Testing for Membership to the IFRA and the NBU Classes of Distributions

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Abstract

This paper provides test procedures to determine whether the probability distribution underlying a set of non-negative valued samples belongs to the ***I**ncreasing **F**ailure **R**ate **A**verage (IFRA)* class or the ***N**ew **B**etter than **U**sed (NBU)* class. Membership of a distribution to one of these classes is known to have implications which are important in reliability, queuing theory, game theory and other disciplines. Our proposed test is based on the Kolmogorov-Smirnov (K-S) distance between an empirical cumulative hazard function and its best approximation from the class of distributions constituting the null hypothesis. It turns out that the least favorable distribution, which produces the largest probability of Type I error of each of the tests, is the exponential distribution. This fact is used to produce an appropriate cut-off or p-value. Monte Carlo simulations are conducted to check small sample size (i.e., significance) and power of the test. Usefulness of the test is illustrated through the analysis of a set of monthly family expenditure data collected by the National Sample Survey Organization of the Government of India.

1 Introduction

Reliability assessment of single-unit and multi-component systems has been the subject of much research over the last few decades. Probabilistic modeling of the lifetime of a system is often a crucial aspect of reliability assessment (Roused and Hoyland, 2004). Apart from parametric modeling of lifetime distributions, non-parametric classes of life distributions exhibiting certain ageing properties have also been used as frameworks for reliability analysis (Lai and Xie, 2006). These aging properties include *increasing failure rate (IFR)*, *increasing failure rate average (IFRA)*, *new better than used (NBU)*, among others (Pham, 2003; Samaniego, 2007).

A distribution function F supported on $[0, \infty)$ is said to be IFR if the corresponding failure rate λ, defined by the relation $\lambda(t) = \frac{d}{dt}[-\log F(t)]$, is a non-decreasing function. As $\lambda(t)dt$ may be regarded as the conditional probability of failure in the age range $[t, t + dt)$ given survival till age t, the IFR property indicates a higher propensity of failure at older age. On the other hand, F is said to be IFRA if the failure rate λ has the property that its average value over the range $[0, t]$ is a non-decreasing function of t, i.e.,

$$\frac{d}{dt}\left[\frac{\int_0^t \lambda(u)du}{t}\right] \geq 0 \quad \text{for all } t > 0. \tag{1}$$

This property is implied by the IFR property, i.e., every IFR distribution function is also IFRA. The IFR and IFRA classes can also be described by the geometric shape of the cumulative hazard function Λ, defined by the relation $\Lambda(t) = \int_0^t \lambda(u)du$. A distribution is IFR if and only if Λ is convex, and is IFRA if and only if $\Lambda(t)/t$ is non-decreasing. The latter property is described as the star-shape, which means that the graph of the function is intersected by any straight line through origin at most once, and from above. If a cumulative hazard function is convex, it is also star shaped, but the converse is not true.

The IFR and the IFRA properties of a life distribution are notions of aging. Weaker Notions of aging can also be found in the literature (see, e.g., Barlow and Proschan (1981), Klefsjö (1983)). For example, a distribution F is called NBU if $[1 - F(t + x)]/[1 - F(x)] \leq 1 - F(t)$ for all $t \geq 0$ and all $x > 0$, i.e., if the lifetime of a unit of age x is stochastically smaller than that of a new unit, for any $x > 0$. The geometric interpretation of this property is that the function Λ is super-additive. Corresponding to each type of aging, there is a corresponding notion of negatively aging, that is obtained by reversing the direction of

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the defining inequality. The negatively aging property corresponding to IFR is decreasing failure rate (DFR). Likewise, the decreasing failure rate average (DFRA) and new worse than used (NWU) properties are the negatively aging counterpart of IFRA and NBU, respectively. The exponential distribution is found to lie at the intersection of all positively and negatively aging classes (Pham, 2003).

Membership to the IFR, the IFRA or the NBU class of distributions brings several benefits. For example, the distribution function can be bounded from above and below in terms of its mean or a quantile. Barlow and Proschan (1981) provided several other useful properties of these classes, relating to reliability of a single unit system, a coherent system, a system subject to cumulative shocks and so on.

Apart from reliability, researchers from various fields, including queuing theory, expert systems, game theory and economics, have historically shown interest in the probabilistic consequences of a distribution belonging to one of these classes. As an example, consider an M/GI/1/n queue, which is used in the analysis of operating systems of computers as well as various expert systems. Explicit expressions for the distribution of the number of losses during a busy period is known when the service time distribution is exponential. When this distribution is NBU, the number of losses is known to be stochastically smaller than that in the exponential case (Abramov, 2006). Before the queue starts operating, it may be possible to obtain samples from the service time distribution, and thus determine whether it is NBU. If so, the results in the exponential case can be used as a worst-case scenario.

In a game theoretic context, membership of some size distributions to such classes can ensure the existence of a Nash equilibrium, and this fact has applications in designing expert systems (see Agah et al., 2004; Zhao and Atkins, 2009). Economists and econometricians have also worked with size distributions with aging properties (see, e.g., Ohn et al., 2004; Moldovanu et al., 2007; Hoppe et al., 2011). Further, aging properties of an income/expenditure distribution have been linked with certain characteristics of the corresponding Lorenz curve, used for studying income inequalities (Chandra and Singpurwala, 1981; Klefsjö, 1984). All these results are contingent on membership of the underlying distribution to the class. Therefore, ascertaining this membership is an important task.

