University of West Bohemia
Faculty of Applied Sciences
Department of Mathematics

DIPLOMA THESIS

# Generalized hamiltonian cycles

Pilsen, 2012 Adam Kabela

Západočeská univerzita v Plzni
Fakulta aplikovaných věd
Katedra matematiky

DIPLOMOVÁ PRÁCE

# Zobecněné hamiltonovské kružnice

v Plzni, 2012

Adam Kabela

## Declaration

I hereby declare that this Diploma Thesis is the result of my own work and that all external sources of information have been duly acknowledged.

Pilsen, 21th May 2012 ......................................

Adam Kabela

## Acknowledgment

I would like to thank my supervisor RNDr. Jakub Teska, Ph.D. for his valu- able suggestions, comments and corrections. No less thanks for his support and encouragement.

## Abstract

In [7] Chvátal raised a question, whether there is a finite constant $t$ such that every $t$-tough graph contains a hamiltonian cycle. Despite some results were presented on this topic, the answer remains unknown. Moreover it is also unknown, whether for some $t$ every $t$-tough graph contains an $r$-trestle. This thesis shows the relations among the hamiltonian cycle, the $r$-trestle and the toughness of a graph and how the relations change considering general graphs, chordal graphs and planar graphs. Some other relations dealing with generalizing hamiltonicity, the toughness of a graph and also the hamiltonicity and forbidden subgraphs are mentioned.

As the original results we present chordal graphs and chordal planar graphs with high toughness and no $r$-trestle. These graphs improves the upper bound on the shortness exponent of the class of 1-tough chordal planar graphs shown in [13]. We also present a sufficient condition on forbidden subgraphs for a graph to have an $r$-trestle.

## Keywords

hamiltonian cycle, $r$-trestle, toughness, chordal graphs, planar graphs, shortness exponent, forbidden subgraphs

# Abstrakt

Jedna z nezodpovězených otázek, na téma hamiltonovskosti a tuhosti grafu, je otázka, kterou vyslovil Chvátal v článku [7]. Existuje taková konstanta t, že každý t-tuhý graf je hamiltonovský? Bez odpovědi zůstává také otázka, zda existuje takové $t$, že každý $t$-tuhý graf obsahuje $r$-trestle. V této práci ukážeme různé souvislosti mezi hamiltonovskými kružnicemi, $r$-trestly a tuhostí grafu. Budeme sledovat, jak se tyto souvislosti mění, pokud namísto obecných grafů uvažujeme chordální grafy nebo rovinné grafy. Zmíníme další možná zobecnění hamiltonovských kružnic a jejich vzájemné vztahy, a v neposlední řadě také souvislost hamiltonovských kružnic a $r$-trestlů se zakázanými podgrafy.

Práce obsahuje nové výsledky související s touto problematikou. Konkrétně jsou to chordální grafy a chordální rovinné grafy s relativně velkou tuhostí, které nemají $r$-trestle. Tyto grafy dávají horní odhad pro shortness exponent třídy 1-tuhých chordálních rovinných grafů. Tento výsledek vylepšuje horní odhad publikovaný v článku [13]. Ukážeme také dvojici zakázaných podgrafů, která zajistí, že daný graf má $r$-trestle.



# Klíčová slova

hamiltonovská kružnice, $r$-trestle, tuhost, chordální grafy, rovinné grafy, shortness exponent, zakázané podgrafy

# Contents

Chapter 1

# Introduction

## 1.1 Goals of the thesis

The thesis deals with hamiltonian properties of a graph, mainly a hamiltonian cycle and an $r$-trestle. We show how these properties cohere with toughness and connectivity of a graph. We also mention the relations with forbidden subgraphs and the relations with some other generalizations of hamiltonicity.

The two main goals of the thesis are

- to depict in detail the relations among toughness of a graph and its hamiltonian properties.
- to present original results on this topic.

We cite results dealing with these topics and discuss them to make a detail picture of the issue. In order to provide a closer look into the issue we state a number of propositions which help to understand the used terms and the relations among the cited results.

In Chapter 1 we show several ways of generalizing hamiltonian cycle and the relations among them. In Chapter 2 we mention some major results dealing with hamiltonian graphs and graphs with an $r$-trestle. Also some results considering toughness and connectivity of a graph are shown. Chapter 3 is the main chapter of the thesis. We consider general graphs, chordal graphs and planar graphs and show the relationship among hamiltonian properties and toughness and connectivity in these classes of graphs. Finally, in Chapter 4 some results dealing with forbidden graphs and hamiltonicity are mentioned.

As the original results we present chordal graphs and chordal planar graphs with high toughness and no $r$-trestle. We also present a sufficient condition on forbidden subgraphs for a graph to have an $r$-trestle. In particular, Bauer, Broersma and Veldman in [5] constucted nonhamiltonian graphs with toughness arbitrary close to $\frac{9}{4}$ and nonhamiltonian chordal graphs with toughness arbitrary close to $\frac{7}{4}$. In [18] Teska and Kužel obtained graphs with toughness greater than 1 and no $r$-trestle. In Theorem 3.13 we present chordal graphs with toughness greater than 1 and no $r$-trestle.

Böhme, Harant and Tkáč in [13] showed that every chordal planar graph with toughness greater than 1 is hamiltonian. In order to show that there exists a nonhamiltonian 1-tough chordal planar graph they also showed in [13] that the shortness exponent of the class of 1-tough chordal planar graphs is at most $\log_9 8$. In Theorem 3.27 we show there are 1-tough chordal planar graphs which do not even have any r-trestle. These graphs also improves the upper bound on the shortness exponent of the class of 1-tough chordal planar graphs, showing the shortness exponent of this class is at most $\frac{1}{2}$.

Gao in [25] showed that every 3-connected planar graph has a 6-trestle. As a corollary of this result every planar graph with toughness greater than 1 has a 6-trestle. By Theorem 3.27 the result in this corollary is the best possible.

Goodman and Hedetniemi in [30] showed that every 2-connected $(K_{1,3}, Z_1)$-free graph is hamiltonian. In Theorem 4.6 we present an extension of this result, showing every 2-connected $(K_{1,r}, Z_1)$-free graph has an $(r-1)$-trestle.

## 1.2 Basic definitions

All graphs considered in this thesis are finite undirected graphs with neither loops nor multiple edges.

A *graph* is a pair $G = (V(G), E(G))$ consisting of a nonempty finite set $V(G)$ and a set $E(G)$ which is a set of different 2-element subsets of $V(G)$.

We say the elements of $V(G)$ are *vertices* and the elements of $E(G)$ are *edges* of the graph $G$, $|V(G)|$ and $|E(G)|$ denote the *number of vertices* and the *number of edges*. Let $u, v \in V(G)$ and let an edge be a 2-element subset containg $u$ and $v$, we denote the edge $uv \in E(G)$, we say vertices $u, v$ are *adjacent* in the graph $G$. We denote $N(v)$ the set of all vertices adjacent to vertex $v$. We say the integer $|N(v)|$ is a *degree* of vertex $v$ denoted $d(v)$. Let $v$ be a vertex such that $d(v) = |V(G)| - 1$ we say $v$ is an universal vertex of the graph $G$. The integer $\delta(G)$ is the *minimum degree* of the graph $G$, $\Delta(G)$ is the *maximum degree* of $G$

$$\delta(G) = \min_{v \in V(G)} (d(v)),$$

$$\Delta(G) = \max_{v \in V(G)} (d(v)).$$

Let G $= (V(G), E(G))$ and $G_0 = (V(G_0), E(G_0))$ be two graphs. If $V(G_0) \subset V(G)$ and $E(G_0) \subset E(G)$, we say the graph $G_0$ is a *subgraph* of the graph $G$ or the graph $G$ *contains* the graph $G_0$ or just $G$ has $G_0$. If $G_0$ is a subgraph of $G$ such that an edge $uv \in E(G_0)$ if and only if $uv \in E(G)$ we say $G_0$ is an *induced subgraph* of $G$, we say the subgraph $G_0$ is *induced by vertices* $V(G_0)$. If $G_0$ is a subgraph of $G$ and $V(G_0) = V(G)$, we say $G_0$ is a *spanning subgraph* of $G$.

Let G $= (V(G), E(G))$ and $G_0 = (V(G_0), E(G_0))$ be two graphs. Let $\varphi$ be a a bijection $\varphi : V(G) \to V(G_0)$ such that $uv \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(G_0)$. We say $\varphi$ is an *isomorphism*, the graphs $G$ and $G_0$ are *isomorphic*, denoted $G = G_0$. Let $\mathbb{F}$ be a class of graphs $\mathbb{F} = \{H_1, H_2, ..., H_k\}$. We say a graph G is $\mathbb{F}$-*free* or $(H_1, H_2, ..., H_k)$-*free* if the graph $G$ contains no induced subgraph isomorphic to any of the graphs $H_1, H_2, ..., H_k$. In particular, for $\mathbb{F} = \{H\}$ we say the graph $G$ is $H$-free. The graphs $H_1, H_2, ..., H_k$ are called *forbidden subgraphs*.

Let us mention the notation we will use. Let $G = (V(G), E(G))$ be a graph, by $G - uv$ we mean the graph $(V(G), E(G) - uv)$. Similary $G \cup uv = (V(G), E(G) \cup uv)$. Let $V_1$, $V_2$ be subsets of $V(G)$ and $G - V_1$ denotes the subgraph of $G$ induced by vertices $V(G) - V_1$, $V_1 \cup V_2$ denotes the subgraph of $G$ induced by vertices $V_1 \cup V_2$.

Let $G = (V(G), E(G))$ be a graph. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and let $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$, we say $G$ is a *path* from $v_1$ to $v_n$, denoted $v_1v_2...v_n$ or $P_n$. Let $G = (V(G), E(G))$ be a graph $|V(G)| > 2$. Let

$V(G) = \{v_1, v_2, ..., v_n\}$ and let $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$, we say $G$ is a *cycle*, denoted $C_n$, integer $n$ is the *length of the cycle* $C_n$.

Let $\Sigma$ be a class of graphs, denote $c(H)$ the length of the longest cycle in the graph $H$. The *shortness exponent* $\sigma(\Sigma)$ of the class of graphs $\Sigma$ is defined as follows.

$$\sigma(\Sigma) = \liminf_{H_n \subset \Sigma} \frac{\log c(H_n)}{\log |V(H_n)|},$$

where the lim inf is taken over all sequences of graphs $H_n \subset \Sigma$ such that $|V(H_n)| \to \infty$ as $n \to \infty$.

Let $G$ be a graph, such that for each pair $u, v \in V(G)$, $uv \in E(G)$, we say $G$ is a *complete graph*. We denote the complete graph $K_n$, $n = |V(G)|$. If a graph $G$ is not a complete graph, we say $G$ is a *non-complete graph*. Let $G$ be a graph and $G_0$ its subgraph such that $G_0$ is a complete graph, we say $G_0$ is a *clique*. Let $v$ be a vertex such that $N(v)$ is a clique, we say $v$ is a *simplicial vertex*.

Let $G_1$ be a graph, we say the graph $G_1$ has a *perfect elimination ordering*, if there exists an ordering $(v_1, v_2, ..., v_{|V(G_1)|})$ of vertices of the graph $G_1$ such that $v_1$ is a simplicial vertex of the graph $G_1$, and for $i = 2, 3, ..., |V(G_1)|$ denote the graph $G_i = G_{i-1} - v_{i-1}$ and $v_i$ is a simplicial vertex of the graph $G_i$.

