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Betti Numbers of Semialgebraic Sets Defined by Quantifier-Free Formulae*

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Abstract. Let X be a semialgebraic set in $\mathbb{R}^n$ defined by a Boolean combination of atomic formulae of the kind $h * 0$ where $* \in \{>, \geq, =\}$, $\deg(h) < d$, and the number of distinct polynomials h is k. We prove that the sum of Betti numbers of X is less than $O(k^2 d)^n$.

Let an algebraic set $X \subset \mathbb{R}^n$ be defined by polynomial equations of degrees less than d. The well-known results of Oleinik, Petrovskii [8], [9], Milnor [6], and Thom [12] provide the upper bound

$$\mathrm{b}(X) \leq d(2d-1)^{n-1}$$

for the sum of Betti numbers $\mathrm{b}(X)$ of X (with respect to the singular homology). In a more general case of a set X defined by a system of k non-strict polynomial inequalities of degrees less than d, the sum of Betti numbers does not exceed $O(kd)^n$.

These results were later extended and refined. Basu [1] proved that if a semialgebraic set X is *basic* (i.e., X is defined by a system of equations and strict inequalities), or is defined by a Boolean combination (with no negations) of only non-strict or of only strict inequalities, then

$$\mathrm{b}(X) \leq O(kd)^n,$$

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where k is the number of distinct polynomials in the defining formula (this is a relaxed form of Basu's bound, for a more precise description see [1], [2].) Papers [7] and [13] imply that if X is compact and is defined by an arbitrary Boolean combination of equations or inequalities, then

$$\mathrm{b}(X) \leq O(kd)^{2n}.$$

The purpose of this note is to prove a bound for an arbitrary semialgebraic set defined by an arbitrary Boolean formula. More precisely, let X be a semialgebraic set in $\mathbb{R}^n$ defined by a Boolean combination of atomic formulae of the kind $h * 0$ where $* \in \{>, \geq, =\}$, $\deg(h) < d$, and the number of distinct polynomials h is k.

Theorem 1. *The sum of Betti numbers of X is less than $O(k^2d)^n$.*

We deduce Theorem 1 from the following result.

Proposition 2 [1]. *Let the Boolean combination which defines X contain only non-strict inequalities and no negations. Then the sum of Betti numbers of X is less than $O(kd)^n$.*

Since sums of Betti numbers of sets X and $X \cap \{x_1^2 + \cdots + x_n^2 < \Omega\}$ coincide for a large enough $\Omega \in \mathbb{R}$ (see Lemma 1 of [1]), we assume in what follows that X is bounded.

Definition 3. For a given finite set $\{h_1, \ldots, h_k\}$ of polynomials h_i define its $(h_1, \ldots, h_k)$-*cell* (or just *cell*) as a semialgebraic set in $\mathbb{R}^n$ of the kind

$$\{h_{i_1} = \cdots = h_{i_{k_1}} = 0, h_{i_{k_1+1}} > 0, \ldots, h_{i_{k_2}} > 0, h_{i_{k_2+1}} < 0, \ldots, h_{i_k} < 0\}, \qquad (1)$$

where $i_1, \ldots, i_{k_1}, \ldots, i_{k_2}, \ldots, i_k$ is a permutation of $1, \ldots, k$.

Obviously, for a given set of polynomials any two distinct cells are disjoint. According to [4] and [5], the number of all non-empty $(h_1, \ldots, h_k)$-cells is at most $(kd)^{O(n)}$, but we do not need this bound in what follows. Observe that both X and the complement $\widetilde{X} = \mathbb{R}^n \backslash X$ are disjoint unions of some non-empty $(h_1, \ldots, h_k)$-cells.

Example 4. Let $X := \{(x, y) \in \mathbb{R}^2 | \, x^2y^2 > 0 \vee x^2 + y^2 = 0\}$, i.e., X is the plane $\mathbb{R}^2$ minus the union of the coordinate axes plus the origin. There are nine (x^2y^2, x^2+y^2)-cells among which exactly three,

$$\{x^2y^2 = x^2 + y^2 = 0\}, \ \{x^2y^2 > 0, x^2 + y^2 > 0\}, \ \text{and} \ \{x^2y^2 = 0, x^2 + y^2 > 0\},$$

are non-empty. The union of the first two of these cells is X.

Introduce the following partial order on the set of all cells. Let $\Gamma \prec \Gamma'$ iff the cell Γ' is obtained from the cell Γ by replacing at least one of the equalities $h_j = 0$ in Γ by either $h_j > 0$ or $h_j < 0$. Thus the minimal cell with respect to $\prec$ is $\Gamma_{\min} := \{h_1 = \cdots = h_k = 0\}$. Clearly, the cells having the same number p of equations are not pairwise comparable with respect to $\prec$, we say that these cells are *on the level* $k-p+1$. In particular, $\Gamma_{\min}$ is the only cell on level 1.