Several tests of hypotheses have been proposed for this purpose. Typically the exponential distribution is regarded as the null hypothesis, while membership to the IFR, the IFRA or the NBU class of distributions is posed as the alternative (Klefsjö, 1983; Kumazawa, 1987; Bandyopadhyay and Basu, 1989; Link, 1989; Jammalamadaka et al., 1990; Tiwari and Zalkikar, 1992; Ahmad, 1994). The applicability of these tests is somewhat limited by the fact that non-membership to the intended class of distributions is not considered. For instance, a distribution with non-monotone aging property is classified by the above test either as the exponential distribution or as belonging to an aging class. A test for membership would be ideally suited as a complement to the above tests. In such a test, the null hypothesis is membership to a proposed class (e.g., IFR, IFRA, NBU), while the alternative hypothesis is non-membership to that class. Tenga and Santner (1984) and Santner and Tenga (1984) proposed a test of membership to the class of IFR distributions. However, tests of membership to other aging classes have not been considered.

In this paper, we propose a test of membership to the IFRA class of distributions, and another test of membership to the NBU class of distributions.

The inequalities, which are applicable for the members of the IFRA or the NBU classes of distributions, become useful when the aging hypothesis is established through (i) a conventional test of exponentiality against IFRA or NBU, and (ii) a test of membership to the IFRA/NBU class against non-membership, proposed here. Since the targeted hypothesis happens to be the null hypothesis of the proposed test, one might wonder whether any non-rejection could be due to shortage of samples. The small sample power of the proposed test is demonstrated through Monte Carlo simulations, and also through the analysis of a data set, for which the plot of the estimated cumulative hazard function appears to largely conform to the general shape implied by the null hypothesis.

2 Main Results

2.1 Testing for membership to the IFRA class

Let $\mathbb{F}$ be the class of all distributions supported on $[0, \infty)$. Let $\mathbb{I}$ be the class of IFRA distributions. Consider testing of the hypotheses,

$$\begin{aligned} \mathrm{H}_0: \quad & F \in \mathbb{I}, \\ \mathrm{H}_1: \quad & F \in \mathbb{F} - \mathbb{I}. \end{aligned} \tag{2}$$

As mentioned in Section 1, the cumulative hazard function, $\Lambda(t)$, of $F \in \mathbb{I}$ is star-shaped. We build the test statistic using this geometric property of an IFRA distribution.

Given that $F \in \mathbb{I}$, Wang (1987) gave an estimator of Λ which is star-shaped and studied its property. We describe the estimator below and construct a test based on it.

Definition 1. The greatest star-shaped minorant (GSM) of a nondecreasing function g over $[0, \infty)$ is

$$\tilde{g}(t) \;=\; \sup\{h(t) : h(t) \leq g(t) \text{ for } 0 \leq t < \infty, \text{ where } h(t) \text{ is star-shaped on } [0, \infty)\}.$$

The GSM of g is essentially a star-shaped approximation of g that has the smallest supremum norm of the approximation error. Let $\mathbf{T} = \{T_1\ T_2\ \cdots\ T_n\}$ be an n-vector of ordered failure times. The empirical distribution function computed from the data vector $\mathbf{T}$ is given by

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(T_i < t),$$

where I is the indicator function. The empirical cumulative hazard function is defined as

$$\Lambda_n(t) = \begin{cases} -\log(1-F_n(t)) & \text{if } t < T_n, \\ \infty & \text{if } t \geq T_n. \end{cases} \quad (3)$$

The function $\Lambda_n(t)$ is a non-decreasing step function with value $-\log\left(1-\frac{i}{n}\right)$ at T_i for $i = 1,\ldots,n-1$. Wang's (1987) estimator of $\Lambda(t)$ is the GSM of $\Lambda_n(t)$, given by

$$\tilde{\Lambda}_n(t) = \begin{cases} \lambda_i t & \text{if } T_{i-1} \leq t < T_i,\ i = 1,\ldots,n, \\ \infty & \text{if } t \geq T_n, \end{cases}$$

where $\lambda_i = \min_{i \leq k \leq n} \dfrac{\Lambda_n(T_{k-1})}{T_k}$ and $T_0 = 0$.

A natural test statistic for (2) can be based on the Kolmogorov-Smirnov (K-S) distance between $\Lambda_n(t)$ and $\tilde{\Lambda}_n(t)$, i.e.,

$$\sup_{0<t<T_n} |\Lambda_n(t) - \tilde{\Lambda}_n(t)| = \max_{1\leq i<n} |\Lambda_n(T_i) - \tilde{\Lambda}_n(T_i)|.$$

It may be observed that the estimates of $\Lambda(t)$ for larger values of t are based on fewer number of observations and hence have larger variance (Gill and Schumacher, 1987). The difference between $\Lambda_n(t)$ and $\tilde{\Lambda}_n(t)$ at the right tail may thus have unduly large influence on the supremum. This imbalance may be corrected by considering a weighted Kolmogorov-Smirnov statistic, given by

$$KSI_n(\mathbf{T}) \quad = \quad \max_{1\leq i<n} w_{i,n}|\Lambda_n(T_i) - \tilde{\Lambda}_n(T_i)|, \quad (4)$$

where $w_{i,n} \geq 0$ for $1 \leq i < n$ and $n \geq 1$.

