Supplementary Material

A Proof of Lemmas

Proof of Lemma 4. Let $t = \min\{|t_1 + \cdots + t_l|, \tau\} \in [0, \tau]$, then it suffices to show that $p(|t_1|) + \cdots + p(|t_l|) \geq p(t)$. Note that we have $|t_1| + \cdots + |t_l| \geq |t_1 + \cdots + t_l| \geq t$. Moreover, since $p(0) = 0$ and $p(\cdot)$ is concave on $[0, \tau]$, we must have $p(\cdot)$ being subadditive, i.e., for any $s_1, \ldots, s_l \geq 0$ such that $s_1 + \cdots + s_l \leq \tau$, we have $p(s_1) + \cdots + p(s_l) \geq p(s_1 + \cdots + s_l)$. Combining both facts, we have

$$\sum_{i=1}^{l} p(|t_i|) \geq \sum_{i=1}^{l} p\left(\frac{t}{|t_1| + \cdots + |t_l|} \cdot |t_i|\right) \geq p\left(\sum_{i=1}^{l} \frac{t}{|t_1| + \cdots + |t_l|} \cdot |t_i|\right) = p(t),$$

where the first inequality is due to monotonicity and the second is due to subadditivity of $p(\cdot)$. ■

Proof of Lemma 5. According to the conditions for $p(\cdot)$, there exists $\tau_2 < \tau$ such that $p(\cdot)$ is twice continuously differentiable on $[\tau_2, \tau]$. We first show that there exists $\tau_0 \in (\tau_2, \tau)$ such that $p(\cdot)$ is concave but not linear on $[0, \tau_0]$. If otherwise, $p(\cdot)$ must be a linear function on $[0, \tau)$, then since $p(\cdot)$ is continuous at $t = \tau$ where continuity follows from concavity, we must have $p(\cdot)$ is a linear function on $[0, \tau]$, which contradicts with that $p(\cdot)$ is not linear on $[0, \tau]$. In the following, we show that this τ_0 satisfies the conditions in the lemma.

We first show that $C_1 > 0$. If otherwise, we have $\frac{p(\tau_0/3)-p(0)}{\tau_0/3} \leq \frac{p(\tau_0)-p(2\tau_0/3)}{\tau_0/3}$. Since $p(t)$ is concave, this must imply that $p(t)$ is linear on $[0, \tau_0]$, which contradicts with that $p(\cdot)$ is not linear on $[0, \tau_0]$.

Before proving the result, we first introduce two auxiliary functions. For any $s \in [0, \tau_0]$, define $\widetilde{\epsilon}(s) := p(\widetilde{t} - s) + p(s) - p(\widetilde{t})$ and $\epsilon(s) := p(\tau_0 - s) + p(s) - p(\tau_0)$. Note that they have the following properties:

(i) $C_1 = \frac{\epsilon(\tau_0/3)}{\tau_0/3}$;

(ii) $\epsilon(s) \leq \widetilde{\epsilon}(s)$: this is due to $\widetilde{\epsilon}(s) - \epsilon(s) = (p(\tau_0) - p(\tau_0 - s)) - (p(\widetilde{t}) - p(\widetilde{t} - s)) \geq 0$;

(iii) $\epsilon(s)/s$ is non-increasing in s: this is due to

$$\frac{\epsilon(s)}{s} = \frac{p(s) - p(0)}{s} - \frac{p(\tau_0) - p(\tau_0 - s)}{s}$$

where $\frac{p(s)-p(0)}{s}$ is non-increasing while $\frac{p(\tau_0)-p(\tau_0-s)}{s}$ is non-decreasing;

(iv) Combining (i) – (iii) above, for any $s \in (0, \tau_0/3]$, we have

$$p(s) \geq p(s) + p(\widetilde{t} - s) - p(\widetilde{t}) = \widetilde{\epsilon}(s) \geq \epsilon(s) \geq C_1 s.$$

When $s = \tau_0/3$, this implies that $p(\tau_0/3) + p(\widetilde{t} - \tau_0/3) - p(\widetilde{t}) \geq C_1 \cdot \tau_0/3 > C_1\delta$.

Now we prove the last statement of Lemma 5. Suppose $t_1 + \cdots + t_l = \widetilde{t}$, and $p(|t_1|) + \cdots + p(|t_l|) - p(\widetilde{t}) < C_1\delta$. Without loss of generality, we assume $t_1 \geq t_2 \geq \cdots \geq t_l$. Now it suffices to show that $|\widetilde{t} - t_1| < \delta$, $t_2 < \delta$, and $t_l > -\delta$.

Denote $T = \{t_1, \ldots, t_l\}$. For any $S \subseteq T$, we use $\sigma(S)$ to denote the sum of all the elements of S. Now we show that $\sigma(S) > -\delta$ for any S. If otherwise, then $\sum_{S^c} t_i \geq \widetilde{t} + \delta \geq \widetilde{t}$, and we have

$$C_1\delta > \sum_S p(|t_i|) + \sum_{S^c} p(|t_i|) - p(\widetilde{t}) \geq p(\delta) + p(\widetilde{t}) - p(\widetilde{t}) \geq C_1\delta,$$

where the second inequality is due to Lemma 4 and the monotonicity of $p(\cdot)$, and the third one is due to (iv) above. This is a contradiction. Note that by having $S = \{t_l\}$, this result implies that $t_l > -\delta$. Also, by considering the complement of a subset, we have $\sigma(S) = \sigma(T) - \sigma(S^c) < \widetilde{t} + \delta < \tau$ for any $S \subseteq T$. This has two implications. First, according to Lemma 4, we have $\sum_S p(|t_i|) \geq p(|\sum_S t_i|)$; second, by letting $S = \{t_1\}$, we have $t_1 < \widetilde{t} + \delta$.

Now we show that $t_1 > \widetilde{t} - \delta$, by sequentially showing that $t_1 > \tau_0/3$, $t_1 > \widetilde{t} - \tau_0/3$, and then $t_1 > \widetilde{t} - \delta$. If $t_1 \leq \tau_0/3$, then we have $|t_i| \leq \tau_0/3$ for any i. Then we can divide T into two sets T_1 and T_2 such that $|\sigma(T_1) - \sigma(T_2)| \leq \tau_0/3$, thus $\sigma(T_1), \sigma(T_2) \in (\widetilde{t}/2 - \tau_0/6, \widetilde{t}/2 + \tau_0/6) \subseteq (\tau_0/3, \widetilde{t} - \tau_0/3)$. Now we have

$$C_1\delta > p\left(\left|\sum_{t_i \in T_1} t_i\right|\right) + p\left(\left|\sum_{t_i \in T_2} t_i\right|\right) - p(\widetilde{t}) \geq p(\tau_0/3) + p(\widetilde{t} - \tau_0/3) - p(\widetilde{t}) > C_1\delta,$$

which is a contradiction. Note that here the first inequality is due to Lemma 4, and the second one is due to the concavity of $p(\cdot)$.

Now we show that $t_1 > \widetilde{t} - \tau_0/3$. If otherwise, since we have proved that $t_1 \geq \tau_0/3$, we have $t_1 \in [\tau_0/3, \widetilde{t} - \tau_0/3]$. Now by letting $T_1 = \{t_1\}$ and $T_2 = T - T_1$, we have $\sigma(T_1), \sigma(T_2) \in (\tau_0/3, \widetilde{t} - \tau_0/3)$, and contradiction arises in the same way as in the previous case.