Let

$$1 \gg \varepsilon_1 \gg \delta_1 \gg \varepsilon_2 \gg \delta_2 \gg \cdots \gg \varepsilon_k \gg \delta_k > 0,$$

where $\gg$ stands for "sufficiently greater than". The set X_1 is the result of the following inductive construction.

Let $\Sigma_{\ell,1}, \ldots, \Sigma_{\ell,t_\ell}$ be all cells on the level ℓ which lie in X. Let $\Delta_{\ell,1}, \ldots, \Delta_{\ell,r_\ell}$ be all cells on the level ℓ which have the empty intersection with X. For any cell

$$\Sigma_{\ell,j} := \{h_{i_1} = \cdots = h_{i_{k-\ell+1}} = 0, h_{i_{k-\ell+2}} > 0, \ldots, h_{i_{k_1}} > 0, h_{i_{k_1+1}} < 0, \ldots, h_{i_k} < 0\}$$

on the level $\ell \le k$ introduce the set

$$\begin{aligned}\widehat{\Sigma}_{\ell,j} \;:=\; \{&h_{i_1}^2 \le \varepsilon_\ell, \ldots, h_{i_{k-\ell+1}}^2 \le \varepsilon_\ell,\\ &h_{i_{k-\ell+2}} \ge 0, \ldots, h_{i_{k_1}} \ge 0, h_{i_{k_1+1}} \le 0, \ldots, h_{i_k} \le 0\}.\end{aligned}$$

Additionally, for any cell

$$\Sigma_{k+1,j} := \{h_{i_1} > 0, \ldots, h_{i_{k_1}} > 0, h_{i_{k_1+1}} < 0, \ldots, h_{i_k} < 0\}$$

on the level $k+1$ let

$$\widehat{\Sigma}_{k+1,j} := \{h_{i_1} \ge 0, \ldots, h_{i_{k_1}} \ge 0, h_{i_{k_1+1}} \le 0, \ldots, h_{i_k} \le 0\}.$$

For any cell

$$\Delta_{\ell,j} := \{h_{i_1} = \cdots = h_{i_{k-\ell+1}} = 0, h_{i_{k-\ell+2}} > 0, \ldots, h_{i_{k_1}} > 0, h_{i_{k_1+1}} < 0, \ldots, h_{i_k} < 0\}$$

on the level $\ell \le k$ introduce the set

$$\begin{aligned}\widehat{\Delta}_{\ell,j} \;:=\; \{&h_{i_1}^2 < \delta_\ell, \ldots, h_{i_{k-\ell+1}}^2 < \delta_\ell,\\ &h_{i_{k-\ell+2}} > 0, \ldots, h_{i_{k_1}} > 0, h_{i_{k_1+1}} < 0, \ldots, h_{i_k} < 0\}.\end{aligned}$$

Let

$$X_{k+1} := X \cup \bigcup_j \widehat{\Sigma}_{k+1,j}.$$

Assume that $X_{\ell+1}$ is constructed. Let

$$X_\ell := \left(X_{\ell+1} \setminus \bigcup_j \widehat{\Delta}_{\ell,j}\right) \cup \bigcup_j \widehat{\Sigma}_{\ell,j}.$$

On the last step of the induction we obtain set X_1.

Example 4 (continued). In Example 4 we have

$$\Gamma_{\min} = \Sigma_{1,1} = \Sigma_{1,t_1} = \{x^2y^2 = x^2 + y^2 = 0\}.$$

Choose the following sub-indices for the non-empty cells:

$$\Delta_{2,1} := \{x^2y^2 = 0, x^2 + y^2 > 0\},$$
$$\Sigma_{3,1} := \{x^2y^2 > 0, x^2 + y^2 > 0\}.$$

Then

$$\widehat{\Sigma}_{1,1} = \{(x^2y^2)^2 \le \varepsilon_1, (x^2 + y^2)^2 \le \varepsilon_1\},$$
$$\widehat{\Delta}_{2,1} = \{(x^2y^2)^2 < \delta_2, x^2 + y^2 > 0\},$$
$$\widehat{\Sigma}_{3,1} = \{x^2y^2 \ge 0, x^2 + y^2 \ge 0\}.$$