Now, we give the universal least favorable distribution for $KSI_n(\mathbf{T})$ when $F \in \mathbb{I}$.

Theorem 1. Let $\mathbf{T}$ be an n-vector of order statistics from the distribution $F \in \mathbb{I}$. Then

$$P[KSI_n(\mathbf{T}) \geq t] \leq P[KSI_n(\mathbf{X}) \geq t] \quad \text{for all } \ t \geq 0, \quad (5)$$

where $\mathbf{X}$ is an n-vector of order statistics from the unit exponential distribution.

It follows from Theorem 1 that a size α (i.e., a significance level α) test (2)of H_0 vs H_1 is given by

$$\phi_I(T) = \begin{cases} 1 & \text{if } KSI_n(\mathbf{T}) \geq k_\alpha, \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

where the cut-off level, k_α, is chosen as the solution of $P[KSI_n(\mathbf{X}) \geq k_\alpha] = \alpha$, $\mathbf{X}$ being an n-vector of order statistics from the unit exponential distribution. The cut-off level k_α is difficult to obtain analytically. However, it can be easily computed through Monte Carlo simulations.

We now present a result on the consistency of the proposed test.

Theorem 2. If the weights $w_{i,n}$ for $i = 1,\ldots,n$ and $n \geq 1$, used in (4), are such that $0 < m < w_{i,n} < M < \infty$, then the test ϕ_I given in (6) is consistent against any $F \in \mathbb{F} - \mathbb{I}$ having finite mean.

The test (6) can easily be shown to be unbiased against DFRA alternatives.

We end this subsection by outlining an analogous test for the DFRA class. Let $\mathbb{D}$ be the class of DFRA distributions, and consider the hypotheses

$$\begin{aligned} \mathrm{H}_0: &\quad F \in \mathbb{D}, \\ \mathrm{H}_1: &\quad F \in \mathbb{F} - \mathbb{D}. \end{aligned} \quad (7)$$

A formal test for this problem may be devised along the lines of the test proposed above. It is known that F is DFRA if and only if $\Lambda(t)/t$ is a decreasing function (Barlow and Proschan, 1981). We refer to this property as the inverse star-shaped property. We define *least inverse star-shaped majorant (LISM)* of a function g, analogous to the greatest star-shaped minorant, as

$$\begin{aligned} \hat{g}(t) \ = \ \inf\{h(t) : h(t) \geq g(t) \text{ for } 0 \leq t < \infty, \\ \text{where } h(t) \text{ is inverse star-shaped on } [0,\infty)\}. \end{aligned}$$

A size α test of H_0 vs. H_1 based on a weighted K-S statistics is given by

$$\phi_D(T) = \begin{cases} 1 & \text{if } KSD_n(\mathbf{T}) \geq k_\alpha, \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

where

$$KSD_n(\mathbf{T}) \quad = \quad \max_{1\leq i<n} w_{i,n}|\Lambda_n(T_i) - \hat{\Lambda}_n(T_i)| \quad (9)$$

for $1 \leq i < n$ and $n \geq 1$, and k_α is chosen as the solution to $P[KSD_n(\mathbf{X}) \geq k_\alpha] = \alpha$, $\mathbf{X}$ being an n-vector of order statistics from the unit exponential distribution. This test can be shown to be consistent against any $F \in \mathbb{F} - \mathbb{D}$, and it is also unbiased against any $F \in \mathbb{I}$.

2.2 Testing for membership to the NBU class

Let $\mathbb{N}$ be the class of NBU distributions. Consider the problem of testing the hypotheses

$$\begin{aligned} \mathrm{H}_0: &\quad F \in \mathbb{N}, \\ \mathrm{H}_1: &\quad F \in \mathbb{F} - \mathbb{N}. \end{aligned} \quad (10)$$

As mentioned in Section 1, the distribution function $F \in \mathbb{N}$ is characterized by the super-additive property of the corresponding Λ function, i.e.,

$$\Lambda(t_1 + t_2) \geq \Lambda(t_1) + \Lambda(t_2), \text{for} \quad t_1,\ t_2 \geq 0.$$

Given $F \in \mathbb{N}$, we estimate $\Lambda(t)$ by

$$\breve{\Lambda}_n(t) = \begin{cases} \inf\limits_{0 \leq s \leq t+s < T_n} [\Lambda_n(t+s) - \Lambda_n(s)] & \text{if } t < T_n, \\ \infty & \text{if } t \geq T_n. \end{cases} \tag{11}$$

Note that $\breve{\Lambda}_n$ is a super-additive step function which has the smallest supremum norm between Λ_n and any super-additive function g such that $g(t) \leq \Lambda_n(t)$ for $t \in [0, \infty)$ (Boyles and Samaniego, 1984). The function $\breve{\Lambda}_n$ is a right continuous step function with jumps at $T_r - T_s$ for some r and s, where $0 \leq s < r \leq r$ with $T_0 = 0$.

