Correction to "Topology design for optimal network coherence"

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***Abstract*—We provide a correction to a subtle error in our paper "Topology design for optimal network coherence", which appeared in the Proceedings of the European Control Conference [3].**

Our paper "Topology design for optimal network coherence" published in the Proceedings of the European Control Conference [3] contains an incorrect proof for part of Theorem 3. We give a counterexample[1] that invalidates the result.

Consider a network with underlying weighted undirected graph $G = (V, E, w)$ where $V = \{1, ..., n\}$ is a set of nodes, $E \subseteq V \times V$ is a set of edges, and $w \in \mathbf{R}^{|E|}$ is a set of nonnegative weights associated with each edge. Suppose a scalar state variable is associated with each node and the network has consensus dynamics modeled by the stochastic differential equation

$$dx(t) = -Lx(t)dt + dW \tag{1}$$

where L is the weighted Laplacian matrix and dW is a vector of independent Gaussian white noise stochastic processes. Network coherence quantifies the steady-state variance of state fluctuations with respect to the expected average state value. and can be considered as a measure of robustness of the consensus process to the additive noise. It is proportional to the trace of the pseudo-inverse of the Laplacian matrix, $\mathbf{tr}(L_{\mathcal{E}}^{\dagger})$.

The paper [3] contained the following statement regarding submodularity of network coherence with respect to edge subsets:

Theorem 1. *Let $G = (V, E, w_E)$ be a given connected weighted graph, let $\mathcal{E} \subseteq V \times V \setminus E$ with weights $w_{\mathcal{E}}$, and let $L_{\mathcal{E}}$ be the weighted graph Laplacian matrix associated with the edge set $E \cup \mathcal{E}$. Then the set function $f : V \times V \setminus E \to \mathbf{R}$ defined by $f(\mathcal{E}) = -\mathbf{tr}(L_{\mathcal{E}}^{\dagger})$ is submodular.*

Unfortunately, further investigation revealed that the proof of this claim contains a subtle error. It effectively relies on a statement that for two positive definite matrices P and Q, $P^{-1} \succeq Q^{-1}$ implies that $P^{-2} \succeq Q^{-2}$. However, this is incorrect in general, since the partial ordering of positive semidefinite matrices is not necessarily preserved by squaring (or by any matrix power greater than one).

The following counterexample demonstrates that, unfortunately, the result is also incorrect, not just the proof. Consider an underlying path graph on 5 nodes with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$, so that the associated (unweighted) Laplacian matrix is

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Consider the additional edge subsets $S_1 = \{(1, 3), (2, 4)\}$ and $S_2 = \{(2, 4), (3, 5)\}$, so that $S_1 \cup S_2 = \{(1, 3), (2, 4), (3, 5)\}$ and $S_1 \cap S_2 = \{(2, 4)\}$. We have

$$\begin{aligned} \mathbf{tr}(L_{S_1}^{\dagger}) = 2.05, \quad \mathbf{tr}(L_{S_2}^{\dagger}) = 2.05 \\ \mathbf{tr}(L_{S_1 \cup S_2}^{\dagger}) = 1.3905, \quad \mathbf{tr}(L_{S_1 \cap S_2}^{\dagger}) = 2.667, \end{aligned} \tag{2}$$

so that

$$\mathbf{tr}(L_{S_1}^{\dagger}) + \mathbf{tr}(L_{S_2}^{\dagger}) - [\mathbf{tr}(L_{S_1 \cup S_2}^{\dagger}) + \mathbf{tr}(L_{S_1 \cap S_2}^{\dagger})] = 0.0429.$$

This violates the definition of submodularity (stated in Definition 1 of [3]), so the set function defined in Theorem 1 is not submodular.

Even though the worst-case theoretical performance guarantee of the greedy algorithm associated with submodularity is lost due to this error, all of the methodological developments and numerical experiments for designing network to optimize coherence remains valid. There may be some alternative explanation for the effectiveness of the greedy algorithm in this setting.

The error originated from a similar argument made in [2] in the context of network controllability, and also affects a result in [1] in the context of network rigidity.

References

[1] I. Shames and T.H. Summers. Rigid network design via submodular set function optimization. *IEEE Transactions on Network Science and Engineering*, 2(3):84–96, 2015.

[2] T.H. Summers, F. Cortesi, and J. Lygeros. On controllability and submodularity in complex dynamical networks. *IEEE Transactions on Control of Network Systems*, 3(1):91–101, 2016.

[3] T.H. Summers, I. Shames, J. Lygeros, and F. Dörfler. Topology design for optimal network coherence. In *European Control Conference*, pages 575–580. IEEE, 2015.

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[1] Strictly speaking, we provide strong numerical evidence supporting incorrectness of the result that relies on accuracy of numerical computations and correctness of the source code of either MATLAB or NumPy. It is not too difficult to generate other numerical counterexamples, so that the evidence becomes overwhelming