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THE CHERN–SIMONS–LANDAU–GINZBURG THEORY OF THE FRACTIONAL QUANTUM HALL EFFECT*

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This paper gives a systematic review of a field theoretical approach to the fractional quantum Hall effect (FQHE) that has been developed in the past few years. We first illustrate some simple physical ideas to motivate such an approach and then present a systematic derivation of the Chern–Simons–Landau–Ginzburg (CSLG) action for the FQHE, starting from the microscopic Hamiltonian. It is demonstrated that all the phenomenological aspects of the FQHE can be derived from the mean field solution and the small fluctuations of the CSLG action. Although this formalism is logically independent of Laughlin's wave function approach, their physical consequences are equivalent. The CSLG theory demonstrates a deep connection between the phenomena of superfluidity and the FQHE, and can provide a simple and direct formalism to address many new macroscopic phenomena of the FQHE.

1. Introduction

The remarkable discovery of the integer and the fractional quantum Hall effect (IQHE and FQHE)[1] opened an exciting new chapter in condensed matter physics. Since then, tremendous progress has been made in understanding this spectacular effect. The theoretical concept[2] developed in this field not only gives a complete explanation of the experimental facts, but also serves as a paradigm in understanding other kinds of strongly correlated systems.

The first successful theory of the FQHE was developed by Laughlin,[3] who proposed a variational wave function to describe a correlated, incompressible electron liquid at filling factors $\nu = \frac{1}{2k+1}$. It was later shown that this class of wave functions are exact for a certain type of short ranged interactions.[4,5] The incompressibility of the Laughlin state leads to plateaus of Hall resistivity ρ_{xy} at $\nu = \frac{1}{2k+1}$, while the absence of gapless excitations leads to vanishing longitudinal resistivity ρ_{xx} at zero temperature. The Laughlin state also supports novel quasiparticle excitations which carry fractional charge and obey fractional statistics. Haldane,[4] Halperin[6] and recently Jain[7] have constructed hierarchical wave functions which explain the FQHE at other filling factors as well.

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Even though the wave function approach gives a satisfactory description of the phenomenon, many theorists feel the urgency to understand FQHE in a general context of other strongly correlated phenomena. In particular, identifying an order parameter of the FQH state would significantly deepen our theoretical understanding and, a Landau–Ginzburg-like field theoretical description would not only capture the basic essence of the phenomenon in all its simplicity and beauty, but might also lead to new experimental predictions which are inaccessible in the wave function approach. This point of view was especially emphasized by Girvin.[2] The possible existence of a Landau–Ginzburg-like description of the FQHE is also suggested by the cooperative ring exchange theory of the FQHE,[8] in which the FQHE is understood in a similar fashion as Feynman's theory of the λ transition in the Bose superfluid.

An important first step towards this direction is made by a deep observation of Girvin and MacDonald.[9] Performing a singular gauge transformation on the Laughlin's wave function, they obtain a bosonic wave function and show that it has algebraic off-diagonal-long-range-order (ODLRO) in the quantum Hall liquid (QHL) phase. They argue that this ODLRO property captures the fundamental correlation of the Laughlin wave function and it should be viewed as an order parameter of the QHL. They also gave heuristic arguments about the possible form of a Landau–Ginzburg theory. Although their action was later shown[10] to be inappropriate in describing the phenomenology of the FQHE, the ideas presented in this work plays an important role for later developments.

A complete and first principle construction of the Landau–Ginzburg theory is given by Zhang, Hansson and Kivelson[10] and later extended by Lee and Zhang.[11] Starting from the microscopic Hamiltonian of interacting electrons in an external magnetic field, they mapped the problem exactly onto an interacting boson problem which has an additional gauge interaction described by the Chern–Simons term. From the mean field solution of this Chern–Simons–Landau–Ginzburg (CSLG) theory, they show that (1) stable uniform mean field solutions are obtained only at filling factor $\nu = \frac{1}{2k+1}$ and the solutions describe an incompressible liquid. (2) Near these filling factors there are Hall plateaus in ρ_{xy}, and ρ_{xx} vanishes. (3) The cyclotron mode is described by the topologically trivial phase fluctuations, while the magnetoroton excitations are described by the topological vortices of the theory. (4) The vortices carry fractional charge and obey fractional statistics. A full quantum theory of the vortices can be constructed by performing a duality transformation of the original model. (5) Neglecting the vortex contribution to the ground state, one can derive Laughlin's wave function and Girvin–MacDonald's ODLRO directly from this approach. (6) A hierarchy scheme similar to that of Halperin can be easily constructed to explain other fractions. Therefore, this approach constitutes a complete description the FQHE. Although this theory is logically independent of the Laughlin's wave function theory, they lead to the same physical consequences.

In parallel to that, Read[12] constructed an order parameter of the Laughlin's wave function which is a composite of an electron creation operator and $1/\nu$ Laughlin quasi-hole operators, and showed that this operator has a non-vanishing expectation value between the Laughlin ground states. Subsequently, Rezayi and Haldane[13] computed this order parameter numerically and found that it is only finite in the QHL phase and vanishes when the QHL becomes unstable. In this sense, similar to the ODLRO of Girvin and MacDonald, this order parameter correctly captures the basic correlations in the Laughlin's wave function, although their equivalence is not obvious since the former correlation decays algebraically while the latter has a finite expectation value. The construction of a Landau–Ginzburg theory based on this order parameter is met with one serious difficulty. Since the order parameter is a composite of an electron and $1/\nu$ quasi-hole, it is electrically neutral, therefore, it does not couple minimally to the external electromagnetic field, in sharp contrast to the CSLG action of Zhang, Hansson and Kivelson. Read's action was derived under the condition of constant external fields, and it is not known at present how it generalizes to the case of time dependent and spatially varying external fields. Therefore, the electromagnetic response and the QHL phenomenology cannot yet be addressed systematically in this formalism.

The plan of this review is to summarize the basic ideas of the CSLG theory in terms of simple physical pictures in Sec. 2. We give a detailed microscopic derivation of the CSLG action in Sec 3 and discuss the mean field solutions and the quantum Hall phenomenology in Sec. 4. Section 5 is devoted to a discussion of the cyclotron mode, Kohn's theorem and the collective excitation of the CSLG theory, while the vortex excitations, duality transformations and the magneto roton excitations are discussed in Secs. 7 and 8. We give a derivation of Laughlin's wave function and the Girvin–MacDonald ODLRO from the CSLG theory in Sec. 6. In the conclusion, we address some of the open problems in FQHE and CSLG theory.

2. The Basic Physical Picture of the CSLG Theory

Consider a system of two-dimensional electron gas subjected to an external magnetic field B in the $-\hat{z}$ direction, and assume that the Zeeman splitting is large enough so that the electrons spins are all polarized. The basic starting point of the CSLG theory[10] is to map the interacting electron gas into a bosonic problem with additional gauge interaction.[14] In this representation, an electron is viewed as a composite of a charged boson and a flux tube with odd number of fundamental flux unit $\phi_0 = hc/e$ attached to it (see Fig. 1). This can be accomplished by introducing a statistical gauge field $\mathbf{a}(x)$ which is determined by the particle density $\rho(x)$ by the relation:

$$\nabla \times \mathbf{a}(x) = (2k+1)\phi_0 \rho(x) \tag{2.1}$$

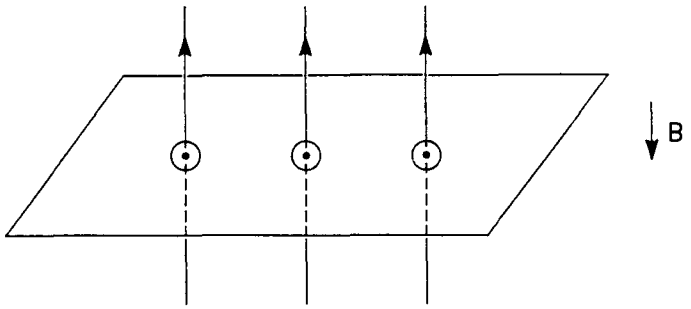


Fig. 1. A two-dimensional electron system subjected to a uniform external magnetic field. Each electron can be viewed as a boson which carries an odd number of flux quanta.

where k is an integer. ($\nabla \times \mathbf{a}(x)$ is a vector always pointing in the $\hat{z}$ directions, for discuission of two-dimensional physics, it can be viewed as a scalar). From (2.1) we see that a unit charge at the origin induces a statistical flux of odd multiples of the flux quanta.

$$\oint \mathbf{a} \cdot \mathbf{d}l = (2k+1)\phi_0 \ . \tag{2.2}$$

As one interchanges two of these boson-flux-tube composites, a Bohm–Aharonov phase factor of

$$\exp\left(i \frac{e}{\hbar c} \int_0^{\pi} \mathbf{a} \cdot \mathbf{d}l\right) = e^{i(2k+1)\pi} = -1$$

is obtained, which correctly reproduces the original statistics of the electron. With this representation, it is straightforward to see why filling factor $\nu = \frac{1}{2k+1}$ is special. Each boson not only "sees" the external magnetic field whose flux density ρ_A is $\frac{1}{\nu}$ times the particle density, i.e., $\rho_A = \frac{1}{\nu}\rho$, but also "sees" the statistical gauge field $\mathbf{a}(x)$ due to the flux tubes carried by other bosons. Since the statistical flux density is $\rho_a = (2k+1)\rho$ from (2.1), it cancels the effect of the external field on the average, if $\nu = \rho/\rho_A = \frac{1}{2k+1}$, in which case the boson "sees" no net field and can form a Bose–Einstein condensate. The Meissner effect of this charge Bose condensate in turn leads to the incompressibility of the original electron system. To see this, one notices that from (2.1), any change in the local density would necessarily induce a change in the statistical gauge field, resulting in a net flux in the same region. However, it is not possible for a net flux to penetrate a charged Bose superfluid because of the Meissner effect. Consequently the particle density has to be kept uniform, or the fluid is incompressible.

While this "bosonic picture" naturally explains the incompressibility at the special filling factors of $\nu = \frac{1}{2k+1}$, it seems to lead to a puzzle about the Hall effect.

According to our previous reasoning, due to the cancellation of the statistical field and the external magnetic field, the bosons see no net field. Consequently, there is no Lorentz force acting on it. How does the Hall effect arise in this picture? This puzzle can be intuitively explained as follows. Each boson carries both a charge e and flux quanta of $(2k+1)\phi_0$. As such a composite object moves, it not only carries a charge current $I_c = e\frac{dN}{dt}$, but also a vortex current of $I_v = (2k+1)\phi_0 \frac{dN}{dt}$ (see Fig. 2). Such a vortex current induces a transverse voltage drop of $V_H = \frac{1}{c}(2k+1)\phi_0 \frac{dN}{dt} = (2k+1)\frac{h}{e}\frac{dN}{dt}$, according to Faraday's law. The resulting Hall resistance is therefore given by $R_H = V_H/I_c = (2k+1)h/e^2$, just as one observes in the FQHE experiments. However, this "explanation" only serves as a pictorial illustration, the detailed derivation of the Hall conductivity is given in Sec. 4.