As in the case of testing for the IFRA class, we consider the weighted K-S statistics for testing (10), given as

$$KSN_n(\mathbf{T}) \quad = \quad \sup_{0<t<T_n} w_n(t)|\Lambda_n(t) - \breve{\Lambda}_n(t)|, \tag{12}$$

where $w_n(t)$ is a positive weight function. Let $A = \{t : t \text{ is jump point of } \Lambda_n(t) \text{ or } \breve{\Lambda}_n(t)\}$. Note that the cardinality of A is finite. The test statistics (12) can be viewed as

$$KSN_n(\mathbf{T}) \quad = \quad \max_{t_i \in A} \; w_n(t_i)|\Lambda_n(t_i) - \breve{\Lambda}_n(t_i)|.$$

As before, the least favorable distribution for (12) turns out to be the exponential distribution.

Theorem 3. Let $\mathbf{T}$ be an n-vector of order statistics from the distribution $F \in \mathbb{N}$. Then

$$P[KSN_n(\mathbf{T}) \geq t] \leq P[KSN_n(\mathbf{X}) \geq t] \quad \text{for all } t \geq 0, \tag{13}$$

where $\mathbf{X}$ is an n-vector of order statistics from the unit exponential distribution.

A size α test for (10) is given by

$$\phi_N(T) = \begin{cases} 1 & \text{if } KSN_n(\mathbf{T}) \geq k_\alpha, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

where k_α is chosen as the solution of $P[KSN_n(\mathbf{X}) \geq k_\alpha] = \alpha$, $\mathbf{X}$ being an n-vector of order statistics from the unit exponential distribution.

Theorem 4. If the weights $w_n(t)$, used in (12), are such that $0 < m < w_n(t) < M < \infty$, then the test ϕ_N given in (14) is consistent against any $F \in \mathbb{F} - \mathbb{N}$ having finite mean.

The test (14) can be shown to be unbiased against NWU alternatives.

A test of membership to the NWU class may be developed in a similar manner.

3 The Case of Censored Data

We begin by extending the results of Section 2 to the case of type I and type II right-censored data. In the case of type I right-censoring, observation takes place from age 0 to a pre-determined point of time. Thus, apart from the times of observed failure, one also records the time till which the remaining samples did not have any failure. In the case of type II right-censoring, observation begins with all samples at age 0 and is continued till a pre-determined number of failures is observed. This implies that the exact times of the earliest few failures are observed, and it is also known that the remaining failures occur later than the last observed failure.

We first consider type II censored data. Let $T_1 < T_2 < \cdots < T_n$ be order statistics from $F \in \mathbb{F}$. We observe $\mathbf{T}'$, the vector of r smallest failure times, $T_i' = T_i$ for $i = 1, \ldots, r$. The function F_n will have essentially r jump points. We do not attempt to define it over (T_r, ∞). The empirical cumulative hazard function Λ_n defined by (3) is modified as follows:

$$\Lambda_n(t) = -\log(1 - F_n(t)) \quad \text{if } t \leq T_r. \tag{15}$$

Let $\tilde{\Lambda}_n(t)$ be the GSM of (15). Consider the K-S statistic $KSI_n(\mathbf{T}')$ defined as in (4), in terms of the Type II censored data vector $\mathbf{T}'$. A size α test for membership to the IFRA class, based on $\mathbf{T}'$, is given as

$$\phi_{I2}(\mathbf{T}') = \begin{cases} 1 & \text{if } KSI_n(\mathbf{T}') \geq k_\alpha, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where k_α is chosen as the solution of $P[KSI_n(\mathbf{X}') \geq k_\alpha] = \alpha$, $\mathbf{X}'$ is the vector of the smallest r out of n ordered samples from the unit exponential distribution.

It can be shown, along the lines of the proof of Theorem 2, that the test (16) is consistent under the assumption of Theorem 2 (see also Santner and Tenga, 1984). It is also an unbiased test against DFRA alternatives.

We now consider Type I censored data for testing (2). Let $T_1, T_2, \ldots, T_n$ be n iid samples from distribution function $F \in \mathbb{F}$. We observe the times $T_i^* = \min(T_i, C)$ for $i = 1, 2, \ldots, n$, where C is a common right-censoring time for all the observations. We also observe the indicators of the events $T_i \leq C$ for $i = 1, 2, \ldots, n$. Let R be the number of observed failures. Note that R is a random variable. Let $\mathbf{T}^*$ be the ordered set of the observed failure times. As in case of Type II censored data, we modify the empirical hazard function by replacing r with R in (15). Again consider the

K-S statistic $KSI_n(\mathbf{T}^*)$ defined as in (4), in terms of the Type I censored data vector $\mathbf{T}^*$. A size α test for the IFRA class, based on $\mathbf{T}^*$, is given by

$$\phi_{I1}(T^*) = \begin{cases} 1 & \text{if } KSI_n(\mathbf{T}^*) \geq k_\alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

where k_α is chosen as the solution of $P[KSI_n(\mathbf{X}^*) \geq k_\alpha] = \alpha$, $\mathbf{X}^*$ being a vector of samples smaller than C, out of a total of n samples from the unit exponential distribution. The test (17) is a consistent test under the assumptions of Theorem 2.

