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BETTI NUMBERS OF SEMIALGEBRAIC AND SUB-PFAFFIAN SETS

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Abstract

Let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \ldots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu) \in X_\nu)\},$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\cdots+n_\nu}$ being a difference between a finite CW-complex and its subcomplex. We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν.

In important particular cases of semialgebraic and semi-Pfaffian sets defined by quantifier-free formulae with polynomials and Pfaffian functions respectively, upper bounds on Betti numbers of X_ν are well known. Our results allow to extend the bounds to sets defined with quantifiers, in particular to sub-Pfaffian sets.

Introduction

Well-known results of Petrovskii, Oleinik [**16**], [**15**], Milnor [**13**], and Thom [**19**] provide an upper bound for the sum of Betti numbers of a semialgebraic set defined by a Boolean combination of polynomial equations and inequalities. A refinement of these results can be found in [**1**]. For semi-Pfaffian sets the analogous bounds were obtained by Khovanskii [**11**] (see also [**23**]). In this paper we describe a reduction of estimating Betti numbers of sets defined by formulae with quantifiers to a similar problem for sets defined by a quantifier-free formulae.

More precisely, let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \ldots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu) \in X_\nu)\}, \qquad (0.1)$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\cdots+n_\nu}$ being a difference between a finite CW-complex and one of its subcomplexes. For instance, if $\nu = 1$ and $Q_1 = \exists$, then X is the projection of X_ν.

We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν. In conjunction with Petrovskii-Oleinik-Thom-Milnor's result this implies a new upper bound for semialgebraic sets defined by formulae with quantifiers, which is significantly better than a bound following from the cylindrical cell decomposition approach. In conjunction with Khovanskii's result our method produces an analogous upper bound for restricted sub-Pfaffian sets defined by formulae with quantifiers. Apparently in this case no general upper bounds were previously known.

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Throughout the paper each topological space is assumed to be a difference between a finite CW-complex and one of its subcomplexes.

EXAMPLE 1. The closure X of the interior of a compact set $Y \subset [-1,1]^n$ is homotopy equivalent to

$$X_{\varepsilon,\delta} = \{\mathbf{x} | \, \exists \mathbf{y} (\|\mathbf{x}-\mathbf{y}\| \le \delta) \, \forall \mathbf{z} (\|\mathbf{y}-\mathbf{z}\| < \varepsilon) \, (\mathbf{z} \in Y)\}$$

for small enough $\delta, \varepsilon > 0$ such that $\delta \gg \varepsilon$. Representing $X_{\varepsilon,\delta}$ in the form (0.1), we conclude that X is homotopy equivalent to $X_{\varepsilon,\delta} = \{\mathbf{x} | \, \exists \mathbf{y} \, \forall \mathbf{z} \, X_2\}$, where

$$X_2 = \{(\mathbf{x},\mathbf{y},\mathbf{z}) | \, (\|\mathbf{x}-\mathbf{y}\| \le \delta \wedge (\|\mathbf{y}-\mathbf{z}\| \ge \varepsilon \vee \mathbf{z} \in Y))\}$$

is a closed set in $[-1,1]^{3n}$. Our results allow to bound from above Betti numbers of X in terms of Betti numbers of X_2.

1. *A spectral sequence associated with a surjective map*

DEFINITION 1. A continuous map $f: X \to Y$ is *locally split* if for any $y \in Y$ there is an open neighbourhood U of y and a section $s: U \to X$ of f (i.e., s is continuous and $fs = \mathrm{Id}$). In particular, a projection of an open set in $\mathbb{R}^n$ on a subspace of $\mathbb{R}^n$ is always locally split.

DEFINITION 2. For two maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the *fibered product* of X_1 and X_2 is defined as

$$X_1 \times_Y X_2 := \{(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2 | \, f_1(\mathbf{x}_1) = f_2(\mathbf{x}_2)\}.$$

THEOREM 1. *Let $f: X \to Y$ be a surjective cellular map. Assume that f is either closed or locally split. Then for any Abelian group G, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y,G)$ with*

$$E^1_{p,q} = H_q(W_p, G) \tag{1.1}$$

where

$$W_p = \underbrace{X \times_Y \ldots \times_Y X}_{p+1 \text{ times}} \tag{1.2}$$

In particular,

$$\dim H_k(Y,G) \le \sum_{p+q=k} \dim H_q(W_p, G), \tag{1.3}$$

for all k.

For a locally split map f, this theorem can be derived from [**5**], Corollary 1.3. We present below a proof for a closed map f.

REMARK 1. In the sequel we use Theorem 1 only for projections of either closed or open sets in $[-1,1]^n$. If f is a projection of an open set, then (1.3) easily follows from the analogous result for closed maps which will be proved below, without references to [**5**]. Indeed, for an open set Z define its *shrinking* $S(Z)$ as the closed set $Z \backslash N(\partial Z)$ where N denotes an open neighbourhood. For a small enough $N(\partial Z)$, the set Z is homotopy equivalent to $S(Z)$ (recall that Z is a difference between a finite

CW–complex and a subcomplex). Let X be open and $S(X)$ be its shrinking with a sufficiently small $N(\partial X)$. It induces shrinkings $S(Y) = f(S(X))$ and $S(W_p) = S(X)\times_{S(Y)}\ldots\times_{S(Y)}S(X)$ which are homotopy equivalent to Y and W_p respectively. The statement for open sets X and Y follows from the statement for closed sets applied to $f : S(X) \to S(Y)$.

Definition 3. For a sequence $(P_0, \ldots, P_p)$ of topological spaces, their *join* $P_0 * \ldots * P_p$ can be defined as follows. Let $\Delta^p = \{s_0 \geq 0, \ldots, s_p \geq 0,\ s_0+\ldots+s_p = 1\}$ be the standard p-simplex. Then $P_0 * \ldots * P_p$ is the quotient space of $P_0 \times \ldots \times P_p \times \Delta^p$ over the following relation:

$$(x_0, \ldots, x_p, s) \sim (x'_0, \ldots, x'_p, s) \text{ if } s = (s_0, \ldots, s_p) \text{ and } x_i = x'_i \text{ whenever } s_i \neq 0. \tag{1.4}$$

Given a continuous surjective map $f_i : P_i \to Y$ for each $i = 0, \ldots, p$, the *fibered join* $P_0 *_Y \ldots *_Y P_p$ is defined as the quotient space of $P_0 \times_Y \ldots \times_Y P_p \times \Delta^p$ over the relation (1.4).