The analogy of the FQHE and boson superfluidity can be pursued even further. In a two-dimensional superfluid, there are topological excitations in the form of a vortex where the phase of the order parameter twists by $2\pi m$ as one encircles the origin of the vortex (see Fig. 3). In a neutral 2-*D* superfluid, such a vortex would cost logarithmically divergent energy. However, the vortex energy is finite in a *charged* 2-*D* superfluid. The finite energy requirement uniquely fixes the gauge flux associated with the vortex to be integer multiples of the flux quantum, i.e., $\oint \mathbf{a}\cdot\mathbf{dl} = m\phi_0$. The same arguments apply to the system of charged bosons interacting with Chern–Simons gauge fields. From (2.1), one sees that the quantization of the statistical gauge flux immediately implies that the vortices with unit vorticity $m = \pm 1$ carry fractional charge $\pm\frac{1}{2k+1}$, the same as one deduces from the Laughlin's wave function approach. Since each vortex carry one unit of flux ϕ_0 and $\frac{1}{2k+1}$ unit of charge, the Bohm–Aharonov phase factor associated with interchanging a pair of such vortices is $\exp(i\frac{\pi}{2k+1})$, i.e., they obey fractional statistics.[6,15]

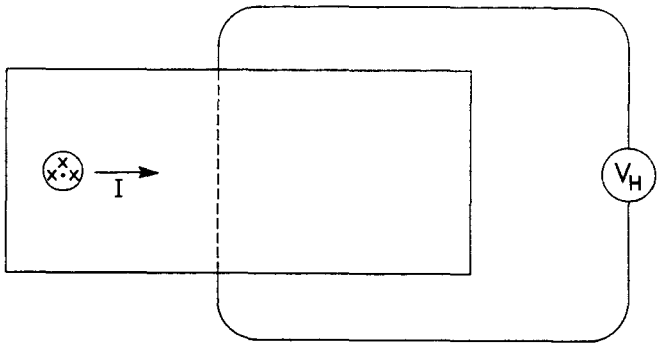


Fig. 2. The boson-flux-tube composite not only carries a charge current I, but also a vortex current. The vortex current induces a transverse voltage drop V_H which leads to the Hall effect.

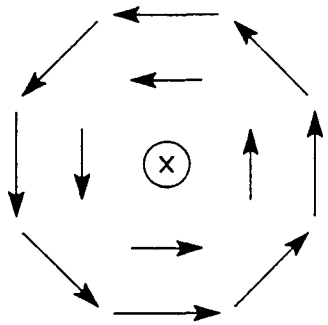
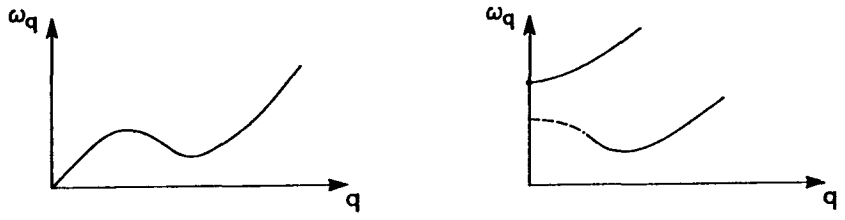


Fig. 3. Schematic illustration of a vortex, the phase of the order parameter twists by 2π around the origin.

These fractionally charged vortices play a crucial role in explaining the plateaus of the Hall conductance. We recall that the magnetic field in a type-II superconductor penetrates the system in the form of a regular or disordered vortex lattice. The supercurrent still flows without dissipation as long as these localized vortices are pinned by the impurities. An analogous situation happens in the FQHE. As one changes the filling factor from the ideal values of $\nu = \frac{1}{2k+1}$, the system accommodates the density excess (deficiency) in the form of localized vortices, in a similar way a type-II superconductor accommodates excess magnetic field. These localized vortices also form a regular or disorder lattices pinned by the impurities. Therefore, the low energy transport properties in a state with pinned vortices is identical to that in a state without vortices, i.e., at $\nu = \frac{1}{2k+1}$. In particular, this leads to a plateau in the Hall conductance and the vanishing of ρ_{xx} in the vicinity of $\nu = \frac{1}{2k+1}$.

In addition to the analogy between the vortex excitation of a charged superfluid and the quasi-particles in the FQH liquid, there is also a similar correspondence between the collective excitations of both systems. The spectrum of collective excitations of a neutral superfluid consists of a phonon branch for small momentum and a roton branch for intermediate momentum of the order of the inverse interparticle spacing (see Fig. 4a). In a charged superfluid, the phonon mode of the neutral superfluid is pushed to the plasmon frequency ω_p, whereas the roton mode is more or less unaffected by the long-ranged Columb interaction and constitutes the lowest lying excitation of the system. For momentum much less than the inverse interparticle spacing, the lowest lying excitations of the system consists of a pair of rotons with nearly opposite momentum, giving rise to an excitation spectrum as pictured in Fig. 4b. Of course, such a roton pair also exists in a neutral superfluid, but since there is a phonon branch lying below in energy, it decays quickly into phonons, and it is not sensible to think of them as elementary excitations. Only in a charged superfluid, where the phonon mode is pushed to the plasma frequency, such roton-pair excitation can maintain their integrity. The collective modes of the



(a) (b)

Fig. 4. Schematic illustration of the elementary excitation of a neutral superfluid, it consists of a phonon and a roton branch. b) In a charged superfluid, the phonon excitation is pushed to the plasma frequency, while the roton branch remains unaffected. The dotted line indicates the excitations due to a *pair* of rotons with nearly opposite momenta.

quantum Hall liquid is qualitatively the same as in Fig. 4b, where the plasma frequency ω_p is replaced by the cyclotron frequency ω_c and represent the inter-Landau level transitions, and the lower branch represents the magneto-roton excitations within the same Landau level.[16] The entire branch of the magneto-roton excitations can be interpreted as composites of vortices in the CSLG theory,[11] as we shall see in Sec. 8.

We see from the qualitative arguments given in this section that there indeed exists a deep analogy between the phenomenon of superfluidity and the FQHE. In the following sections, we shall build the necessary mathematical formalism to give a quantitative description of these ideas.

3. Derivation of the CSLG Action

In this section we shall give a microscopic derivation of the CSLG action. The idea is to perform a singular gauge transformation on the original interacting electron problem and map it onto a interacting boson problem with Chern-Simons gauge fields.

We start from the microscopic Hamiltonian of a two-dimensional system of polarized electrons:

$$H = \frac{1}{2m} \sum_i \left[\mathbf{p}_i - \frac{e}{c} \mathbf{A}(x_i) \right]^2 + \sum_i e A_0(x_i) + \sum_{i<j} V(x_i - x_j) \tag{3.1}$$

where **A** is the vector potential of the external magnetic field, which in symmetric gauge can be expressed as

$$A_\alpha = \frac{1}{2} B \varepsilon_{\alpha\beta} x_\beta \tag{3.2}$$

A_0 is the scalar potential of the external electric field,

$$E_\alpha = -\partial_\alpha A_0$$

and $V(x)$ is the interaction between the electrons. We could study a general class of interactions, but the physically relevant one is the Columb potential

$$V(\mathbf{x}) = \frac{e^2}{|\mathbf{x}|} \ . \tag{3.3}$$

In this paper, we use $i = 1, \ldots, N$ to denote the labels of N electrons, $\alpha = 1, 2$ (or x, y) denotes the 2-dimensional space indices and $\mu = 0, 1, 2$ denotes the space time indices.

Since the electrons are assumed to be completely polarized, their space wave function $\Psi(x_1 \ldots x_N)$ must be totally antisymmetric, according to the Pauli principle. The Schrödinger equation

$$H\Psi(x_1 \ldots x_N) = E\Psi(x_1 \ldots x_N) \tag{3.4}$$

together with this total antisymmetry requirement defines the quantum eigenvalue problem of our interest.

Now we shall perform a singular gauge transformation to map this problem into a bosonic one. In order to do this, let us first *define a bosonic problem* and then prove its equivalence with the electron problem of our interest. Consider a new Hamiltonian:

$$H' = \frac{1}{2m} \sum_i \left[\mathbf{p}_i - \frac{e}{c} \mathbf{A}(x_i) - \frac{e}{c} \mathbf{a}(x_i) \right]^2 + \sum_i eA_0(x_i) + \sum_{i<j} V(x_i - x_j) \ . \tag{3.5}$$

Every symbol in H' has the same meaning as in H, except the new vector potential $\mathbf{a}$, which describes a *gauge interaction* among the particles and is given by

$$\mathbf{a}(x_i) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \sum_{j \neq i} \nabla \alpha_{ij} \tag{3.6}$$

where $\phi_0 = hc/e$ is the unit of flux quantum, θ is a parameter unspecified at the moment and α_{ij} is the angle sustained by the vector connecting particles i and j with an arbitrary vector specifying a reference direction, say the $\hat{x}$ axis. The Hamiltonian H' defines a Schrödinger's equation

$$H'\phi(x_1 \ldots x_N) = E'\phi(x_1 \ldots x_N) \tag{3.7}$$

and we *define* $\phi(x_1 \ldots x_N)$ to be a bosonic wave function, i.e. totally symmetric under the exchange of coordinates. As defined, the problems (3.4) and (3.7) are completely different from each other, however, we shall prove the following theorem to establish their connection:

Theorem. For $\theta = (2k+1)\pi$, (3.4) and (3.7) define the same quantum eigenvalue problem.

Proof. We start from (3.6) and define a unitary transformation

$$\tilde{\phi}(x_1 \ldots x_N) = U\,\phi(x_1 \ldots x_N)\,; \quad U = \exp\left(-i \sum_{i<j} \frac{\theta}{\pi} \alpha_{ij} \right) . \tag{3.8}$$

it is easy to check that

$$U\left[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(x_i) - \frac{e}{c}\mathbf{a}(x_i) \right] U^{-1} = \mathbf{p}_i - \frac{e}{c}\mathbf{A}(x_i)$$

and consequently, $UH'U^{-1} = H$, where H is nothing but the Hamiltonian defined in (3.1). Therefore, if $\phi(x_1 \ldots x_N)$ obeys the Schrödinger's equation (3.7), $\tilde{\phi}$ $(x_1 \ldots x_N)$ obeys the Schrödinger's equation

$$H\tilde{\phi}(x_1 \ldots x_N) = E'\,\tilde{\phi}(x_1 \ldots x_N) \ . \tag{3.9}$$

Furthermore, since $\phi(x_1 \ldots x_N)$ is a totally symmetric wave function, $\tilde{\phi}(x_1 \ldots$ $x_N)$ is totally antisymmetric, if $\theta = (2k+1)\pi$. This fact can be simply checked from (3.8) and note that $\alpha_{ij} = \alpha_{ji} + \pi$, from the definition of the angle α_{ij}. Thus $\tilde{\phi}(x_1 \ldots x_N)$ obeys the same Schrödinger's equation as (3.4) and is a totally antisymmetric wave function, its eigenvalue spectrum E' coincide with that of the original electron problem E. The equivalence of problems (3.4) and (3.7) is thus established for $\theta = (2k+1)\pi$. Q.E.D.