It is important to note that, under both the censoring schemes, the least favorable distribution is based on n order statistics of the unit exponential distribution. The test based on censored data has less power in comparison to that based on complete data. It can be seen that the test (17) is unbiased against DFRA alternatives.

The tests for membership to the NBU class of distributions can be similarly extended to the cases of Type I and Type II censored data.

4 Simulation of Performance

In order to study performance of the test ϕ_I given in Section 2 through Monte Carlo simulations, we generate data from the Weibull distribution having cumulative hazard function

$$\Lambda(t) = (\lambda t)^a,$$

where $\lambda > 0$ is a scale parameter and $a > 0$ is a shape parameter. This distribution is IFRA for $a \geq 1$ and DFRA for $a \leq 1$. The case of $a = 1$ corresponds to the exponential distribution. We choose two types of weights:

1. Inverse hazard weight i.e., $w_{i,n}^{(1)} = \frac{1}{\Lambda_n(T_i)}$,
2. Constant weight i.e., $w_{i,n}^{(2)} = 1$.

We simulate Weibull failure times for the parameter values $\lambda = 1$ and $a = 1.1$, 1.4 and 1.6 to study the empirical size (i.e., the empirical significance level) of the test (6), for sample sizes $n = 10$, 100 and 1000. Table 1 shows the empirical size based on 10^5 simulation runs for the nominal 5% level of significance.

The empirical size is found to be less than the nominal level (0.05), even for sample size $n = 10$. The test becomes more conservative for larger values of a, as the underlying distribution departs further from the worst case null distribution, which corresponds to $a = 1$.

In order to study the empirical power (i.e., the complement of the empirical probability of Type II error) of the test (6), we simulate Weibull failure times for the parameter values $\lambda = 1$ and $a = 0.4$, 0.6 and 0.9, for sample sizes $n = 10$, 100 and 1000. Table 2 shows the empirical power based on 10^5 simulation runs for the nominal 5% level of significance.

Table 1: Empirical size (significance level) of Test ϕ_I

Parameters	$n = 10$	$n = 100$	$n = 1000$
$a = 1.1, w^{(1)}$	0.0228	0.0199	0.0212
$a = 1.1, w^{(2)}$	0.0316	0.0298	0.0294
$a = 1.4, w^{(1)}$	0.0029	0.0030	0.0034
$a = 1.4, w^{(2)}$	0.0072	0.0059	0.0061
$a = 1.6, w^{(1)}$	0.0007	0.0010	0.0012
$a = 1.6, w^{(2)}$	0.0024	0.0018	0.0023

Table 2: Empirical power of Test ϕ_I

Parameters	$n = 10$	$n = 100$	$n = 1000$
$a = 0.9, w^{(1)}$	0.1043	0.1690	0.2663
$a = 0.9, w^{(2)}$	0.0775	0.0840	0.0856
$a = 0.6, w^{(1)}$	0.5693	0.9972	1
$a = 0.6, w^{(2)}$	0.2558	0.3631	0.4439
$a = 0.4, w^{(1)}$	0.9229	1	1
$a = 0.4, w^{(2)}$	0.5219	0.7933	0.9594

It is found that the power of the test increases as the sample size increases from 10 to 1000. As expected, the power also increases as the underlying distribution moves away from the worst case null distribution, which corresponds to $a = 1$. Table 2 shows that the weighted Kolmogorov-Smirnov test has much larger power than the unweighted Kolmogorov-Smirnov test.

5 A Data Analytic Example

We consider the monthly household expenditure (in Indian Rupees) distribution in the State of West Bengal of India, based on the response from 7877 households canvassed during the 61st round of nationwide survey conducted by the National Sample Survey Organization (NSSO) of the Government of India (NSSO, 2007). The graph of the empirical cumulative hazard function computed from these data, shown in Figure 1, exhibits the opposite of the star-shaped pattern, in the sense that straight lines through origin would generally intersect it from below. Thus, it would be of interest to check whether the NSSO expenditure data set could have originated from a distribution belonging to the DFRA class.

If we test for the hypotheses

$$\begin{aligned} \mathrm{H}_0 &: \quad F \text{ is exponential,} \\ \mathrm{H}_1 &: \quad F \in \mathbb{D}, \end{aligned}$$

by using the test based on the scaled Total Time on Test transform (Ahmad, 1994), the null hypothesis is rejected with a very small one-sided p-value (less than 0.0001). We can complement this finding by testing the hypotheses

$$\begin{aligned} \mathrm{H}_0 : &\quad F \in \mathbb{D}, \\ \mathrm{H}_1 : &\quad F \in \mathbb{F} - \mathbb{D}. \end{aligned}$$

Figure 1 shows the overlaid plots of the estimated cumulative hazard function $\Lambda_n(t)$ and its LISM, computed for the NSSO expenditure data. Even though the two graphs are visually somewhat close, the formal test statistics, corresponding to the inverse hazard weight function, is 1276.14 whereas the 5% cut-off level turns out to be 4.474. Thus, we have to reject the hypothesis that the data had originated from a DFRA distribution.