The inductive construction proceeds as follows. Since $\Sigma_{3,1}$ is the only non-empty cell on level 3, we get $X_3 = X \cup \widehat{\Sigma}_{3,1} = X$. Next, since $\Delta_{2,1}$ is the only non-empty cell on level 2, we get $X_2 = X_3 \backslash \widehat{\Delta}_{2,1}$ (i.e., X_2 is $\mathbb{R}^2$ minus an open δ_2-neighbourhood of the union of the coordinate axes). Finally, $X_1 = X_2 \cup \widehat{\Sigma}_{1,1}$, or, in terms of polynomial inequalities,

$$X_1 = (X \backslash \{(x^2y^2)^2 < \delta_2, x^2 + y^2 > 0\}) \cup \{(x^2y^2)^2 \le \varepsilon_1, (x^2 + y^2)^2 \le \varepsilon_1\}. \qquad (2)$$

Thus, X_1 is the plane $\mathbb{R}^2$ minus an open neighbourhood of the union of the coordinate axes plus a larger closed neighbourhood of the origin. Obviously, X_1 can be defined by a Boolean formula without negations, involving the same polynomials as in (2), and having only non-strict inequalities. It is easy to see that X and X_1 are homotopy equivalent.

Returning to the general case, one can prove that X and X_1 are weakly homotopy equivalent. For our purposes the following weaker statement will be sufficient.

Lemma 5. *The sum of Betti numbers of X coincides with the sum of Betti numbers of X_1.*

Proof. For every m, $1 \le m \le k + 1$, define a set Y^m using the inductive procedure similar to the one used for defining X_1. The difference is that the base step of the induction starts at some level m rather than specifically at the level $k + 1$. More precisely, let $Y^{k+1} := X_1$. For any $m \le k$, let

$$Z_m^{m,1} := X \backslash \bigcup_j \widehat{\Delta}_{m,j} \quad \text{and} \quad Z_m^{m,2} := Z_m^{m,1} \cup \bigcup_j \widehat{\Sigma}_{m,j}.$$

This concludes the base of the induction.

On the induction step, suppose that $Z_{\ell+1}^{m,s}$ is defined, where $m - 1 \ge \ell \ge 1$, $s = 1, 2$. Define

$$Z_\ell^{m,s} := \Big(Z_{\ell+1}^{m,s} \backslash \bigcup_j \widehat{\Delta}_{\ell,j} \Big) \cup \bigcup_j \widehat{\Sigma}_{\ell,j}.$$

Let $Y^m := Z_1^{m,2}$.

For every m, $1 \le m \le k + 1$, define the set Y'^m by the procedure similar to the definition of Y^m, replacing in each $\widehat{\Sigma}_{\ell,j}$ the inequalities $h_{i_1}^2 \le \varepsilon_\ell, \ldots, h_{i_{k-\ell+1}}^2 \le \varepsilon_\ell$ by $h_{i_1}^2 < \varepsilon_\ell, \ldots, h_{i_{k-\ell+1}}^2 < \varepsilon_\ell$, respectively, and in each $\widehat{\Delta}_{\ell,j}$ the inequalities $h_{i_1}^2 <$

$\delta_\ell, \dots, h^2_{i_{k-\ell+1}} < \delta_\ell$ by $h^2_{i_1} \le \delta_\ell, \dots, h^2_{i_{k-\ell+1}} \le \delta_\ell$, respectively. Denote the results of the replacements by $\widehat{\Sigma}'_{\ell,j}$ and $\widehat{\Delta}'_{\ell,j}$, respectively.

We show by induction on m that $\mathrm{b}(Y^m) = \mathrm{b}(Y'^m)$ and that $\mathrm{b}(X) = \mathrm{b}(Y^m) = \mathrm{b}(Y'^m)$. It will follow, in particular, that the sum of Betti numbers of X does not exceed the sum of Betti numbers of $X_1 = Y^{k+1}$.

For the base case of $m = 1$, let first $\Gamma_{\min} \neq \emptyset$ and $\Gamma_{\min} \cap X = \emptyset$ (i.e., $\Gamma_{\min} = \Delta_{1,1} = \Delta_{1,r_1}$), then

$$Y^1 = X \backslash \widehat{\Delta}_{1,1} = X \backslash \{h_1^2 < \delta_1, \dots, h_k^2 < \delta_1\}.$$