Now that we have established the exact equivalence between these two problems, we are going to work directly with the boson representation in all our subsequent considerations. We take the next step to second quantize the bosonic Hamiltonian (3.5) by introducing the bosonic field operators

$$[\phi(x), \phi^{\dagger}(y)] = \delta(x-y)$$

and (3.5) becomes

$$\begin{aligned} H = &\int d^2x \phi^{\dagger}(x) \left[\frac{1}{2m} \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(x) - \frac{e}{c}\mathbf{a}(x) \right)^2 + eA_0(x) \right] \phi(x) \\ &+ \frac{1}{2} \int d^2x d^2y \delta\rho(x)\, V(x-y) \delta\rho(y) \end{aligned} \tag{3.10}$$

in second quantized notations. Here $\rho(x) = \phi^{\dagger}(x)\phi(x)$ is the density operator, $\delta\rho(x) = \rho(x) - \bar{\rho}$ is the deviation from the average density $\bar{\rho}$, it is introduced in the last term of (3.9) so that a thermodynamic limit can be defined in the case where

$V(x)$ is a long-ranged potential. In the second quantized notation, the relation (3.6) is expressed as

$$a^{\alpha}(x) = -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\varepsilon^{\alpha\beta}\int d^2y \frac{x^{\beta}-y^{\beta}}{|x-y|^2}\rho(y) \tag{3.11}$$

which can be considered as the solution of the differential equation

$$\varepsilon^{\alpha\beta}\partial_{\alpha}a_{\beta}(x) = \phi_0\frac{\theta}{\pi}\rho(x) \tag{3.12}$$

in the Coulomb gauge $\partial^{\alpha}a_{\alpha}(x) = 0$.

Equation (3.12) specifies the statistical gauge field $\mathbf{a}(x)$ at a given time. In order to obtain its dynamics, we take the time derivative of (3.12) and use the equation of continuity $\partial_t\rho(x, t) + \partial_{\alpha}j_{\alpha}(x, t) = 0$:

$$\varepsilon^{\alpha\beta}\partial_{\alpha}\dot{a}_{\beta}(x, t) = \phi_0\frac{\theta}{\pi}\dot{\rho}(x, t) = -\phi_0\frac{\theta}{\pi}\partial_{\alpha}j^{\alpha}$$

which gives

$$\varepsilon^{\alpha\beta}\dot{a}_{\beta}(x, t) = -\phi_0\frac{\theta}{\pi}j^{\alpha} \tag{3.13}$$

up to a constant. Equations (3.12) and (3.13) completely determine the statistical gauge field $\mathbf{a}(x, t)$ in terms of the bosonic matter fields $\phi(x)$. These equations can be viewed as the analog of the Maxwell's equation which determines the electromagnetic fields in terms of the matter density and current. The Maxwell's equation can be derived from a simple action principle, namely

$$\mathscr{L} = \frac{1}{4}F_{\mu\nu}^2 - A_{\mu}j^{\mu} \ . \tag{3.14}$$

One can ask the question whether there is an action principle for the statistical gauge field $a_{\mu}(x)$ so that Eqs. (3.12) and (3.13) can be derived in an analogous fashion. The answer is indeed yes, and the action is the beautiful Chern–Simons term[17]

$$\mathscr{L} = \frac{1}{2}\left(\frac{\pi}{\theta}\right)\frac{1}{\phi_0}\varepsilon^{\mu\nu\rho}a_{\mu}\partial_{\nu}a_{\rho} - a_{\mu}j^{\mu} \ . \tag{3.15}$$

The three vector $a_{\mu} = (a_0, \mathbf{a})$ contains a time component a_0 which is introduced here as a Lagrangian multiplier field and the Chern–Simons term is

$$\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho = \varepsilon^{\alpha\beta} a_0 \partial_\alpha a_\beta - \varepsilon^{\alpha\beta} a_\alpha \partial_t a_\beta + \varepsilon^{\alpha\beta} a_\alpha \partial_\beta a_0 \tag{3.16}$$

when expressed in terms of the space-time components. Functional variation of (3.15) with respect of a_0 gives Eq. (3.12) and the variation with respect of **a** gives

$$\varepsilon^{\alpha\beta}(\partial_\beta a_0 - \partial_t a_\beta) = \phi_0 \frac{\theta}{\pi} j^\alpha$$

which is identical to Eq. (3.13) in the gauge choice of $a_0 = 0$.

Although the Chern–Simons term is expressed in terms of the gauge potential rather than the field strength, it is actually gauge invariant up to a surface term. To see this, consider the gauge variation $\delta a_\mu = \partial_\mu \Lambda(x, t)$, where $\Lambda(x, t)$is a space-time dependent gauge parameter. It is easy to see that the variation of the Chern–Simons term is

$$\delta \mathscr{L} \propto \varepsilon^{\mu\nu\rho} \partial_\mu [(\partial_\nu \Lambda) a_\rho]$$

which is a total derivative and vanishes upon integrating over a closed surface.

From (3.10) and (3.15) one can easily formulate the problem into a path integral form, with an action given by

$$S = S_a + S_\phi = \int d^3x \mathscr{L}_a + \int d^3x \mathscr{L}_\phi \tag{3.17}$$

where

$$\mathscr{L}_a = \frac{e\pi}{2\theta\phi_0} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

and

$$\mathscr{L}_\phi = \phi^\dagger (i\hbar\partial_t - e(A_0 + a_0))\phi - \frac{1}{2m} \left| \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} - \frac{e}{c} \mathbf{a} \right) \phi \right|^2 - \frac{1}{2} \int d^2y \, \delta\rho(x) V(x - y) \delta\rho(y) \ . \tag{3.18}$$

All the thermodynamic properties as well as electromagnetic response of the system is completely contained in the path integral

$$Z[A_\mu] = \int [da_\mu] [d\phi] \exp\left(iS_a[a_\mu] + iS_\phi[A_\mu + a_\mu, \phi] \right) \ . \tag{3.19}$$

This completes our microscopic derivation of the CSLG action (3.18) and (3.19). We note that so far, it is an *exact representation* of the original electron problem

(3.1) and (3.4) for $\theta = (2k+1)\pi$. In the subsequent sections we shall analyze the path integral (3.19) within mean field approximation and show that all the phenomenology of FQHE can be extracted from it.

4. The Mean Field Solution and the Phenomenology of FQHE

In the last section, we derived from the microscopic Hamiltonian the desired form of the CSLG action. It seems that we have transformed an interacting electron problem into a rather complicated boson problem with gauge interactions, and at this stage it is unclear what one gains from such a transformation. However, we shall see in this section that the CSLG action has a great advantage in that it can be treated within the standard mean field approximation. From this mean field solution emerges a deep and beautiful connection between the phenomenology of superfluidity and that of the FQHE.

We shall seek mean field solution of the CSLG action in the presence of an external magnetic field

$$\varepsilon^{\alpha\beta}\partial_\alpha A_\beta = -B \ . \tag{4.1}$$

One might guess the simplest form of the uniform mean field solution to be:

$$\phi(x) = \sqrt{\bar{\rho}} \ , \quad \mathbf{a}(x) = -\mathbf{A}(x) \ , \quad a_0(x) = 0 \tag{4.2}$$

where $\bar{\rho}$ is the average particle density. It can be easily checked that this solution indeed satisfies all the equations of motion derived from the CSLG action, except that of a_0, which is only satisfied under a special condition. To see this, we note that the a_0 equation of motion has the form:

$$\varepsilon^{\alpha\beta}\partial_\alpha a_\beta = \phi_0 \frac{\theta}{\pi}\rho \ . \tag{4.3}$$

When one substitutes the proposed mean field solution (4.2) into (4.3), one finds that: $B = \phi_0 \frac{\theta}{\pi}\bar{\rho}$. Since the magnetic flux density ρ_A is given by B/ϕ_0, one can write this equation as

$$\nu \equiv \frac{\bar{\rho}}{\rho_A} = \frac{\pi}{\theta} = \frac{1}{2k+1} \ . \tag{4.4}$$

Since the value of $\theta = (2k+1)\pi$ is required by the fermion to boson mapping, as explained in the last section, one therefore sees that the uniform mean field solution (4.2) is only possible when the filling factor is $\nu = \frac{1}{2k+1}$, precisely as those fractions where one observes the FQHE! The physical origin for this is explained in Sec 2. The mean field solution $\phi(x) = \sqrt{\bar{\rho}}$ describes a boson superfluid state.

Since the bosonic field sees a combination of the statistical gauge field $\mathbf{a}(x)$ and the external field $\mathbf{A}(x)$, such a uniform superfluid state is only possible when there is no net field present, i.e., $\mathbf{a} + \mathbf{A} = 0$. On the other hand, $\mathbf{a}(x)$ is tied to the particle density through (4.3), such cancellation occurs only when $\nu = \frac{1}{2k+1}$.

In order to extract the electromagnetic response of this novel state, we shall adopt a strategy of "divide and conquer". From (3.19) we see that the path integral can be performed in two stages. We first integrate over the bose field ϕ, in order to obtain an effective action for the gauge field $\delta a_\mu \equiv A_\mu + a_\mu$, i.e.,

$$\exp\left(i \int d^3 \mathscr{L}_{\text{eff}}(\delta a_\mu)\right) \equiv \int [d\phi] \exp\left(i \int d^3x S_\phi[\delta a_\mu, \phi]\right) . \tag{4.5}$$

This path integral describes the response of a bose superfluid to an "external" field δa_μ. When the mean field conditions are satisfied, i.e., $\nu = \frac{1}{2k+1}$, the average of δa_μ vanishes, and only small fluctuations of δa_μ remain. In the spirit of the linear response theory, we expand $\mathscr{L}_{\text{eff}}$ to quadratic order.

$$\mathscr{L}_{\text{eff}} = \frac{1}{2} \delta a_0(-q, -\omega)\pi_0(q, \omega)\delta a_0(q, \omega) + \frac{1}{2} \delta a_\alpha(-q, -\omega)\pi_{\alpha\beta}(q, \omega)\delta a_\beta(q, \omega) \tag{4.6}$$

where $\pi_0(q, \omega)$ and $\pi_{\alpha\beta}(q, \omega) \equiv \pi_1(q, \omega)\delta_{\alpha\beta} + \pi_2(q, \omega)q_\alpha q_\beta$ are the response functions of the superfluid. Later we shall obtain these response functions approximately from our CSLG action, but at this stage we would like to keep our discussion general and demonstrate how the superfluid nature of the bose condensate describes the phenomenology of the FQHE. The static response of a superfluid is generally described by a finite compressibility $\tilde{k}$ and a finite superfluid density $\tilde{\rho}_s$:

$$\tilde{k} = \frac{1}{e^2} \lim_{q \to 0} \pi_0(q, \omega = 0) \quad \text{and} \quad \tilde{\rho}_s = -\frac{m}{e^2} \lim_{q \to 0} \pi_1(q, \omega = 0) = \frac{m}{e^2} \lim_{q \to 0} \pi_2(q, \omega = 0) . \tag{4.7}$$

(See, for example, Pines and Nozieres[18] and Schrieffer[19] for these general definitions of a superfluid.) The last equality in (4.7) is a consequence of the gauge invariance.