This conclusion is quite the opposite of what one can infer only from a test of exponentiality against the DFRA alternative. This fact underscores the utility of the procedure developed in this paper.

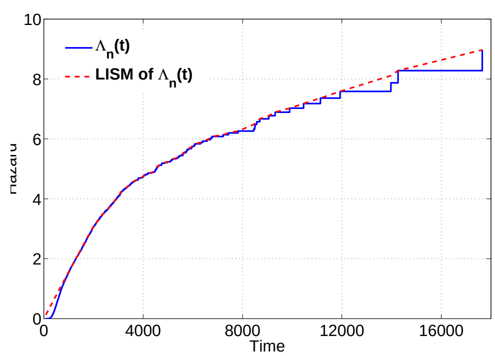

Figure 1: Empirical cumulative hazard function Λ_n (blue solid line) and least inverse star-shaped majorant (LISM) of Λ_n (red dashed curve) of the monthly house hold expenditure data.

The rejection of the DFRA hypothesis gives rise to the question: could the same conclusion be reached through the proposed test if the sample size had been smaller? In order to answer this question, we considered a random subsample of size 100 from the expenditure data. Figure 2 shows the plot of the cumulative hazard function $\Lambda_n(t)$ estimated from the sub-sample, and the corresponding LISM. The plot shows some degree of non-conformity between the two graphs. The formal test statistics is 42.78 corresponding to the inverse hazard weight function. The corresponding 5% cut-off level are 4.766, indicating rejection of the null hypothesis even at the smaller sample size.

6 Concluding Remarks

The test procedures developed in this paper, together with the standard tests of exponentiality, should provide a firm basis for making use of the inequalities known to hold for the IFRA, NBU, DFRA and NWU classes of life distributions.

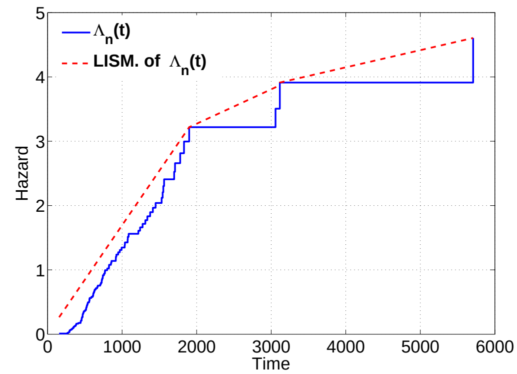

Figure 2: Empirical cumulative hazard function Λ_n (blue solid line) and least inverse star-shaped majorant (LISM) of Λ_n (red dashed curve) based on a random sample of size 100 of the monthly house hold expenditure data.

Instead of using the supremum of the difference between the estimated cumulative hazard function and its shape-restricted approximation, one could also use the integral of this difference. Use of the TTT (*total time-on-test transformation*) plot (as attempted by Tenga and Santner, 1984) as a basis for the test may not be possible, as there is no characterization of the IFRA and the NBU classes in terms of the shape of the TTT plot. Extension of the proposed tests to the case of randomly right censored data can be an area of useful future work.

Acknowledgement

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7 Appendix: Proofs

To proceed with the proof of Theorem 1, we first prove a lemma.

Lemma 1. Let $a = t_1 < t_2 < \cdots < t_n = b$ and $0 < \eta_1 < \eta_2 < \ldots < \eta_n < A$, for some real number a, b and A. Let g be a nondecreasing, right continuous step function defined over $[a, b]$, as

$$g(t) = \begin{cases} \eta_i & \text{if } t_i \le t < t_{i+1} \text{ for } j = 1, 2, ..., n-1, \\ \eta_n & \text{if } t = t_n. \end{cases}.$$

Let h be a strictly increasing function defined over $[a, b]$ such that $\frac{h(t)}{t}$ is decreasing. Let G be a nondecreasing, right continuous step function defined over $[h(a), h(b)]$

with values $\eta_1, \eta_2, \ldots, \eta_n$ at successive jump points $h(t_1), h(t_2), \ldots, h(t_n)$ respectively. Then the GSM's $\tilde{g}$ and $\tilde{G}$ of g and G, respectively, satisfy

$$\tilde{G}(h(t)) \geq \tilde{g}(t).$$

Proof. The GSM of the function $g(\cdot)$ is given by

$$\tilde{g}(t) = \begin{cases} \min\limits_{i \leq k \leq n} \dfrac{\eta_{k-1}}{t_k} \cdot t & \text{if } t_i \leq t < t_{i+1}; \\ & \quad \text{for } i = 1, \ldots, n-1, \\ \eta_n & \text{if } t = t_n, \end{cases}$$

where $\eta_0 = \eta_1$. Note that $\tilde{g}(\cdot)$ is a piecewise linear function. Since $h(\cdot)$ is an increasing function, to establish

$$\tilde{G}(h(t)) \geq \tilde{g}(t),$$

it would suffice to show that $\tilde{G}(h(t_i)) \geq \tilde{g}(t_i)$ for $i = 1, \ldots, n$. Thus, we have to show that

$$\left(\min_{i \leq k \leq n} \frac{\eta_{k-1}}{h(t_k)}\right) h(t_i) \geq \left(\min_{i \leq k \leq n} \frac{\eta_{k-1}}{t_k}\right) t_i,$$