Definition 4. For a space Z, 1-*st suspension* of Z is defined as the suspension (see [**12**]) of $(Z \sqcup \{point\})$. For an integer $p > 0$, the p-th iteration of this operation will be called *p-th suspension of Z*.

Lemma 1. *Let $f_i : P_i \to Y$, $i = 0, \ldots, p$, be continuous surjective maps and $P = P_0 *_Y \ldots *_Y P_p$ their fibered join. There is a natural map $F : P \to Y$ induced by the maps $f_0, \ldots, f_p$. For a point $y \in Y$ the fiber $F^{-1}y$ coincides with the join $f_0^{-1}y * \ldots * f_p^{-1}y$ of the fibers of f_i.*

There is a natural map $\pi : P \to \Delta^p$. The fiber of π over an interior point of Δ^p is $P_0 \times_Y \ldots \times_Y P_p$. For each $i = 0, \ldots, p$, there is a natural embedding

$$\phi_i : P(i) = P_0 *_Y \ldots *_Y P_{i-1} *_Y P_{i+1} *_Y \ldots *_Y P_p \to P. \tag{1.5}$$

Its image coincides with $\pi^{-1}(\{s_i = 0\})$, and the space $P/\left(\bigcup_i \phi_i(P(i))\right)$ is homotopy equivalent to the p-th suspension of $P_0 \times_Y \ldots \times_Y P_p$.

Proof. Directly follows from Definitions 3, 4. □

Definition 5. Let $f : X \to Y$ be a surjective continuous map. Its *join space* $J^f(X)$ is the quotient space of the disjoint union of spaces

$$J^f_p(X) = \underbrace{X *_Y \ldots *_Y X}_{p+1 \text{ times}}, \quad p = 0, 1, \ldots, \tag{1.6}$$

identifying $J^f_{p-1}(X)$ with each of its images $\phi_i(J^f_{p-1}(X))$ in $J^f_p(X)$ for $i = 0, \ldots, p$, where ϕ_i is defined in (1.5). When Y is a point, we write $J_p(X)$ instead of $J^f_p(X)$ and $J(X)$ instead of $J^f(X)$.

Lemma 2. *Let $\phi : J_p(X) \to J(X)$ be the natural map induced by the maps ϕ_i. Then $\phi(J_{p-1}(X))$ is contractible in $\phi(J_p(X))$.*

Proof. Let x be a point in X. For $t \in [0, 1]$, the maps

$$g_t(x, x_1, \ldots, x_p, s) \mapsto (x, x_1, \ldots, x_p, (1 - t + ts_0, ts_1, \ldots, ts_p))$$

define a contraction of $\phi_0(J_{p-1}(X))$ to the point $x \in X$ where X is identified with its embedding in $J_p(X)$ as $\pi^{-1}(1, 0, \dots, 0)$. It is easy to see that the maps g_t are compatible with the equivalence relations in Definition 5 and define a contraction of $\phi(J_{p-1}(X))$ to a point in $\phi(J_p(X))$. $\square$

LEMMA 3. *The join space $J(X)$ is homologically trivial.*

Proof. Any cycle in $J(X)$ belongs to $\phi(J_p(X))$ for some p, while according to Lemma 2 $\phi(J_p(X))$ is contractible in $J(X)$. Hence the cycle is homologous to 0. $\square$

Proof of Theorem 1. Let f be closed. Let $F : J^f(X) \to Y$ be the natural map induced by f. Then F is also closed. Its fiber $F^{-1}y$ over a point $y \in Y$ coincides with the join space $J(f^{-1}y)$ which is homologically trivial according to Lemma 3. It follows that $\tilde{\bar{H}}^*(J(f^{-1}y)) \cong 0$, where $\bar{H}^*$ is the Alexander cohomology ([**18**], p. 308), since $\bar{H}^*(Z)) \cong H^*(Z)$ for any locally contractible space Z ([**18**], p. 340), in particular for a difference between CW-complex and a subcomplex.

Vietoris-Begle theorem ([**18**], p. 344) applied to $F : J^f(X) \to Y$, implies

$$\bar{H}^*(J^f(X), G) \cong \bar{H}^*(Y, G)$$

and therefore

$$H_*(J^f(X), G) \cong H_*(Y, G).$$

By Lemma 1, the space $J^f_p(X)/\left(\bigcup_{q<p} J^f_q(X)\right)$ is homotopy equivalent to the p-th suspension of W_p. Theorem 1 follows now from the spectral sequence associated with filtration of $J^f(X)$ by the spaces $J^f_p(X)$. $\square$

REMARK 2. For a map f with 0-dimensional fibers, a similar spectral sequence, "image computing spectral sequence" was applied to problems in theory of singularities and topology by Vassiliev [**20**], Goryunov-Mond [**8**], Goryunov [**7**], Houston [**10**], and others. For *proper* maps an analogous "cohomological descent spectral sequence" appears in [**4**].

REMARK 3. A continuous map $f : X \to Y$ is called *compact-covering* if any compact set in Y is an image of a compact set in X. This condition includes both the closed and the locally split cases and may be more convenient for applications. For a surjective cellular compact-covering $f : X \to Y$ Theorem 1 is also true. A proof will appear elsewhere.

2. *Alexander's duality and Mayer-Vietoris inequality*

Let

$$I^n_i := \bigcap_{1 \le j \le n} \{-i \le x_j \le i\} \subset \mathbb{R}^n.$$

Define the "thick boundary" $\partial I^n_i := I^n_{i+1} \setminus I^n_i$. The following lemma is a version of Alexander's duality theorem.