From the response of the bose condensate ϕ, which is characterized by $\mathscr{L}_{\text{eff}}(\delta a_\mu)$, one can then perform the second path integral over the statistical gauge field a_μ to obtain the response of the entire system to external electromagnetic fields A_μ, i.e.,

$$Z[A_\mu] \equiv \exp\left(i \int d^3x \mathscr{L}_A(A_\mu)\right) = \int [da_\mu] \exp\left(i \int d^3x [\mathscr{L}_a(a_\mu) + \mathscr{L}_{\text{eff}}(\delta a_\mu)]\right) . \tag{4.8}$$

Since both $\mathscr{L}_a$ and $\mathscr{L}_{\text{eff}}$ are quadratic in a_μ, such a Gaussian integral can be performed exactly. At this moment, we shall only look at the static, i.e., $\omega = 0$ response. Within the transverse gauge $\partial_\alpha a_\alpha = \partial_\alpha A_\alpha = 0$ and set the units $\hbar = c = 1$ one obtains after some algebra

$$\begin{aligned}\mathscr{L}_A = &\frac{1}{2}\left(\frac{e^2}{2\theta}\right)^2 A_0(-q)\frac{\pi_0 q^2}{\left(\frac{e^2}{2\theta}\right)^2 q^2 - \pi_0\pi_1}A_0(q)\\ &+\frac{1}{2}\left(\frac{e^2}{2\theta}\right)^2 A_\alpha(-q)\frac{\pi_1 q^2}{\left(\frac{e^2}{2\theta}\right)^2 q^2 - \pi_0\pi_1}A_\alpha(q)\\ &-\frac{e^2}{2\theta}\varepsilon^{\alpha\beta}A_0(-q)\frac{\pi_0\pi_1}{\left(\frac{e^2}{2\theta}\right)^2 q^2 - \pi_1\pi_0}iq_\alpha A_\beta(q)\ . \end{aligned} \tag{4.9}$$

This Lagrangian is the central result of this section, it encodes all the information about the phenomenology of the FQHE. The electromagnetic response of the system are derived from the correlation functions

$$\frac{\delta^2 S_A}{\delta A_\mu(-q)\,\delta A_\nu(q)} \equiv D_{\mu\nu(q)}\ .$$

The first term in (4.9) characterizes the density-density response, and yields information about the compressibility of the system:

$$k = \frac{1}{e^2}\lim_{q\to 0} D_{00}(q) = \lim_{q\to 0}\frac{\tilde{k}q^2}{q^2 + (2\pi/v)^2\,\tilde{k}\tilde{\rho}_s/m}\ .$$

From the behavior of the bose condensate (4.7), one immediately sees that this quantity vanishes, i.e., $k = 0$, the system is an incompressible liquid! Central to this conclusion is the finiteness of the superfluid density of the bose system, i.e., $\tilde{\rho}_s \neq 0$. The physical reason why a finite superfluid density of the bose condensate necessarily lead to the incompressibility of the original electron system was explained in Sec. 2, and shall not be repeated here. If $\tilde{\rho}_s = 0$, one sees from above that a finite compressibility arise in general.

The second term in (4.9) describes the current-current response of the system. One natural question is whether the superfluidity of the bose system, $\tilde{\rho}_s \neq 0$, would also lead to superfluid behavior of the original electron system. The answer is no, since $D_{\alpha\alpha}(q) \propto q^2$, is a behavior characteristic of an insulator and it vanishes in the long wavelength limit $q \to 0$.

The third term in (4.9) describes the quantum Hall effect. Since we are interested in the current response to a *uniform* electric field, one can set $q = 0$ in the denominator of (4.9), and obtain

$$j_\alpha(x) \equiv \frac{\delta S_A}{\delta A_\alpha(x)} = \frac{e^2}{2\theta}\varepsilon^{\alpha\beta}\partial_\beta A_0 = \frac{e^2}{2\theta}\varepsilon^{\alpha\beta}E_\beta \tag{4.10}$$

from which we conclude that Hall conductivity is

$$\sigma_H = \frac{e^2}{2\theta} = \frac{1}{2k+1}\frac{e^2}{2\pi} \equiv \frac{1}{2k+1}\frac{e^2}{h} \tag{4.11}$$

where we set $\theta = (2k+1)\pi$ and restored the units. How the Hall effect arises in this boson description was explained pictorially in Sec. 2.

In the above discussions, we assumed a most general form of the bose response (4.6) and showed how the superfluid behavior (4.7) lead to the phenomenology of the FQHE. To complete this section, we shall now obtain these response functions directly from the CSLG action by an approximate evaluation of the path integral (4.4). From the form of $\mathscr{L}_\phi$ given in (3.18), we decompose the bose field $\phi(x) \equiv \sqrt{\rho(x)}e^{i\theta(x,t)}$ into an amplitude and a phase part. It can be shown that the amplitude gradient generally lead to higher derivative terms in the effective action and shall be neglected in the lowest order. One thus obtains

$$\mathscr{L}_\phi \approx -\delta\rho(\partial_t\theta + e\delta a_0) - \frac{\bar{\rho}}{2m}(\nabla\theta - e\delta\mathbf{a})^2 - \frac{1}{2}\delta\rho(x)V(x-y)\delta\rho(y) \ . \tag{4.12}$$

After integrating over the amplitude variation $\delta\rho$ one obtains an effective action purely for the phase degrees of freedom:

$$\mathscr{L}_\phi = \frac{1}{2}\frac{1}{V(q)}[-i\omega\theta(-q,-\omega) - e\delta a_0(-q,-\omega)][i\omega\theta(q,\omega) - e\delta a_0(q,\omega)]$$

$$-\frac{\bar{\rho}}{2m}[-iq_\alpha\theta(-q,-\omega) - e\delta a_\alpha(-q,-\omega)][iq_\alpha\theta(-q,-\omega) - e\delta a_\alpha(q,\omega)] \ . \tag{4.13}$$

From the definitions (4.6) one finds

$$\pi_0(q,\omega=0) = \frac{e^2}{V(q)}; \quad \pi_{\alpha\beta}(q,\omega=0) = \frac{e^2\bar{\rho}}{m}\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right) \ . \tag{4.14}$$

Within these approximations, the system indeed has a finite superfluid density $\tilde{\rho}_s = \bar{\rho}$. Therefore, all the previous discussions about the FQHE phenomenology is valid.

From the above discussions, we see that the superfluid phase of the boson field $\phi(x)$ corresponds to the incompressible quantum Hall phase of the original electron system. This approach provides a basic formalism in addressing the stability of the FQH phase and the possible transitions into the various insulating phases. In the CSLG theory, these transitions corresponds to the superfluid to insulator transition of the ϕ boson. This approach is outlined in a recent preprint by Lee, Kivelson and Zhang.[20]

5. The Cyclotron Mode and Kohn's Theorem

In the previous section, we studied the static response within the CSLG theory and showed that it leads to the most important phenomenological aspect of a Hall liquid: incompressibility and fractional Hall conductivity. In the following, we shall study the dynamic response within the CSLG theory. For electrons in a partially filled Landau level, there are two kinds of collective excitations: the inter-Landau level transition, which has an energy scale of $\omega_c = eB/mc$, shall be studied in this section, while the intra-Landau level transition, which has an energy scale of the Coulomb energy, shall be addressed in Sec. 8.

In a translationally invariant system, the center of mass motion is decoupled from the relative coordinates and is therefore independent of the interparticle interaction. When such a system is subjected to a magnetic field, the center of mass executes a cyclotron motion with frequency $\omega_c = eB/mc$. This simple physical observation can be transformed into an exact theorem, as shown by Kohn.[21].

Starting from the microscopic Hamiltonian (3.1), with $A_0 = 0$ and A_α given by (3.2), one observes that the operator

$$\pi^\alpha \equiv \sum_i p_i^\alpha - \frac{e}{c} A^\alpha(x_i) \tag{5.1}$$

has the following commutator with the Hamiltonian:

$$[H, \pi^\alpha] = i\hbar\omega_c \varepsilon^{\alpha\beta} \pi^\beta \tag{5.2}$$

which is independent of the interaction. From this one can construct eigen-operators

$$\pi^\pm = \frac{1}{\sqrt{2}} \frac{l_0}{\hbar} (\pm i\pi_x + \pi_y), \quad [H, \pi^\pm] = \pm \hbar\omega_c \pi^\pm \tag{5.3}$$

where $l_0 = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length.

Such operators generate a class of exact eigenstates of the Hamiltonian with eigenvalues spaced by $\hbar\omega_c$. Since if $|\psi\rangle$ is an eigenstate of H, i.e., $H|\psi\rangle = E|\psi\rangle$ with eigenvalue E, then $\pi^\pm|\psi\rangle$ is also an eigenstate with eigenvalue $E \pm \hbar\omega_c$.

From this argument, one sees that the ground state $|\psi_0\rangle$ must obey the constraint $\pi^- |\psi_0\rangle = 0$, and $\pi^+ |\psi_0\rangle$ is an exact eigenstate with one cyclotron quantum.

The relation (5.3) makes a prediction about the density-density correlation function of the system. Defining the density operator $\rho_q = \Sigma_i e^{iq^\alpha x_i^\alpha}$,one finds that

$$\dot{\rho}_q = \frac{i}{\hbar}[H, \rho_q] = iq^\alpha j_q^\alpha \tag{5.4}$$

where $j_q^\alpha = \frac{1}{2m}\Sigma_i\{p_i^\alpha - \frac{e}{c}A_i^\alpha, e^{iq\alpha x_i^\alpha}\}$ is the current operator. (5.4) implies a relationship between the matrix elements

$$\langle n|\rho_q|0\rangle(E_n - E_0) = \hbar q^\alpha \langle n|j_q^\alpha|0\rangle \tag{5.5}$$

where $|0>$ and $|n>$ are the ground state and the excited states with energies E_0 and E_n respectively. Using (5.5) and the expansion of $j_q^\alpha = \frac{1}{m}\pi^\alpha + o(q)$ for small q, one obtains for the time order density-density correlation

$$\begin{aligned}
\rho(q,\omega) &\equiv \sum_n \left[\frac{\langle 0|\rho_q|n\rangle\langle n|\rho_{-q}|0\rangle}{\omega - E_n + E_0 + i\delta} - \frac{\langle 0|\rho_{-q}|n\rangle\langle n|\rho_q|0\rangle}{\omega + E_n - E_0 - i\delta}\right] \\
&= \sum_n \frac{\hbar^2 q^\alpha q^\beta}{(E_n - E_0)^2}\left[\frac{\langle 0|j_q^\alpha|n\rangle\langle n|j_{-q}^\beta|0\rangle}{\omega - E_n + E_0 + i\delta} - \frac{\langle 0|j_{-q}^\alpha|n\rangle\langle n|j_q^\beta|0\rangle}{\omega + (E_n - E_0) - i\delta}\right] \\
&= \sum_n \frac{\hbar^2 q^\alpha q^\beta}{m^2(E_n - E_0)^2}\langle 0|\pi^\alpha|n\rangle\langle n|\pi^\beta|0\rangle \\
&\quad \times \left[\frac{1}{\omega - E_n + E_0 + i\delta} - \frac{1}{\omega + E_n - E_0 - i\delta}\right] + o(q^3) \\
&= \frac{\mathbf{q}^2}{2} l_0^2 \left[\frac{1}{\omega - \hbar\omega_c + i\delta} - \frac{1}{\omega + \hbar\omega_c - i\delta}\right] + o(q^3)\ ,
\end{aligned} \tag{5.6}$$

where the last equation is obtained by using the fact that π^α is a linear combination of the *eigenoperators* $\pi^\pm$ in (5.3), therefore, only the state with exactly one cyclotron quantum can contribute to the sum over intermediate states.