for $i = 1, \ldots, n$. The proof is completed by observing that for $k \geq i$,

$$\frac{\eta_{k-1}}{h(t_k)} h(t_i) \geq \frac{\eta_{k-1}}{t_k} t_i,$$

since $\frac{h(t)}{t}$ is a decreasing function. □

Proof of Theorem 1. For ordered failure times $T_1, T_2, \ldots, T_n$ sampled from the distribution F, define the transformed variable $X_i = h(T_i)$, where

$$h(t) = F^{-1}(1 - e^{-t}).$$

Thus, the transformed variables $X_1, X_2, \ldots, X_n$ have the unit exponential distribution. Observe that the transformation h is the inverse of the cumulative hazard function Λ. Consequently

$$\frac{h(t)}{t} = \frac{h(t)}{\Lambda[h(t)]}.$$

As $F \in \mathbb{I}$, $\Lambda(t)/t$ is increasing in t and $\frac{h(t)}{t}$ is decreasing in t. According to Lemma 1, the GSM of the empirical cumulative hazard function based on $\mathbf{X}$ is greater than that based on $\mathbf{T}$, i.e.,

$$\tilde{\Lambda}_n(X_i) \geq \tilde{\Lambda}_n(T_i).$$

Thus, we have

$$\begin{aligned} KSI_n(\mathbf{T}) &= \max_{1 \leq i \leq n} w_{i,n} |\Lambda_n(T_i) - \tilde{\Lambda}_n(T_i)| \\ &\geq \max_{1 \leq i \leq n} w_{i,n} |\Lambda_n(X_i) - \tilde{\Lambda}_n(X_i)| \\ &= KSI_n(\mathbf{X}). \end{aligned}$$

It follows that

$$P[KSI_n(\mathbf{T}) \geq t] \leq P[KSI_n(\mathbf{X}) \geq t] \quad \text{for all } t \geq 0.$$

This completes the proof. □

Proof of Theorem 2. In order to show that the test (6) is consistent, we need to show that the power of the test converges to 1 as the sample size n tends to infinity. Note that for any fixed α, the cut-off level k_α depends on the sample size n. We now denote it by $k_{\alpha,n}$ and show that it converges to zero as n tends to infinity for any fixed α. Observe that $k_{\alpha,n}$ is obtained by solving

$$P[KSI_n(\mathbf{X}) \geq k_\alpha] = \alpha,$$

where $\mathbf{X}$ is an n-vector of order statistics from the unit exponential distribution. We have

$$\begin{aligned} KSI_n(\mathbf{X}) &= \max_{1 \leq i \leq n} w_{i,n} |\Lambda_n(X_i) - \tilde{\Lambda}_n(X_i)| \\ &\leq M \sup_{0 < t < X_n} |\Lambda_n(t) - \tilde{\Lambda}_n(t)| \end{aligned}$$

If the underlying hazard function is star-shaped, $K^* = \sup\limits_{0<t<X_n} |\Lambda_n(t) - \tilde{\Lambda}_n(t)|$ converges to 0 in probability (Wang, 1984; 1987). This implies that for any fixed α, the cut-off level $k_{\alpha,n}$ tends to zero.

To complete the proof, we now show that the power of the test converges to 1, i.e.,

$$\lim_{n \to \infty} P\left[KSI_n(\mathbf{T}) \geq k_{\alpha,n} | F \in \mathbb{F} - \mathbb{I}\right] = 1.$$

Thus we have to show, for some $\delta > 0$, as $n \to \infty$

$$P\left[\sup_{0<t<X_n} |\Lambda_n(t) - \tilde{\Lambda}_n(t)| \geq \delta \;\middle|\; F \in \mathbb{F} - \mathbb{I}\right] \to 1.$$

For this purpose, it is enough to show that

$$P\left[|\Lambda_n(t) - \tilde{\Lambda}_n(t)| \geq \delta \text{ for some } t > 0 \;\middle|\; F \in \mathbb{F} - \mathbb{I}\right] \to 1. \tag{18}$$

Note that $\Lambda_n(t) \to \Lambda(t)$ and $\tilde{\Lambda}_n(t) \to \tilde{\Lambda}(t)$ as n tends to infinity, where $\tilde{\Lambda}$ is the GSM of Λ (Wang, 1984). If F is not IFRA, we have $|\Lambda(t) - \tilde{\Lambda}(t)| > 0$ for some t, say t^*. The proof is completed by observing that the probability statement (18) holds for this t^*. □

We now present another lemma before proving Theorem 3.