Lemma 4. (Alexander's duality) *If $X \subset I_i^n$ is an open set in I_i^n, then for any $q \in \mathbb{Z}$, $q \leq n-1$,*

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \partial I_i^n, \mathbb{R}). \tag{2.1}$$

If $X \subset I_i^n$ is a closed set in I_i^n, then for any $q \in \mathbb{Z}$, $q \leq n-1$,

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup closure(\partial I_i^n), \mathbb{R}). \tag{2.2}$$

Proof. For definiteness let X be closed. Compactifying $\mathbb{R}^n$ at infinity as $\mathbb{R}^n \cup \infty \simeq S^n$, we have, by Alexander's duality [**12**],

$$\tilde{H}_q(S^n \setminus (X \cup closure(\partial I_i^n), \mathbb{R}) \cong \tilde{H}_{n-q-1}((X \cup closure(\partial I_i^n), \mathbb{R}).$$

The first group is isomorphic to $H_q(I_i^n \setminus X, \mathbb{R})$ when $q > 0$, and to $\tilde{H}_0(I_i^n \setminus X, \mathbb{R}) + \mathbb{R} \cong H_0(I_i^n \setminus X, \mathbb{R})$ when $q = 0$. Combining these two cases, we obtain (2.2). □

Lemma 5. (Mayer-Vietoris inequality) *Let $X_1, \ldots, X_m \subset I_1^n$ be all open or all closed in I_1^n. Then*

$$\mathrm{b}_i\Big(\bigcup_{1 \leq j \leq n} X_j\Big) \leq \sum_{J \subset \{1,\ldots,n\}} \mathrm{b}_{i-|J|+1}\Big(\bigcap_{j \in J} X_j\Big)$$

and

$$\mathrm{b}_i\Big(\bigcap_{1 \leq j \leq n} X_j\Big) \leq \sum_{J \subset \{1,\ldots,n\}} \mathrm{b}_{i+|J|-1}\Big(\bigcup_{j \in J} X_j\Big),$$

where b_i is the ith Betti number.

Proof. A well-known corollary to Mayer-Vietoris sequence. □

3. *Thom-Milnor's and Khovanskii's bounds*

Necessary definitions regarding semi-Pfaffian and sub-Pfaffian sets can be found in [**11**], [**6**]. In this paper we consider only *restricted* sub-Pfaffian sets.

To apply our results to semialgebraic sets and to restricted sub-Pfaffian sets, defined by formulae with quantifiers, we need the following known upper bounds on Betti numbers for sets defined by quantifier-free formulae.

Let $X = \{\varphi\} \subset I_1^n$ be a semialgebraic set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$, f being polynomials in n variables with coefficients in $\mathbb{R}$, $\deg(f) < d$. We will refer to the sequence (n, s, d) as to *format* of φ. It follows from [**19**], [**13**], [**1**] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \leq O(sd)^n. \tag{3.1}$$

If $X = \{\varphi\}$ is a *compact* semialgebraic set, where φ is a Boolean combination with no negations of s atomic formulae of the kind either $f \geq 0$ or $f > 0$, f being polynomials in n variables, $\deg(f) < d$, then a combination of results from [**19**], [**13**], [**1**] and [**14**], [**22**] implies that that the sum of Betti numbers of X also satisfies (3.1).

Now let $X = \{\varphi\} \subset I_1^n$ be a semi-Pfaffian set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$, f being Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain with coefficients in $\mathbb{R}$. The sequence $(n, s, \alpha, \beta, \rho)$ is called *format* of φ. It follows from [**11**], [**23**] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \le s^n 2^{\rho(\rho-1)/2} O(n\beta + \min\{n, \rho\}\alpha)^{n+\rho}. \tag{3.2}$$

Let $X \subset I_1^{n_0}$ be a semialgebraic set defined by a formula

$$Q_1 \mathbf{x}_1 Q_2 \mathbf{x}_2 \dots Q_\nu \mathbf{x}_\nu F(\mathbf{x}_0, \mathbf{x}_1, \dots \mathbf{x}_\nu), \tag{3.3}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \ne Q_{i+1}$, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i}) \in I_1^{n_i}$, and F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f > 0$, where f's are polynomials with real coefficients of degrees less than d. The *cylindrical algebraic decomposition* technique from [**3**], [**21**] allows to bound from above the number of cells in a representation of X as a difference between a CW-complex and its subcomplex. In particular,

$$\mathrm{b}(X) \le (sd)^{2^{O(n)}}. \tag{3.4}$$

A better upper bound can be obtained as follows. According to [**2**] (which refines [**9**], [**17**]), there exists a Boolean combination

$$\psi(\mathbf{x}_0) = \bigvee_{1 \le i \le I} \bigwedge_{1 \le j \le J_i} (g_{i,j}(\mathbf{x}_0) *_{i,j} 0),$$

such that $X = \{\psi(\mathbf{x}_0)\}$. Here

$$*_{i,j} \in \{=, <, >\}, \quad g_{i,j} \in \mathbb{R}[\mathbf{x}_0], \quad \deg(g_{i,j}) < d^{\prod_{i \ge 1} O(n_i)},$$

$$I < s^{(n_0+1)\prod_{i \ge 1}(n_i+1)} d^{(n_0+1)\prod_{i \ge 1} O(n_i)},$$

$$J_i < s^{\prod_{i \ge 1}(n_i+1)} d^{\prod_{i \ge 1} O(n_i)}.$$

Applying (3.1) to $X = \{\psi(\mathbf{x}_0)\}$, we get

$$\mathrm{b}(X_0) \le s^{O(n_0^2 \prod_{i \ge 1} n_i)} d^{O(n_0^2)\prod_{i \ge 1} O(n_i)} \le (sd)^{O(n_0^2)\prod_{i \ge 1} O(n_i)} \tag{3.5}$$

4. *Basic notation*

Let $X = \widetilde{X_0} = I_1^{n_0} \setminus X_0 \subset I_1^{n_0}$ be a set defined by a formula (0.1). For example, X could be a sub-Pfaffian or a semialgebraic set defined by (3.3), where F is a quantifier-free Boolean formula with no negations. For definiteness assume that $Q_1 = \exists$ and X is open in $I_1^{n_0}$.

Define

$$X_i := \{(\mathbf{x}_0, \dots, \mathbf{x}_i) |\, Q_{i+1}\mathbf{x}_{i+1} Q_{i+2}\mathbf{x}_{i+2} \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}$$

for odd i and

$$X_i := I_1^{n_0+\dots+n_i} \setminus \{(\mathbf{x}_0, \dots, \mathbf{x}_i) | Q_{i+1}\mathbf{x}_{i+1} Q_{i+2}\mathbf{x}_{i+2} \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}$$

for even i. Then $\pi_i(X_i) = \widetilde{X_{i-1}}$, where $\pi_i : \mathbb{R}^{n_0+\dots+n_i} \to \mathbb{R}^{n_0+\dots+n_{i-1}}$ and tilde denotes the complement in $I_1^{n_0+\dots+n_{i-1}}$.

For a set $I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1}$ define $\partial(I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$ as

$$(I_{i+1}^{m_i} \times I_i^{m_{i-1}} \times \dots \times I_2^{m_1}) \setminus (I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$$

for even i and as the closure of this difference for odd i.