Let $G$ be a graph. If there exists a partition of $V(G)$ into two subsets $V_1$ and $V_2$ such that $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ and if $E(G) \subset V_1 \times V_2$, then we say $G$ is a *bipartite graph*. We say the sets $V_1$ and $V_2$ are *partities* of the graph $G$. A bipartite graph $G$ such that $E(G) = V_1 \times V_2$ is called a *complete bipartite graph*. We denote $K_{m,n}$ the complete bipartite graph with $|V_1| = n$ and $|V_2| = m$. We denote $S(K_{1,r})$ a subdivision of the graph $K_{1,r}$, formed by inserting a vertex of degree 2 on each edge of $K_{1,r}$.

We say that a graph $G$ is *connected* if for every pair of vertices $u$ and $v$, $G$ contains a path from $u$ to $v$. We say that a graph $G$ is *k-connected* if for every pair of vertices $u$ and $v$, $G$ contains a $k$ paths from $u$ to $v$, such that no two of these paths contain a common edge. The maximal integer $\kappa(G)$ such that a graph $G$ is $\kappa(G)$-connected is called the *connectivity* of the graph $G$. We say that a graph $G$ is *locally connected* if for every $v \in V(G)$, $N(v)$ is a connected graph. If $G$ is a connected graph and a subgraph $G - uv$ is not a connected graph, we say the edge $uv$ is a *bridge*. A maximal connected subgraph of the graph $G$ is called a *component* of

G. Denote $\omega(G)$ the *number of components* of the graph $G$. Let $G$ be a non-complete graph and let

$$\tau(G) = \min_{S \subset V(G)} (\frac{|S|}{\omega(G-S)}),$$

such that $\omega(G-S) \geq 2$. For a complete graph $K_n$ let $\tau(K_n) = \infty$. The number $\tau(G)$ is the *toughness* of the graph $G$. For $t \leq \tau(G)$ we say the graph is *$t$-tough*.

A *tree* is a connected graph that contains no cycle. A *spanning tree* is a spanning subgraph which is a tree. A *walk* (of length k) in a graph $G$ is a non-empty alternating sequence $v_1 e_1 v_2 e_2 ... e_{k-1}$ such that for $i = 1, 2, ..., k-1$, $v_i, v_k \in V(G)$ and $e_i, \in E(G)$ and $e_i = v_i v_{i+1}$. If $v_1 = v_k$ we call this sequence a *closed walk*. A *spanning walk* is a walk that contains every vertex of the graph $G$. Let $k \geq 1$ be an integer, the complete graph $K_k$ is the smallest *$k$-tree*, and a graph $G$ is a *$k$-tree* if and only if it contains a simplicial vertex $v$ with degree $k$ such that $G - v$ is a *$k$-tree*. Clearly, 1-trees are just trees.

Let $G = (V(G), E(G))$ be a graph and let $F$ be a set of different 2-element subsets of $V(G)$ such that $uv \in F$ if and only if there exists a vertex $x \in V(G)$ such that $u \in N(x)$ and $v \in N(x)$. We say the graph $G^2 = (V(G), E(G) \cup F)$ is the *square of the graph $G$*. We say the graph $G$ is a *$k$-regular graph* if $d(v) = k$ for every $v \in V(G)$. Let $G$ be a graph, such that if an induced subgraph $S$ of $G$ is a cycle, then $S = C_3$, we say $G$ is a *chordal graph*. Let $\{I_1, I_2, ..., I_n\}$ be the set of real intervals. Let $G$ be a graph, $V(G) = \{v_1, v_2, ..., v_n\}$ such that $v_i v_j \in E(G)$ if and only if $I_i \ I_j \neq \emptyset$, $i \neq j$, we say $G$ is an *interval graph*. Let $G$ be a graph, if $G$ can be drawn on the plane in such a way that no two edges meet in a point other than a common vertex, then we say $G$ is a *planar graph*. Let $G$ be a planar graph, we say $G$ is a *maximal planar graph* if for every pair $u, v \in V(G)$ such that $uv \notin E(G)$ the graph $G_0 = (V(G), E(G) \cup uv)$ is not planar.

We define only terms that are further used in this thesis. More definitions can be found in [1].

## 1.3 Generalizing hamiltonicity

There are several ways of generalizing hamiltonian cycle. In this section we define some of them, show how they generalize the hamiltonian cycle and show the relations among them.

### 1.3.1 Various generalizations

A *hamiltonian cycle* is a spanning subgraph which is a cycle, a *hamiltonian path* is a spanning subgraph which is a path. If a graph $G$ contains a hamiltonian cycle, we say $G$ has a hamiltonian cycle or $G$ is *hamiltonian.* If a graph $G$ contains a hamiltonian path, we say $G$ is *traceable.* Otherwise we say $G$ is *nonhamiltonian* or *nontraceable.* An *$r$-trestle* is a 2-connected spanning subgraph with maximum degree at most $r$. If a graph $G$ contains an $r$-trestle, we say $G$ has an $r$-trestle or $G$ is the graph with an $r$-trestle. A 2-trestle is exactly a hamiltonian cycle. Refering to the fact whether a graph has a hamiltonian cycle or an $r$-trestle we will also use the words *hamiltonicity* and *hamiltonian properties.*

A $k$-factor is a $k$-regular spanning subgraph. A hamiltonian cycle is a special case of a 2-factor, such that the 2-factor consists of 1 component. A *$k$-walk* is a closed spanning walk which enters every vertex of a graph at most $k$ times. A 1-walk is exactly a hamiltonian cycle. A *spanning $k$-tree* is a spanning tree with maximum degree at most $k$. The spanning 2-tree is exactly the hamiltonian path. Notice that in this section spanning $k$-tree always means a spanning tree with maximum degree at most $k$. In Chapter 3 we will also mention k-trees. Those $k$-trees in Chapter 3 are entirely different graphs from the spanning $k$-trees mentioned here.

### 1.3.2 Relations among generalizations

Let $G$ be a graph which has an $r$-trestle $T$. By definition the graph $T$ is a 2-connected spanning subgraph of the graph $G$ and $\Delta(T) \leq r$. Clearly, $\Delta(T) \leq r+1$, so the graph $T$ is also an $(r+1)$-trestle of the graph $G$.

**Proposition 1.1** *Let $G$ be a graph which has an $r$-trestle. Then the graph $G$ has an $(r+1)$-trestle.*

Or less formally, by Proposition 1.1 we have the following chain of implications.

2-trestle (hamiltonian cycle) $\Rightarrow$ 3-trestle $\Rightarrow$ 4-trestle $\Rightarrow$ ...

Similary, if a graph has a $k$-walk it also has a $(k+1)$-walk, if a graph contains a spanning $k$-tree it contains a spanning $(k+1)$-tree as well. Jackson and Wormald in [32] showed the following relation between a $k$-walk and spanning $k$-tree.

**Theorem 1.2 [32]**
*(1) If $G$ contains a spanning $k$-tree, then $G$ has a $k$-walk.*
*(2) If $G$ has a $k$-walk, then $G$ contains a spanning $(k+1)$-tree.*

1-walk (hamiltonian cycle) $\Rightarrow$ spanning 2-tree (hamiltonian path) $\Rightarrow$
$\Rightarrow$ 2-walk $\Rightarrow$ spanning 3-tree $\Rightarrow$ 3-walk $\Rightarrow$ ...

The following result was performed by Kaiser, Kužel, Li and Wang in [33].

**Theorem 1.3 [33]** *Let $G$ be a graph such that $G$ contains no bridge. Then $G$ has a $\lceil \frac{\Delta(G)+1}{2} \rceil$-walk.*

Let $G$ be a graph which has an $r$-trestle $T$. The graph $T$ is 2-connected, so it contains no bridge. By Theorem 1.3 the graph $T$ has an $\lceil \frac{r+1}{2} \rceil$-walk. The graph $T$ is a spanning subgraph of the graph $G$, so the graph $G$ has an $\lceil \frac{r+1}{2} \rceil$-walk.

$r$-trestle $\Rightarrow$ $\lceil \frac{r+1}{2} \rceil$-walk

Jackson and Wormald in [32] also showed a necessary condition on toughness for a graph to have a $k$-walk.

**Theorem 1.4 [32]** *If $G$ has a $k$-walk, then $G$ is $\frac{1}{k}$-tough*

Notice that every graph with a spanning $k$-tree by Theorem 1.2 has a $k$-walk, so the necessary condition in Theorem 1.4 holds for a spanning $k$-tree. A similar necessary condition for an $r$-trestle is mentioned in Section 3.1. The following result was obtained by Win in [35]

**Theorem 1.5 [35]** *If $G$ is connected, $k \geq 2$ and for any subset $S$ of $V(G)$, $\omega(G - S) \leq (k-2)|S| + 2$, then $G$ has a spanning $k$-tree.*

For $k \geq 3$ let $G$ be a $\frac{1}{k-2}$-tough graph, so $G$ is a connected graph. By definition of toughness $\omega(G-S) \leq (k-2)|S|$ for every $S \subset V(G)$ such that $\omega(G-S) \geq 2$. Hence $\omega(G-S) \leq (k-2)|S| + 2$, for every $S \subset V(G)$. By putting together Theorems 1.5 and 1.2 the following corollary is obtained.

**Corollary 1.6** *For $k \geq 3$ let $G$ be a $\frac{1}{k-2}$-tough graph, then the graph $G$ has a spanning $k$-tree and a $k$-walk.*

Ellingham and Zha in [21] showed the following.

**Theorem 1.7 [21]** *Every 4-tough graph has a 2-walk.*

In this thesis we focus on hamiltonian cycles and $r$-trestles, we will also see a couple of results considering a 2-factor. Since toughness ensures some properties similar to hamiltonicity, it is reasonable to ask whether certain toughness of a graph ensures the graph has a hamiltonian cycle, or whether for given $r$ the graph has an $r$-trestle. For general graphs the answer is unknown. Yet some relations among toughness hamiltonian cycle and $r$-trestle are known, as we will see further on in the thesis. First of all, let us have a look on toughness and hamiltonicity separately.

# Chapter 2

# Hamiltonicity and toughness

## 2.1 Hamiltonian cycle and r-trestle

In Section 1.3 we showed various ways to generalize hamiltonicity. From now on in the thesis we focus on the hamiltonian cycle and the $r$-trestle. First of all, let us mention some of the major results dealing with hamiltonian graphs and graphs with an $r$-trestle. These presented results show that certian conditions on the degree of vertices of a graph ensure the graph is hamiltonian, or ensure the graph has an $r$-trestle. It is also known that the problem whether a graph is hamiltonian and the problem whether a graph has an $r$-trestle are both NP-complete. The NP-complexity is sketched in Subsection 2.1.2. Now let's show that sufficiently high degree of vertices implies hamiltonicity.

### 2.1.1 Degree conditions

We start with a condition on the minimal degree, the following well-known result was showed by Dirac in [9].

**Theorem 2.1 [9]** *Let $G$ be a graph such that $|V(G)| \geq 3$ and $\delta(G) \geq \frac{|V(G)|}{2}$, then $G$ is hamiltonian.*

The result in Theorem 2.1 was improved by Ore. In fact, Dirac's Theorem 2.1 is a corollary of the following Ore's Theorem 2.2.

**Theorem 2.2 [4]** *Let $G$ be a graph such that $|V(G)| \geq 3$ and such that for all pairs of distinct nonadjacent vertices $x$ and $y$, $d(x)+d(y) \geq |V(G)|$, then $G$ is hamiltonian.*

Further studies led to the result performed by Bondy and Chvátal in [10]. It further extends Ore's Theorem 2.2.