Equation (5.6) is the exact content of Kohn's theorem. It states that to order q^2, the density-density correlation function $\rho(q, \omega)$ must contain a pole at exactly $\omega_c = \frac{\rho B}{mc}$, without any renormalization corrections from the interparticle interactions. Although the physical origin of Kohn's theorem is extremely simple, it nevertheless places strong constraints on any approximation schemes. In the following, we shall compare our CSLG theory with this exact theorem.

To calculate the dynamical density-density correlation function, one follows the same strategy as in the previous section, where we calculated the static response. It turns out that for long wavelength, it suffices to take the static part of $\mathscr{L}_{\text{eff}}$ as given in (4.14) and combine it with the dynamic Chern–Simons term $\mathscr{L}_a$.

$$\mathscr{L}_a + \mathscr{L}_{\text{eff}} = \frac{e^2}{2\theta} a_0(-q, -\omega)\varepsilon^{\alpha\beta} i q_\alpha a_\beta(-q, -\omega) - \frac{e^2}{4\theta}\varepsilon^{\alpha\beta} a_\alpha(-q, -\omega) i\omega\, a_\beta(q, \omega)$$
$$+ \frac{e^2}{2}\delta a_0(-q, -\omega)\frac{1}{V(q)}\delta a_0(q, \omega)$$
$$- \frac{e^2\bar{\rho}}{2m}\delta a_\alpha(-q, -\omega)\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)\delta a_\beta(q, \omega)\ . \tag{5.7}$$

Integrating out a_0 and a_α, and restricting to lowest order terms in $|\mathbf{q}|$, one obtains the following effective action for A_0:

$$\frac{1}{2}e^2 A_0(-q, -\omega)\frac{\frac{\bar{\rho}}{m}\mathbf{q}^2}{\omega^2 - \omega_q^2 + i\delta}A_0(q, \omega) \tag{5.8}$$

where

$$\omega_q = \omega_c + \frac{1}{2}\frac{\bar{\rho}}{eB}q^2 V(q)\ . \tag{5.9}$$

Restoring the units, we see that (5.8) is in exact agreement with expression (5.6) of Kohn's theorem. For short range interactions, $\lim_{q\to 0} V(q) = \text{const.}$, we see from (5.9) that the cyclotron mode disperses quadratically,whereas for the long ranged Coulomb interaction, $V(q) \propto \frac{1}{q}$, we see it disperses linearly (see Fig. 5). This behavior is in perfect agreement with the RPA calculations performed by Kallin and Halperin.[22]

The collective mode found in (5.8) and (5.9) is due to the topologically trivial fluctuations of the bose condensate. It is well known that, while a neutral superfluid supports gapless phonon excitation, in the presence of long-ranged interactions, this excitation can be pushed to the plasmon energy ω_p due to the Anderson-Higgs mechanism. In the present context, it is precisely the coupling of the bose field ϕ to the Chern-Simons field a_μ which is responsible for (5.9). In the original paper by Zhang, Hansson and Kivelson,[10] this mode was misidentified as an intra-Landau level excitation. This misidentification was and corrected by Lee and Zhang.[11] In Sec 8, we shall find another branch of excitations, due to topologically non-trivial excitations, which lies below the dispersion curve in

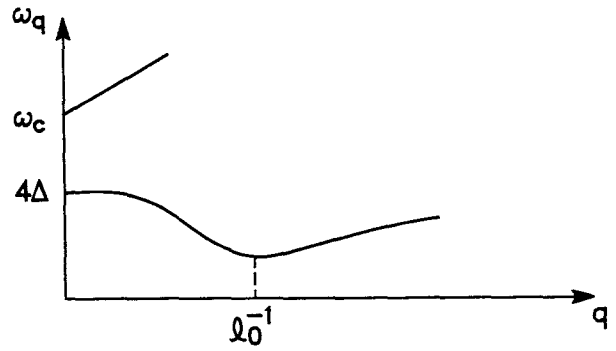


Fig. 5. Schematic illustration of the elementary excitation of a fractional quantum Hall liquid. The cyclotron mode is due to the inter-Landau level excitations, and has an energy scale of ω_c. The magneto-roton branch is due to the intra-Landau level excitations, and has an energy of the order of the Coulomb energy. In the CSLG theory, the excitations with $q < l_0^{-1}$ is interpreted in terms of a quadrupolar configuration of the vortices, while the excitations with $q > l_0^{-1}$ is viewed as due to the vortex-antivortex pair.

(5.9), and are associated with the intra-Landau level excitations. By Kohn's theorem proved in (5.6), the contribution due to the intra-Landau level mode to the density correlation must vanish faster than q^2 in the $q \to 0$ limit.

6. Derivation of Laughlin's Wave Function and Algebraic ODLRO

As we have seen in the previous section, the CSLG theory not only explains well all the phenomenological aspects of FQHE, it is also in full agreement with microscopic results like Kohn's theorem. In this section, we shall further demonstrate its success by deriving Laughlin's microscopic wave function and the algebraic ODLRO of Girvin and MacDonald, directly from the CSLG theory.[23] In order to achieve this goal, we shall first start from the CSLG action (3.18) and integrate out the statistical gauge field a_μ to obtain an effective action for the boson field $\phi(x)$.

First integrating out the a_0 field in (4.19) yields in the transverse gauge:

$$a_\alpha(q, \omega) = \frac{2\theta}{e} \varepsilon^{\alpha\beta} \frac{iq_\beta}{q^2} \rho(q, \omega) \ . \tag{6.1}$$

Inserting this expression into (3.18) gives an effective action for the boson field:

$$\mathscr{L} = \phi^\dagger(-q, -\omega)\left[\omega - \frac{1}{2m} q^2\right]\phi(q, \omega) - \frac{1}{2}\delta\rho(-q, -\omega) V(q) \delta\rho(q, \omega)$$

$$- 2\theta\delta\rho(-q, -\omega)\varepsilon^{\alpha\beta} \frac{iq_\alpha}{q^2} j^\beta(q, \omega) - \frac{2\bar{\rho}\theta^2}{m} \delta\rho(-q, -\omega) \frac{1}{q^2} \delta\rho(q, \omega) \tag{6.2}$$

where $j^\alpha(x) = \frac{ie}{2m}(\phi^\dagger \nabla^\alpha \phi - \phi \nabla^\alpha \phi^\dagger)$ is the paramagnetic current. In deriving the last term in (6.2), we neglected a three body contribution. Arguments can be given that such a contribution is unimportant for long wavelength physics.[23]

In order to proceed further we make a crucial approximation that we neglect vortex like configuration in the boson field $\phi(x)$. It will be shown in the next section that vortices have finite creation energy, therefore, for the ground state properties that we are interested in now, their contribution is negligible. Under such an assumption, the paramagnetic current is purely longitudinal, i.e., $j^\alpha \propto \nabla^\alpha \theta$, where θ is the phase of the boson field θ, the third term in (6.2) vanishes. We therefore arrive at an extremely simple effective action for the boson field ϕ, which in addition to the $V(x)$ interaction in the original Hamiltonian, also contains a logarithmic interaction mediated by the statistical gauge field. In the first quantized language, the last term in (6.2) can be expressed as:

$$\begin{aligned} &\frac{\bar{\rho}}{2m}\frac{2\pi}{v^2}\int dxdy\,\delta\rho(x)\ln|x-y|\,\delta\rho(y) \\ &=\frac{1}{m}\frac{1}{l_0^2}\left[\sum_{i<j}\ln|x_i-x_j|^{1/v}-\frac{1}{4l_0^2}\sum_i|x_i|^2\right] \end{aligned} \tag{6.3}$$

where we substituted $\delta\rho(x)=\Sigma_i\delta(x-x_i)-\bar{\rho}$ and $l_0^{-2}=2\pi\bar{\rho}/v$.

We are now in a position to derive the algebraic ODLRO correlation function. Writing $\phi(x)=\sqrt{\rho(x)}\,e^{i\theta(x)}$ in (6.2) and neglect the amplitude variations, one obtains

$$\begin{aligned} \mathscr{L}=&+\delta\rho(-q,-\omega)i\omega\theta(q,\omega)-\frac{\bar{\rho}}{2m}\theta(-q,-\omega)q^2\theta(q,\omega) \\ &-\frac{1}{2}\delta\rho(-q,-\omega)\left[V(q)+\frac{4\pi^2\bar{\rho}}{mv^2}\frac{1}{q^2}\right]\delta\rho(q,\omega)\ . \end{aligned} \tag{6.4}$$

Integrating over the amplitude $\delta_\rho(q,\omega)$, we arrive at an effective Lagrangian for the θ phase field only:

$$\mathscr{L}=\frac{1}{2}\theta(-q,-\omega)\left[\frac{\omega^2}{V(q)+\dfrac{4\pi^2\bar{\rho}}{mv^2}\dfrac{1}{q^2}}-\frac{\bar{\rho}}{m}q^2\right]\theta(q,\omega)\ . \tag{6.5}$$

The time-ordered propagator resulting from this quadratic Lagrangian is thus given by

$$\langle\theta(-q,-\omega)\theta(q,\omega)\rangle=\frac{\dfrac{2\pi}{v}\omega_c\dfrac{1}{q^2}+V(q)}{\omega^2-\omega_q^2+i\delta} \tag{6.6}$$

where $\omega_q^2=\omega_c^2+\dfrac{\bar{\rho}}{m}q^2V(q)$ is the same as the one given by (6.9).

From (6.7) one obtains the static correlation function:

$$\langle\theta(-q)\theta(q)\rangle = -i\int\frac{d\omega}{2\pi}\langle\theta(-q,-\omega)\theta(q,\omega)\rangle = -\frac{1}{2\nu}\frac{2\pi}{q^2}+o\left(\frac{1}{q}\right) \tag{6.7}$$

and therefore also the order parameter correlation function:

$$\begin{aligned}\langle\phi^\dagger(x)\phi(y)\rangle &= \bar{\rho}\langle e^{i\theta(x)-i\theta(y)}\rangle \approx \bar{\rho}\, e^{\langle\theta(x)\theta(y)\rangle}\\ &= \bar{\rho}\,|x-y|^{-1/(2\nu)}\ .\end{aligned} \tag{6.8}$$

We see that because of the long-ranged logarithmic interaction in the effective boson Lagrangian (6.4), the conventional ODLRO is modified to an algebraic ODLRO.

The ODLRO exponent is simply $\frac{1}{2\nu}$, unmodified by the interaction $V(q)$ as one can see from (6.7). (6.8) is an extremely important result, first derived by Girvin and MacDonald from the Laughlin's wave function. It is quite astonishing that exactly the same result is obtained from the CSLG theory, which so far has made no reference to the Laughlin's wave function.