Now we show that $t_1 > \widetilde{t} - \delta$, which is equivalent to showing that $\widetilde{t}_2 = t_2 + \cdots + t_l = \widetilde{t} - t_1 < \delta$. If $\widetilde{t}_2 \geq \delta$, then due to subadditivity, concavity, and (iv) above, we have

$$C_1\delta > p(|t_1|) + p(|\widetilde{t}_2|) - p(\widetilde{t}) \geq p(\widetilde{t} - \delta) + p(\delta) - p(\widetilde{t}) \geq C_1\delta,$$

which is a contradiction.

Now to complete the proof, the only last thing we need to show is that $t_2 < \delta$. If $t_2 \geq \delta$, then due to subadditivity and concavity, we have

$$C_1\delta > p(|t_2|) + p(|\widetilde{t} - t_2|) - p(\widetilde{t}) \geq p(\delta) + p(\widetilde{t} - \delta) - p(\widetilde{t}) \geq C_1\delta,$$

which is a contradiction. ■

Proof of Lemma 6. According to Lemma 5, $p(\cdot)$ is twice continuously differentiable on $[\tau_0, \tau]$, thus there exists $K > 0$ such that $p''(t) \geq -K$ for any $t \in [\tau_0, \tau]$. Now we take $\underline{\theta} = \frac{1+K}{q(q-1)\min\{\tau_0^{q-2}, \tau^{q-2}\}}$, and $\underline{\mu} = \frac{p(\widehat{\tau})+\underline{\theta}\widehat{\tau}^q+1}{\underline{\theta}\cdot|\tau_0-\widehat{\tau}|^q}$, and verify the results in the lemma.

For the first result, we have for any $t \in [\tau_0, \tau]$,

$$g''_{\theta,\mu}(t) = p''(t)+\theta q(q-1)t^{q-2}+\mu q(q-1)|\widehat{\tau}-t|^{q-2} \geq -K+\underline{\theta}q(q-1)t^{q-2}+0 \geq 1,$$

thus $g''_{\theta,\mu}(t) \geq 1$ for any $t \in [\tau_0, \tau]$.

Now we show the result of unique minimizer. Since $g_{\theta,\mu}(t)$ is strictly increasing on $[\widehat{\tau}, +\infty)$, any global minimizer must lie in $(-\infty, \widehat{\tau}]$. Moreover, for any $t \in (-\infty, \tau_0]$, we have

$$g_{\theta,\mu}(t) > 0+0+\theta\underline{\mu}\cdot|\tau_0-\widehat{\tau}|^q = \theta/\underline{\theta}\cdot(p(\widehat{\tau})+\underline{\theta}\widehat{\tau}^q+1) \geq p(\widehat{\tau})+\theta\widehat{\tau}^q+1 = g_{\theta,\mu}(\widehat{\tau})+1, \tag{A.1}$$

thus any global minimizer must lie within $(\tau_0, \widehat{\tau}] \subseteq (\tau_0, \tau)$. Now since $g''(t) \geq 1$ for any $t \in (\tau_0, \tau)$, we know that $g(\cdot)$ is strictly convex thereon, thus the global minimizer of $g_{\theta,\mu}(t)$ on $[\tau_0, \tau]$ exists and is unique. Denote the minimizer on $[\tau_0, \tau]$ by $t^*(\theta, \mu)$, then according to the previous discussion, $t^*(\theta, \mu)$ must also be the global minimizer of $g_{\theta,\mu}(t)$ on .

Now we show the last statement. Suppose that $g_{\theta,\mu}(\bar{t}) < h(\theta, \mu) + \delta^2$ for some $\delta \in (0, \bar{\delta})$. We first consider the case where $\bar{t} \in [\tau_0, \tau]$. According to the mean-value theorem, there exists $\widetilde{t}$ between $\bar{t}$ and $t^*(\theta, \mu)$ such that

$$g_{\theta,\mu}(\bar{t}) = g_{\theta,\mu}(t^*(\theta, \mu)) + \frac{1}{2}g''(\widetilde{t})(\bar{t} - t^*(\theta, \mu))^2 \geq h(\theta, \mu) + \frac{1}{2}(\bar{t} - t^*(\theta, \mu))^2$$

Therefore, a necessary condition for $g_{\theta,\mu}(\bar{t}) < h(\theta, \mu)+\delta^2$ is that $|\bar{t}-t^*(\theta, \mu)| < \delta$. Note that this implies $g_{\theta,\mu}(\tau) \geq h(\theta, \mu) + \delta^2$. Now to complete the proof, we only need to show that $g_{\theta,\mu}(t) \geq h(\theta, \mu) + \delta^2$ for any $t \in (-\infty, \tau_0] \cup [\tau, +\infty)$. The inequality with $t \in (-\infty, \tau_0]$ has been proved in (A.1). And for any $t \in [\tau, +\infty)$, we have $g_{\theta,\mu}(t) \geq g_{\theta,\mu}(\tau) \geq h(\theta, \mu) + \delta^2$. Therefore, the proof is complete. ■

Proof of Lemma 7. We take $\underline{\widehat{\mu}} = \max\left\{1 + p'(\tau_0), \frac{p(\widehat{\tau})+1}{\widehat{\tau}-\tau_0}\right\}$ and verify the results in the lemma. Note that we have $p(\cdot)$ being twice continuously differentiable on $[\tau_0, \tau]$ thus $p'(\tau_0)$ is well-defined.

For any $t \in [\tau_0, \widehat{\tau})$, we have $g'_{0,\mu}(t) = p'(t) - \mu \leq p'(\tau_0) - \underline{\widehat{\mu}} \leq -1$; and for any $t \in (\widehat{\tau}, \tau]$, we have $g'_{0,\mu}(t) = p'(t) + \mu \geq 0 + \underline{\widehat{\mu}} \geq 1$. Therefore, the first property in Lemma 7 holds.

Now we show the result of unique minimizer. Since $g_{0,\mu}(t)$ is strictly increasing on $[\widehat{\tau}, +\infty)$, any global minimizer must lie in $(-\infty, \widehat{\tau}]$. Moreover, for any $t \in (-\infty, \tau_0]$, we have

$$g_{0,\mu}(t) \geq 0 + \underline{\widehat{\mu}} \cdot |\tau_0 - \widehat{\tau}| \geq p(\widehat{\tau}) + 1 = g_{0,\mu}(\widehat{\tau}) + 1, \tag{A.2}$$

thus any global minimizer must lie within $(\tau_0, \widehat{\tau}]$. Now since $g'_{0,\mu}(t) < -1$ for any $t \in [\tau_0, \widehat{\tau})$, the global minimizer of $g_{0,\mu}(\cdot)$ is $\widehat{t}^*(0, \mu) = \widehat{\tau}$ and is unique.