**Theorem 2.3 [10]** *Let $x$ and $y$ be distinct nonadjacent vertices of a graph $G$ such that $d(x) + d(y) \geq |V(G)|$. Then $G + xy$ is hamiltonian if and only if $G$ is hamiltonian.*

Considering an $r$-trestle and a minimum degree condition, the result in Theorem 2.4 was obtained by Jendroľ, Kaiser, Ryjáček and Schiermeyer in [14]. It is an extension of Dirac's Theorem 2.1.

**Theorem 2.4 [14]** *Let $G$ be a 2-connected graph such that $\delta(G) \geq \frac{2|V(G)|}{r+2}$, then $G$ has an $r$-trestle.*

More results related with hamiltonicity can be found in Chapters 3 and 4, in particular in Chapter 3 results considering the relations between toughness and hamiltonicity in general graphs and in special classes of graphs, in Chapter 4 results that show come conditions on forbidden subgraphs ensure hamiltonicity. Now let's have a look at the NP-complexity.

### 2.1.2 NP-complexity

In the following paragraphs we will briefly mention the NP-complexity, proper definitions and more on this topic can be found in [2].

Consider the following decision problem. Let's have a graph $G$ and we should decide whether the graph $G$ is hamiltonian or not. In fact to show the problem is NP-complete it is enough to consider 3-connected 3-regular planar graphs instead of general graphs.

**Planar hamiltonian cycle problem**
INSTANCE: A 3-connected 3-regular planar graph $G$.
QUESTION: Is the graph $G$ hamiltonian?

In [23] Garey, Johnson and Tarjan showed that the planar hamiltonian cycle problem is NP-complete.

**Theorem 2.5 [23]** *The planar hamiltonian cycle problem is NP-complete.*

However every 3-connected planar graph has a 6-trestle and every 4-connected planar graph is hamiltonian. The hamiltonicity of planar graphs is depicted in detail in Section 3.4. Let's have a similar decision problem, but this time consider a 3-connected graph $G$ which does not have to be planar and an r-trestle.

**r-trestle problem**
INSTANCE: A 3-connected graph G and an integer $r \geq 3$.
QUESTION: Does the graph $G$ have an r-trestle?

Teska and Kužel showed that the r-trestle problem is NP-complete, the result can be found in [18].

**Theorem 2.6 [18]** *The r-trestle problem is NP-complete.*

A similar decision problems considering the connectivity and toughness of a graph are mentioned in the next Section. The problem to decide for given t whether a given graph is t-tough is NP-hard.

## 2.2 Connectivity and toughness

First of all, let us remind the definiton of *toughness*. Denote $\omega(G)$ the number of components of the graph $G$. Let $G$ be a non-complete graph and let

$$\tau(G) = \min_{S \subset V(G)} (\frac{|S|}{\omega(G-S)}),$$

such that $\omega(G-S) \geq 2$. For a complete graph $K_n$ let $\tau(K_n) = \infty$. $\tau(G)$ is the *toughness* of the graph $G$. For $t \leq \tau(G)$ we say the graph is *t-tough*.

Notice for a graph $G$ which is not connected $S = \emptyset$, $\tau(G) = 0$, so the graph $G$ is 0-tough. An example of a non-complete graph with high toughness is the graph $K_n^-$ obtained from the complete graph $K_n$ by leaving 1 edge. The graph $K_n^-$ is $\frac{n-2}{2}$-tough and $(n-2)$-connected.

### 2.2.1 Complexity

In the previous section the NP-complexity of the problems whether a graph is hamiltonian and whether a graph has an $r$-trestle were mentioned. In this section we consider the following decision problems. Let's have a graph $G$ and an integer $k$ and we should decide whether the graph $G$ is $k$-connected. Similary, we have a graph $G$ and a positive rational number $t$ and we should decide whether the graph $G$ is $t$-tough. Whereas the $k$-connected problem can be solved in polynomial time, the $t$-tough problem is NP-hard.

**t-tough problem**
INSTANCE: A graph $G$ and a positive rational number $t$
QUESTION: Is the graph $G$ $t$-tough?

The following result was shown by Bauer, Hakimi and Schmeichel in [26].

**Theorem 2.7 [26]** *The $t$-tough problem is NP-hard.*

### 2.2.2 Toughness and connectivity in relation

The toughness of a graph provides certain connectivity of a graph. On the other hand there are graphs with high connectivity and arbitrary low toughness. The relation between toughness and connectivity is depicted in the following Propositions 2.8 and 2.9.

Consider a non-complete $t$-tough graph $G$, by definition $|S| \geq \omega(G - S) \cdot t$ for every $S \subset V(G)$ such that $\omega(G - S) \geq 2$. So in particular $|S| \geq \omega(G - S) \cdot t \geq 2t$, hence to obtain at least 2 components at least $k$ vertices $k \geq 2t$ have to be left from the graph $G$.

**Proposition 2.8** *Every non-complete $t$-tough graph is $k$-connected, $k \geq 2t$.*

On the other hand, consider the complete bipartite graph $K_{k,nk+1}$. By leaving fewer then k vertices from the graph $K_{k,nk+1}$ the obtained graph remains connected, hence the graph $K_{k,nk+1}$ is $k$-connected. By leaving all $k$ verticies of the smaller partity of the graph $K_{k,nk+1}$ each of the remained

$nk+1$ vertices is isolated. The obtained graph has $nk+1$ components, hence the graph $K_{k,nk+1}$ is not $\frac{1}{n}$-tough.

**Proposition 2.9** *For every integer $k \geq 1$ and for every number $t > 0$ there exists a $k$-connected graph which is not $t$-tough.*

# Chapter 3

# Toughness related to hamiltonian properties

By definition the toughness of a complete graph is $\infty$. Clearly, the graphs $K_1, K_2$ are not hamiltonian and the graphs $K_n$ for $n \geq 3$ are hamiltonian. In Chapter 3 we consider only non-complete graphs. So everytime we say a graph we mean a non-complete graph.

As it was mentioned in Section 1.3, certian toughness of a graph ensures the graph has a $k$-walk or the graph has a spanning $k$-tree. In this chapter we will see certain toughness also ensures the graph has a $k$-factor. These spanning subgraphs are somewhat similar to a hamiltonian cycle. The straightaway question is: Does toughness ensure hamiltonicity? In this chapter we will see it does for some special classes of graphs. However, for general graphs it is not known.

## 3.1 Hamiltonicity implies toughness

Let's have a graph $G$ and its spanning subgraph $S$. Notice that if $S$ is t-tough then $G$ is also t-tough. Clearly, every cycle is 1-tough, so a graph which contains a spanning cycle is 1-tough. Therefore to be 1-tough is the necessary condition for a graph to be hamiltonian. (Or recall the hamiltonian cycle is the 1-walk and consider Theorem 1.4.) Later on, we will refer to this simple claim, therefore let us state it formally.

**Proposition 3.1** *Every hamiltonian graph is* 1*-tough.*

An extention of Proposition 3.1 was performed by Tkáč and Voss in [6]. It shows the necessary condition on toughness for a graph to have an $r$-trestle.

**Theorem 3.2 [6]** *Let $r$ be an integer, $r \geq 2$. Every graph with an $r$-trestle is $\frac{2}{r}$-tough.*

However, not every 1-tough graph is hamiltonian. For example, consider Petersen's graph. It is not hard to see Petersen's graph is $\frac{4}{3}$-tough and it is not hamiltonian. Similary not every $\frac{2}{r}$-tough graph has an $r$-trestle. In fact, for $r \geq 4$ and $2 \leq k \leq \frac{r}{2}$ the graph $K_{1,k}$ is clearly $\frac{2}{r}$-tough and is not even 2-connected, hence it has no $r$-trestle. We will see more examples of graphs which are not hamiltonian and graphs with no $r$-trestle further on in the thesis.

## 3.2 Toughness implies hamiltonicity

First of all, notice that high connectivity of a graph does not ensure the graph is hamiltonian. In other words, there are graphs with high connectivity which contain no hamiltonian cycle, there also are such graphs with no $r$-trestle. For example, consider a complete bipartite graph $K_{k,nk+1}$. As it was mentioned in the paragraph linked to Proposition 2.9 the graph $K_{k,nk+1}$ is k-connected and it is not $\frac{1}{n}$-tough. Hence by Proposition 3.1 it is not hamiltonian for $n \geq 1$, also by Theorem 3.2 it has no $r$-trestle for $n \geq \frac{r}{2}$.

In the previous paragraph we saw that the graphs with high connectivity are not necessarily hamiltonian, also by Propositions 2.8 and 2.9 we know toughness is a stronger property than connectivity and in Section 1.3 we saw the toughness ensures some properties similar to hamiltonicity. So does the toughness ensure hamiltonicity? In [7] Chvátal conjectured the following.

**Conjecture 3.3 [7]** *There is a finite constant $t$ such that every $t$-tough graph is hamiltonian.*

There has been a lot of reserch on this topic. Despite of that it is still unknown whether the Conjecture 3.3 is true. Moreover it is not known whether there are any finite constants $t$ and $r$ such that every t-tough graph has an $r$-trestle. The following Conjecture 3.4 was stated by Tkáč and Voss in [6].

**Conjecture 3.4 [6]** *For every integer $r \geq 2$, there is a finite constant $t$ such that every $t$-tough graph has an $r$-trestle.*

Yet it was shown by Enomoto, Jackson, Katerinis and Saito in [16] that certain toughness of a graph ensures the graph has a k-factor.

**Theorem 3.5 [16]** *Let $G$ be a $k$-tough graph with $|V(G)| \geq k+1$ and $|V(G)|k$ even. Then $G$ has a $k$-factor.*

Theorem 3.6 also performed in [16] shows the result in Theorem 3.5 is the best possible.

**Theorem 3.6 [16]** *Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k-\epsilon)$-tough graph $G$ with $|V(G)| \geq k+1$ and $|V(G)|k$ even which has no $k$-factor.*

A hamiltonian cycle itself is a 2-factor of a graph. Due to Theorem 3.5 a 2-tough graph has a 2-factor and by Theorem 3.6 there exists a graph with toughness arbitrary close to 2 which has no 2-factor.

It was believed 2 might be the value of toughness to ensure hamiltonicity of a graph. In other words, Chvátal's Conjecture 3.3 was specified to the following Conjecture 3.7.

**Conjecture 3.7** *Every 2-tough graph is hamiltonian.*

Anyway, the Conjecture 3.7 was disproved. In [5] Bauer, Broersma and Veldman constructed graphs with toughness arbitrary close to $\frac{9}{4}$ which are nontraceable, hence not hamiltonian.

**Theorem 3.8 [5]** *For every $\epsilon > 0$ there exists a $(\frac{9}{4}-\epsilon)$-tough nontraceable graph.*

Also in [5] nontraceble chordal graphs with toughness arbitrary close to $\frac{7}{4}$ were constructed. In [18] Teska and Kužel obtained graphs with toughness greater than 1 and no r-trestle. In fact, all these graphs are similar, therefore let us sketch the construction of the nontraceable graphs with toughness arbitrary close to $\frac{9}{4}$.