In the following, we shall show that not only can the ODLRO correlation function, but also the Laughlin's wave function itself can be derived from the effective Lagrangian (6.5). As we see from the previous analysis, $V(q)$ does not play an important role for the long-range correlations, we shall set it to zero. The readers can easily convince themselves that restoring $V(q)$ does not modify the following analysis. Defining $\pi_q = \frac{\delta\mathscr{L}}{\delta\theta_q}$, one obtains the Hamiltonian version of (6.5)

$$H = \frac{1}{2}\frac{\bar{\rho}}{m}\sum_q\left[\left(\frac{2\pi}{\nu}\right)^2\frac{1}{q^2}\pi_q\pi_{-q}+q^2\theta_q\theta_{-q}\right] \tag{6.9}$$

which can be quantized by the standard canonical commutation rule

$$[\theta_q,\pi_{q'}] = -i\delta_{q+q'}\ . \tag{6.10}$$

In a representation where the operator π_q is diagonal, θ_q can be represented as a functional derivative, i.e., $\theta_q = -i\frac{\delta}{\delta\pi_{-q}}$ on the wave function $\Psi[\pi_q]$. In this representation, the ground state wave function takes the form:

$$\Psi_0[\pi_q] = \exp\left(\frac{1}{2}\sum_q\frac{2\pi}{\nu}\frac{1}{q^2}\pi_q\pi_{-q}\right) \tag{6.11}$$

which by inspection can be easily seen to satisfy the functional Schrödinger equation $H\Psi_0 = E_0\Psi_0$ with eigenvalue $-\frac{N}{2}\omega_c$. The canonical momentum π_q is in fact

nothing but the density $\delta\rho_q$ of the boson system, since it is the canonical conjugate of the phase field. One can check this by expanding $\phi(x) = \sqrt{\bar{\rho} + \delta\rho(x)}\, e^{i\theta(x)}$. From $[\phi(x), \phi^\dagger(y)] = \delta(x-y)$ it follows $[\theta_q, \delta\rho_{q'}] = -i\,\delta_{q+q'}$, as in (6.10). One therefore identifies π_q with $\delta\rho_q = \Sigma_i e^{iqx_i} - \bar{\rho}$. Inserting this expression into (5.11) one obtains

$$\Psi_0(x_1 \ldots x_N) = \prod_{i<j} |x_i - x_j|^{1/\nu} \exp\left(-\frac{1}{4l_0^2} \sum_i |x_i|^2 \right) \tag{6.12}$$

which is nothing but the modulus of the Laughlin's wave function. Since we have performed a singular gauge transformation (3.8) in order to go from the fermionic representation to the bosonic representation, performing this transformation backwards on (6.12) gives precisely the Laughlin's wave function itself!

The analysis carried out in this section establishes a connection between the CSLG theory and the Laughlin's wave function, and therefore also an interesting analogy of boson superfluidity and algebraic ODLRO with FQHE. The CSLG theory not only captures the macroscopic phenomenology of the FQHE, it even describes microscopic details like the ground state wave function itself!

7. Vortex Excitations and Duality Transformations

A superfluid supports vortex-like configurations as its elementary excitations. The analogy between superfluidity and FQHE we established in the previous sections naturally leads us to the search of similar kind of excitations in the CSLG theory. It turns out that due to the presence of the Chern–Simons term, a vortex does not only have quantized circulation but also fractionally quantized charge. They are naturally identified with the Laughlin's quasi-particle and quasi-hole wave functions.

In (4.2) we identified the uniform solution to the CSLG action at filling factors $\nu = \frac{1}{2k+1}$. In addition to this uniform solution, we can find topologically non-trivial configurations in which the Bose field has a "phase twist" of $\pm 2\pi$ about the origin. The asymptotic ($|\mathbf{x}| \to \infty$) field configuration of a vortex is given by

$$\phi(x) = \sqrt{\bar{\rho}}\; e^{\pm i\alpha(x)} \tag{7.1}$$

$$\delta\mathbf{a}(x) = \pm\frac{\phi_0}{2\pi} \nabla\alpha(x) = \pm\frac{\phi_0}{2\pi}\, \hat{z} \times \frac{\mathbf{x}}{|x|^2} \tag{7.2}$$

$$a_0(x) = 0 \tag{7.3}$$

where α is the angle of $\mathbf{x}$. In a 2-D neutral superfluid, a vortex cost logarithmically divergent energy, since

$$\frac{\hbar^2\bar{p}}{2m}\int d^2x\,|\nabla\alpha|^2 \sim \frac{\hbar^2\bar{p}}{2m}\int d^2x\,\frac{1}{x^2} \sim \frac{\hbar^2\bar{p}}{2m}\ln(R/l_0) \tag{7.4}$$

where R is the system size and l_0 is some short distance cut-off. In a charged superfiuid, however, the asymptotic energy density of a vortex vanishes exponentially by a suitable choice of the transverse gauge field $\delta\mathbf{a}$ as given in (7.2)

$$\left|\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\delta\mathbf{a}\right)\phi\right|^2 \to 0\,, \quad \text{as } |\mathbf{x}| \to \infty\ . \tag{7.5}$$

Therefore, just like vortices in a superconductor, the finite energy requirement-fixes the flux to be quantized, i.e.,

$$\oint \delta\mathbf{a}\cdot\mathbf{d}l = \pm\,\phi_0\ . \tag{7.6}$$

The new ingredient in the CSLG theory is that such quantized flux automatically lead to quantized charge. To see this, we observe that the charge density in the CSLG theory is given by

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} = \frac{\delta S_\phi}{\delta a_0(x)} = -\frac{\delta S_a}{\delta a_0(x)} = e\frac{\nu}{\phi_0}\varepsilon^{\alpha\beta}\partial_\alpha a_\beta\ . \tag{7.7}$$

The excess charge carries by a vortex is therefore

$$Q = \int d^2x\delta\rho(x) = e\frac{\nu}{\phi_0}\oint \delta\mathbf{a}\cdot\mathbf{d}l = \pm\,\frac{e}{2k+1}\ . \tag{7.8}$$

These fractionally charged vortices accommodate excess density of an incompressible fluid in a similar way the vortices in a charged superfluid accommodate excess magnetic field. As explained in the introduction, the pinning of these vortices is crucial for explaining the plateaus in the Hall conductance. The analogy of the vortex dynamics in the superfluid and the FQHE is exploited by Stone,[24] while a full quantum theory of the vortices in the FQHE is developed by Lee and Zhang,[11] using the formalism of the duality transformations developed earlier by Lee and Fisher.[25]

The original CSLG theory is formulated in terms of the charge current j_μ of the ϕ bosons coupled to the Chern-Simons gauge field a_μ. The duality transformation maps this action onto a dual version of the theory which involves the vortex current $\tilde{j}_\mu$ coupled to a dual Chern-Simons field b_μ. If one decomposes the Bose field $\phi(x)$ in terms of an amplitude part, a topologically trivial phase part $e^{i\theta(x)}$ and a topologically non-trivial part ϕ_v

$$\phi(x) = \sqrt{\rho(x)}\, e^{i\theta(x)}\, \phi_\nu(x) \tag{7.9}$$

where $\phi_\nu^\dagger(x)\, \phi_\nu(x) = 1$, then the vortex 3-current $\tilde{j}_\nu$ is defined as

$$\tilde{j}_\nu(x) \equiv \frac{1}{2\pi i}\, \varepsilon_{\mu\nu\rho}\, \partial_\nu(\phi_\nu^\dagger(x)\, \partial_\rho \phi_\nu(x)) \ . \tag{7.10}$$

For a vortex located at the origin, $\phi_\nu(x) = e^{\pm i\alpha(x)}$, (7.10) indeed gives the correct total vorticity

$$\tilde{Q}(x) = \int d^2x\, \tilde{\rho}(x) = \pm \frac{1}{2\pi} \oint \mathbf{dl} \cdot (\nabla\alpha) = \pm 1 \ . \tag{7.11}$$

In general, for a collection of vortices with vorticity $\{q_i\}$ and locations $\{x_i\}$, $\tilde{j}_\mu(x)$ is given by

$$\tilde{\rho}(x) = \sum_i q_i \delta(x - x_i) \quad \text{and} \quad \tilde{j}_\alpha(x) = \sum_i q_i\, \dot{x}_i^\alpha \delta(x - x_i) \ . \tag{7.12}$$

Substituting the decomposition (7.9) into the CSLG Lagrangian (3.18) one obtains

$$\begin{aligned} \mathscr{L}_\phi = {} & i\rho(i\,\partial_t\theta + \phi_\nu^\dagger\, \partial_t\, \phi_\nu) - e\rho\delta a_0 - \frac{\bar{\rho}}{2m}(\nabla_\alpha\theta - i\phi_\nu^\dagger\, \nabla_\alpha\, \phi_\nu - e\delta a_\alpha)^2 \\ & - \frac{1}{2}\, \delta\rho(x)\, V(x-y)\, \delta\rho(y) \end{aligned} \tag{7.13}$$

where we neglected the spatial derivative of the amplitude field $\rho(x)$. Next one introduces a Hubbard-Stratonovich field J_α to decouple the kinetic term in (7.13):

$$-\frac{\bar{\rho}}{2m}(\nabla_\alpha\theta - i\phi_\nu^\dagger\nabla_\alpha\phi_\nu - e\delta a_\alpha)^2 \rightarrow -J_\alpha(\nabla_\alpha\theta - i\phi_\nu^\dagger\nabla_\alpha\phi_\nu - e\delta a_\alpha) + \frac{m}{2\bar{\rho}} J_\alpha J_\alpha \ . \tag{7.14}$$

Substituting (7.14) into (7.13), one sees that the Lagrangian contains the Gaussian field $\theta(x)$ in a linear fashion, so that one can easily integrate it to obtain the constraint

$$\partial_t \rho + \nabla_\alpha J_\alpha = 0 \ . \tag{7.15}$$

This constraint is easily solved by introducing a new gauge field $b_\mu = (b_0, b_\alpha)$:

$$\rho = \varepsilon_{\alpha\beta}\partial_\alpha b_\beta \text{ and } J_\alpha = \varepsilon_{\alpha\beta}(\partial_\beta b_0 - \partial_0 b_\beta) \ . \tag{7.16}$$

(7.16) determines b_μ up to a gauge transformation, i.e., $b_\mu \to b_\mu + \partial_\mu \Lambda$, this freedom can be fixed by choosing a particular gauge, say the Coulomb gauge $\partial_\alpha b_\alpha = 0$.

