we get that Point-Rationalizability is equivalent to Rationalizability in terms of pure strategy profiles.

In this context we identify the set $\mathbb{P}_S$ with $\mathbb{P}_{S_m}$ and so from Theorem 3.8 we know that $\mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_{S_m})$. What can we say about the relation between $\mathbb{P}_{\mathcal{A}}$ and the (Point-)Rationalizable sets in pure strategies? An answer is given in Corollary 4.12.

Corollary 4.12. *In a game* $\boldsymbol{u}$ *where* $S \equiv \Delta$ *we have,*

(i) $\mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_{S_m})$ *and* $\mathbb{P}_{S_m} \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \equiv \left\{ \boldsymbol{m} \in S_m^I : \begin{array}{l} \boldsymbol{m} \text{ is a measurable selection of} \\ i \rightrightarrows B_m(i, \mathbb{P}_{\mathcal{A}}) \end{array} \right\};$

(ii) $\mathbb{P}_{\mathcal{A}} \equiv \bar{A}\big(\mathbb{P}_{S_p}\big)$ *and* $\mathbb{P}_{S_p} \equiv \left\{ \boldsymbol{s} \in S_p^I : \begin{array}{l} \boldsymbol{s} \text{ is a measurable selection of} \\ i \rightrightarrows B_p(i, \mathbb{P}_{\mathcal{A}}) \end{array} \right\}.$

Proof.

Item (i) is the exact same result of Theorem 3.8.

Now note that the best response mappings $B_p(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S_p$ and $B_m(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S_m$ satisfy for $X \subseteq \mathcal{A}$:

$$\cup_{a\in X} B_p(i,a) \subseteq \cup_{a\in X} B_m(i,a) \equiv \cup_{a\in X} \operatorname{co}\{B_p(i,a)\} \subseteq \operatorname{co}\{\cup_{a\in X} B_p(i,a)\}\,,$$

and so we get

$$\tilde{Pr}_p(X) \equiv \int_I B_p(i,X)\,\mathrm{di} \subseteq \int_I B_m(i,X)\,\mathrm{di} \subseteq \int_I \operatorname{co}\{B_p(i,X)\}\,\mathrm{di} \equiv \tilde{Pr}_p(X)\,,$$

where the integral in the middle is $\tilde{Pr}_m(X)$. Point (ii) of the Corollary is then consequence of Theorem 3.8.

■

This is, in a game $\mathbf{u}$ with $S \equiv \Delta \equiv S_m$ we have that the set of Rationalizable Pure Strategies is equal to the set of Point-Rationalizable Pure Strategies, and these sets are paired with the set of Point-Rationalizable States which in turn is paired with the set of Point-Rationalizable Mixed Strategies.

Finally let's note that the hypothesis of Theorem 4.9 are implied by the assumptions on $\mathbf{u}$ when we consider the set S to be the set of mixed strategies of a finite strategy set game. Moreover, since we want to deal with rationalizability in terms of pure strategies, it is not enough to identify S with the finite set of pure strategies, since in that case we would not be asking that the utility functions depended on the mixed strategy profiles through their integral which is crucial for our results.

5 Comments and Conclusions

In this work we have formally introduced the concept of Rationalizability for models that use a continuum of agents. We have proposed a definition for Point-Rationalizable Strategies in the context of general games with a continuum of players, considering the original characterization for games with finite set of players, compact strategy sets and continuous utility functions; as the maximal subset of the strategy profiles set that satisfies being a fixed point of the process of elimination of non-best response strategies. When such models have the particularity that payoffs depend on other players' actions through an aggregate variable that cannot be unilaterally affected, we have defined as well the set of Point-Rationalizable States. This last setting is an important generalization of several models that explore Rational Expectations in economics such as models of currency attacks, stag hunts, standard markets, macroeconomic dynamics and global games.

We have given sufficient conditions that allow the (Point-)Rationalizable sets to be well defined and characterized. As in the case of finite player games, continuity properties of the payoff functions are crucial to assure the convergence of the process of elimination of non-best replies (the *eductive process*) to the rationalizable set. For the continuum of players case, an additional measurability assumption must be made on the mapping that associates players to their payoff functions to be able to have existence of equilibrium. It turns out that this same assumption is sufficient to assure the integrability of the set valued mapping that is used in the eductive process and, in consequence, to obtain the constructive characterization of the different rationalizbale sets introduced throughout the document.