Introduce the following *directed system* of sets. First replace δ_1 in the definition of Y^1 by a parameter and then consider the family of sets as the parameter tends to 0. Denote this directed system by $\{Y^1\}_{\delta_1 \to 0}$. Observe that $\{Y^1\}_{\delta_1 \to 0}$ is a *fundamental covering* of X. Indeed, since any point $x \in X$ does not belong to the *closed* set $\{h_1 = \cdots = h_k = 0\}$, there is a neighbourhood U of x in Y^1 for all small enough δ_1, which is also a neighbourhood of x in X, such that $U \cap \{h_1 = \cdots = h_k = 0\} = \emptyset$. Thus, if for a subset $A \subset X$ the intersection $A \cap Y^1$ is open in Y^1 for any small enough δ_1, then A is open in X. Therefore (see Section 1.2.4.7 of [10]), X is a direct limit of $\{Y^1\}_{\delta_1 \to 0}$. It follows (see Theorem 4.1.7 on p. 162 of [11]) that $H_*(X)$ is the direct limit of $\{H_*(Y^1)\}_{\delta_1 \to 0}$. On the other hand, by Hardt's triviality theorem [3, p. 62, Theorem 5.22] for a small enough positive δ_1 all Y^1 are pairwise homeomorphic. Thus, for a small enough δ_1 we have $\mathrm{b}(X) \le \mathrm{b}(Y^1)$. Moreover, we have $H_*(X) \simeq H_*(Y^1)$ and therefore $\mathrm{b}(X) = \mathrm{b}(Y^1)$. Indeed, due again to Hardt's triviality theorem, for all small enough positive values of δ_1 the inclusion maps in the filtration of spaces Y^1 are homotopic to homeomorphisms and therefore induce *isomorphisms* in the corresponding direct system of groups $H_*(Y^1)$. It follows that the direct limit of groups $\{H_*(Y^1)\}_{\delta_1 \to 0}$ is isomorphic to any of these groups for a fixed small enough positive δ_1.

Observe that a similar argument is applicable to $Y'^1 = X \backslash \{h_1^2 \le \delta_1, \dots, h_k^2 \le \delta_1\}$, therefore $H_*(X) \simeq H_*(Y'^1)$.

Suppose now that $\Gamma_{\min} \neq \emptyset$ and $\Gamma_{\min} \subset X$ (i.e., $\Gamma_{\min} = \Sigma_{1,1} = \Sigma_{1,t_1}$). Then $\Gamma_{\min} \cap \widetilde{X} = \emptyset$, where $\widetilde{X}$ is the complement of X. Replacing in the above proof the set X by $\widetilde{X}$, and δ_1 by ε_1, we get $H_*(\widetilde{X}) \simeq H_*(\widetilde{Y'^1})$. Since X is bounded, by Alexander's duality, $\mathrm{b}(\widetilde{X}) = \mathrm{b}(X) + 1$ and $\mathrm{b}(\widetilde{Y'^1}) = \mathrm{b}(Y'^1) + 1$, hence $\mathrm{b}(X) = \mathrm{b}(Y'^1)$.

Similar argument shows that $\mathrm{b}(X) = \mathrm{b}(Y^1)$.

The case when $\Gamma_{\min} = \emptyset$ is trivial. This concludes the base induction step.

Assume that $\mathrm{b}(X) = \mathrm{b}(Y^m) = \mathrm{b}(Y'^m)$. First let $\bigcup_j \Delta_{m+1,j} \neq \emptyset$, then the family of sets $\{Z_1'^{m+1,1}\}_{\delta_{m+1} \to 0}$ is a fundamental covering of Y'^m. Indeed, by the definition we have

$$Z_{m+1}'^{m+1,1} = X \backslash \bigcup_j \widehat{\Delta}'_{m+1,j}.$$

Take any point $x \in Z_1'^{m+1,1}$. Then x belongs either to

$$\bigcap_j \left(\{h^2_{i_1} > \delta_{m+1}\} \cup \cdots \cup \{h^2_{i_{k-m}} > \delta_{m+1}\}\right)$$

for all non-empty cells

$$\Delta_{m+1,j} = \{h_{i_1} = \cdots = h_{i_{k-m}} = 0, h_{i_{k-m+1}} > 0, \dots, h_{i_k} < 0\}$$

and all sufficiently small δ_{m+1}, or to a set of the kind

$$\{h_{i_1} = \cdots = h_{i_{k-m}} = 0, h^2_{i_{k-m+1}} < \varepsilon_t, \ldots, h^2_{i_{k-t+1}} < \varepsilon_t,\\ h_{i_{k-t+2}} > 0, \ldots, h_{i_{k_1}} > 0, h_{i_{k_1+1}} < 0, \ldots, h_{i_k} < 0\}$$

for some $t \le m$ and a non-empty cell

$$\Sigma_{t,j} = \{h_{i_1} = \cdots = h_{i_{k-t+1}} = 0, h_{i_{k-t+2}} > 0, \ldots, h_{i_k} < 0\} \subset X.$$

In both cases there is a set U which is a neighbourhood of x in $Z_1'^{m+1,1}$ for all sufficiently small δ_{m+1}, and also a neighbourhood of x in Y'^m.