Substituting (7.16) into (7.13) and (7.14), one obtains

$$\begin{aligned}\mathscr{L}_\phi = & -e\delta a_0\, \varepsilon^{\alpha\beta}\, \partial_\alpha b_\beta + e\, \delta a_\alpha\, \varepsilon^{\alpha\beta}(\partial_\beta b_0 - \partial_0 b_\beta) \\ & + \frac{m}{2\bar{\rho}}(\partial_\alpha b_0 - \partial_0 b_\alpha)^2 - \frac{1}{2}(\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta)(x)\, V(x-y)(\varepsilon^{\alpha'\beta'}\partial_{\alpha'}\delta b_{\beta'})(y) \\ & + i\, b_0\, \varepsilon^{\alpha\beta}\partial_\alpha(\phi_\nu^\dagger \partial_\beta \phi_\nu) + i\, b_\alpha\, \varepsilon^{\alpha\beta}[\partial_\beta(\phi_\nu^\dagger \partial_o \phi_\nu) - \partial_0(\phi_\nu^\dagger \partial_\beta \phi_\nu)]\ . \end{aligned} \tag{7.17}$$

From the definitions of the vortex current $\tilde{j}_\mu$ given in (7.10), one sees that the last two terms of (7.17) is precisely of the form of covariant coupling of b_μ to the vortex current $\tilde{j}_\mu$: $2\pi\, b_\mu \tilde{j}_\mu$. Combining $\mathscr{L}_\phi$ and $\mathscr{L}_a$ and integrate out the original gauge field a_μ, one arrives finally at the dual Lagrangian:

$$\begin{aligned}\mathscr{L} = & -\frac{\pi}{\nu}\varepsilon^{\mu\nu\rho} b_\mu\, \partial_\nu\, b_\rho - e\varepsilon^{\mu\nu\rho} A_\mu\, \partial_\nu\, b_\rho + 2\pi\, b_\mu \tilde{j}_\mu \\ & + \frac{m}{2\bar{\rho}}(\partial_\alpha b_0 - \partial_0 b_\alpha)^2 - \frac{1}{2}(\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta)(x) V(x-y)(\varepsilon^{\alpha'\beta'}\partial_{\alpha'}\delta b_{\beta'})(y)\ . \end{aligned} \tag{7.18}$$

As promised, we have transformed the original CSLG Lagrangian with a_μ coupled to the current j_μ of the Bose field to a dual Lagrangian containing b_μ coupled to the vortex current $\tilde{j}_\mu$.

To proceed further, one can integrate out the b_μ field in (7.18) to obtain an effective action for the vortices. First integrating out b_0 in the Coulomb gauge, one obtains

$$\partial_\alpha^2 b_0 = \frac{\bar{\rho}}{m}\left[2\pi\tilde{\rho} - \frac{2\pi}{\nu}\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta\right] \tag{7.19}$$

and

$$\begin{aligned}\mathscr{L} = & -eA_0\, \varepsilon_{\alpha\beta}\, \partial_\alpha\, b_\beta + 2\pi b_\alpha \tilde{j}_\alpha + \frac{m}{2\bar{\rho}}(\partial_0 b_\alpha)^2 \\ & - \frac{1}{4\pi}\frac{\bar{\rho}}{m}\left[2\,\pi\tilde{\rho} - \frac{2\pi}{\nu}\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta\right]\ln|x-y|\left[2\pi\tilde{\rho} - \frac{2\pi}{\nu}\varepsilon^{\alpha'\beta'}\partial_{\alpha'}\delta b_{\beta'}\right] \\ & - \frac{1}{2}(\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta) V(x-y)(\varepsilon^{\alpha'\beta'}\partial_{\alpha'}\delta b_{\beta'})\ . \end{aligned} \tag{7.20}$$

From (7.20) one readily obtains the Laughlin's quasi-hole wave function, in a similar way one derives the Laughlin's ground state wave function in the last

section. To do this, consider a static vortex, $\tilde{\rho}(x) = -\delta(x), j(x) = 0$, located at the origin. From (7.16), one sees that $\varepsilon_{\alpha\beta}\partial_\alpha \delta b_\beta = \delta\rho(x)$ is identified with the density of the fluid. Under these conditions, the Hamiltonian version of (7.20) becomes:

$$H = \frac{1}{2}\frac{\bar{\rho}}{m} q^2 \theta_q \theta_{-q} + \frac{1}{2}\frac{\bar{\rho}}{m}\left[2\pi\tilde{\rho}_{-q} - \frac{2\pi}{\nu}\delta\rho_{-q}\right]\frac{1}{q^2}\left[2\pi\tilde{\rho}_q - \frac{2\pi}{\nu}\delta\rho_q\right] + \frac{1}{2}\delta\rho_{-q}V(q)\delta\rho_q \tag{7.21}$$

where θ_q and $\delta\rho_q$ are canonically conjugate variable, and can be represented by a functional derivative $\theta_q = -i\frac{\delta}{\delta\delta\rho_{-q}}$. As was done in the last section, we shall treat the last terms perturbatively. Then (7.21) differs from (6.9) only by a shift of $\delta\rho_q$, the ground state wave function of (7.21) is therefore given by:

$$\Psi_1[\delta\rho_q] = \exp\left(\frac{1}{2}\sum_q 2\pi\nu\left[\tilde{\rho}_{-q} - \frac{1}{\nu}\delta\rho_{-q}\right]\frac{1}{q^2}\left[\tilde{\rho}_q - \frac{1}{\nu}\delta\rho_q\right]\right) \tag{7.22}$$

with the same eigenvalue $-\frac{N}{2}\omega_c$ as before. Written explicitly in coordinates, (7.22) becomes

$$\Psi_1(x_1 \ldots x_N) = \prod_i |x_i| \prod_{i<j} |x_i - x_j|^{1/\nu} \exp\left(-\frac{1}{4l_0^2}\sum_i |x_i|^2\right) \tag{7.23}$$

which is precisely the modulus of Laughlin's quasi-hole wave function. As we have seen, the energy of Ψ_1 is the same as Ψ_0 for the case $V = 0$. Restoring the interaction term, we get the perturbative energy associated with the quasi-hole

$$\Delta = \frac{2}{N(N-1)}\int dx_1 \ldots dx_N \sum_{i<j} V(x_i - x_j)\{|\Psi_1(x_1 \ldots x_N)|^2 - |\Psi_0(x_0 \ldots x_N)|^2\} . \tag{7.24}$$

Clearly, Δ is only dependent on $V(x)$, and it remains finite in the limit $\hbar\omega_c \to \infty$.

One can carry out a similar analysis for the quasi-particle, with $\tilde{\rho}(x) = +\delta(x)$. The corresponding wave function is given by

$$\Psi_{-1}(x_1 \ldots x_N) = \prod_i |x_i|^{-1} \prod_{i<j} |x_i - x_j|^{1/\nu} \exp\left(-\frac{1}{4l_0^2}\sum_i |x_i|^2\right) . \tag{7.25}$$

Although this wave function describes the correct asymptotic behavior far away from the vortex, it is singular at the origin. This signals a breakdown of the harmonic approximation employed by our analysis, near the origin. While it is necessary to include the non-linear effect to obtain the correct core profile of the quasi-particle, the long distance property remain unmodified.

After the detailed discussion of a single vortex excitation, we now turn to the multivortex configuration and their interactions. In order to do that, we integrate out b_α in (7.20) to obtain an effective action for the low energy dynamics of the vortices only. This can be done simply by setting the fourth term in (7.20) to zero, i.e.,

$$\varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta = \nu \cdot \tilde{\rho} = \nu \sum_i q_i \delta(x - x_i) \ . \tag{7.26}$$

Since we learned from previous sections that derivation from (7.26) describes density oscillation at the energy scale of $\hbar\omega_c$ and can therefore be neglected in the discussion of the low energy dynamics of the vortices. Here x_i denotes the location of the vortices, not to be confused with x_i in (7.23)–(7.25), which denotes the location of the particles. Inserting (7.26) into the first term in (7.20) one obtains

$$-\int eA_0 \varepsilon^{\alpha\beta}\partial_\alpha \delta b_\beta d^2x = -e\nu \sum_i q_i A_0(x_i) \ . \tag{7.27}$$

This term describes the coupling of the vortices to the external scalar potential, and we see that the vortices carry fractional charge $e\nu = \frac{e}{2k+1}$. The second term can be split into two pieces. The first contribution is

$$\begin{aligned} \int 2\pi \bar{b}_\alpha \tilde{j}_\alpha d^2x &= -2\pi \frac{\bar{\rho}}{2} \int d^2x \varepsilon_{\alpha\beta} x^\beta \sum_i q_i \dot{x}_i^\alpha \delta(x - x_i) \\ &= -\pi\bar{\rho} \sum_i q_i \varepsilon^{\alpha\beta} \dot{x}_i^\alpha x_i^\beta \ . \end{aligned} \tag{7.28}$$

This term describes the kinetic energy of the vortices, and we see that the vortices obey Euler dynamics rather than Newtonian dynamics. We shall return to its consequence later in detail. The second contribution is

$$\begin{aligned} \int d^2x 2\pi \delta b_\alpha \, \tilde{j}_\alpha &= \nu \int d^2x d^2y \ \tilde{\rho}(x) \varepsilon^{\alpha\beta} \frac{x^\beta - y^\beta}{|x - y|^2} \tilde{j}^\beta(y) \\ &= \nu \varepsilon^{\alpha\beta} \sum_{i \neq j} q_i q_j \dot{x}_i^\alpha \frac{x_i^\beta - x_j^\beta}{|x_i - x_j|^2} \ . \end{aligned} \tag{7.29}$$

This term describes the fractional statistics of the vortices. To see this, one can simply take a path where vortex i rotates around vortex j by π, and find that the contribution to the action $S = \int dt$ (7.29) is $\pi\nu = \frac{\pi}{2k+1}$.

The last term in (7.20) simply gives the interaction between the vortices

$$\sum_{i<j} v^2 q_i q_j V(x_i - x_j) \ . \tag{7.30}$$

As we argued in the previous discussion about the single vortex, the combined effect of the third, fourth and the last term in (7.20) is to give a self-energy (7.24) to the vortices, i.e.,

$$\sum_i \Delta(q_i) \ . \tag{7.31}$$

Collecting all these terms, we finally arrive at the effective Lagrangian for the vortices

$$\begin{aligned} \mathscr{L} = & \sum_i v q_i A_0(x_i) - \sum_i \Delta(q_i) + \pi \bar{\rho} \sum_i q_i \varepsilon^{\alpha\beta} \dot{x}_i^\alpha x_i^\beta \\ & + v\varepsilon^{\alpha\beta} \sum_{i \neq j} q_i q_j \dot{x}_i^\alpha \frac{x_i^\beta - x_j^\beta}{|x_i - x_j|^2} - \sum_{i<j} v^2 q_i q_j V(x_i - x_j) \ . \end{aligned} \tag{7.32}$$

(7.32) describes a collection of fractionally charged particles, which obey fractional statistics and vortex dynamics, interacting via a two-body potential $V(x)$. It is independent of ω_c and contains all the low energy physics of the FQHE.

8. Magneto-roton Excitations

In the FQHE system, where the lowest Landau level is partially filled, there are two kinds of collective excitations. The inter-Landau level transition has an energy scale of $\hbar\omega_c$; it is discussed in detail in Sec. 5. In addition to that, there exists intra-Landau level transition, with an energy scale given by the Coulomb interaction $V(x)$. This situation is quite similar to the collective excitations in a superfluid. The phonon mode is pushed to the plasma frequency ω_p in a charged superfluid, and the roton represents the low energy excitation. In Feynman's picture, a roton is visualized as a drifting smoke ring in three dimensions, or a vortex-antivortex pair in two dimensions. As we shall see, the intra-Landau level transition, or the magneto-roton excitation of the FQHE is also naturally explained in terms of the vortex-antivortex pairs of the CSLG theory. One severe constraint on this interpretation is the Kohn's theorem proved in Sec. 5. We saw that the cyclotron mode saturate the dipole oscillator strength, the contribution from the magneto-roton to the density-density correlation must therefore vanish like q^4 in the long wavelength limit.