The set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. We have shown that the set of Point-rationalizable States can be obtained, as in the case of finite player games with (Point-)Rationalizable Strategies, by eliminating unreasonable states. Moreover, this set is non-empty, convex and compact.

We have seen as well that for the most important application of Rationalizability in economic models, namely Strong Rationality, it is equivalent in terms of properties and more desirable in terms of tractability to use the state approach rather than the strategy profile approach.

To incorporate standard Rationalizability to our framework, we have formally defined Rationalizable States. We give a similar characterization for this set and we give a sufficient (but not at all general) condition on payoff functions, in order to have equivalence between standard and point Rationalizability.

In the particular case where the strategy sets are finite and payoff functions depend on the integral of the mixed strategy profile, we were able to formally define Rationalizable strategies and we have extended an equivalence result to Rationalizability vs. Point-Rationalizability in terms of pure strategy profiles, which in turn implies that in this setting the three concepts: Rationalizable Pure Strategies, Point-Rationalizable Pure Strategies and Point-Rationalizable States; give the same outcomes.

We have defined a key concept in a unified exploratory framework that encompasses a variety of economic models. With this, we have a general framework on which we can study general properties of equilibria such as (local) eductive stability of equilibria and applications to models with continuum of agents that feature strategic complementarities or substitutes (Cooper (1999), Chamley (2004), Guesnerie (2005), Guesnerie and Jara-Moroni (2007)).

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A Relegated proofs

Proof of Lemma 3.2.

We show first that the mapping $\mathrm{G} : I \rightrightarrows \mathcal{A} \times S$, that associates with each agent $i \in I$ the graph of the best response mapping $B(i, \cdot)$, $\mathrm{G}(i) := \operatorname{gph} B(i, \cdot)$, is measurable.

Take a closed set $C \subseteq \mathcal{A} \times S$. We need to prove that the set

$$\mathrm{G}^{-1}(C) \equiv \{i \in I \ : \ C \cap \operatorname{gph} B(i, \cdot\,) \neq \emptyset\}$$

is measurable. Consider the subset $U \subseteq \mathcal{U}_{S\times\mathcal{A}}$ defined by:

$$U := \{\ g \in \mathcal{U}_{S\times\mathcal{A}} \ : \ \exists\,(a,s) \in C \text{ such that } g(s,a) \geq g(y,a)\ \forall\ y \in S\}$$

note that $\mathbf{u}^{-1}(U) \equiv \mathrm{G}^{-1}(C)$ and so, from the measurability assumption over $\mathbf{u}$, it suffices to prove that U is closed. That is, we have to show that for any sequence $\{g^\nu\}_{\nu\in\mathbb{N}} \subset U$, such that $g^\nu \to g^*$ uniformly $g^* \in U$.

Since the functions g^ν are finite and continuous in $S \times \mathcal{A}$, from Weierstrass' Theorem g^* is continuous and so it belongs to $\mathcal{U}_{S\times\mathcal{A}}$. Moreover, g^ν converges continuously to g^*, that is, for any convergent sequence (a^ν, s^ν) with limit (a^*, s^*), the sequence $g^\nu(s^\nu, a^\nu)$ converges to $g^*(s^*, a^*)$. Indeed, consider any $\varepsilon > 0$. By the continuity of g^* there exists $\bar\nu_1 \in \mathbb{N}$ such that $\forall\ \nu > \bar\nu_1$,

$$2|g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \frac{\varepsilon}{2}.$$

From the uniform convergence of g^ν we get that there exists $\bar\nu_2 \in \mathbb{N}$ such that,

$$|g^\nu(s,a) - g^*(s,a)| < \frac{\varepsilon}{2} \qquad \text{for all } \nu \geq \bar\nu_2 \text{ and } \forall\,(s,a) \in S \times \mathcal{A},$$

in particular this is true for all the elements of the sequence of points. We get then that $\forall$ $\nu \geq \max\{\bar\nu_1, \bar\nu_2\}$,

$$\begin{aligned}|g^\nu(s^\nu, a^\nu) - g^*(s^*, a^*)| &\leq |g^\nu(s^\nu, a^\nu) - g^*(s^\nu, a^\nu)| \\ &\quad + |g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \varepsilon.\end{aligned}$$