Now we show the last statement. Suppose that $g_{0,\mu}(\bar{t}) < h(0,\mu) + \delta^2$ for some $\delta \in (0, \bar{\delta})$. Again we first consider the case where $\bar{t} \in [\tau_0, \tau]$. When $\bar{t} \in [\widehat{\tau}, \tau]$, since $g'_{0,\mu}(t) > 1$, we have $g_{0,\mu}(\bar{t}) - g_{0,\mu}(\widehat{\tau}) \geq \bar{t} - \widehat{\tau}$; when $\bar{t} \in [\tau_0, \widehat{\tau}]$, since $g'_{0,\mu}(t) < -1$, we have $g_{0,\mu}(\bar{t}) - g_{0,\mu}(\widehat{\tau}) \geq \widehat{\tau} - \bar{t}$. Therefore, a necessary condition for $g_{0,\mu}(\bar{t}) < h(0,\mu) + \delta^2$ is that $|\bar{t} - \widehat{\tau}| < \delta^2 < \delta$. Note that this implies $g_{0,\mu}(\tau) \geq h(0,\mu) + \delta^2$. Now to complete the proof, we only need to show that $g_{0,\mu}(t) > h(0,\mu) + \delta^2$ for any $t \in (-\infty, \tau_0] \cup [\tau, +\infty)$. The inequality with $t \in (-\infty, \tau_0]$ has been proved in (A.2). And for any $t \in [\tau, +\infty)$, we have $g_{0,\mu}(t) \geq g_{0,\mu}(\tau) \geq h(0,\mu) + \delta^2$. Therefore, the proof is complete. ■

Proof of Lemma 8. If $q > 1$, then we can find θ and μ such that the properties in Lemma 6 is satisfied; if $q = 1$, then we can set $\theta = 0$ and find μ such that the properties in Lemma 7 is satisfied.

Now we first prove the desired inequality in two cases. In the first case, we suppose that $|\sum_{j=1}^{l} t_j| > \tau$. Then due to Lemma 4, We have $\sum_{j=1}^{l} p(|t_j|) \geq p(\tau)$, thus

$$\sum_{j=1}^{l} p(|t_j|) + \theta \cdot \left|\sum_{j=1}^{l} t_j\right|^q + \mu \cdot \left|\sum_{j=1}^{l} t_j - \widehat{\tau}\right|^q > p(\tau) + \theta\tau^q + \mu|\tau - \widehat{\tau}|^q = g_{\theta,\mu}(\tau) > h(\theta,\mu) + \delta^2 \tag{A.3}$$

where the last inequality is proved in Lemmas 6 and 7. In the second case, we suppose that $|\sum_{j=1}^{l} t_j| \leq \tau$. Then according to Lemma 4, we have $\sum_{j=1}^{l} p(|t_j|) \geq p\left(\left|\sum_{k=1}^{l} t_k\right|\right)$, thus

$$\sum_{j=1}^{l} p(|t_j|) + \theta \cdot \left|\sum_{j=1}^{l} t_j\right|^q + \mu \cdot \left|\sum_{j=1}^{l} t_j - \widehat{\tau}\right|^q \geq g_{\theta,\mu}\left(\sum_{j=1}^{l} t_j\right) \geq h(\theta,\mu), \tag{A.4}$$

where the second inequality is due to Lemmas 6 and 7.

Now we prove the "only if" statement. Suppose we have $t_1, \ldots, t_l \in$ such that (3) holds. Now according to (A.3), we must have $|\sum_{j=1}^{l} t_j| \leq \tau$, and combining (A.4), we have $g_{\theta,\mu}\left(\sum_{j=1}^{l} t_j\right) < h(\theta,\mu) + \delta^2$. Then we have $|\sum_{j=1}^{l} t_j - t^*(\theta,\mu)| < \delta$ according to Lemmas 6 and 7, thus $\widetilde{t} := \sum_{j=1}^{l} t_j \in [\tau_0, \tau]$. Moreover, in order for (3) to hold, we must also have $\sum_{j=1}^{l} p(|t_j|) - p(\widetilde{t}) \leq \delta^2 \leq C_1\delta$. Then according to Lemma 5, we must have $|t_i - \widetilde{t}| < \delta$ for some i while $|t_j| < \delta$ for all $j \neq i$. Now since $|\widetilde{t} - t^*(\theta,\mu)| < \delta$, we have $|t_i - t^*(\theta,\mu)| < 2\delta$, which completes the proof. ■

B Proof of Theorem 2

In this section, we prove the hardness of approximation of Problem 1 for general loss function ℓ. We develop the reduction proof through a series of preliminary lemmas. In particular, our Lemmas B.1,B.2,B.3 establish important properties about the sparse penalty function p, and are analogs to Lemmas 4, 5 and 8, respectively. We have to reprove these lemmas with additional technicalities in order to address the ϵ-approximibility instead of exact solution. Our first lemma gives us a key fact about the nonconvex penalty function p. We use $B(\theta, \delta)$ to denote the interval $(\theta - \delta, \theta + \delta)$.

Lemma B.1. For any penalty function p that satisfies Assumption 2, we have

(i) $p(t)$ is continuous on $(0, \tau]$.

(ii) For any $t_1, ..., t_l \geq 0$, if $\sum_{i=1}^{n} t_i \leq \tau$, then $\sum_{i=1}^{l} p(t_i) \geq p(\sum_{i=1}^{l} t_i)$.

(iii) There exists $a \in [1/2, 1)$ such that when $\sum_{i=1}^{l} t_i \in [a\tau, \tau]$, the above inequality holds as equality if and only if $t_i = t^*$ for some i while $t_j = 0$ for $j \neq i$.

(iv) Denote $\kappa = \min_{t\in[a\tau,\tau]}\{\frac{2p(t/2)-p(t)}{t}\}$. For the constant a given in (iii), we have that $\forall \delta > 0, t_1, \cdots, t_l \in \mathbb{R}, \forall \epsilon \leq \kappa\delta$: if $\sum_{i=1}^{l} t_i = t^* \in [a\tau, \tau]$ and $p(\sum_{i=1}^{l} t_i) + \epsilon \geq \sum_{i=1}^{l} p(t_i)$, then there is at most one i such that $t_i \notin B(0, \delta)$.

Proof. As (i), (ii) and (iii) are proved in Ge et al. (2015), we prove (iv) here. We first prove the lemma when $t_1, \cdots, t_l \geq 0$. We start by proving the case when $l = 2$. For the simplicity of notation, we use t^* to denote $t_1 + t_2$ in the rest of the proof. By (iii), there exists a such that when $t^* \in [a\tau, \tau]$ and $p(t^*) \geq p(t_1) + p(t_2)$, we have $t_1 = 0$ or $t_2 = 0$. It follow that when $t_1 \neq 0, t_2 \neq 0$ and $t^* \in [a\tau, \tau]$, we have $p(t_1 + t_2) < p(t_1) + p(t_2)$. Without loss of genearlity, we assume that $t_1 \leq t_2$. Then, we have

$$\frac{p(t^*) - p(t^* - t_1)}{t_1} < \frac{p(t_1)}{t_1}.$$

Notice that the right term is non-increasing with the increment of t_1 as p is a concave function and the left term is non-decreasing with the increment of t_1 when t^* is fixed. As $t_1 \leq t^*/2$, we have $\frac{p(t_1)}{t_1} \geq k_1(t^*) := \frac{p(t^*/2)}{t^*/2}$ and $\frac{p(t^*)-p(t^*-t_1)}{t_1} \leq k_2(t^*) := \frac{p(t^*)-p(t^*/2)}{t^*/2}$. As p is not linear on $[0, t^*]$, we have $k_1(t^*) > k_2(t^*)$.