Let $p_1, \ldots, p_i$ be some positive integers to be specified later. Define

$$B^i_i := \partial(I^{n_0+(p_1+1)n_1}_{i-1} \times I^{(p_2+1)n_2}_{i-2} \times \ldots \times I^{(p_{i-1}+1)n_{i-1}}_1) \times I^{n_i}_1.$$

For any j, $i < j \le \nu$ define $B^i_j := \widetilde{B^i_{j-1}} \times I^{n_j}_1$, where tilde denotes the complement in the appropriate cube.

Definition 6.

(i) Let $Y \subset I^{n_0}_v \times I^{(p_1+1)n_1}_v \times I^{(p_2+1)n_2}_{v-1} \times \ldots \times I^{(p_{l-1}+1)n_{l-1}}_{v-l+2} \times I^{n_l+\ldots+n_i}_1$, where $1 \le l \le i$, $v \ge i$, and let $J \subset \{(j_l, \ldots, j_i)|\ 1 \le j_k \le p_k + 1,\ l \le k \le i\}$. Then define $\prod^l_{i,J} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}^{(1)}_1, \ldots, \mathbf{x}^{(p_1+1)}_1, \ldots, \mathbf{x}^{(1)}_i, \ldots, \mathbf{x}^{(p_i+1)}_i)|$$

$$\mathbf{x}_0 \in I^{n_0}_v,\ \mathbf{x}^{(m)}_k \in I^{n_k}_{v-k+1}\ (1 \le k \le l-1),$$

$$\mathbf{x}^{(m)}_k \in I^{n_k}_1\ (l \le k \le i),\ (\mathbf{x}_0, \mathbf{x}^{(1)}_1, \ldots, \mathbf{x}^{(p_{l-1}+1)}_{l-1}, \mathbf{x}^{(j_l)}_l, \ldots, \mathbf{x}^{(j_i)}_i) \in Y\}$$

over all $(j_l, \ldots, j_i) \in J$.

(ii) Let $Y \subset I^{n_0}_v \times I^{(p_1+1)n_1}_v \times I^{(p_2+1)n_2}_{v-1} \times \ldots \times I^{(p_{l-1}+1)n_{l-1}}_{v-l+2} \times I^{n_l+\ldots+n_i+n_{i+1}}_1$. Define $\prod^{l,i+1}_{i,J} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}^{(1)}_1, \ldots, \mathbf{x}^{(p_1+1)}_1, \ldots, \mathbf{x}^{(1)}_i, \ldots, \mathbf{x}^{(p_i+1)}_i, \mathbf{x}_{i+1})|$$

$$\mathbf{x}_0 \in I^{n_0}_v, \mathbf{x}^{(m)}_k \in I^{n_k}_{v-k+1} (1 \le k \le l-1), \mathbf{x}^{(m)}_k \in I^{n_k}_1 (l \le k \le i),\ \mathbf{x}_{i+1} \in I^{n_{i+1}}_1,$$

$$(\mathbf{x}_0, \mathbf{x}^{(1)}_1, \ldots, \mathbf{x}^{(p_{l-1}+1)}_{l-1}, \mathbf{x}^{(j_l)}_l, \ldots, \mathbf{x}^{(j_i)}_i, \mathbf{x}_{i+1}) \in Y\}$$

over all $(j_l, \ldots, j_i) \in J$.

(iii) If $l = i$ and $J = \{j|\ 1 \le j \le p_i + 1\}$ we use the notation $\prod^i_i Y$ for $\prod^i_{i,J} Y$.

Lemma 6. *Let*

$$Y \subset I^{n_0}_v \times I^{(p_1+1)n_1}_v \times I^{(p_2+1)n_2}_{v-1} \times \ldots \times I^{(p_{l-1}+1)n_{l-1}}_{v-l+2} \times I^{n_l+\ldots+n_i+n_{i+1}}_1.$$

Then for any $J \subset \{j|1 \le j \le p_{i+1}+1\}$, $J' \subset \{(j_l, \ldots, j_i)|1 \le j_k \le p_k+1, l \le k \le i\}$ *we have*

$$\prod^{i+1}_{i+1,J} \prod^{l,i+1}_{i,J'} Y = \prod^l_{i+1,J' \times J} Y.$$

Proof. Straightforward. □

Definition 7. Let Y, l, i, J be as in Definition 6. Define $\coprod^l_{i,J} Y$ and $\coprod^{l,i+1}_{i,J} Y$ similar to $\prod^l_{i,J} Y$ and $\prod^{l,i+1}_{i,J} Y$ respectively, replacing in Definition 6 "intersection" by "union".

Lemma 7. (De Morgan law)

$$\coprod^l_{i,J} Y = \left(\prod^l_{i,J} \widetilde{Y}\right)^{\sim};$$

$$\coprod^{l,i+1}_{i,J} Y = \left(\prod^{l,i+1}_{i,J} \widetilde{Y}\right)^{\sim},$$

where tildes denote complements in the appropriate cubes.

Proof. Straightforward. □

DEFINITION 8. Let $t_i = n_0 + n_1(p_1+1) + \ldots + n_i(p_i+1)$. Define projection maps

$$\pi_i : \mathbb{R}^{n_0+\ldots+n_i} \to \mathbb{R}^{n_0+\ldots+n_{i-1}}$$

$$(\mathbf{x}_0, \ldots, \mathbf{x}_i) \mapsto (\mathbf{x}_0, \ldots, \mathbf{x}_{i-1}),$$

and for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j+n_{j+1}+\ldots+n_i} \to \mathbb{R}^{t_j+n_{j+1}+\ldots+n_{i-1}}$$

$$(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_j^{(p_j+1)}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_i) \mapsto (\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_j^{(p_j+1)}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_{i-1}).$$

LEMMA 8. *Let*

$$Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \ldots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\ldots+n_i+n_{i+1}}.$$

Then

$$\bigsqcup\nolimits_{i,J}^{l} \pi_{i+1,l-1}(Y) = \pi_{i+1,i}\Big(\bigsqcup\nolimits_{i,J}^{l,i+1} Y\Big).$$

Proof. Straightforward. □

5. *Case of a single quantifier block*

According to Theorem 1,

$$\mathrm{b}_{q_0}(X) = \mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\Big(\prod\nolimits_{1,J_1^1}^{1} X_1\Big), \tag{5.1}$$

where $J_1^1 = \{1, \ldots, p_1+1\}$.