Let $H$ be the graph in Figure 3.1 and x, y its vertices. We say

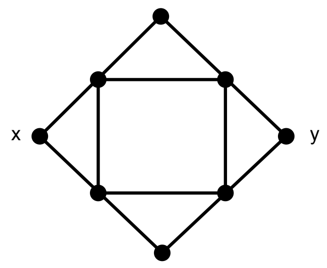


Figure 3.1: building block $H$

the graph $H$ is a building block. For integers $m$ and $l \geq 2$ define the graph $G(H,x,y,l,m)$ as follows. Take $m$ disjoint copies of the graph $H$ denoted $H_1, H_2, ..., H_m$, the vertices $x_i$, $y_i$ are the appropriate vertices of $H_i$. Let $F_m$ be the graph obtained from the disjoint union $H_1 \cup H_2 \cup ... \cup H_m$ by adding all edges such that the graph induced by vertices $\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_m\}$ is a clique $K_{2m}$. Let $T$ be a clique $K_l$ and let $G(H,x,y,l,m)$ be the graph obtained from the disjoint union $T \cup F_m$ by adding all edges $uv$ such that $u \in T$, $v \in F_m$. The graph $G(H,x,y,l,m)$ is $\frac{l+4m}{2m+1}$-tough and for $m \geq 2l+3$ it is nontraceable. The graph $G(H,x,y,l,m)$ is sketched in Figure 3.2.

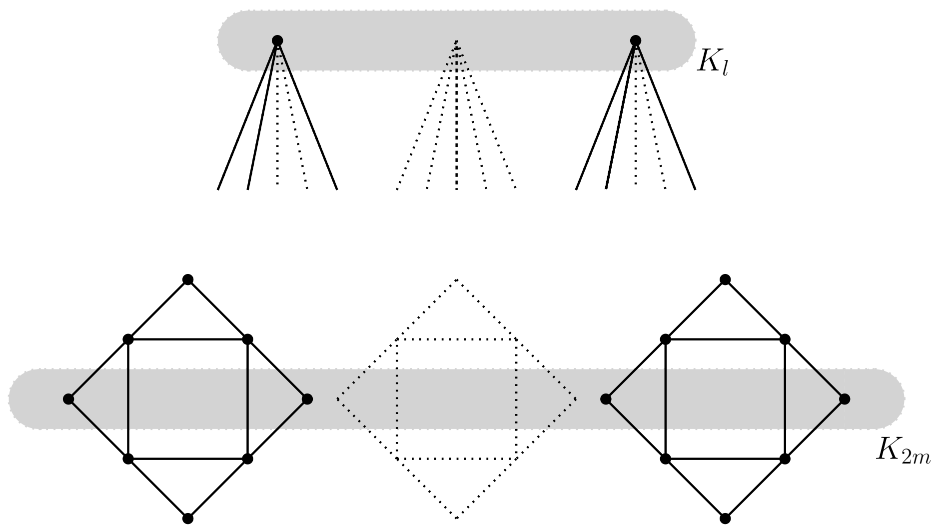


Figure 3.2: the graph $G(H,x,y,l,m)$

Using the same construction, adapting the relation between $m$ and $l$

and using the graph $L^r$ from Figure 3.3 instead of the graph $H$ Teska and Kužel obtained graphs with toughness greater than 1 with no $r$-trestle. The result can be found in [18]

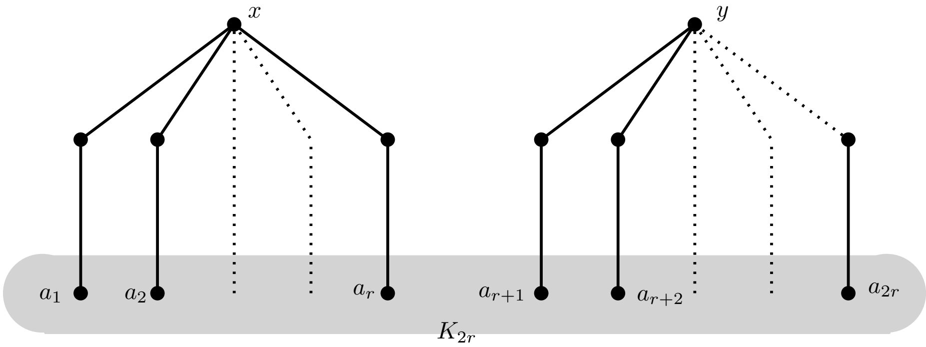


Figure 3.3: building block $L^r$

**Theorem 3.9 [18]** *For every $\epsilon > 0$ and for every integer $r \geq 3$ there exists a $(\frac{r^2+r+1}{r^2} - \epsilon)$-tough graph having no $r$-trestle.*

Although it is not known whether Chvátal's Conjecture 3.3 is true for general graphs, it holds for some special classes of graphs. For example, by [24] it holds for interval graphs, by [3] for chordal graphs, by [12] for planar graphs. In particular it was shown in [3] that every 18-tough chordal graph is hamiltonian. Also by putting together the result from [12] with Proposition 2.8 every planar graph with toughness greater then $\frac{3}{2}$ is hamiltonian. We now focus on showing the relations among hamiltonicity, generalized hamiltonicity and connectivity and toughness in special classes of graphs, namely chordal graphs, k-trees and planar graphs.

## 3.3 Chordal graphs

Let us start this section with a characterization of chordal graphs. Let $G_1$ be a graph, we say the graph $G_1$ has a *perfect elimination ordering*, if there exists an ordering $(v_1, v_2, ..., v_{|V(G_1)|})$ of vertices of the graph $G_1$ such that $v_1$ is a simplicial vertex of the graph $G_1$, and for $i = 2, 3, ..., |V(G_1)|$ denote the graph $G_i = G_{i-1} - v_{i-1}$ and $v_i$ is a simplicial vertex of the graph $G_i$.

Fulkerson and Gross in [34] showed the following characterization of chordal graphs. We will use this characterization further on in the thesis.

**Theorem 3.10 [34]** *Let $G$ be a graph. The graph $G$ is chordal if and only if the graph $G$ has a perfect elimination ordering.*

Let $X_{k,nk+1}$ be the graph obtained from the graph $K_{k,nk+1}$ by adding all $\frac{k(k-1)}{2}$ edges among all pairs of the $k$ vertices of the smaller partity. So the graph $X_{k,nk+1}$ consists of a clique $K_k$ and $nk+1$ simplicial verices each of them has degree $k$ and each of them is adjacent to each vertex of the clique $K_k$. It is easy to see $X_{k,nk+1}$ is a chordal graph. In Subsection 3.3.1 we will see the graph $X_{k,nk+1}$ in fact is a $k$-tree. By the argument from the paragraph linked to Proposition 2.9 the graph $X_{k,nk+1}$ is $k$-connected and is not $\frac{1}{n}$-tough, hence by Proposition 3.1 it is not hamiltonian for $n \geq 1$ and by Theorem 3.2 it has no $r$-trestle for $n \geq \frac{r}{2}$. So for chordal graphs connectivity does not ensure hamiltonicity, on the other hand toughness does.

As we mentioned above, Chvátal's Conjecture 3.3 holds for chordal graphs. In [3] Chen, Jacobson, Kezdy and Lehel showed that an 18-tough chordal graph has a hamiltonian cycle.

**Theorem 3.11 [3]** *Every 18-tough chordal graph is hamiltonian.*

For chordal graphs 18 is the best known value of toughness so far. In [11] it was shown by Bauer, Katona, Kratsch, Veldman that every $\frac{3}{2}$-tough chordal graph has a 2-factor. In [7] Chvátal performed $(\frac{3}{2}-\epsilon)$-tough chordal graphs with no 2-factor.

Anyway, not every $\frac{3}{2}$-tough chordal graph is hamiltonian. In [5] Bauer, Broersma and Veldman obtained nontraceable chordal graphs with toughness arbitrary close to $\frac{7}{4}$ by the construction already mentioned in the paragraph linked to Theorem 3.8 using the graph $M$ from Figure 3.4 instead of the graph $H$.

**Theorem 3.12 [5]** *For every $\epsilon > 0$ there exists a $(\frac{7}{4} - \epsilon)$-tough chordal nontraceable graph.*

The results in Theorems 3.11 and 3.12 are the best known for chordal graphs so far, hence it is not known whether the Conjecture 3.7 holds for

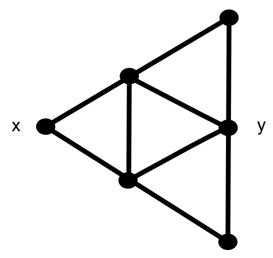


Figure 3.4: building block $M$

chordal graphs.

We now perform a similar result to Theorem 3.12 considering chordal graphs with no r-trestle. By Theorem 3.9 there are graphs with toughness greater than 1 and no $r$-trestle. In Theorem 3.13 we will show there also are chordal graphs with toughness greater than 1 having no $r$-trestle, for every $r \geq 3$.

Notice that the graph $L^r$ in Figure 3.3 is itself a 1-tough graph. In fact, by altering the graph $L^r$ graphs with toughness greater than 1 with no $r$-trestle can be obtained, without using the construction from [5]. Moreover by further altering of the graph, as we will see in Subsection 3.4.2, 1-tough chordal planar graphs with no r-trestle are obtained. Those obtained graphs also provide an upper bound on the shortness exponent of the class of 1-tough chordal planar graphs.

**Theorem 3.13** *For every integer $r \geq 3$ there exists a $\frac{2r+2}{2r+1}$-tough chordal graph having no $r$-trestle.*

*Proof.* Let $r$ be an integer $r \geq 3$, we construct a graph $G$ (see Figure 3.5) as follows. $|V(G)| = 2(2r+1)+2 = 4r+4$, let $K$, $L$, $M$ be subsets of $V(G)$ such that $K \cup L \cup M = V(G)$ and $|K| = 2r+1$, $|L| = 2r+1$, $|M| = 2$. Denote the vertices $K = \{k_1, k_2, ..., k_{2r+1}\}$, $L = \{l_1, l_2, ..., l_{2r+1}\}$, $M = \{m_1, m_2\}$, denote $d_G(v)$ the degree of vertex $v$ in the graph $G$. For $i = 1, 2, ..., 2r+1$, $d_G(k_i) = 2r+3$ and the vertex $k_i$ is adjacent to all other vertices of K and to the vertices $l_i$, $m_1$, $m_2$. In other words, the subgraph induced by $K$ is a clique $K_{2r+1}$, and $k_i l_i, k_i m_1, k_i m_2 \in E(G)$. For $i = 1, 2, ..., 2r+1$, $d_G(l_i) = 3$ and $l_i m_1, l_i m_2 \in E(G)$. Also $m_1 m_2 \in E(G)$. So clearly, $d_G(m_1) = d_G(m_2) = 4r+3$ and the vertices $m_1$, $m_2$ are universal vertices of the graph $G$.

We show the graph $G$ is $\frac{2r+2}{2r+1}$-tough. Let $S$ be a subset of $V(G)$ such that $\omega(G-S) \geq 2$. Denote $K_s = K \cap S$ and $L_s = L \cap S$. The vertices $m_1$, $m_2$ are universal vertices of the graph $G$ and $\omega(G-S) \geq 2$ so $m_1, m_2 \in S$. Denote the graph $R = (G - K_s - m_1 - m_2)$. Notice that for $|K_s| \leq 2r-1$, $\omega(R) = |K_s|$ and for $|K_s| \geq 2r$, $\omega(R) = 2r+1$. Also notice that $\omega(R - L_s) \leq \omega(R)$. So $\omega(G-S) \leq |K_s|$ for $|K_s| \leq 2r-1$ and $\omega(G-S) \leq 2r+1$ for $|K_s| \geq 2r$. Clearly $|S| = |K_s| + |L_s| + 2$. So all in all, $|S| \geq \omega(G-S)+1$, therefore $\frac{|S|}{\omega(G-S)} \geq \frac{\omega(G-S)+1}{\omega(G-S)}$, hence the graph $G$ is $\frac{2r+2}{2r+1}$-tough.