On the other hand, we can see that when $p(t_1 + t_2) + \epsilon \geq p(t_1) + p(t_2)$,

$$\frac{p(t_1 + t_2) - p(t_2)}{t_1} + \frac{\epsilon}{t_1} \geq \frac{p(t_1)}{t_1}.$$

Assume $t_1 < t_2$, we have $k_2(t^*) + \epsilon/t_1 \geq k_1(t^*)$ [1]. As a result $t_1 \leq \frac{\epsilon}{k_1(t^*)-k_2(t^*)}$. Note that k_1 and k_2 are defined on a closed interval $[a\tau, \tau]$ by (iii), giving us that $\min_{t\in[a\tau,\tau]}(k_1(t)-k_2(t)) > 0$. Therefore, $\exists a \in (0, 1), \forall \delta > 0, \exists \epsilon_0 = \min_{t\in[a\tau,\tau]}(k_1(t)-k_2(t)) \cdot \delta, \forall \epsilon < \epsilon_0$, if $t_1 + t_2 = t^* \in [a\tau, \tau]$ and $p(t_1 + t_2) + \epsilon \geq p(t_1) + p(t_2)$, then $t_1 \leq \frac{\epsilon}{k_1(t^*)-k_2(t^*)} \leq \delta$. Therefore, there is at most one i such that $t_i \notin B(0, \delta)$.

Now consider the case when $l > 2$ and $t_1, \ldots, t_l \geq 0$. If there are more than one i such that $t_i \notin B(0, \delta)$, assume t_1 and t_2 are two of them. By (ii), we have

$$\sum_{i=1}^{l} p(t_i) \geq p(t_1) + p\left(\sum_{i=2}^{l} t_i\right).$$

[1] For the case when $t_1 = 0$, (iv) holds trivially.

If $t_1+\sum_{i=2}^n t_i \in [a\tau,\tau]$ and $p(t_1+\sum_{i=2}^l t_i)+\epsilon \geq \sum_{i=1}^l p(t_i) \geq p(t_1)+p(\sum_{i=2}^l t_i)$, either t_1 or $\sum_{i=2}^n t_i$ should be inside $B(0,\delta)$. This is contradictory to our assumption that both t_1 and t_2 are outside $B(0,\delta)$. To this point, we prove (iv) when $t_1,\cdots,t_l \geq 0$.

Next, we prove the lemma when $t_1,\cdots,t_l$ could be smaller than 0. Suppose $t^*=\sum_{i=1}^l t_i \in [a\tau,\tau]$ and $p(t^*)+\epsilon \geq \sum_{i=1}^l p(t_i)$. We consider two cases separately. In the first case, assume that there is one $t_i \leq -\delta$ and one $t_j \geq \delta$. Without loss of generality, we assume that $t^*>0$. Then we can choose $\alpha=\delta, \beta=t^*-\alpha$ and get

$$p(\alpha+\beta)+\epsilon = p(t^*)+\epsilon \geq \sum_{i\in\{j:t_j<0\}} p(t_i) + \sum_{i\in\{j:t_j>0\}} p(t_i) \geq p(\alpha)+p(\beta),$$

which is a contradiction to the previous proof that only one of α,β could be outside of $B(0,\delta)$ as δ is smaller than $t^*/2$ by our choice. We then proceed to the case when there is one $t_i \geq \delta$ and one $t_j \geq \delta$. Suppose that $\alpha=t_i\geq t_j=\beta$. If $\alpha+\beta>t^*$, we set $\alpha'=\delta+\frac{t^*-2\delta}{\alpha+\beta-2\delta}\cdot(\alpha-\delta)$ and $\beta'=\delta+\frac{t^*-2\delta}{\alpha+\beta-2\delta}\cdot(\beta-\delta)$. It is easy to verify that

$$p(\alpha'+\beta')+\epsilon = p(t^*)+\epsilon \geq \sum_{i=1}^l p(t_i) \geq p(\alpha)+p(\beta) \geq p(\alpha')+p(\beta'),$$

which is a contradiction. If $\alpha+\beta<t^*$, we can verify that

$$p(\alpha+\beta+t^*-\alpha-\beta)+\epsilon = p(t^*)+\epsilon \geq \sum_{i=1}^l p(t_i) \geq p(\alpha)+p(\beta)+p(t^*-\alpha-\beta),$$

which is also a contradiction. To this point, we prove the case that $t_1,\cdots,t_l$ could be smaller than 0, which completes the proof of the lemma.

Remark. In the proof of (iv), our choice of ϵ is linear to δ given δ. However, in the case of L_0, ϵ could be any constant smaller than 1 no matter what δ is. This property of L_0 has wide applications in statistical problems. Actually, suppose that penalty function is indexed by δ and p_δ satisfies

$$p_\delta(\delta)-p_\delta(a\tau)+p_\delta(a\tau-\delta) \geq C \tag{B.1}$$

for some constant C, then $\forall\delta>0$ and $\epsilon\leq C$, the proposition stated in (iv) holds. To prove this, just note that if $p(t_1+t_2)-p(t_2)+\epsilon>p(t_1)$ and $t_1>\delta$, then $p(t_1)-p(t_1+t_2)+p(t_2)>p(\delta)-p(a\tau)+p(a\tau-\delta)\geq C$ which is a contradiction to that ϵ should be smaller than C. ■

Lemma B.1 states the key properties of the penalty function p. Property (iv) is of special interest. It indicates that if we can manipulate the sum of non-negative variables to let it lie within $[a\tau,\tau]$ while minimizing the penalty function, we can be sure that only one variable has positive value.

Our second lemma explores the relationship between the penalty function p and the loss function ℓ.

Lemma B.2. Let Assumption 2 hold. Let $f(\cdot)$ be a convex function with a unique minimizer $\widehat{\tau} \in (a\tau, \tau)$ and $\frac{f(\widehat{\tau}\pm x)-f(\widehat{\tau})}{x^N} \geq C(0 < x < \bar{\delta})$ for some $N \in \mathbb{Z}^+, \bar{\delta} \in \mathbb{R}^+, C \in \mathbb{R}^+$. Define

$$g_\mu(t) = p(|t|) + \mu \cdot f(t),$$

where $\mu > 0$. Let $h(\mu)$ be the minimum value of $g_\mu(\cdot)$. We have $\forall \delta < \bar{\delta}, \mu_\delta > \frac{p(|\widehat{\tau}|)2^N}{C\delta^N}, \exists \epsilon_0 = \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N - p(|\widehat{\tau}|)$: if t satisfies $h(\mu_\delta) + \epsilon_0 \geq g_{\mu_\delta}(t) \geq h(\mu_\delta)$, then $t \in [\widehat{\tau} - \delta/2, \widehat{\tau} + \delta/2]$.