Let $\nu = 1$, then (3.3) turns into $\exists \mathbf{x}_1 F(\mathbf{x}_0, \mathbf{x}_1)$, where $X_1 = \{F(\mathbf{x}_0, \mathbf{x}_1)\}$ and $F(\mathbf{x}_0, \mathbf{x}_1)$ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$.

5.1. *Polynomial case*

Suppose that X_1 is semialgebraic, with f's being polynomials of degrees $\deg(f) < d$. For any $k \le \dim(X)$, we bound the Betti number $\mathrm{b}_k(X)$ from above in the following way. Observe that $\prod_{1,J_1^1}^{1} X_1$ is an open set in $I_1^{n_0+(p_1+1)n_1}$ definable by a Boolean combination with no negations of $(p_1+1)s$ atomic formulae of the kind $g > 0$, $\deg(g) < d$ in $t_1 = n_0 + (p_1+1)n_1$ variables.

According to (3.1), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\Big(\prod\nolimits_{1,J_1^1}^{1} X_1\Big) \le O(p_1 sd)^{n_0+(p_1+1)n_1}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} O(p_1 sd)^{n_0+(p_1+1)n_1} \le (ksd)^{O(n_0+kn_1)}.$$

5.2. *Pfaffian case*

Suppose that $X_1 \subset I_1^n$ is sub-Pfaffian, with f's being Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain. Observe that $\prod^1_{1,J_1^1} X_1$ is an open set definable by a Boolean combination with no negations of $(p_1+1)s$ atomic formulae of the kind $g > 0$, where g are Pfaffian functions in an open domain contained in $I_1^{n_0+(p_1+1)n_1}$ of degrees (α, β), order $(p_1+1)\rho$ in $n_0+(p_1+1)n_1$ variables, having a common Pfaffian chain. According to (3.2), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\Big(\prod{}^1_{1,J_1^1} X_1\Big) \le ((p_1+1)s)^{n_0+(p_1+1)n_1} 2^{(p_1+1)\rho((p_1+1)\rho-1)/2} \cdot$$

$$\cdot O((n_0+p_1n_1)\beta + \min\{p_1\rho, n_0+p_1n_1\}\alpha)^{n_0+(p_1+1)(n_1+\rho)}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} \mathrm{b}_{q_1}\Big(\prod{}^1_{1,J_1^1} X_1\Big) \le$$

$$k((k+1)s)^{n_0+(k+1)n_1} 2^{(k+1)\rho((k+1)\rho-1)/2} \cdot$$

$$\cdot O((n_0+kn_1)\beta + \min\{k\rho, n_0+kn_1\}\alpha)^{n_0+(k+1)(n_1+\rho)}.$$

Let $d > \alpha+\beta$. Relaxing the obtained bound, we get

$$\mathrm{b}_k(X) \le (ks)^{O(n_0+kn_1)} 2^{(O(k\rho))^2} ((n_0+kn_1)d)^{O(n_0+kn_1+k\rho)}.$$

6. *Cases of two and three quantifier blocks*

In this section we obtain a generalization of (5.1) to the case of two and three blocks of quantifiers, as a preparation for cumbersome general formulae in the next section. The case of three quantifier blocks is considered separately also because of a technical difficulty that appears first in that case (see the discussion after (6.1)).

Recall that

$$\pi_i : \mathbb{R}^{n_0+\ldots+n_i} \to \mathbb{R}^{n_0+\ldots+n_{i-1}},$$

for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j+n_{j+1}+\ldots+n_i} \to \mathbb{R}^{t_j+n_{j+1}+\ldots+n_{i-1}}.$$

Let $\nu = 3$, then the original formula becomes $\exists \mathbf{x}_1 \forall \mathbf{x}_2 \exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)$. Thereby,

$$X_1 = \{\forall \mathbf{x}_2 \exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)\},\ \widetilde{X_2} = \{\exists \mathbf{x}_3 ((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in X_3)\},$$

$X = \widetilde{X_0}$ is open in $I_1^{n_0}$.

According to Theorem 1,

$$\mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\Big(\prod{}^1_{1,J_1^1} X_1\Big).$$

Applying in succession Lemma 7 (De Morgan law), Lemma 4 (Alexander's duality), definitions of π_2 and $\pi_{2,1}$, and Lemma 8 we get

$$\mathrm{b}_{q_1}\Big(\prod{}^1_{1,J_1^1} X_1\Big) = \mathrm{b}_{q_1}\Big(\widetilde{\Big(\bigsqcup{}^1_{1,J_1^1} \widetilde{X_1}\Big)}\Big) \le$$

$$\le \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup{}^1_{1,J_1^1} \widetilde{X_1} \cup \partial I_1^{t_1}\Big) = \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup{}^1_{1,J_1^1} \pi_2(X_2) \cup \pi_{2,1}(\partial I_1^{t_1} \times I_1^{n_2})\Big) =$$

$$= \mathrm{b}_{t_1-q_1-1}\Big(\pi_{2,1}\Big(\bigsqcup{}^{1,2}_{1,J_1^1} X_2 \cup \partial I_1^{t_1} \times I_1^{n_2}\Big)\Big).$$

Due to Theorem 1, the last expression does not exceed

$$\sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod\nolimits_2^2\Big(\bigsqcup\nolimits_{1,J_1^1}^{1,2} X_2 \cup \partial I_1^{t_1} \times I_1^{n_2}\Big)\Big) =$$

$$= \sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod\nolimits_2^2\Big(\bigsqcup\nolimits_{1,J_1^1}^{1,2} X_2 \cup B_2^2\Big)\Big).$$

In a case of sub-Pfaffian or semialgebraic X it is now possible to estimate

$$\mathrm{b}_{q_2}\Big(\prod\nolimits_2^2\Big(\bigsqcup\nolimits_{1,J_1^1}^{1,2} X_2 \cup B_2^2\Big)\Big)$$

via the format of X_2. This completes the description of the case of two quantifier blocks. We now proceed to the case of three blocks.

Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander's duality),

$$\mathrm{b}_{q_2}\Big(\prod\nolimits_2^2\Big(\bigsqcup\nolimits_{1,J_1^1}^{1,2} X_2 \cup B_2^2\Big)\Big) = \mathrm{b}_{q_2}\Big(\widetilde{\Big(\bigsqcup\nolimits_2^2\Big(\prod\nolimits_{1,J_1^1}^{1,2} \widetilde{X_2} \cap \widetilde{B_2^2}\Big)\Big)}\Big) = \tag{6.1}$$

$$= \mathrm{b}_{t_2-q_2-1}\Big(\bigsqcup\nolimits_2^2\Big(\prod\nolimits_{1,J_1^1}^{1,2} \widetilde{X_2} \cap \widetilde{B_2^2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big).$$

From this point we could have proceeded in a "natural" way similar to the just considered case of two blocks, namely, replacing in the previous expression the set $\widetilde{X_2}$ by $\pi_3(X_3)$, then carrying the projection operator to the left to obtain an expression of the kind $\mathrm{b}_{t_2-q_2-1}(\pi_{3,2}(\ldots))$, and after that applying Theorem 1. However, carrying the projection operator through the symbol $\prod_{1,J_1^1}^{1,2}$ (which corresponds to an intersection of some cylindrical sets) would require an introduction of $p_1 n_2$ new variables. This would result in a significantly higher upper bound for $\mathrm{b}_{q_0}(X)$. Instead we reduce intersections to unions, then carrying the projection operator to the left does not require new variables.

More precisely, by Lemma 5 (Mayer-Vietoris inequality) expression (6.1) does not exceed

$$\sum_{1 \le k_2 \le p_2+1} \quad \sum_{\hat{J}_2^2 \subset \{1,\ldots,p_2+1\},\ |\hat{J}_2^2|=k_2}$$

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod\nolimits_{2,\hat{J}_2^2}^{2}\Big(\prod\nolimits_{1,J_1^1}^{1,2} \widetilde{X_2} \cap \widetilde{B_2^2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big).$$

(We estimate a Betti number of the union of cylindrical sets from the definition of the symbol $\bigsqcup_2^2$ by a sum of Betti numbers of intersections of various combinations of these sets.)

By Lemma 6,

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod\nolimits_{2,\hat{J}_2^2}^{2}\Big(\prod\nolimits_{1,J_1^1}^{1,2} \widetilde{X_2} \cap \widetilde{B_2^2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big) =$$

$$= \mathrm{b}_{t_2-q_2-k_2}\Big(\prod\nolimits_{2,J_1^1 \times \hat{J}_2^2}^{1} \widetilde{X_2} \cap \prod\nolimits_{2,J_1^2 \times \hat{J}_2^2}^{2} \widetilde{B_2^2} \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big),$$

with $J_1^2 = \{1\}$. By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1 \le s_2 \le q_2+k_2+1} \quad \sum_{J_2^1 \subset J_1^1 \times \hat{J}_2^2,\ J_2^2 \subset J_1^2 \times \hat{J}_2^2,\ |J_2^1|+|J_2^2|=s_2}$$

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup\nolimits_{2,J_2^1}^{1} \widetilde{X_2} \cup \bigsqcup\nolimits_{2,J_2^2}^{2} \widetilde{B_2^2} \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big),$$

taking into the account that

$$\dim\Big(\bigsqcup\nolimits^{1}_{2,J^1_2}\widetilde{X_2}\cup\bigsqcup\nolimits^{2}_{2,J^2_2}\widetilde{B^2_2}\cup\partial(I^{t_1}_2\times I^{n_2(p_2+1)}_1)\Big)\le t_2$$

and therefore

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup\nolimits^{1}_{2,J^1_2}\widetilde{X_2}\cup\bigsqcup\nolimits^{2}_{2,J^2_2}\widetilde{B^2_2}\cup\partial(I^{t_1}_2\times I^{n_2(p_2+1)}_1)\Big)=0$$

for $s_2>q_2+k_2+1$.

We have

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup\nolimits^{1}_{2,J^1_2}\widetilde{X_2}\cup\bigsqcup\nolimits^{2}_{2,J^2_2}\widetilde{B^2_2}\cup\partial(I^{t_1}_2\times I^{n_2(p_2+1)}_1)\Big)=$$

$$=\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup\nolimits^{1}_{2,J^1_2}\pi_3(X_3)\cup\bigsqcup\nolimits^{2}_{2,J^2_2}\pi_{3,1}(B^2_3)\cup\pi_{3,2}(\partial(I^{t_1}_2\times I^{n_2(p_2+1)}_1)\times I^{n_3}_1)\Big)=$$

$$=\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\pi_{3,2}\Big(\bigsqcup\nolimits^{1,3}_{2,J^1_2}X_3\cup\bigsqcup\nolimits^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big).$$

Due to Theorem 1 the last expression does not exceed

$$\sum_{p_3+q_3=t_2-q_2-k_2+s_2-1}\mathrm{b}_{q_3}\Big(\prod\nolimits^3_3\Big(\bigsqcup\nolimits^{1,3}_{2,J^1_2}X_3\cup\bigsqcup\nolimits^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big).$$

In case of a sub-Pfaffian or a semialgebraic X it is now possible to estimate

$$\mathrm{b}_{q_3}\Big(\prod\nolimits^3_3\Big(\bigsqcup\nolimits^{1,3}_{2,J^1_2}X_3\cup\bigsqcup\nolimits^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big)$$

via the format of X_3.

7. *Arbitrary number of quantifiers*

THEOREM 2. *For any i the Betti number $\mathrm{b}_{q_0}(X)$ does not exceed*

$$\sum_{p_1+q_1=q_0}\ \sum_{p_2+q_2=t_1-q_1-1}\ \sum_{1\le k_2\le p_2+1}\ \sum_{\hat{J}^2_2\subset\{1,\dots,p_2+1\},\ |\hat{J}^2_2|=k_2} \tag{7.1}$$

$$\sum_{1\le s_2\le q_2+k_2+1}\ \sum_{J^1_2\subset J^1_1\times\hat{J}^2_2,\ J^2_2\subset J^2_1\times\hat{J}^2_2,\ |J^1_2|+|J^2_2|=s_2}\ \sum_{p_3+q_3=t_2-k_2+s_2-1}\cdots$$

$$\cdots\sum_{1\le k_{i-1}\le p_{i-1}+1}\ \sum_{\hat{J}^{i-1}_{i-1}\subset\{1,\dots,p_{i-1}+1\},\ |\hat{J}^{i-1}_{i-1}|=k_{i-1}}\ \sum_{1\le s_{i-1}\le q_{i-1}+k_{i-1}+1}$$

$$\sum_{J^1_{i-1}\subset J^1_{i-2}\times\hat{J}^{i-1}_{i-1},\dots,\ J^{i-1}_{i-1}\subset J^{i-1}_{i-2}\times\hat{J}^{i-1}_{i-1},\ |J^1_{i-1}|+\dots+|J^{i-1}_{i-1}|=s_{i-1}}\ \sum_{p_i+q_i=t_{i-1}-q_{i-1}-k_{i-1}+s_{i-1}-1}$$

$$\mathrm{b}_{q_i}\Big(\prod\nolimits^i_i\Big(\bigsqcup\nolimits^{1,i}_{i-1,J^1_{i-1}}X_i\cup\bigcup_{2\le r\le i-1}\bigsqcup\nolimits^{r,i}_{i-1,J^r_{i-1}}B^r_i\cup B^i_i\Big)\Big).$$