We show the graph $G$ is chordal. Notice that for $i = 1, 2, ..., 2r+1$, $d_G(l_i) = 3$ and $l_i k_i, l_i m_1, l_i m_2 \in E(G)$. A subgraph induced by vertices $\{l_i, k_i, m_1, m_2\}$ is a clique $K_4$. Also a subgraph induced by vertices $\{k_1, k_2, ..., k_{2r+1}, m_1, m_2\}$ is a clique $K_{2r+3}$. So the ordering

$$(l_1, l_2, ..., l_{2r+1}, m_1, m_2, k_1, k_2, ..., k_{2r+1})$$

is a perfect elimination ordering of the graph $G$. Hence by Theorem 3.10 $G$ is a chordal graph.

We show the graph $G$ has no $r$-trestle. $d_G(l_i) = 3$ and the vertex $l_i$ is adjacent to vertices $k_i$, $m_1$, $m_2$ and $|L| = 2r+1$. In every 2-connected subgraph $T$ of the graph $G$ either $d_T(m_1) \geq r+1$ or $d_T(m_2) \geq r+1$. So the graph $G$ has no $r$-trestle.

□

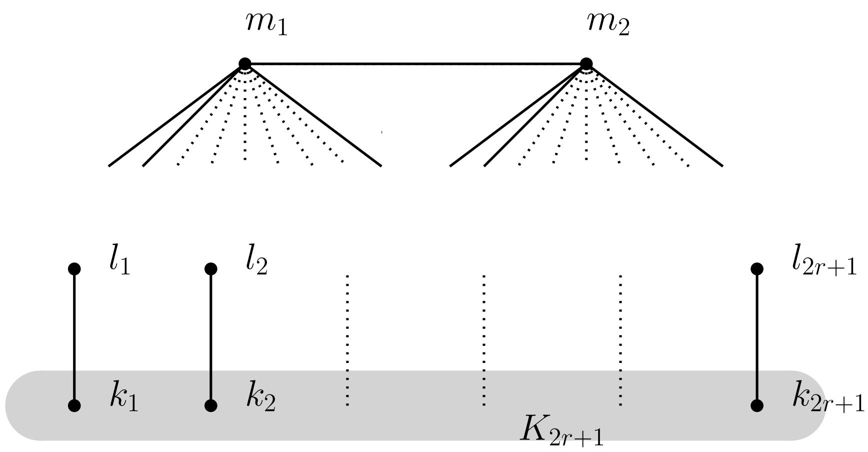


Figure 3.5: more than 1-tough chordal graph $G$ having no $r$-trestle

The graph $G$ in Figure 3.5 is also similar to the $(\frac{3}{2} - \epsilon)$-tough chordal graphs with no 2-factor from [7].

In the next subsection we make a mention of a subclass of chordal graphs called k-trees.

### 3.3.1 Briefly on k-trees

First of all we remind the definition of a *k-tree*. Let $k \geq 1$ be an integer, the complete graph $K_k$ is the smallest *k-tree*, and a graph $G$ is a *k-tree* if and only if it contains a simplicial vertex $v$ with degree $k$ such that $G - v$ is a *k-tree*. Clearly, 1-trees are just trees.

Notice that in this chapter a k-tree always means a graph as it is defined in the paragraph above. In Section 1.3 spanning k-trees were mentioned. Those spanning k-trees in Section 1.3 are entirely different graphs from the k-trees considered here.

Clearly, by definition every k-tree has a perfect elimination ordering so by Theorem 3.10 every k-tree is a chordal graph.

**Proposition 3.14** *Let $G$ be a k-tree, then $G$ is chordal graph.*

Recall the graph $X_{k,nk+1}$ mentioned at the beginning of Section 3.3. The graph $X_{k,nk+1}$ is a $k$-tree. So for every integer $k \geq 1$ and for every integer $r \geq 2$ there exists a $k$-tree with no $r$-trestle.

The following results were obtained by Broersmaa, Xiongc and Yoshimoto in [15].

**Theorem 3.15 [15]** *Let $G$ be a k-tree. Then $G$ is hamiltonian if and only if $G$ contains a spanning subgraph which is a 1-tough 2-tree.*

**Theorem 3.16 [15]** *Let $k \geq 2$ and let $G$ be a $\frac{k+1}{3}$-tough k-tree, then $G$ is hamiltonian.*

By Proposition 3.14 every k-tree is a chordal graph. Notice that for $k \geq 53$ a better value of toughness ensuring the k-tree is hamiltonian is obtained by Theorem 3.11.

## 3.4 Planar graphs

As it was mentioned above every planar graph with toughness greater than $\frac{3}{2}$ is hamiltonian, so Conjecture 3.7 holds for planar graphs. Moreover, unlike general graphs, certain connectivity of planar graphs also ensures hamiltonian properties.

Let us remind two well-known theorems on planar graphs, Kuratowski's theorem and Euler's formula. The definitions of a minor and a face of plane graph can be found in [1].

**Theorem 3.17 [36]** *Let $G$ be a graph. $G$ is planar if and only if $G$ contains neither $K_5$ nor $K_{3,3}$ as a minor.*

**Theorem 3.18** *Let $G$ be a connected planar graph with $f$ faces. Then $|V(G)| + f = |E(G)| + 2$.*

Let $\mathbb{P}$ be the class of planar graphs, by Theorem 3.17 for $n \geq 5$ the graph $K_n \notin \mathbb{P}$. Denote the class of graphs $\mathbb{P}_0 = \mathbb{P} - \{K_1, K_2, K_3, K_4\}$. As it was mentioned in the first paragraph of Chapter 3, we consider only non-complete graphs in Chapter 3. So in this section, when we say planar graph, we mean a graph from $\mathbb{P}_0$.

Let us show that, Euler's formula implies the planar graphs have bounded toughness. Notice each face is built by at least 3 edges also each edge appears in at most 2 faces, hence $f \leq \frac{2}{3}|E(G)|$. Using this inequality and the following relation $\sum\limits_{v \in V(G)} d(v) = 2|E(G)|$ Corollary 3.19 is obtained.

**Corollary 3.19** *Let $G$ be a connected planar graph. Then* $6|V(G)| - 12 \geq \sum\limits_{v \in V(G)} d(v)$.

Let $G$ be a planar graph. For $|V(G)| \leq 6$ the graph is $G$ at most 4-connected. By Corollary 3.19 the graph $G$ has $\delta(G) \leq 5$. For $|V(G)| \geq 7$ let $v$ be the vertex of $G$ such that $d(v) \leq 5$. Denote $N(v)$ the vertices adjacent to $v$, $|N(v)| \leq 5$. The graph $G - N(v)$ has at least 2 components. Therefore the graph $G$ is at most 5-connected, hence by Proposition 2.8 at most $\frac{5}{2}$-tough.

**Corollary 3.20** *Let $G$ be a planar graph and let $t > \frac{5}{2}$. The graph $G$ is* *not t-tough.*

Is there a $\frac{5}{2}$-tough planar graph? It is not hard to see the graph of icosahedron in Figure 3.6 is $\frac{5}{2}$-tough. Notice that every $\frac{5}{2}$-tough planar graph by Corollary 3.19 has at least 12 vertices. In fact, the graph of icosahedron has 12 vertices and it is a 5-regular graph, hence the graph of icosahedron is the smallest $\frac{5}{2}$-tough planar graph.

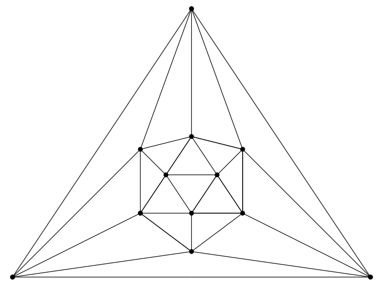

Figure 3.6: the graph of icosahedron

The graph of icosahedron is also used in [**?**] for the construction of a 3-connected planar graph with no 5-trestle, it will be mentioned further on in this section in Theorem 3.22. So unlike general graphs most of planar graphs have limited connectivity and therefore by Proposition 2.8 also limited toughness. Also unlike general graphs for planar graphs some connectivity ensures hamiltonian properties of a graph, let's see this in more detail.

### 3.4.1 4-connected, 3-connected, 2-connected

Now we show, that certain connectivity ensures hamiltonian properties of a planar graph. In fact, 4-connected, 3-connected and 2-connected planar graphs are 3 classes of graphs with different hamiltonian properties. The following result was shown by Tutte in [12].

**Theorem 3.21 [12]** *Every 4-connected planar graph is hamiltonian.*

Tutte's graph from Figure 3.7 is a 3-connected planar graph which is not hamiltonian. Although the Tutte's is a 3-regular graph, so clearly it has a 3-trestle.

In [8] Barnette showed a 3-connected planar graph with no 5-trestle. This graph was obtained from the graph of icosahedron (Figure 3.6) by

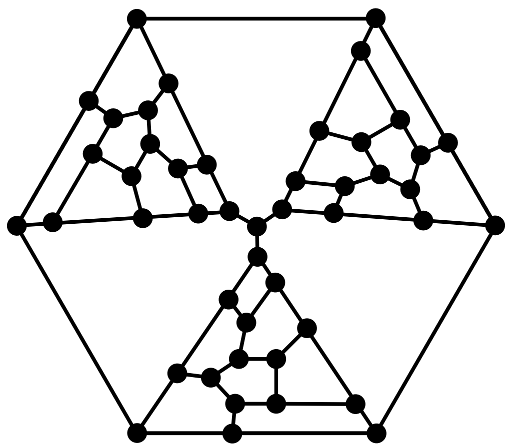

Figure 3.7: Tutte's graph

using the following construction.

For a 3-connected planar graph $G$ define a graph $K^1(G)$ as follows, place a new vertex inside each face of G, join each of these new vertices to each vertex of the face the new vertex lies in. Define a graph $K^n(G) = K^1(K^{n-1}(G))$. It is not hard to see $K^n(G)$ is a 3-connected planar graph.

**Theorem 3.22 [8]** *Let $I$ be the graph of icosahedron. The graph $K^3(I)$ has no 5-trestle.*

Notice $K^3(I)$ is a maximal planar graph. Gao in [25] showed every 3-connected planar graph has a 6-trestle and this result is by Theorem 3.22 the best possible, also for every $r \geq 2$ there exists a 2-connected planar graph with no $r$-trestle.

**Theorem 3.23 [25]** *Let $G$ be a 3-connected planar graph. Then $G$ has a 6-trestle.*

In fact in [25] Gao proved every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle. These items are not mentioned in the thesis, the definitions of projective plane, torus and Klein bottle can be found in [1].

As it was mentioned at the beginning of Section 3.3 for every $r \geq 2$ the graph $X_{2,r+1}$ in Figure 3.8 has no $r$-trestle. Notice $X_{2,r+1}$ is a 2-connected planar graph.

**Proposition 3.24** *For every integer $r \geq 2$ there exists a 2-connected planar graph with no $r$-trestle.*

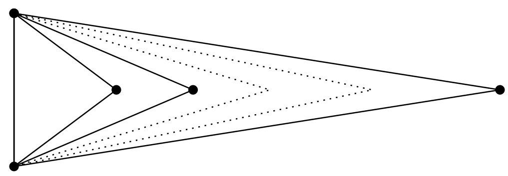

Figure 3.8: the graph $X_{2,r+1}$

Finally, notice that just based on what has been mentioned in the thesis so far it is clear that for planar graphs some connectivity implies toughness unlike for general graphs. In particular a 4-connected planar graph is by Theorem 3.21 hamiltonian, hence by Proposition 3.1 it is 1-tough. Similary a 3-connected planar graph by Theorem 3.23 has a 6-trestle, so by Thorem 3.2 it is $\frac{1}{3}$-tough.