Proof. First, we can see that when $t > \widehat{\tau} + \delta/2$, we have

$$\begin{aligned} g_{\mu_\delta}(t) &\geq p(|\widehat{\tau}|) + \mu_\delta \cdot f(t) > p(|\widehat{\tau}|) + \mu_\delta \cdot f(\widehat{\tau} + \delta/2) \geq p(|\widehat{\tau}|) + \mu_\delta \cdot f(\widehat{\tau}) + \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N \\ &= g_{\mu_\delta}(\widehat{\tau}) + \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N \geq h(\mu_\delta) + \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N \geq h(\mu_\delta) + \epsilon_0, \end{aligned}$$

by the definition of $f(\cdot)$. When $t < \widehat{\tau} - \delta/2$, we have

$$\begin{aligned} g_{\mu_\delta}(t) &\geq \mu_\delta \cdot f(t) > \mu_\delta \cdot f(\widehat{\tau} - \delta/2) \geq \mu_\delta \cdot f(\widehat{\tau}) + \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N \\ &= \mu_\delta \cdot f(\widehat{\tau}) + \frac{p(|\widehat{\tau}|)2^N}{C\delta^N} \cdot C \cdot \left(\frac{\delta}{2}\right)^N + \left(\mu_\delta - \frac{p(|\widehat{\tau}|)2^N}{C\delta^N}\right) \cdot C \cdot \left(\frac{\delta}{2}\right)^N \\ &\geq h(\mu_\delta) + \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N - p(|\widehat{\tau}|). \end{aligned}$$

Therefore, when we choose $\epsilon_0 = \mu_\delta \cdot C \cdot \left(\frac{\delta}{2}\right)^N - p(|\widehat{\tau}|)$, point t satisfying $h(\mu_\delta) + \epsilon_0 \geq g_{\mu_\delta}(t) \geq h(\mu_\delta)$ must lie in $[\widehat{\tau} - \delta/2, \widehat{\tau} + \delta/2]$. ∎

Lemma B.3. Let Assumption 2 hold and let $f(\cdot)$ be a convex function with a unique minimizer $\widehat{\tau} \in (a\tau, \tau)$ and $\frac{f(\widehat{\tau}\pm x)-f(\widehat{\tau})}{x^N} \geq C_1(0 < x < \bar{\delta})$ for some $N \in \mathbb{Z}^+, \bar{\delta} \in \mathbb{R}^+, C_1 \in \mathbb{R}^+$. Let $h(\mu)$ be the minimum value of $g_\mu(x) = p(|x|) + \mu \cdot f(x)$, then we have

(i) $\forall \mu \in \mathbb{Z}^+, t_1, ..., t_n \in \mathbb{R} : \sum_{j=1}^n p(|t_j|) + \mu \cdot f\left(\sum_{j=1}^n t_j\right) \geq h(\mu)$.

(ii) $\exists \kappa = \min_{t\in[a\tau,\tau]}\{\frac{2p(t/2)-p(t)}{t}\}, \forall \delta \leq \min\{\bar{\delta}, 4\tau - 4\widehat{\tau}, 4\widehat{\tau} - 4a\tau, p(\widehat{\tau})/\kappa\}, \exists \mu = \frac{p(|\widehat{\tau}|)4^{N+1}}{C_1\delta^N}, \epsilon_0 = \kappa \cdot \frac{\delta}{n}, \forall \theta \in [\widehat{\tau} - \delta/4, \widehat{\tau} + \delta/4]$: if $t_1, ..., t_n \in \mathbb{R}$ satisfy

$$h(\mu) + \epsilon_0 \geq \sum_{j=1}^n p(|t_j|) + \mu \cdot f\left(\sum_{j=1}^n t_j\right) \geq h(\mu), \tag{B.2}$$

then $t_i \in B(\theta, \delta)$ for one i and $t_j \in B(0, \delta)$ for all $j \neq i$.

Proof. We first prove (i). We consider two cases separately. In the first case, we suppose that $|\sum_{j=1}^n t_j| > \tau$. Then we have

$$\sum_{j=1}^n p(|t_j|) \geq \sum_{j=1}^n p\left(\frac{\tau}{\sum_{k=1}^n |t_k|} \cdot |t_j|\right) \geq p\left(\sum_{j=1}^n \frac{\tau}{\sum_{k=1}^n |t_k|} \cdot |t_j|\right) \geq p(\tau),$$

where the first inequality is inferred by the monotonicity of p and the second inequality is due to (ii) of Lemma B.1. Thus, we have

$$\sum_{j=1}^n p(|t_j|) + \mu \cdot f\left(\sum_{j=1}^n t_j\right) > \min\{p(\tau) + \mu \cdot f(\tau), p(\tau) + \mu \cdot f(-\tau)\} \geq h(\mu).$$

As a result, we can see that (i) holds when $|\sum_{j=1}^n t_j| > \tau$. In the second case, we suppose $|\sum_{j=1}^n t_j| \leq \tau$ and obtain

$$\sum_{j=1}^n p(|t_j|) \geq \sum_{j=1}^n p\left(\frac{|\sum_{k=1}^n t_k|}{\sum_{k=1}^n |t_k|} |t_j|\right) \geq p\left(\sum_{j=1}^n \frac{|\sum_{k=1}^n t_k|}{\sum_{k=1}^n |t_k|} |t_j|\right) \geq p\left(\left|\sum_{j=1}^n t_j\right|\right),$$

where the second inequality is due to (ii) of Lemma B.1. It follows that

$$\sum_{j=1}^n p(|t_j|)+\mu\cdot f\left(\sum_{j=1}^n t_j\right) \geq p\left(\left|\sum_{j=1}^n t_j\right|\right)+\mu\cdot f\left(\sum_{j=1}^n t_j\right) = g_\mu\left(\sum_{j=1}^n t_j\right) \geq h(\mu). \tag{B.3}$$

which completes our proof of (i).

We then prove (ii). Assume equation (B.2) holds. If $\sum_{j=1}^n t_j > \tau$, we can see that by choosing $\epsilon_0 \leq g_\mu(\tau) - g_\mu(\widehat{\tau})$, we have

$$\sum_{j=1}^n p(|t_j|) + \mu \cdot f\left(\sum_{j=1}^n t_j\right) > g_\mu(\tau) = g_\mu(\widehat{\tau}) + g_\mu(\tau) - g_\mu(\widehat{\tau}) \geq h(\mu) + \epsilon_0.$$

We will show later that our choice of ϵ_0 is indeed smaller than $g_\mu(\tau) - g_\mu(\widehat{\tau})$. We will also show later that equation (B.2) cannot hold when $\sum_{j=1}^n t_j < -\tau$ under our choice of parameters. Thus, if equation (B.2) holds, then $|\sum_{j=1}^n t_j| \leq \tau$, which implies that

$$p\left(\left|\sum_{j=1}^n t_j\right|\right) + \mu \cdot f\left(\sum_{j=1}^n t_j\right) \leq h(\mu) + \epsilon_0, \tag{B.4}$$

by equation (B.2) and the first inequality of (B.3), and

$$\sum_{j=1}^n p(|t_j|) \leq p\left(\left|\sum_{j=1}^n t_j\right|\right) + \epsilon_0, \tag{B.5}$$

due to equation (B.2) and equation (B.3). Note that we just need to prove the case when δ is sufficiently small. Thus, we assume in the following paper that δ is smaller than $\bar{\delta}, 4\tau - 4\widehat{\tau}, 4\widehat{\tau} - 4a\tau$.