Proof. Induction on i. Suppose (7.1) is true. Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander's duality),

$$\mathrm{b}_{q_i}\Big(\prod\nolimits^i_i\Big(\bigsqcup\nolimits^{1,i}_{i-1,J^1_{i-1}}X_i\cup\bigcup_{2\le r\le i-1}\bigsqcup\nolimits^{r,i}_{i-1,J^r_{i-1}}B^r_i\cup B^i_i\Big)\Big)=$$

$$= \mathrm{b}_{q_i}\Big(\widetilde{\Big(\bigsqcup\nolimits_i^i\Big(\prod\nolimits^{1,i}_{i-1,J^1_{i-1}}\widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod\nolimits^{r,i}_{i-1,J^r_{i-1}} \widetilde{B^r_i} \cap \widetilde{B^i_i}\Big)\Big)}\Big) \le$$

$$\le \mathrm{b}_{t_i-q_i-1}\Big(\bigsqcup\nolimits_i^i\Big(\prod\nolimits^{1,i}_{i-1,J^1_{i-1}}\widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod\nolimits^{r,i}_{i-1,J^r_{i-1}} \widetilde{B^r_i} \cap \widetilde{B^i_i}\Big) \cup$$

$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big).$$

By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\le k_i\le p_i+1} \quad \sum_{\hat{J}^i_i\subset\{1,\ldots,p_i+1\},\ |\hat{J}^i_i|=k_i}$$

$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits^i_{i,\hat{J}^i_i}\Big(\prod\nolimits^{1,i}_{i-1,J^1_{i-1}}\widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod\nolimits^{r,i}_{i-1,J^r_{i-1}} \widetilde{B^r_i} \cap \widetilde{B^i_i}\Big) \cup$$

$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big),$$

where, by Lemma 6,

$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits^i_{i,\hat{J}^i_i}\Big(\prod\nolimits^{1,i}_{i-1,J^1_{i-1}}\widetilde{X_i} \cap \bigcap_{2\le r\le i-1} \prod\nolimits^{r,i}_{i-1,J^r_{i-1}} \widetilde{B^r_i} \cap \widetilde{B^i_i}\Big) \cup$$

$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big) =$$

$$= \mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits^1_{i,J^1_{i-1}\times\hat{J}^i_i}\widetilde{X_i} \cap \bigcap_{2\le r\le i} \prod\nolimits^r_{i,J^r_{i-1}\times\hat{J}^i_i} \widetilde{B^r_i} \cup$$

$$\cup\, \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big),$$

where $J^i_{i-1} = \{1\}$. By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\le s_i\le q_i+k_i+1} \quad \sum_{J^1_i\subset J^1_{i-1}\times\hat{J}^i_i,\ \ldots,\ J^i_i\subset J^i_{i-1}\times\hat{J}^i_i,\ |J^1_i|+\ldots+|J^i_i|=s_i}$$

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits^1_{i,J^1_i}\widetilde{X_i} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits^r_{i,J^r_i}\widetilde{B^r_i} \cup\ \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big).$$

We have

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits^1_{i,J^1_i}\widetilde{X_i} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits^r_{i,J^r_i}\widetilde{B^r_i} \cup\ \partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i})\Big) =$$

$$= \mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits^1_{i,J^1_i}\pi_{i+1}(X_{i+1}) \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits^r_{i,J^r_i}\pi_{i+1,r-1}(B^r_{i+1}) \cup$$

$$\cup\, \pi_{i+1,i}(\partial(I_i^{n_0+(p_1+1)n_1} \times \ldots \times I_1^{(p_i+1)n_i}) \times I_1^{n_{i+1}})\Big) =$$

$$= \mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\pi_{i+1,i}\Big(\bigsqcup\nolimits^{1,i+1}_{i,J^1_i}X_{i+1} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits^{r,i+1}_{i,J^r_i}B^r_{i+1} \cup B^{i+1}_{i+1}\Big)\Big).$$

Due to Theorem 1 the last expression does not exceed

$$\sum_{p_{i+1}+q_{i+1}=t_i-q_i-k_i+s_i-1}$$

$$\mathrm{b}_{q_{i+1}}\Big(\prod\nolimits_{i+1}^{i+1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1,i+1} X_{i+1} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^{r,i+1} B_{i+1}^r \cup B_{i+1}^{i+1}\Big)\Big).$$

□

8. *Upper bounds for sub-Pfaffian sets*

We first estimate from above the number of additive terms in (7.1). These terms can be partitioned into $i-1$ groups of the kind

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j\subset\{1,\dots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

$$\sum_{1\le s_j\le q_j+k_j+1} \quad \sum_{J_j^1\subset J_{j-1}^1\times\hat{J}_j^j,\dots,\ J_j^j\subset J_{j-1}^j\times\hat{J}_j^j,\ |J_j^1|+\dots+|J_j^j|=s_j}$$

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1},$$

where $1\le j\le i-1$.

The number of terms in

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j\subset\{1,\dots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

is 2^{p_j+1}. The number of terms in

$$\sum_{1\le s_j\le q_j+k_j+1} \quad \sum_{J_j^1\subset J_{j-1}^1\times\hat{J}_j^j,\dots,\ J_j^j\subset J_{j-1}^j\times\hat{J}_j^j,\ |J_j^1|+\dots+|J_j^j|=s_j}$$

does not exceed $2^{j(q_j+k_j+1)}$. The number of terms in

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1}$$

does not exceed t_j+1.

It follows that the total number of terms in jth group does not exceed

$$2^{p_j+1+j(q_j+k_j+1)}(t_j+1)\le 2^{O(jt_{j-1})}.$$

Since $t_j=n_0+n_1(p_1+1)+\ldots+n_j(p_j+1)$, $p_l\le t_{l-1}$, and therefore $t_j\le 2^j n_0n_1\dots n_j$, the number of terms in jth group does not exceed $2^{O(j2^j n_0n_1\dots n_{j-1})}$. It follows that the total number of terms in (7.1) does not exceed $2^{O(i^2 2^i n_0n_1\dots n_{i-2})}$.