Also notice toughness ensures connectivity by Proposition 2.8. Since some connectivity of planar graphs ensures hamiltonian properties, clearly some toughness of planar graph ensures hamiltonian properties. In particular every planar graph with toughness greater than $\frac{3}{2}$ is 4-connected by Proposition 2.8 and by Theorem 3.21 hamiltonian. So conjecture 3.7 holds for planar graphs. Let's have a closer look on the toughness and hamiltonicity of planar graphs.

### 3.4.2 Toughness, hamiltonicity, shortness exponent

As it was mentioned in Section 3.3, every 18-tough chordal graph is hamiltonian. For chordal planar graphs the value of toughness is lower. In [13] Böhme, Harant and Tkáč showed that every chordal planar graph with toughness greater than 1 has a hamiltonian cycle.

**Theorem 3.25 [13]** *For every $\epsilon > 0$, every $(1 + \epsilon)$-tough chordal planar graph is hamiltonian.*

Let us remind the definiton of the *shortness exponent* of a class of graphs. Let $\Sigma$ be a class of graphs, denote $c(H)$ the length of the longest cycle in the graph $H$. The *shortness exponent* $\sigma(\Sigma)$ of the class of graphs $\Sigma$ is defined as follows.

$$\sigma(\Sigma) = \liminf_{H_n \subset \Sigma} \frac{\log c(H_n)}{\log |V(H_n)|},$$

where the lim inf is taken over all sequences of graphs $H_n \subset \Sigma$ such that $|V(H_n)| \to \infty$ as $n \to \infty$.

It is also shown in [13] that the shortness exponent of the class of 1-tough chordal planar graphs is at most $\log_9 8$, hence there exists a 1-tough chordal planar graph which is not hamiltonian. So by Theorem 3.26 the result in Theorem 3.25 is the best possible.

**Theorem 3.26 [13]** *The shortness exponent of the class of all 1-tough chordal planar graphs is at most* $\log_9 8$.

Those 1-tough chordal planar nonhamiltonian graphs in [13] are all 3-connected, so by Theorem 3.23 they have a 6-trestle.

We perform a better upper bound on the shortness exponent of the class of 1-tough chordal planar graphs. By altering the graph from Theorem 3.13 we show the shortness exponent of the class of 1-tough chordal planar graphs is at most $\frac{1}{2}$. We also show, which might be more interesting in the context of this thesis, there exist 1-tough chordal planar graphs with no r-trestle.

**Theorem 3.27** *For every integer $r \geq 2$ there exists a 1-tough chordal planar graph having no $r$-trestle. Furthermore the shortness exponent of the class of 1-tough chordal planar graphs is at most $\frac{1}{2}$.*

*Proof.* Let $r$ be an integer $r \geq 2$, we construct a graph $H$ (see Figure 3.9 as follows. $|V(H)| = r + (r+1) + (r+1)^2 + 1 = r^2 + 4r + 3$, let $J$, $K$, $L$, $M$ be subsets of $V(H)$ such that $J \cup K \cup L \cup M = V(G)$ and $|J| = r$, $|K| = r+1$, $|L| = (r+1)^2$, $|M| = 1$. Let $L_1, L_2, ..., L_{r+1}$ be subsets of $L$ such that $L_1 \cup L_2 \cup ... \cup L_{r+1} = L$ and $|L_1| = |L_2| = ... = |L_{r+1}| = r+1$. Denote the vertices $J = \{j_1, j_2, ..., j_r\}$, $K = \{k_1, k_2, ..., k_{r+1}\}$, for $i = 1, 2, ..., r+1$, $L_i = \{l_{i,1}, l_{i,2}, ..., l_{i,r+1}\}$, $M = \{m\}$.

Denote $d_H(v)$ the degree of vertex $v$ in the graph $H$. $d_H(j_1) = d_H(j_r) = 4$, for $i = 2, 3, ..., r-1$, $d_H(j_i) = 5$. A subgraph induced by $J$ is a path $j_1j_2...j_r$ and for $i = 1, 2, ..., r$, $j_ik_i, j_ik_{i+1}, j_im \in E(H)$. $d_H(k_1) = d_H(k_{r+1}) = r + 4$, for $i = 2, 3, ..., r$, $d_H(k_i) = r + 6$. A subgraph induced by $K$ is a path $k_1k_2...k_{r+1}$ and for $i = 1, 2, ..., r+1$ and $j = 1, 2, ..., r+1$, $k_il_{i,j}, k_im \in E(H)$. For $i = 1, 2, ..., r+1$, $d_H(l_{i,1}) = d_H(l_{i,r+1}) = 3$. For $i = 1, 2, ..., r+1$ and $j = 2, 3, ..., r$, $d_H(l_{i,j}) = 4$. For $i = 1, 2, ..., r+1$ a subgraph induced by $L_i$ is a path $l_{i,1}l_{i,2}...l_{i,r+1}$ and for $i = 1, 2, ..., r+1$ and $j = 1, 2, ..., r+1$, $l_{i,j}m \in E(H)$. So clearly, $d_H(m) = r^2 + 4r + 2$ and the vertex $m$ is an universal vertex of the graph $H$. The graph $H$ is pictured in Figure 3.9. Clearly, $H$ is a planar graph for every $r \geq 2$.

We show the graph $H$ is 1-tough. Let $S$ be a subset of $V(H)$ such that $\omega(H - S) \geq 2$. The vertex $m$ is an universal vertex of the graph $H$ and $\omega(H - S) \geq 2$ so $m \in S$. Denote $J_s = J \cap S$, $K_s = K \cap S$, $L_s = L \cap S$. Denote $R$ the graph obtained from the graph $H$ by leaving the vertex $m$ and all vertices of $K_s$. For every $k_i \in K_s$ the path $l_{i,1}l_{i,2}...l_{i,r+1}$ is a component of the graph $R$. Denote each of these components $P_i$, $k_i \in K_s$. The graph $R$ consists of $|K_s| + 1$ components. $|K_s|$ of these components are the paths $P_i$. Denote $D$ the 1 remaining component. Notice $\omega(D - J_s) \leq |J_s| + 1$. Also notice $\omega(D - J_s - (D \cap L_s)) = \omega(D - J_s)$. Finally, $\omega(P_i - (P_i \cap L_s)) \leq |P_i \cap L_s| + 1$ for every path $P_i$. So $\omega(H - S) \leq |K_s| + |J_s| + |L_s| + 1$ and $|S| = |K_s| + |J_s| + |L_s| + 1$. The graph $H$ is 1-tough.

We show the graph $H$ is chordal. Consider the vertex $l_{1,1}$, $d_H(l_{1,1}) = 3$ and $l_{1,1}k_i, l_{1,1}m, l_{1,1}l_{1,2} \in E(H)$. A subgraph induced by vertices $\{k_1, l_{1,1}, l_{1,2}, m\}$ is a clique $K_4$. So the vertex $l_{1,1}$ is a simplicial vertex of the graph $H$, similary the vertex $l_{1,2}$ is a simplicial vertex of the graph $(H - l_{1,1})$, the vertex $l_{1,3}$ is a simplicial vertex of the graph $(H - (l_{1,1} \cup l_{1,2}))$ and so on. So the vertex $l_{i,j}$ is a simplicial vertex of the graph $(H - (L_1 \cup L_2 \cup ... \cup L_{i-1}) - (l_{i,1} \cup l_{i,2} \cup ... \cup l_{i,j-1}))$.

Denote $H - L$ the graph obtained from the graph $H$ by leaving all vertices of $L$. $d_{H-L}(k_1) = 3$ and $k_1j_1, k_1k_2, k_1m \in E(H - L)$. A subgraph induced by vertices $\{k_1, j_1, k_2, m\}$ is a clique $K_4$. So the vertex $k_1$ is a simplicial vertex of the graph $H - L$. $d_{H-L-k_1}(j_1) = 3$ and $j_1k_2, j_1j_2, k_1m \in E(H - L - k_1)$. A subgraph induced by vertices $\{j_1, k_2, j_2, m\}$ is a clique $K_4$. With the same argument continue considering graphs $(H - L - k_1 - j_1), (H - L - k_1 - j_1 - k_2), ..., (H - L - k_1 - j_1 - ... - k_{r-1} - j_{r-1})$. Finally, a graph $(H - L - k_1 - j_1 - ... - k_{r-1} - j_{r-1} - k_r)$ is nothing else but a subgraph induced by vertices $\{j_r, k_{r+1}, m\}$ which is a clique $K_3$. So the ordering

$$(l_{1,1}, l_{1,2}, ..., l_{1,r+1}, l_{2,1}, l_{2,2}, ..., l_{2,r+1}, ..., l_{r+1,1}, l_{r+1,2}, ..., l_{r+1,r+1},$$
$$k_1, j_1, k_2, j_2, ..., k_r, j_r, k_{r+1}, m)$$

is a perfect elimination ordering of the graph $H$. Hence by Theorem 3.10 $H$ is a chordal graph.

We show the graph $H$ has no $r$-trestle. For $i = 1, 2, ..., r + 1$ and $j = 1, 2, ..., r + 1$, every vertex $l_{i,j} \in L_i$ is adjacent to some other vertices of $L_i$ and to the vertices $k_i$, $m$ and no other vertices. So in every 2-connected subgraph $T$ of the graph $H$ $d_T(m) \geq r + 1$, hence the graph $H$ has no $r$-trestle.

We show the shortness exponent of the class of 1-tough chordal planar graphs is at most $\frac{1}{2}$. Denote $c(H)$ the length of the longest cycle in the graph $H$. For $i = 1, 2, ..., r+1$ and $j = 1, 2, ..., r+1$, every vertex $l_{i,j} \in L_i$ is adjacent to some other vertices of $L_i$ and to the vertices $k_i$, $m$ and no other vertices. So any cycle $C$ subgraph of $H$ does not contain more than $2|L_1|$ vertices from $L$.

$$c(H) \leq 2|L_1| + |J| + |K| + |M| = 2(r + 1) + (r + 1) + r + 1 = 4r + 4$$

Let $(H_n)_{n=1}^{\infty}$ be a sequence of graphs such that for $n = 1, 2, ...$ the graph $H_n$ is the graph $H$ constructed according to $r = n + 1$. As it was shown above in this proof $H_n$ is a 1-tough chordal planar graph, notice $|V(H_n)| \to \infty$ as $n \to \infty$. Denote $\Lambda$ the class of 1-tough chordal planar graphs.

$$\frac{\log c(H_n)}{\log |V(H_n)|} \leq \frac{\log(4r + 4)}{\log(r^2 + 4r + 3)} = \frac{\log 4n}{\log(n^2 + 2n)}$$

$$\sigma(\Lambda) = \liminf_{H_n \subset \Lambda} \frac{\log c(H_n)}{\log |V(H_n)|} \leq \lim_{n\to\infty} \frac{\log 4n}{\log(n^2 + 2n)} \leq \lim_{n\to\infty} \frac{\log 4n}{\log n^2} =$$
$$= \lim_{n\to\infty} \frac{\log 4 + \log n}{2 \log n} = \frac{1}{2}$$

So the shortness exponent of the class of 1-tough chordal planar graphs is at most $\frac{1}{2}$.