Consider the case when equation (B.4) holds. By Lemma B.3, if we choose $\mu = \frac{p(|\widehat{\tau}|)4^{N+1}}{C\delta^N}$ and $\epsilon_1 = 3p(|\widehat{\tau}|)$, then all of the points t such that $h(\mu) + \epsilon_1 \geq g_\mu(t) \geq h(\mu)$ lie in $[\widehat{\tau} - \delta/4, \widehat{\tau} + \delta/4]$. Thus, we have $\sum_{j=1}^n t_j \in [a\tau, \tau]$ and $\sum_{j=1}^n t_j \in B(\theta, \frac{\delta}{2})$ for all $\theta \in [\widehat{\tau} - \delta/4, \widehat{\tau} + \delta/4]$. Note that $g_\mu(t)$ is non-increasing when $t < 0$, meaning that equation (B.2) cannot hold under our choice of ϵ_1 when $\sum_{j=1}^n t_j \leq -\tau$.

On the other hand, if equation (B.2) holds, equation (B.5) should also hold. By (iv) of Lemma B.1, for the same δ, $\exists \epsilon_2 = \min_{t \in [a\tau, \tau]}(k_1(t) - k_2(t)) \cdot \frac{\delta}{2n-2}$, there is at most one i such that $t_i \notin B(0, \frac{\delta}{2n-2})$. As $\sum_{j=1}^n t_j \in B(\theta, \frac{\delta}{2})$, we have $t_i \in B(\theta, \delta)$ for all $i = 1, \cdots, n$. Observe that $g_\mu(\tau) - g_\mu(\widehat{\tau})$ is always larger than ϵ_1. Also, $\epsilon_1 > \epsilon_2$ if δ is sufficiently small. Therefore, $\exists \kappa = \min_{t \in [a\tau, \tau]}(k_1(t) - k_2(t))/2, \forall \delta \leq \min\{\bar{\delta}, 4\tau - 4\widehat{\tau}, 4\widehat{\tau} - 4a\tau\, p(\widehat{\tau})/\kappa\}, \exists \mu = \frac{p(|\widehat{\tau}|)4^{N+1}}{C\delta^N}, \epsilon = \kappa \cdot \frac{\delta}{n}, \forall \theta \in [\widehat{\tau} - \delta/4, \widehat{\tau} + \delta/4]:$ if $h(\mu) + \epsilon \geq g_\mu(\sum_{j=1}^n t_i)$, then $t_i \in B(\theta, \delta)$ for some i while $t_j \in B(0, \delta)$ for all $j \neq i$. ■

Now we are ready to prove the main theorem.

Proof of Theorem 2. Suppose that we are given the input to the 3-partition problem, i.e., $3m$ positive integers $s_1, ..., s_{3m}$. Assume *without loss of generality* that all s_i's are upper bounded by some polynomial function $M(m)$. This restriction on the input space does not weaken our result, because the 3-partition problem is strongly NP-hard.

In what follows, we construct a reduction from the 3-partition problem to Problem 1. We assume without loss of generality that $\frac{1}{4m}\sum_{j=1}^{3m} s_j < s_i < \frac{1}{2m}\sum_{j=1}^{3m} s_j$ for all $i = 1, \ldots, n$. Such condition can always be satisfied by adding a sufficiently large integer to all s_i's.

Step 1: The Reduction

The first reduction is developed through the following steps.

1. For the interval $[a\tau, \tau]$ determined by p, we choose $\{b_{1i}\}_{i=1}^{k_1}$ such that $\ell_1(y) = \frac{1}{\lambda}\sum_{i=1}^{k_1} \ell(y, b_{1i})$ satisfies Assumption 2 with constants $C, N, \bar{\delta}$ and has a unique minimizer $\widehat{\tau}$ inside the interval $(a\tau, \tau)$. Let $\kappa = \min_{t \in [a\tau, \tau]}\{\frac{2p(t/2) - p(t)}{t}\}$. Let $\delta \leq \{\frac{a\tau}{9m \cdot M(m)}, \bar{\delta}, 4\tau - 4\widehat{\tau}, 4\widehat{\tau} - 4a\tau, p(\widehat{\tau})/\kappa\}$, $\mu \geq \frac{p(|\widehat{\tau}|)4^{N+1}}{C_1\delta^N}$ and $\epsilon = \kappa \cdot \frac{\delta}{3m}$ such that Lemma B.3 is satisfied. Note that $\epsilon \geq \frac{C_3}{m^2 \cdot M(m)}$ for some constant C_3 by our construction.

2. For the μ and ϵ chosen in the previous step, all the minimizers of $g_\mu(x) = p(|x|) + \mu \cdot \ell_1(x)$ lie in $[\widehat{\tau} - \delta/4, \widehat{\tau} + \delta/4]$ by Lemma B.3. By the Lipschitz continuity of $p(|x|), f(x)$ and thus $g_\mu(x)$ on $[a\tau, \tau]$, there exists $\delta_\epsilon = \frac{\epsilon}{6mK}$ (K is the Lipschitz constant) such that we can find in polynomial time an interval $[\theta_1, \theta_2]$ where $\theta_2 - \theta_1 = \delta_\epsilon$ and $g_\mu(x) - g_\mu(t^*) < \frac{\epsilon}{6m}$ for $x \in [\theta_1, \theta_2]$. This interval can be find in polynomial time as $g_\mu(x)$ is Lipschitz continuous.

3. By Assumption 1, for the interval $[\theta_1, \theta_2]$, we choose $\{b_{2i}\}_{i=1}^{k_2}$ to construct a loss function $\ell_2 : \mathbb{R} \mapsto \mathbb{R}$ in polynomial time with regard to $1/\delta_\epsilon$ such that $\ell_2(y) = \frac{1}{\lambda}\sum_{i=1}^{k_2} \ell(y, b_{2i})$ has a unique minimizer at $\widetilde{t} \in [\theta_1, \theta_2]$. We choose

$$\nu = \left\lceil \epsilon / \max\left(\ell_2(\widetilde{t} + 2\delta m) - \ell_2(\widetilde{t}), \ell_2(\widetilde{t} - 2\delta m) - \ell_2(\widetilde{t})\right)\right\rceil + 1,$$

and construct function $f : \mathbb{R}^{3m\times m} \mapsto \mathbb{R}$ where

$$f(x) = \lambda \cdot \sum_{i=1}^{3m}\sum_{j=1}^{m} p\left(|x_{ij}|\right) + \lambda\mu \cdot \sum_{i=1}^{3m} \ell_1\left(\sum_{j=1}^{m} x_{ij}\right) + \lambda\nu \cdot \sum_{j=1}^{m} \ell_2\left(\sum_{i}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} x_{ij}\right). \tag{B.6}$$

Note that by (iii) of Assumption 1, ν is polynomial in $\max(\lceil \frac{1}{\delta_\epsilon} \rceil, \lceil \theta_2 \rceil)$. In the rest of the paper, we ignore the $\lceil \theta_2 \rceil$ term in the bound as it can be upperbounded by τ, which can be taken as a constant in the reduction.

4. Let $\Phi_1 = 3m \cdot p(|\widetilde{t}|) + \mu \cdot 3m \cdot \ell_1(\widetilde{t}) - \frac{\epsilon}{2}$ and $\Phi_2 = \nu \cdot m \cdot \ell_2(\widetilde{t})$. We claim that

   (i) If there exists z such that

   $$\Phi_1 + \Phi_2 + \epsilon \geq \frac{1}{\lambda} f(z) \geq \Phi_1 + \Phi_2,$$

   then we obtain a feasible assignment for the 3-partition problem as follows: If $z_{ij} \in B(\widetilde{t}, \delta)$, we assign number i to subset j.