We have to show then that there exists a point $(a,s) \in C$ such that $g^*(s,a) \geq g^*(y,a)$ $\forall\ y \in S$. Since $g^\nu \in U$, we have for each $\nu \in \mathbb{N}$, points $(a^\nu, s^\nu) \in C$ such that $g^\nu(s^\nu, a^\nu) \geq g^\nu(y, a^\nu)\ \forall\ y \in S$. Let $(a^*, s^*) \in C$ be the limit of a convergent subsequence of $\{(a^\nu, s^\nu)\}_{\nu\in\mathbb{N}}$, which without loss of generality we can take to be the same sequence. We see that (a^*, s^*) is the point we are looking for since for a fixed $y \in S$, continuous convergence implies that in the limit

$$g^*(s^*, a^*) \geq g^*(y, a^*)\,.$$

We conclude then that $g^* \in U$. Thus, U is closed and since $\mathbf{u}$ is a measurable mapping, $\mathbf{u}^{-1}(U)$ is measurable.

With this in mind, consider a closed set $X \subseteq \mathcal{A}$ and the mapping $i \rightrightarrows B(i, X)$. Applying Theorem 14.13 in Rockafellar and Wets (1998) to the constant mapping $i \rightrightarrows X$ along with G above, we get that the correspondence $i \rightrightarrows B(i, X)$ is measurable and has closed values (hence compact since S is compact).

■

With Lemma 3.2 we can now prove Theorem 2.1.

Proof of Theorem 2.1:
Consider the correspondence $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$ defined by

$$\Gamma(a) := \int_I B(i,a)\,\mathrm{di}.$$

Note that a fixed point of Γ defines an equilibrium of the game $\mathbf{u}$. Lemma 3.2 implies that for all $a \in \mathcal{A}$, $\Gamma(a) \neq \emptyset$. By definition, for all $a \in \mathcal{A}$, $\Gamma(a)$ is convex. Under our assumptions, the correspondences $B(i,\cdot) : \mathcal{A} \rightrightarrows S$ are u.s.c. and from Aumann (1976) so is Γ. This last assertion implies as well that $\Gamma(a)$ is compact $\forall\, a \in \mathcal{A}$. Applying Kakutani's fixed point Theorem we get that there exists $a^* \in \mathcal{A}$ such that $a^* \in \Gamma(a^*)$.

■

Proof of Lemma 4.2.
We write U as the composition of two functions:

$$(y,\mu) \in Y \times \mathcal{P}(X) \quad \to (u(y,\cdot),\mu) \in C_{\mathrm{b}}(X) \times \mathcal{P}(X)$$
$$\text{and } (f,\mu) \in C_{\mathrm{b}}(X) \times \mathcal{P}(X) \to \int_X f(x)\,\mathrm{d}\mu(x)$$

If we endow $C_{\mathrm{b}}(X)$ with the sup norm topology and $\mathcal{P}(X)$ with the weak* topology, from Corollary 15.7 in Aliprantis and Border (1999) we get that $(f,\mu) \to \int f \mathrm{d}\mu$ is continuous on $C_{\mathrm{b}}(X) \times \mathcal{P}(X)$.

Therefore, the result will follow from the continuity of the function

$$(y,\mu) \to \mathcal{F}(y,\mu) = (u(y,\cdot),\mu)\,.$$

Note first that this function is defined component to component by functions that depend each only on one variable, this is $\mathcal{F}(y,\mu) = (\mathcal{F}_1(y), \mathcal{F}_2(\mu))$, and second that $\mathcal{F}_2$ is the identity. Thus, we only need to prove that $\mathcal{F}_1 : Y \to C_{\mathrm{b}}(X)$ is continuous for the sup norm topology in $C_{\mathrm{b}}(X)$.

Let $y^\nu \to y$ and take $\varepsilon > 0$.

Since $Y \times X$ is compact and u is in $C_{\mathrm{b}}(Y \times X)$, this function is as well uniformly continuous. Thus, $\exists\, \delta > 0$ (that depends only on ε) such that

$$\|(y,x) - (y',x')\| < \delta \quad \Longrightarrow \quad |u(y,x) - u(y',x')| < \frac{\varepsilon}{3}$$

Since X is compact $\exists\, \{x_1,\ldots,x_N\} \subset X$ such that $X \subseteq \cup_{i=1}^N \mathrm{B}(x_i,\delta)$. This is, for any $x \in X$ there exists x_i in the previous set such that $x \in \mathrm{B}(x_i,\delta)$.