□

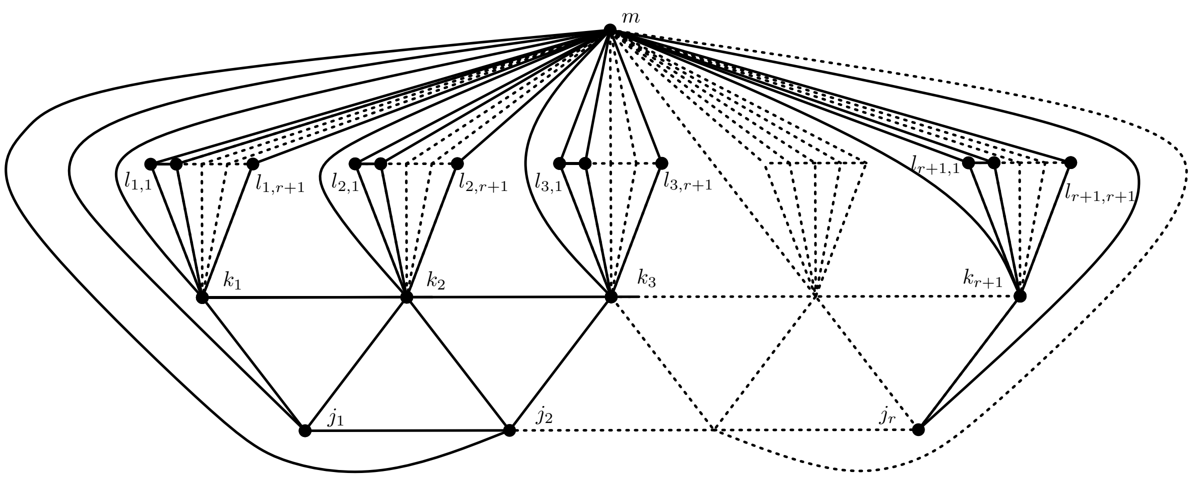


Figure 3.9: 1-tough chordal planar graph $H$ having no $r$-trestle

Before we say more about the toughness and hamiltonicity of planar graphs that are not chordal, let us mention two more results dealing with shortness exponent of certain classes of planar graphs. It was shown by Chen and Yu in [29] that the shortness exponent of the class of 3-connected planar graphs is $\log_3 2$.

**Theorem 3.28 [29]** *The shortness exponent of the class of 3-connected planar graphs is* $\log_3 2$.

The best known upper bound on the shortness exponent of the class 1-tough maximal planar graphs was obtained by Tkáč in [27].

**Theorem 3.29 [27]** *The shortness exponent of the class of 1-tough maximal planar graphs is at most* $\log_6 5$.

Hence there are 1-tough maximal planar graphs with no hamiltonian cycle. In fact, we will see that by Theorem 3.33 there are maximal planar graphs with toughness arbitrary close to $\frac{3}{2}$ which are not hamiltonian.

As it was mentioned above in Theorem 3.25 every chordal planar graph with toughness greater than 1 is hamiltonian. In fact, a planar graph which is not chordal and has toughness greater than 1 still has hamiltonian properties. Due to Proposition 2.8 a graph with toughness greater than 1 is 3-connected and by Theorem 3.23 every 3-connected planar graph has a 6-trestle.

**Corollary 3.30** *For every $\epsilon > 0$, every $(1+\epsilon)$-tough planar graph has a 6-trestle.*

By Theorem 3.27 there exists a 1-tough chordal planar graph with no $r$-trestle, for every $r \geq 2$. Although Theorem 3.23 originally considers not only planar graphs and although by Propositions 2.8 and 2.9 toughness is a stronger property then connectivity, interestingly the result in Corollary 3.30 is the best possible.

We show a similar corollary for planar graphs with toughness greater than $\frac{3}{2}$. By Proposition 2.8 every graph with toughness greater than $\frac{3}{2}$ is 4-connected and by Theorem 3.21 every 4-connected planar graph is hamiltonian.

**Corollary 3.31** *For every $\epsilon > 0$, every $(\frac{3}{2}+\epsilon)$-tough planar graph is hamiltonian.*

So by Corollary 3.31 the Conjecture 3.7 holds for planar graphs. In [28] Harant performed $\frac{3}{2}$-tough planar graph with no hamiltonian cycle. Notice that by Theorem 3.32 the result in Corollary 3.31 is the best possible.

**Theorem 3.32 [28]** *There exists a $\frac{3}{2}$-tough planar graph which is not hamiltonian.*

The nonhamiltonian $\frac{3}{2}$-tough planar graphs in [28] are regular graphs of degree 3, 4, and 5. However, those graphs are not maximal planar graphs. By Theorem 3.29 there are 1-tough maximal planar graphs with no hamiltonian cycle. In [17] Owens showed there are maximal planar graphs with toughness arbitrary close to $\frac{3}{2}$ which are not hamiltonian. In fact, those graphs do not even have a 2-factor.

**Theorem 3.33 [17]** *For every $\epsilon > 0$, there is a $(\frac{3}{2}-\epsilon)$-tough maximal planar graph which is not hamiltonian.*

Notice again the results in Corollaries 3.30 and 3.31 are the best possible. This means speaking of planar graphs and their hamiltonian properties toughness and connectivity are somehow similar. However not all of the results shown in this section are necessarily the best possible. In the next Subsection we mention some questions that remain open.

### 3.4.3 Summary of open questions

First of all, remind that it is not known whether Chvátal Conjecture 3.3 is true for general graphs, anyway it holds for planar graphs. In general a lot remains unknown on the topic of hamiltonicity related to toughness. Considering planar graphs in Section 3.4 we saw several results. Moreover we saw the results in some theorems and corollaries are the best possible. Let us just briefly sum up what is unknown in the context of what was said about hamiltonicity of planar graphs.

By Corollary 3.31 every planar graph with toughness greater than $\frac{3}{2}$ is hamiltonian. By Theorem 3.32 there exists a $\frac{3}{2}$-tough planar graph which is not hamiltonian and by Theorem 3.33 there are maximal planar nonhamiltonian graphs with toughness arbitrary close to $\frac{3}{2}$. Do all $\frac{3}{2}$-tough maximal planar graphs have a 2-factor? Does every $\frac{3}{2}$-tough planar graph have a 2-factor? Is every $\frac{3}{2}$-tough maximal planar graph hamiltonian?

By Theorem 3.23 every 3-connected planar graph has a 6-trestle. By Proposition 3.24 there are 2-connected planar graphs with no $r$-trestle for every $r \geq 2$. By Theorem 3.22 there exists a 3-connected planar graph with no 5-trestle, this graph also is a maximal planar graph. Do all 3-connected chordal planar graphs have a 5-trestle, 4-trestle, 3-trestle?

By Corollary 3.30 every planar graph with toughness greater than 1 has a 6-trestle. By Theorem 3.27 there exists a 1-tough chordal planar graph with no $r$-trestle for every $r \geq 2$. Is there $1 < t \leq \frac{3}{2}$ such that every $t$-tough planar graph has a 5-trestle, 4-trestle, 3-trestle?

By Theorem 3.28 the shortness exponent of the class of 3-connected planar graphs is $\log_3 2$. Every graph with toughness greater than 1 is 3-connected by Proposition 2.8. Also every maximal planar graph is 3-connected (the proof can be found in [1]). So by Theorem 3.28 we have lower bounds on the shortness exponent of the class of planar graphs with toughness greater than 1 and the class of maximal planar graphs. The upper bound on the shortness exponent of the class of 1-tough maximal planar graphs was mentioned in Theorem 3.29. So all in all, the shortness exponent of the class of 1-tough maximal planar graphs is somewhere between $\log_3 2$ and $\log_6 5$. In Theorem 3.27 we showed the shortness exponent of the class of 1-tough chordal planar graphs is not greater than $\frac{1}{2}$.

Chapter 4

# Forbidden subgraphs and hamiltonicity

This chapter continues on the topic of forbidden subgraphs and hamiltonicity, which is the topic of author's Bachelor Thesis. In the Bachelor Thesis results considering the square of $S(K_{1,4})$-free trees and 3-trestle and the square of $S(K_{1,r})$-free trees and $r$-trestle were obtained. In this chapter we mention several sufficient conditions on the forbidden subgraphs for a graph to be hamiltonian or to have an $r$-trestle.

Let us recall the definition of an $\mathbb{F}$-*free* graph. Let $\mathbb{F}$ be a class of graphs $\mathbb{F} = \{H_1, H_2, ..., H_k\}$. We say a graph G is $\mathbb{F}$-*free* or $(H_1, H_2, ..., H_k)$-*free* if the graph $G$ contains no induced subgraph isomorphic to any of the graphs $H_1, H_2, ..., H_k$. In particular, for $\mathbb{F} = \{H\}$ we say the graph $G$ is $H$-free. The graphs $H_1, H_2, ..., H_k$ are called *forbidden subgraphs*.

First of all, consider $\mathbb{F} = \{H\}$. The following result was performed by Oberly and Sumner in [31]

**Theorem 4.1** *[31] Let $G$ be a connected, locally connected, $K_{1,3}$-free graph, $|V(G)| \geq 3$, then $G$ is hamiltonian.*

Balakrishnan and Paulraja in [19] showed that every 2-connected chordal graph is locally connected. By putting together this fact with Theorem 4.1 the following result was obtained.

**Theorem 4.2** *[19] Let $G$ be a 2-connected, $K_{1,3}$-free chordal graph, then $G$ is hamiltonian.*

Notice that a 2-connected $K_{1,2}$-free graph ($K_{1,2} = P_3$) is a complete graph $K_n$. Clearly, for $n \geq 3$ the graph $K_n$ is hamiltonian. In [22] Kužel and Teska showed a 2-connected $K_{1,r}$-free graph has an $r$-trestle.

**Theorem 4.3 [22]** *Every 2-connected $K_{1,r}$-free graph has an $r$-trestle.*

In the next Section we mention results dealing with $\mathbb{F} = \{H_1, H_2\}$.

## 4.1 Forbidden pairs

We say $\mathbb{F} = \{H_1, H_2\}$ is a *forbidden pair of graphs*. All the results mentioned in this chapter so far were considering graphs $K_{1,3}$ and $K_{1,r}$. Also each forbidden pair consists of one of the graphs $K_{1,3}$ or $K_{1,r}$ and some other graph.

For example, every 2-connected $(K_{1,3}, Z_1)$-free graph is hamiltonian. This was shown by Goodman and Hedetniemi in [30].

**Theorem 4.4 [30]** *Let $G$ be a 2-connected $(K_{1,3}, Z_1)$-free graph. Then $G$ is hamiltonian.*

More research on the topic of forbidden pairs and hamiltonicity led to the result performed by Faudree and Gould in [20]. The graphs $Z_1$, $Z_2$, $Z_3$, $B$, $N$ and $W$ are shown in Figure 4.1.

**Theorem 4.5 [20]** *Let $R$ and $S$ be connected graphs ($R, S \neq P_3$) and $G$ a 2-connected graph $|V(G)| \geq 10$. Then $G$ is $(R, S)$-free implies $G$ is hamiltonian if and only if $R = K_{1,3}$ and $S$ is one of the graphs $C_3$, $P_4$, $P_5$, $P_6$, $Z_1$, $Z_2$, $Z_3$, $B$, $N$ or $W$.*

We show an extension of Theorem 4.4 considering a $(K_{1,r}, Z_1)$-free graph and an $(r-1)$-trestle.