   (ii) If the 3-partition problem has a solution, we have $\frac{1}{\lambda}\min_x f(x) \leq \Phi_1 + \Phi_2 + \frac{\epsilon}{2}$.

5. Choose $r = \left\lceil \left(\frac{2(3m\cdot\lambda\cdot\mu\cdot k_1 + m\cdot\lambda\cdot\nu\cdot k_2)^{c_1}(3m^2)^{c_2}}{\epsilon/\kappa}\right)^{1/(1-c_1-c_2)} \right\rceil$ where c_1 and c_2 are two arbitrary constants that $c_1 + c_2 < 1$. Construct the following instance of Problem 1:

$$\min_{x^{(1)},\cdots,x^{(r)}\in\mathbb{R}^{3m\times m}} \sum_{q=1}^{r} f(x^{(q)}) = \min_{x^{(1)},\cdots,x^{(r)}\in\mathbb{R}^{3m\times m}} \lambda \cdot \sum_{q=1}^{r}\sum_{i=1}^{3m}\sum_{j=1}^{m} p(|x_{ij}^{(q)}|) + \lambda\mu \sum_{q=1}^{r}\sum_{i=1}^{3m}\sum_{t=1}^{k_1} \ell\left(\sum_{j=1}^{m} x_{ij}^{(q)}, b_{1t}\right) + \lambda\nu \sum_{q=1}^{r}\sum_{j=1}^{m}\sum_{t=1}^{k_2} \ell\left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} x_{ij}^{(q)}, b_{2t}\right), \tag{B.7}$$

   where the input data are coefficients of x and the values $b_{11}, \ldots, b_{1t}, b_{21}, \ldots, b_{2t}$. The variable dimension d is $r \cdot 3m^2$ and the sample size n is $\lambda \cdot \mu \cdot r \cdot 3m \cdot k_1 + \lambda \cdot \nu \cdot r \cdot m \cdot k_2$. The input size is polynomial with respect to m. Our choice of r is the solution to $\epsilon r = 2\kappa n^{c_1} d^{c_2}$ where $\kappa = \min_{t\in[a\tau,\tau]}\{\frac{2p(t/2)-p(t)}{t}\}$.

   The parameters μ, ν, δ, r, d are bounded by polynomial functions of m. Computing their values also takes polynomial time. The parameter k_1 and k_2 is a constant determined by the loss function ℓ and is not related to m. As a result, the reduction is polynomial.

6. Let $z^{(1)}, \cdots z^{(r)} \in \mathbb{R}^{3m\times m}$ be a $\lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$-optimal solution to problem (B.16) such that $\sum_{i=1}^{r} f(z^{(i)}) \leq \min_{x^{(1)},\cdots,x^{(r)}} \sum_{i=1}^{r} f(x^{(i)}) + \lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$. We claim that

   (iii) If the approximate solution $z^{(1)}, \cdots z^{(r)}$ satisfies

$$\frac{1}{\lambda} \sum_{i=1}^{r} f(z^{(i)}) \leq r\Phi_1 + r\Phi_2 + 2\kappa n^{c_1} d^{c_2}, \tag{B.8}$$

   we can choose one $z^{(i)}$ such that $\Phi_1 + \Phi_2 + \epsilon \geq \frac{1}{\lambda} f(z^{(i)}) \geq \Phi_1 + \Phi_2$ and obtain a feasible assignment: If $z_{ij}^{(i)} \in B(\widetilde{t}, \delta)$, we assign number i to subset j. If the $\lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$-optimal solution $z^{(1)}, \cdots z^{(r)}$ does not satisfy (B.8), the 3-partition problem has no feasible solution.

We have constructed a polynomial reduction from the 3-partition problem to finding an $\lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$-optimal solution to problem (B.16). In what follows, we prove that the reduction works.

Step 2: Proof of Claim (i)

We begin with the proof (i). By our choice of μ and Lemma B.3(i), we can see that for all $x \in \mathbb{R}^{3m\times m}$,

$$\sum_{i=1}^{3m} \sum_{j=1}^{m} p(|x_{ij}|) + \mu \cdot \sum_{i=1}^{3m} \ell_1 \left(\sum_{j=1}^{m} x_{ij} \right) \geq 3m \cdot p(|t^*|) + \mu \cdot 3m \cdot \ell_1(t^*) \geq \Phi_1,$$

where the last inequality is due to that $g_\mu(\widetilde{t}) - g_\mu(t^*) < \frac{\epsilon}{6m}$. By the fact $\widetilde{t} = \text{argmin}_t \ell_2(t)$, we have for all $x \in \mathbb{R}^{3m\times m}$ that

$$\nu \cdot \sum_{j=1}^{m} h \left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} x_{ij} \right) \geq \nu \cdot m \cdot \ell_2(\widetilde{t}) = \Phi_2.$$

Thus we always have $\min_z \frac{1}{\lambda} f(z) \geq \Phi_1 + \Phi_2$. Now if there exists z such that $\Phi_1 + \Phi_2 + \epsilon \geq \frac{1}{\lambda} f(z) \geq \Phi_1 + \Phi_2$, we must have

$$\Phi_1 + \epsilon \geq \sum_{i=1}^{3m} \sum_{j=1}^{m} p(|z_{ij}|) + \mu \cdot \sum_{i=1}^{3m} h \left(\sum_{j=1}^{m} z_{ij} \right) \geq \Phi_1, \tag{B.9}$$

and

$$\Phi_2 + \epsilon \geq \nu \cdot \sum_{j=1}^{m} h \left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij} \right) \geq \Phi_2. \tag{B.10}$$

In order for equation (B.9) to hold, we have that for all i,

$$p(|\widetilde{t}|) + \mu \cdot \ell_1(\widetilde{t}) + \frac{\epsilon}{2} \geq \sum_{j=1}^{m} p(|z_{ij}|) + \mu \cdot \ell_1 \left(\sum_{j=1}^{m} z_{ij} \right) \geq p(|t^*|) + \mu \cdot \ell_1(t^*).$$

Consider an arbitrary i. By Lemma B.3(ii) and $g_\mu(\widetilde{t}) - g_\mu(t^*) < \frac{\epsilon}{6m}$, we have $z_{ij} \in B(\widetilde{t}, \delta)$ for one j while $z_{ik} = 0$ for all $k \neq j$. If $z_{ij} \in B(\widetilde{t}, \delta)$, we assign number i to subset j. As $\delta < a\tau/2 \leq \widetilde{t}/2$, $B(\widetilde{t}, \delta)$ and $B(0, \delta)$ are not overlapping. Thus each number index i is assigned to exactly one subset index j. Therefore the assignment is feasible.