Lemma 2. Let $a = t_1 < t_2 < \cdots < t_n = b$ and $0 < \eta_1 < \eta_2 < \ldots < \eta_n < A$, for some real number a, b and A. Let g be a nondecreasing, right continuous step function defined over $[a, b]$, as

$$g(t) = \begin{cases} \eta_i & \text{if } t_i \leq t < t_{i+1} \text{ for } j = 1, 2, \ldots, n-1, \\ \eta_n & \text{if } t = t_n. \end{cases}.$$

Let h be a strictly increasing function defined over $[a,b]$, such that $h(t)$ is sub-additive, i.e., $h(t_1 + t_2) \leq h(t_1) + h(t_2)$. Let G be a nondecreasing, right continuous step function defined over $[h(a), h(b)]$ with values $\eta_1, \eta_2, \ldots, \eta_n$ at successive jump points $h(t_1), h(t_2), \ldots, h(t_n)$ respectively. Then $\breve{g}$ and $\breve{G}$, defined as in (11) for g and G, respectively, satisfy

$$\breve{G}(h(t)) \geq \breve{g}(t).$$

Proof. The proof is similar to that of Lemma 1. □

Proof of Theorem 3. Since $F \in \mathbb{N}$, the inverse hazard function h as defined in the proof of Theorem 1 is subadditive. The proof proceeds along the lines of that of Theorem 1, by using Lemma 2. □

Proof of Theorem 4. The proof is similar to that of Theorem 2. □

References

V. M. Abramov (2006). Stochastic inequalities for single-server loss queueing systems. *Stoch. Proc. Appln.*, Vol. 24, 12051221.

I. A. Ahmad (1994). A class of statistics useful in testing increasing failure rate average and new better than used life distributions. *J. Statist. Plann. Inf.*, Vol. 41, 141-149.

A. Agah, S. K. Das, K. Basu and M. Asadi (2004). Intrusion detection in sensor networks: A non-cooperative game approach. In *Proceedings of the Third IEEE International Symposium on Network Computing and Applications (NCA'04)*, 343-346.

D. Bandyopadhyay and A. P. Basu (1989). A note on tests for exponentiality by Deshpande. *Biometrika*, Vol. 76, 403-405.

R. E. Barlow and F. Proschan (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt, Rinehart and Winston, New York.

R. A. Boyles and F. J. Samaniego (1984). Estimating a survival curve when new is better than used. *Oper. Res.*, Vol. 32, 732-740.

M. Chandra and N. Singpurwala (1981). Relationships between some notions which are common to reliability theory and economics. *Math. Op. Res.*, Vol. 6, 113-121.

R. D. Gill and M. Schumacher (1987). A simple test of the proportional hazards assumption. *Biometrika*, Vol. 74, 289-300.

H. C. Hoppe, B. Moldovanu and E. Ozdenoren (2011). Coarse matching with incomplete information. *Economic Th.*, Vol. 47, 75-104.

S. R. Jammalamadaka, R. C. Tiwari and J. N. Zalkikar (1990). Testing for exponentiality against IFRA alternatives using a U-statistic process. *Revista Brasileira de Probabilidade e Estatistica*, Vol. 4, 147-159.

B. Klefsjö (1983). Some tests against aging based on the total time on test transform. *Commun. Statist. Theor. Meth.*, Vol. 12, 907-927.

B. Klefsjö (1984). Reliability interpretations of some concepts from economics. *Naval Res. Logistics*, Vol. 31, 301-308.

Y. Kumazawa (1987). On testing whether new is better than used using randomly censored data. *Ann. Statist.*, Vol. 15, 420-426.

C. D. Lai and M. Xie (2006). *Stochastic aging and Dependence for Reliability*, Springer, New York.

W. A. Link (1989). Testing for exponentiality against monotone failure rate average alternatives. *Commun. Statistics - Theor. Meth.*, Vol. 18, 3009-3017.

B. Moldovanu, A. Sela and X. Shi (2007). Competing auctions with endogenous quantities. *J. Economic Th.*, Vol. 141, 1-27.

NSSO (2007). *Household Consumption of Various Goods and Services in India, 2004-05 Vol. I: Major States and All-India*, Report No. 509 (61/1.0/2), National Sample Survey Organization, Government of India, downloaded on July 23, 2011 from http://mospi.nic.in/rept%20_%20pubn/509_P1_final.pdf.

J. Ohn, L. W. Taylor and A. Pagan (2004). Testing for duration dependence in economic cycles. *Econometrics J.*, Vol. 7, 528-549.

H. Pham (2003). *Handbook of Reliability Engineering*. Springer-Verlag, London.

M. Rousand and A. Hoyland (2004). *System Reliability Theory: Models, Statistical Methods and Applications*, Second edition, Wiley, Chichester.

F. Samaniego (2007). *System Signatures and Their Applications in Engineering Reliability*, Springer Science, New York.

T. J. Santner and R. Tenga (1984). Testing goodness of fit to the increasing failure rate family with censored data. *Naval Res. Logistics*, Vol. 31, 631-646.

R. Tenga and T. J. Santner (1984). Testing goodness of fit to the increasing failure rate family. *Naval Res. Logistics*, Vol. 31, 617-630.

R. C. Tiwari and J. N. Zalkikar (1992). A class of efficient tests for increasing-failure-rate-average distribution under random censoring. *IEEE Trans. Reliab.*, Vol. 41, 602-607.

J. L. Wang (1987). Estimators of a distribution function with increasing failure rate average. *J. Statist. Planning Inf.* Vol. 16, 415-427.

J. L. Wang (1987). Estimating IFRA and NBU survival curves based on censored data. *Scand. J. Statist.*, Vol. 14, 199-210.

X. Zhao and D. R. Atkins (2009). Transshipment between competitive retailers. *IIE Transactions*, Vol. 41, 665-676.