**Theorem 4.6** *Let $r$ be an integer, $r \geq 3$. Let $G$ be a 2-connected $(K_{1,r}, Z_1)$-free graph. Then $G$ has an $(r-1)$-trestle.*

*Proof.* Assume to the contrary the graph $G$ has no $(r-1)$-trestle. The graph $G$ is 2-connected so for any two of vertices $G$ contains a cycle, hence $G$ contains an induced subgraph which has an $(r-1)$-trestle. Let $H$ be the induced subgraph of $G$ such that $H$ has an $(r-1)$-trestle $T$ and suppose to

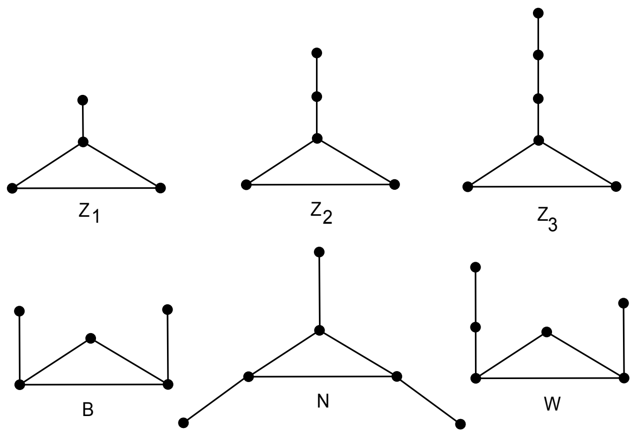


Figure 4.1: the graphs $Z_1$, $Z_2$, $Z_3$, $B$, $N$ and $W$

the contrary $H$ is the largest subgraph with this property. In other words, for every induced subgraph $S$ of the graph $G$ such that $|V(S)| > |V(H)|$ the graph $S$ has no $(r-1)$-trestle.

We supposed $G$ has no $(r-1)$-trestle, so there exists a vertex of $G$ which is not in $T$. Let $x$ be a vertex such that $x \in (V(G) - V(T))$ and such that there exists a vertex $a \in T$ such that $xa \in E(G)$. The graph $G$ is 2-connected so in the graph $G - ax$ there exists a path from the vertex $x$ to some vertex of $T$ which is not the vertex $a$. Let $P$ be the path from the vetex $x$ to a vertex $b \in (V(T) - a)$ such that, for every vertex $y \in (V(P) - b)$, $y \notin V(T)$.

Denote $d_T(a)$ the degree of vertex $a$ in the graph $T$. If $d_T(a) \leq r-2$ and $d_T(b) \leq r-2$, then the graph $(T \cup P \cup ax)$ is an $(r-1)$-trestle of a graph induced by vertices $(V(H) \cup V(P))$. This graph is larger than the graph $H$ which is a contradiction.

Hence $d_T(a) = r-1$ or $d_T(b) = r-1$. Suppose $d_T(a) = r-1$. Denote $N_T(a)$ vertices adjacent to the vertex $a$ in the graph $T$. We show the graph $G$ contains an edge $ux$, $u \in N_T(a)$. Let $F$ be a subgraph of the graph $G$ induced by vertices $(N_T(a) \cup a \cup x)$. $G$ is a $K_{1,r}$-free graph so $F \neq K_{1,r}$. So the graph $G$ contains either an edge $ux$, $u \in N_T(a)$ or an edge $uv$, $u, v \in N_T(a)$. Suppose $uv \in E(G)$. Let $D$ be a subgraph of the graph $G$ induced by vertices $\{u, v, a, x\}$. The graph $G$ is $Z_1$-free, hence $D \neq Z_1$.

So either $ux \in E(G)$ or $vx \in E(G)$. We showed the graph $G$ contains an edge $ux$, $u \in N_T(a)$. So the graph $((T - au) \cup ax \cup ux)$ is an $(r-1)$-trestle of a graph induced by vertices $(V(H) \cup x)$. This graph is larger than the graph $H$ which is a contradiction. The graph $G$ has an $(r-1)$-trestle.

(If $d_T(a) \leq r - 2$ and $d_T(b) = r - 1$, then the argument holds just consider the vertex $b$ instead of the vertex $a$ and a vertex $y$ instead of the vertex $x$. $y$ is the vertex adjacent to the vertex $b$ in the graph $P$.)

□

# Chapter 5

# Conclusion

The thesis deals with with hamiltonian properties of a graph, mainly a hamiltonian cycle and an $r$-trestle. Primarily, in Chapter 3 the relationship among the hamiltonian cycle, the $r$-trestle and the toughness of a graph is disscussed. It is shown how these relations change when chordal graphs or planar graphs are considered. However at first, the context of the issue is sketched in Chapters 1 and 2. In Chapters 1 various generalizations of hamiltonicity and the relations among them are mentioned. Chapter 2 shows some results dealing with hamiltonian graphs and graphs with an $r$-trestle. Also results considering the toughness and connectivity are mentioned. At last, Chapter 4 shows some relations among the hamiltonian cycle, the $r$-trestle and forbidden subgraphs.

Original results are presented in Theorems 3.13, 3.27 and 4.6. Following the results from [5] and [18], which show graphs with high toughness and no hamiltonian cycle or no $r$-trestle, in Theorem 3.13 we showed chordal graphs with toughness greater than 1 and no $r$-trestle. By [13] every chordal planar graph with toughness greater than 1 is hamiltonian. Also by [13] the shortness exponent of the class of 1-tough chordal planar graphs is at most $\log_9 8$, so there are nonhamiltonian 1-tough chordal planar graphs. In Theorem 3.27 we showed there exist 1-tough chordal planar graphs with no $r$-trestle and the shortness exponent of the class of 1-tough chordal planar graphs is at most $\frac{1}{2}$. By a corollary of [25] every planar graph with toughness greater than 1 has a 6-trestle. By Theorem 3.27 the result of this corollary is the best possible. In Theorem 4.6 we showed every 2-connected $(K_{1,r}, Z_1)$-free graph has an $(r-1)$-trestle. It is an extension of the result from [30].

Speaking of hamiltonian properties of planar graphs, we have already mentioned some questions that remain open in Subsection 3.4.3. Besides that, there are unanswered questions considering hamiltonicity of general graphs and chordal graphs. For example, is there a finite constant $t$ such that every $t$-tough graph is hamiltonian, or such that every $t$-tough graph has some $r$-trestle? By [3] every 18-tough chordal graph is hamiltonian. Is there a finite constant $t < 18$ that ensures every $t$-tough chordal graph is hamiltonian, or every $t$-tough chordal graph has some $r$-trestle?

# Bibliography


[1] R. Diestel, Graph Theory, Springer Verlag, New York (2000).

[2] A. Schrijver, Combinatorial Optimization - Polyhedra and Efficiency, Springer Verlag, Berlin (2003).

[3] G. Chen, H. S. Jacobson, A. E. Kezdy, J. Lehel, Tough enough chordal graphs are hamiltonian, Networks 31 (1998), 29-38.

[4] O. Ore, A Note on Hamilton Circuits, Amer. Math. Monthly 67 (1960), 55.

[5] D. Bauer, H. J. Broersma, H. J. Veldman, Not every 2-tough graph is hamiltonian, Discrete Appl. Math. 99 (2000), 317-321.

[6] M. Tkáč, H. J. Voss, On k-trestles in polyhedral graphs, Discuss. Math. Graph Theory 22 (2002), 193-198.

[7] V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973), 215-228.

[8] D. Barnette, 2-connected spanning subgraphs of planar 3-connected graphs, J. Combin. Theory (B) 61 (1994), 210-216.

[9] G. A. Dirac, Some Theorems on Abstract Graphs, Proc London Math. Soc. 2 (1952), 69-81.

[10] J. A. Bondy, V. Chvátal, A Method in Graph Theory, Discrete Math. 15 (1976), 111-136.

[11] D. Bauer, G. Y. Katona, D. Kratsch, H. J. Veldman, Chordality and 2-factors in tough graphs, Discrete Appl. Math 99 (2000), 323–329.

[12] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956), 99-116.

[13] T. Böhme, J. Harant, M. Tkáč, More than 1-tough chordal planar graphs are hamiltonian, J. Graph Theory 32 (1999), 405-410.

[14] S. Jendroľ, T. Kaiser, Z. Ryjáček, I. Schiermeyer, A Dirac theorem for trestles, Discrete Math. 312 (2012), 2000-2004.

[15] H. Broersmaa, L. Xiongc, K. Yoshimoto, Toughness and hamiltonicity in k-trees, Discrete Math. 307 (2007), 832–838.

[16] H. Enomoto, B. Jackson, P. Katerinis, A. Saito, Toughness and the existence of k-factors, J. Graph Theory 9 (1985), 87-95.

[17] P. J. Owens, Nonhamiltonian maximal planar graphs with high toughness, Tatra Mountains Math. Publ. 18 (1999), 89–103.

[18] J. Teska, Factors and cycles in graphs, PhD Thesis (2008)

[19] R. Balakrishnan, P. Paulraja, Chordal graphs and some of their derived graphs, Congr. Numer. 53 (1986), 71–74.

[20] R. J. Faudree, R. J. Gould, Characterizing Forbidden Pairs for Hamiltonian Properties, Discrete Math. 173 (1997), 45-60.

[21] M. N. Ellingham, X. Zha, Toughness, trees, and walks, J. Graph Theory 33 (2000), 125-137.

[22] R. Kužel, J. Teska, On 2-Connected Spanning Subgraphs with Bounded Degree in K1,r-Free Graphs, Graphs and Combinatorics 27 (2011), 199-206.

[23] M. R. Garey, D. S. Johnson, R. E. Tarjan, The planar Hamiltonian circuit problem is NP-complete, SIAM J. Comput. 5 (1976), 704-714.

[24] J. M. Kiel, Finding Hamiltonian circuits in interval graphs, Inf. Process. Lett 20 (1985), 201–206.

[25] Z. Gao, 2-connected covering of bounded degree in 3-connected graphs, J. Graph Theory 20 (1995), 327-338.

[26] D. Bauer, S. L. Hakimi, E. Schmeichel, Recognizing tough graphs is NP-hard, Discrete Appl. Math. 28 (1990), 191–195.

[27] M. Tkáč, On the shortness exponent of 1-tough, maximal planar graphs, Discrete Math. 154 (1996), 321–328.

[28] J. Harant, Toughness and nonhamiltonicity of polyhedral graphs, Discrete Math. 113 (1993), 249–253.

[29] G. Chen, X. Yu, Long cycles in 3-connected graphs, J. Combin. Theory (B) 86 (2002), 80–99.

[30] S. Goodman, S. Hedetniemi, Sufficient conditions for a graph to be hamiltonian, J. Combin. Theory (B) 16 (1974), 175-180.

[31] D. J. Oberly, D. P. Sumner, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979), 351-356.

[32] B. Jackson, N. C. Wormald, k-walks of graphs, Australas. J. Combin. 2 (1990), 135-146.

[33] T. Kaiser, R. Kužel, H. Li, G. Wang, A note on k-walks in bridgeless graphs, Graphs Comb. 23 (2007), 303-308.

[34] D. R. Fulkerson, O. A. Gross, Incidence matrices and interval graphs, Pacific J. Math. 15 (1965), 835–855.

[35] S. Win, On connection between the existence of k-trees and the toughness of a graph, Graphs and Combinatorics 5 (1989), 201-205.

[36] K. Kuratowski, Sur le probleme des courbes gauches en topologie, Fund. Math. 15 (1930), 271-283.