Finally, since $y^\nu \to y$, there exist for each x_i numbers $\bar{\nu}_i$ such that

$$\|(y^\nu,x_i) - (y,x_i)\| < \delta$$

for all $\nu \geq \bar{\nu}_i$.

All together gives, for $\nu \geq \max\{\bar{\nu}_i \ : \ i \in \{1, \dots, N\}\}$ and $x \in X$:

$$\begin{aligned}|u(y^\nu, x) - u(y, x)| &< |u(y^\nu, x) - u(y^\nu, x_i)| + |u(y^\nu, x_i) - u(y, x_i)| \\ &\qquad + |u(y, x_i) - u(y, x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

We conclude that $u(y^\nu, \cdot)$ converges to $u(y, \cdot)$ for the sup norm topology, which completes the proof.

■

We use this Lemma to prove Theorem 4.5:

Proof of Theorem 4.5.

We will show that:

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \mathbb{R}'_{\mathcal{A}}$$

Theorem 4.3 assures that $\tilde{R}(\mathbb{R}'_{\mathcal{A}})$ is correctly defined.

Suppose that $a \in \tilde{R}(\mathbb{R}'_{\mathcal{A}})$. By the definition of $\tilde{R}$, there exists a measurable selection $\mathbf{s} : I \to S$ of $i \ \rightrightarrows \ \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{R}'_{\mathcal{A}}$ is a Borel set and $\mathbb{R}'_{\mathcal{A}} \subseteq \tilde{R}^t(\mathcal{A})$, which are as well Borel sets $\forall\ t \geq 0$, we have that $\mathcal{P}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)$ are well defined and, $\forall\ t \geq 0$, $\forall\ i \in I$, $\mathbb{B}(i, \mathcal{P}(\mathbb{P}'_{\mathcal{A}})) \subseteq \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)\Big)$. So $\mathbf{s}$ is a selection of $i \ \rightrightarrows \ \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)\Big)$ and then $a \in \tilde{R}^{t+1}(\mathcal{A})$ $\forall\ t \geq 0$, which means that $a \in \mathbb{R}'_{\mathcal{A}}$. This proves that $\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathbb{R}'_{\mathcal{A}}$.

For the other inclusion, we consider again a sequence of set valued mappings $F^t : I \ \rightrightarrows \ S$, $t \geq 0$, whose p-lim sup limit will be again very helpful. Consider then $\forall\ i \in I$,

$$\begin{aligned}F^0(i) &:= S \\ F^t(i) &:= \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^{t-1}(\mathcal{A})\Big)\Big) \quad t \geq 1\end{aligned}$$

We have now $\forall\ t \geq 0$,

$$\tilde{R}^t(\mathcal{A}) \equiv \int_I F^t(i) \ \mathrm{di}.$$

We know that $\mathcal{A} \equiv \int_I F^0$ is non empty and compact. From Proposition 4.1 we get that so are the sets $\tilde{R}^t(\mathcal{A})$ for all $t \geq 1$.

From Lemma 4.2, we get that $\forall\ i \in I$ the mapping $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \ \rightrightarrows \ S$ is u.s.c. and, as a consequence, the set $\mathbb{B}(i, \mathcal{P}(X))$ is compact for any compact subset $X \subseteq \mathcal{A}$. So the correspondences F^t are compact valued. From the proof of Theorem 3.6 we get that they all have a measurable selection, since for a closed set X, $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$ and from Lemma 3.2 the mappings $i \rightrightarrows B(i, X)$ are measurable and hence have a measurable selection.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t:

$$F(i) := \left(\text{p-}\limsup_t F^t\right)(i) \equiv \limsup_t F^t(i)\,.$$

So now let us take a point $a \in \mathbb{R}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t\in\mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976) we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t\in\mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int \mathbf{s}^t$.

To show that $F(i) \subseteq \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, we use that the weak* topology in $\mathcal{P}(\mathcal{A})$ is metrizable and the upper semi continuity of $\mathbb{B}(i, \cdot\,) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ to give an argument that follows the one at the end of the proof of Theorem 3.6.

■