We first discuss the quantization of the vortex dynamics. The equation of motion derived from (7.32) has the form

$$2\pi\bar{\rho}\varepsilon^{\alpha\beta}\dot{x}_i^\beta = -\nu^2 \sum_{j\neq i} q_j \frac{\partial V(x_i - x_j)}{\partial x_i^\alpha} \,. \tag{8.1}$$

Here we have neglected the effect of the fractional statistics for the moment since it does not enter the equation of motion. This equation of motion can also be derived from a Hamiltonian

$$H = \nu^2 \sum_{i<j} q_i q_j \, V(x_i - x_j) + \sum_i \Delta(q_i) \tag{8.2}$$

if one postulates the following commutation relation for the particle coordinates:

$$[x_i^\alpha, x_j^\beta] = i\delta_{ij}\varepsilon^{\alpha\beta}\frac{q_i}{2\pi\bar{\rho}} \,. \tag{8.3}$$

Using the Heisenberg equation of motion for the operators $\dot{x}_i^\alpha = -i[x_i^\alpha, H]$, one easily recovers (8.1) from the Hamiltonian approach. In fact, similar procedures of quantizing the vortices was first introduced in the context of superfluid.[26]

From (8.1) a sum rule can be derived for the density correlation function. For small q, the vortex density operator

$$\tilde{\rho}(q) = \sum_i q_i e^{i\mathbf{q}\cdot\mathbf{x}_i} \tag{8.4}$$

can be expanded in powers of q as

$$\tilde{\rho}(q) = Q + iq^\alpha D^\alpha - \frac{1}{2} q^\alpha q^\beta Q_{\alpha\beta} + \cdots \tag{8.5}$$

where $Q = \Sigma_i q_i$ is the total charge and $D^\alpha = \Sigma_j q_j x_j^\alpha$ the total dipole moment. Both quantities are conserved by the equations of motion (8.1) and their contribution to the density fluctuations therefore vanish. $Q^{\alpha\beta} = \Sigma_i q_i x_i^\alpha x_i^\beta$ is the total quadrupole moment of the system, its contribution to the density fluctuation vanish like q^4 in the small q limit, i.e.,

$$\begin{aligned} \langle \tilde{\rho}(q, t)\tilde{\rho}(-q, 0)\rangle &= \langle Q(t)Q(0)\rangle + \frac{1}{2} q^2 \langle D^\alpha(t) D^\alpha(0)\rangle \\ &\quad + \frac{1}{4}(q_\alpha q_\beta)^2 \langle Q^{\alpha\beta}(t) Q^{\alpha\beta}(0)\rangle + \cdots \\ &\approx q^4 + \cdots \,. \end{aligned} \tag{8.6}$$

From (8.6) ones sees that the interpretation of the intra-Landau level excitations in terms of the vortices is indeed consistent with the Kohn's theorem, since their contribution to the density fluctuation vanish faster than q^2.

The form of the magneto-roton dispersion can be studied directly with the vortex Hamiltonian (8.2). Since the vortices have finite creation energy, we can first restrict ourself to the low energy sector of a vortex-antivortex pair. Denote their coordinates by (x_1, y_1) and (x_2, y_2), from (8.3) we see that the only non-vanishing commutators are

$$[x_1, y_1] = il_0^2/v \quad \text{and} \quad [x_2, y_2] = -il_0^2/v \ . \tag{8.7}$$

In terms of the relative coordinates $x = x_1 - x_2$, $y = y_1 - y_2$ and the center of mass coordinates $X = \frac{1}{2}(x_1 + x_2)$, $Y = \frac{1}{2}(y_1 + y_2)$, (8.7) becomes

$$[x, Y] = il_0^2/v \quad \text{and} \quad [y, X] = -il_0^2/v \ . \tag{8.8}$$

In quantum mechanics, the total momentum P_x is identified with the canonical conjugate variable of the center of mass coordinate X, i.e., $[P_x, X] = -i$, and is the generator of the center of mass translation. From (8.8), one sees that in the present case, P_x is naturally identified with the relative y coordinate of the vortex-antivortex pair, i.e.,

$$P_X = yvl_0^{-2} \quad \text{and similarly} \quad P_Y = -xvl_0^2 \ . \tag{8.9}$$

This rather peculiar expression for the total momentum in fact has a classical counterpart. Classically, a charge dipole drifts in a magnetic field with constant velocity, the Lorentz force on the drifting pair is balanced by the electrostatic attraction. The drifting velocity is therefore uniquely determined by their separation.

Because of (8.3), each vortex has an intrinsic size of the order of l_0, a picture of the vortex-antivortex pair only makes sense as long as their relative separation is larger than their intrinsic size, which according to (8.9) gives a center of mass momentum P_X or P_Y larger than l_0^{-1}. For this range of total momentum, the energy of the vortex-antivortex pair is simply given by (8.2), since the Hamiltonian is diagonal in terms of the relative coordinates. For the case where total momentum is in the x direction

$$\omega(q_x) = E(y = q_x l_0^2/v) = 2\Delta - \frac{v^2 e^2}{y} = 2\Delta - \frac{v^3 e^2}{q_x l_0^2} \tag{8.10}$$

which gives a dispersion of the intra-Landau level excitation as pictured in Fig. 5 for $q_x > l_0^{-1}$. q_x denotes the eigenvalue of P_x.

For $q_x < l_0^{-1}$, the vortex-antivortex picture breaks down, since their separation would be less than their intrinsic size. A configuration of the vortices with a total dipole moment smaller than l_0 necessarily involves a quadrupole-like configuration, see Figs. 6a and 6b. This interpretation has two consequences, first, it

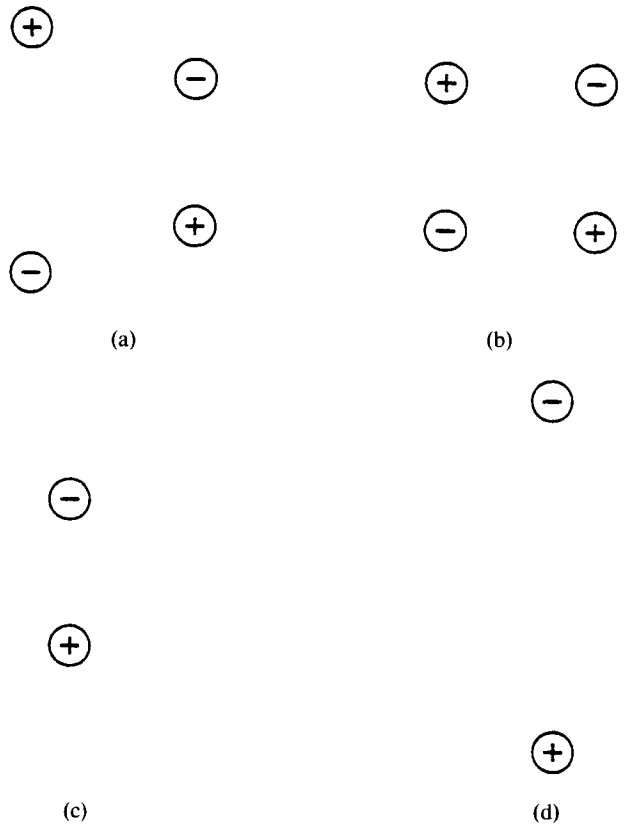


Fig. 6. Schematic illustration of the vortex configurations giving rise to the magneto-roton excitations. a) A quadrupolar arrangement of the vortices with no net dipole moment. It contributes to the $q = 0$ excitation. b) A distorted quadrupolar arrangement, with a small net dipole moment. It contributes to the excitations in the momentum range $0 < q < l_0^{-1}$. c) A vortex-antivortex pair separated by l_0^{-1}. It contributes to the excitations around the roton minimum. d) Far separated vortex-antivortex pair, which contributes to the excitations with $q > l_0^{-1}$.

naturally explains the fact that the vortex contribution to the density correlation vanish like q^4 in the small q limit because of the vanishing dipole moments of these configurations. Secondly, since there are four vortices involved, the energy at $q = 0$ is of the order of 4Δ plus a correction due to their mutal Coulomb energy, which should be roughly twice the energy at the roton minimum. The spectrum obtained from the numerical diagonalization indeed supports this observation.[27] To summarize, at $q_x = 0$, the vortex configuration involves a quadrupolar arrangement (see Fig. 6a) and the energy is of the order of 4Δ. As q_x increases, the quadrupole configuration distorts to acquire a net total dipole moment in the y direction (see Fig. 6b), and the Coulomb energy decreases. For $q_x \sim l_0^{-1}$, one pair of vortex-antivortex annihilate and disappear into the vacuum (see Fig. 6c). Within this range of q_x, the energy decreases monotonically. As q_x increases further, so does the dipole moment of the remaining vortex-antivortex pair. The

energy increases according to (8.10). The dispersion curve for the entire momentum range is pictured in Fig. 5, and is in qualitative agreement with the single-mode approximation [16] and the exact diagonalization.[27]

9. Conclusions

In this paper, we intended to give a self-contained review of the CSLG theory of the FQHE, and communicate the most basic ideas to the readers. Due to the limitation of the space, many important further developments was not reviewed here. Here we summarize the relevant references for the convenience of the readers.

This review mainly concentrated on the FQHE at the filling factors $\nu = \frac{1}{2k+1}$. A hierarchy scheme similar to that of Halperin can be easily constructed. This was done by Zhang, Hansson and Kivelson,[28]and by Lee and Fisher.[25] A hierarchy scheme similar to that of Jain can also be constructed, as was done by Shapere and Wilczek,[29]and also by Lee, Kivelson and Zhang.[30] The relationship between these two schemes was explained by Read.[31]Lee and Kane,[32] Balatsky and Fradkin[33] extended the spinless CSLG theory to the case of singlet FQHE. Wen and Niu,[34] Fröhlich and Zee,[35] discussed various global topological aspects of the CSLG theory. Stone[24] and Haldane[36] studied the detailed dynamics of the vortex excitation. Sakita, Sheng and Su[37] applied the collective coordinate formalism to the CSLG theory. Lopez and Fradkin[38] used the fermionic representation to construct a Chern-Simons theory for the FQHF. Wen,[39] Haldane,[40] Stone,[41] Lee and Wen[42] applied the CSLG theory to the edge states of the FQHE. Recently, Lee, Kivelson and Zhang[20] constructed a formalism to relate various phase transitions in the two-dimensional electron gas in a magnetic field to the superfluid to insulator transition in a two-dimensional Bose gas.

We hope to have demonstrated in this review that the CSLG theory is a complete, first principle theory of the FHQE. While it is logically independent of Laughlin's wave function approach, it leads to the same phenomenological consequences. In fact, a direct connection of these two theories can be established, as we have shown in Sec. 6. CSLG theory demonstrates a deep connection of the FQHE with superfluidity, and it possesses an appealing mathematical structure through the introduction of the Chern–Simons term. CSLG may have a great advantage in the study of macroscopic phenomena associated with the FQHE, like the transition between the FQHE phase, the Wigner crystal phase and the localized glasslike phase. It is also naturally suited for addressing various transport properties at the finite temperature. Some of these questions have been investigated recently,[30,20] but it is clear that these directions are still wide open. The interesting mathematical structure that emerges from the CSLG theory may also help us in understanding other problems in the strongly correlated electron systems.

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