We claim that every subset sum must equal to $\sum_{i=1}^{3m} s_i/m$. Assume that the jth subset sum is greater than or equal to $\sum_{i=1}^{3m} s_i/m + 1$. Let $I_j = \{i \mid z_{ij} \in B(\widetilde{t}, \delta)\}$. Thus, $\sum_{i \in I_j} s_i \geq \sum_{i=1}^{3m} s_i/m + 1$. As a result, we have

$$\begin{aligned} \sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij} &\geq \sum_{i \in I_1} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} (\widetilde{t} - \delta) + \sum_{i \in I_2} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} (-\delta) \\ &\geq \frac{\sum_{i=1}^{3m} s_i/m + 1}{\sum_{i=1}^{3m} s_i/m} \widetilde{t} - \delta m = \widetilde{t} + \frac{\widetilde{t}}{\sum_{i=1}^{3m} s_i/m} - \delta m. \end{aligned}$$

Because $s_i \leq M(m)$ for all i and $\delta = \frac{a\tau}{9m \cdot M(m)}$, we have

$$\frac{\widetilde{t}}{\sum_{i=1}^{3m} s_i/m} - \delta m \geq \frac{a\tau}{3m \cdot M(n)} m - \delta m = 2\delta m > 0.$$

Since h is a convex function with minimizer y^*, we apply the preceding inequalities and further obtain

$$\ell_2 \left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij} \right) \geq \ell_2(\widetilde{t} + 2\delta m).$$

By our construction of ν and Assumption 1(iii), we further have

$$\nu \cdot \left(\ell_2 \left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij} \right) - \ell_2(\widetilde{t}) \right) \geq \nu \cdot \big(\ell_2(\widetilde{t} + 2\delta m) - \ell_2(\widetilde{t}) \big) > \epsilon. \quad \text{(B.11)}$$

However, in order for equation (B.10) to hold, we have that for all j,

$$\nu \cdot \ell_2(\widetilde{t}) + \epsilon \geq \nu \cdot \ell_2 \left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij} \right) \geq \nu \cdot \ell_2(\widetilde{t}),$$

yielding a contradiction to (B.11). We could prove similarly that it is not possible for any subset sum to be strictly smaller than $\frac{1}{m} \sum_{i=1}^{3m} s_i$. Therefore, the sum of every subset equals to $\sum_{i=1}^{3m} s_i/m$. Finally, using the assumption that $\frac{1}{4m} \sum_{i=1}^{3m} s_i < s_i < \frac{1}{2m} \sum_{i=1}^{3m} s_i$, each subset has exactly three components. Therefore the assignment is indeed a solution to the 3-partition problem.

Step 3: Proof of Claim (ii)

Suppose we have a solution to the 3-partition problem. Now we construct z to the optimization problem such that $f(z) \leq \Phi_1 + \Phi_2 + \frac{\epsilon}{2}$. For all $1 \leq i \leq 3m$, if number i

is assigned to subset j, let $z_{ij} = \widetilde{t}$ and $z_{ik} = 0$ for all $k \neq j$. We can easily verify that

$$\sum_{i=1}^{3m}\sum_{j=1}^{m} p\left(|z_{ij}|\right) + \mu \cdot \sum_{i=1}^{3m} \ell_1\left(\sum_{j=1}^{m} z_{ij}\right) = 3m \cdot \left(p(\widetilde{t}) + \mu \cdot \ell_1(\widetilde{t})\right) = \Phi_1 + \frac{\epsilon}{2},$$

Also, we have

$$\nu \cdot \sum_{j=1}^{m} \ell_2\left(\sum_{i=1}^{3m} \frac{s_i}{\sum_{i'=1}^{3m} s_{i'}/m} z_{ij}\right) = \nu \cdot m \cdot \ell_2(\widetilde{t}) = \Phi_2.$$

Therefore,

$$\frac{1}{\lambda} f(z) \leq \Phi_1 + \Phi_2 + \frac{\epsilon}{2}. \tag{B.12}$$

which completes the proof of (ii).

Step 4: Proof of Claim (iii)

Suppose that the $\lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$-optimal solution satisfies (B.8), i.e., $\frac{1}{\lambda}\sum_{i=1}^{r} f(z^{(i)}) \leq r\Phi_1 + r\Phi_2 + 2\kappa n^{c_1} d^{c_2}$. It follows that there exists at least one term $z^{(i)}$ such that

$$\frac{1}{\lambda} f(z^{(i)}) \leq \Phi_1 + \Phi_2 + \frac{2\kappa n^{c_1} d^{c_2}}{r} \leq \Phi_1 + \Phi_2 + \epsilon. \tag{B.13}$$

where the second inequality equality uses $\epsilon r = 2\kappa n^{c_1} d^{c_2}$. Therefore, by claim (ii), we can find a solution to the 3-partition problem.

Suppose that the 3-partition problem has a solution. By claim (ii), there exists z such that $\frac{1}{\lambda} f(z) \leq \Phi_1 + \Phi_2 + \frac{\epsilon}{2}$. Thus we have

$$\min_{x^{(1)},\cdots,x^{(r)}} \frac{1}{\lambda}\sum_{i=1}^{r} f(x^{(i)}) \leq \frac{r}{\lambda} f(z) \leq r\Phi_1 + r\Phi_2 + \kappa n^{c_1} d^{c_2}. \tag{B.14}$$

Thus if $z^{(1)}, \cdots z^{(r)}$ is a $\lambda \cdot \kappa \cdot n^{c_1} d^{c_2}$-optimal solution to (B.16), we have

$$\frac{1}{\lambda}\sum_{i=1}^{r} f(z^{(i)}) \leq \min_{x^{(1)},\cdots,x^{(r)}} \frac{1}{\lambda}\sum_{i=1}^{r} f(x^{(i)}) + \kappa n^{c_1} d^{c_2} \leq r\Phi_1 + r\Phi_2 + 2\kappa n^{c_1} d^{c_2} \tag{B.15}$$

implying that the relation (B.8) must hold. If (B.8) is not satisfied, the 3-partition problem has no solution. ■

Remark. When the loss function is L_2 loss, we can move $\lambda\mu$ and $\lambda\nu$ of equation (B.16) into the loss. Specifically, we have

$$\min_{x^{(1)},\cdots,x^{(r)} \in \mathbb{R}^{3m\times m}} \sum_{q=1}^{r} f(x^{(q)}) = \min_{x^{(1)},\cdots,x^{(r)} \in \mathbb{R}^{3m\times m}} \lambda \cdot \sum_{q=1}^{r}\sum_{i=1}^{3m}\sum_{j=1}^{m} p(|x_{ij}^{(q)}|) +$$
$$\sum_{q=1}^{r}\sum_{i=1}^{3m}\left(\sum_{j=1}^{m}\sqrt{\lambda\mu}\, x_{ij}^{(q)} - \sqrt{\lambda\mu}\, b_1\right)^2 + \sum_{q=1}^{r}\sum_{j=1}^{m}\left(\sum_{i=1}^{3m}\frac{\sqrt{\lambda\nu}\, s_i}{\sum_{i'=1}^{3m} s_{i'}/m} x_{ij}^{(q)} - \sqrt{\lambda\nu}\, b_2\right)^2, \tag{B.16}$$

where μ, ν is chosen such that $\sqrt{\lambda\mu}, \sqrt{\lambda\nu}$ are rational numbers. In this case, the variable dimension is $r \cdot 3m^2$ and the sample size n is $4r \cdot m$. Our choice of r is the solution to $\epsilon r = 2\kappa n^{c_1} d^{c_2}$ which is $r = \left\lceil \left(\frac{2(4m)^{c_1}(3m^2)^{c_2}}{\epsilon/\kappa}\right)^{1/(1-c_1-c_2)} \right\rceil$. The value of r doesn't depend on λ and p, which means that we can plug in any λ, p and the reduction is still polynomial in m. It means that for any choice of λ and p, it is strongly NP hard to find a $\lambda\kappa n^{c_1} d^{c_2}$-optimal solution.

References

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