Thus, for a small enough δ_{m+1} we have $H_*(Y'^m) \simeq H_*(Z_1'^{m+1,1})$. Introduce a set $Z_1'^{m+1,1}(\gamma)$, where $0 < \gamma \ll \delta_{m+1}$, defined by a formula $\varphi(\gamma)$ which is constructed as follows. In the formula φ defining $Z_1'^{m+1,1}$ replace all occurrences of the systems of inequalities of the kind $h^2_{i_1} < \varepsilon_\ell, \ldots, h^2_{i_{k-\ell+1}} < \varepsilon_\ell$ by $h^2_{i_1} \le \varepsilon_\ell - \gamma, \ldots, h^2_{i_{k-\ell+1}} \le \varepsilon_\ell - \gamma$ and all occurrences of the systems inequalities of the kind $h^2_{i_1} \le \delta_\ell, \ldots, h^2_{i_{k-\ell+1}} \le \delta_\ell$ by $h^2_{i_1} < \delta_\ell + \gamma, \ldots, h^2_{i_{k-\ell+1}} < \delta_\ell + \gamma$. The family of sets $\{Z_1'^{m+1,1}(\gamma)\}_{\gamma\to 0}$ is a fundamental covering of $Z_1'^{m+1,1}$, thus for a small enough γ we have

$$H_*(Z_1'^{m+1,1}) \simeq H_*(Z_1'^{m+1,1}(\gamma)).$$

However, the sets $Z_1'^{m+1,1}(\gamma)$ and $Z_1^{m+1,1}$ are homeomorphic due to Hardt's triviality theorem, therefore $H_*(Z_1'^{m+1,1}) \simeq H_*(Z_1^{m+1,1})$. It follows that

$$\mathrm{b}(X) = \mathrm{b}(Y'^m) = \mathrm{b}(Z_1'^{m+1,1}) = \mathrm{b}(Z_1^{m+1,1}).$$

Now let $\bigcup_j \Sigma_{m+1,j} \ne \emptyset$. Note that $\widetilde{X} \cap \bigcup_j \Sigma_{m+1,j} = \emptyset$. As above (but using ε_{m+1} in place of δ_{m+1}), we get

$$\mathrm{b}(\widetilde{X}) = \mathrm{b}(\widetilde{Z}_1^{m+1,2}) = \mathrm{b}(\widetilde{Z}_1'^{m+1,2}).$$

By Alexander's duality we have $\mathrm{b}(\widetilde{X}) = \mathrm{b}(X) + 1$, $\mathrm{b}(\widetilde{Z}_1^{m+1,2}) = \mathrm{b}(Z_1^{m+1,2}) + 1$, and $\mathrm{b}(\widetilde{Z}_1'^{m+1,2}) = \mathrm{b}(Z_1'^{m+1,2}) + 1$, hence in this case the condition

$$\mathrm{b}(X) = \mathrm{b}(Z_1^{m+1,2}) = \mathrm{b}(Z_1'^{m+1,2})$$

is also true.

The case when $\bigcup_j \left(\Delta_{m+1,j} \cup \Sigma_{m+1,j}\right) = \emptyset$ is trivial.

Recalling that $Z_1^{m+1,2} = Y^{m+1}$ and $Z_1'^{m+1,2} = Y'^{m+1}$, we get the required $\mathrm{b}(X) = \mathrm{b}(Y^{m+1}) = \mathrm{b}(Y'^{m+1})$. □

Proof of Theorem 1. According to Lemma 5, it is sufficient to prove the bound for the set X_1 which is defined by a Boolean combination (with no negations) of non-strict inequalities. The atomic polynomials are either of the kind h_i or of the kind $h_i^2 - \delta_j$ or of the kind $h_i^2 - \varepsilon_j$, $1 \le i, j \le k$, hence there is at most $O(k^2)$ pairwise distinct among them. Now the theorem follows from Proposition 2. □

Remark 6. Employing some additional technicalities one can prove that in the construction of set X_1 it is sufficient to use just one sort of constants, i.e., keep $\varepsilon_1, \ldots, \varepsilon_k$ in their positions and replace $\delta_1, \ldots, \delta_k$ by $\varepsilon_1, \ldots, \varepsilon_k$, respectively. This reduces the number of polynomials involved in the description of X_1 and therefore the O-symbol constant in the upper bound of Theorem 1.

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