We now find an upper bound for

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_{\nu}^{\nu}\Big(\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \cup B_\nu^\nu\Big)\Big).$$

Assume that $X_\nu=\{F(\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_\nu)\}$, where F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f>0$, f's are polynomials or Pfaffian functions of degrees less than d or (α,β) respectively. In Pfaffian case, let functions f be defined in an open domain G by the same Pfaffian chain of order ρ. We assume without loss of generality that $I_\nu^{n_0+\ldots+n_\nu} \subset G$.

The set $\bigsqcup^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having $|J^1_{\nu-1}|s \le s_{\nu-1}s \le (2t_{\nu-2}+1)s$ atoms of degrees less than d (for polynomials) or less than (α,β) (for Pfaffian functions) and at most $2t_{\nu-1}+2n_\nu$ linear atoms (defining $I_1^{t_{\nu-1}+n_\nu}$).

For any $2 \le r \le \nu$ the set $B^r_r \subset \mathbb{R}^{t_{r-1}+n_r}$ is defined by a Boolean formula with no negations having $4t_{r-1}+2n_r$ linear atomic inequalities. Therefore, all sets of the kind B^r_j for $j \ge r$ are defined by Boolean formulae with no negations having $4t_{r-1}+2(n_r+\ldots+n_j)$ linear inequalities. In particular, the set $B^r_\nu \subset \mathbb{R}^{t_{r-1}+n_r+\ldots+n_\nu}$ is defined by $4t_{r-1}+2(n_r+\ldots+n_\nu)$ linear atomic inequalities.

For any $2 \le r \le \nu-1$ the set $\bigsqcup^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$(4t_{r-1}+2(n_r+\ldots+n_\nu))|J^r_{\nu-1}|+2t_{\nu-1}+2n_\nu \le$$

$$\le (4t_{r-1}+2(n_r+\ldots+n_\nu))s_{\nu-1}+2t_{\nu-1}+2n_\nu \le$$

$$\le (4t_{r-1}+2(n_r+\ldots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu$$

linear atoms.

It follows that the set $\bigcup_{2\le r\le\nu-1}\bigsqcup^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$((4t_{\nu-1}+2(n_2+\ldots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu)(\nu-2)$$

linear atoms.

The set

$$\prod\nolimits^\nu_\nu\Big(\bigsqcup\nolimits^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le\nu-1}\bigsqcup\nolimits^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big) \subset \mathbb{R}^{t_\nu} \tag{8.1}$$

is defined by a Boolean formula with no negations having at most

$$((2t_{\nu-2}+1)s+2t_{\nu-1}+2n_\nu+((4t_{\nu-1}+2(n_2+\ldots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu)(\nu-2))\cdot$$

$$\cdot(t_{\nu-1}+1) \le s t_{\nu-1}^{O(1)}$$

atoms of degrees less than d for polynomials or less than (α,β) for Pfaffian functions.

Similar calculation shows that, in the Pfaffian case, the set (8.1) is defined by Pfaffian functions having the order at most $\rho(2t_{\nu-2}+1)(t_{\nu-1}+1) \le O(\rho t_{\nu-2}t_{\nu-1})$.

8.1. *Polynomial case*

Let functions f in formula F be polynomials of degrees $\deg(f) < d$. Then, according to (3.1),

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits^\nu_\nu\Big(\bigsqcup\nolimits^{1,\nu}_{\nu-1,J^1_{\nu-1}} X_\nu \cup \bigcup_{2\le r\le\nu-1}\bigsqcup\nolimits^{r,\nu}_{\nu-1,J^r_{\nu-1}} B^r_\nu \cup B^\nu_\nu\Big)\Big) \le$$

$$\le O(ds)^{t_\nu} t_{\nu-1}^{O(t_\nu)} \le (2^\nu d s n_0 n_1 \ldots n_{\nu-1})^{O(2^\nu n_0 n_1 \ldots n_\nu)}.$$

Using (7.1) in case $i=\nu$, we get

$$\mathrm{b}_{q_0}(X) \le (2^{\nu^2} d s n_0 n_1 \ldots n_{\nu-1})^{O(2^\nu n_0 n_1 \ldots n_\nu)}$$

(compare with (3.4) and (3.5)).

8.2. *Pfaffian case*

Let f be Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain. Then, according to (3.2),

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_\nu^\nu\Big(\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \cup B_\nu^\nu\Big)\Big) \le$$

$$\le 2^{O(\rho^2 t_{\nu-2}^2 t_{\nu-1}^2)}(st_{\nu-1})^{O(t_\nu)} O(t_\nu\beta + \min\{t_\nu, \rho\}\alpha)^{O(t_\nu + \rho t_{\nu-2}t_{\nu-1})} \le$$

$$\le 2^{O(\rho^2 2^{4\nu} n_0^4 n_1^4 \dots n_{\nu-2}^4 n_{\nu-1}^2)} s^{O(2^\nu n_0 n_1 \dots n_\nu)}\cdot$$

$$\cdot(2^\nu n_0 n_1 \dots n_\nu(\alpha+\beta))^{O(2^\nu n_0 n_1 \dots n_\nu + \rho 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1})}.$$

Using (7.1) in case $i = \nu$, we get

$$\mathrm{b}_{q_0}(X) \le 2^{O(\nu 2^\nu n_0 n_1 \dots n_\nu + \rho^2 2^{4\nu} n_0^4 n_1^4 \dots n_{\nu-2}^4 n_{\nu-1}^2)} s^{O(2^\nu n_0 n_1 \dots n_\nu)}\cdot$$

$$\cdot(n_0 n_1 \dots n_\nu(\alpha+\beta))^{O(2^\nu n_0 n_1 \dots n_\nu + \rho 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1})}.$$

Introducing the notations:

$$u_\nu := 2^\nu n_0 n_1 \dots n_\nu, \quad v_\nu := 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1},$$

we can rewrite this bound in a more compact form

$$\mathrm{b}_{q_0}(X) \le 2^{O(\nu u_\nu + \rho^2 v_\nu^2)} s^{O(u_\nu)} (u_\nu(\alpha+\beta))^{O(u_\nu + \rho v_\nu)}.$$

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