Name - SHOHAM TEBERIWAL

Roll No — MT23MA7L1004

Reg No — NIAS2300018169

Maths Test Series — Test 7 & Test 8
(optional Next IAS)

Section – A

1a) Given M: maximal ideal of commutative ring with identity

To show: M is prime ideal

Proof:

we know an ideal is prime ideal if for a & b s.t $ab \in I$ then either $a \in I$ or $b \in I$

Now M is maximal (with unity element)

suppose $ab \in M$ but $a \notin M$

$\Rightarrow$ $M + (a)$ is another ideal of ring
$M + (a) \neq M$

$\because$ M is maximal

$\Rightarrow$ $M + (a) = R$

$\Rightarrow$ $\exists\, r$ such that $m + ar = 1$

we show $\frac{R}{M}$ is a field

$\frac{R}{M} = M + a$ $\quad a \notin M$

$(M + a)(M + r) = M + ar = M + 1$

Hence for every element (non-zero) $\frac{R}{M}$ is a field

$\frac{R}{M}$ is a field $\Rightarrow$ $\frac{R}{M}$ is integral domain

we show if

$\frac{R}{M}$ is I D then M is prime ideal

$\frac{R}{M} = M + a \mid a \notin M$

let $ab \in M$ where $a, b \in R$
$a \neq 0,\ b \neq 0$

so; $(M+a)(M+b) = M+ab = M+0$

$\Rightarrow$ for integral domain

either $M+a = M$

or $M+b = M$

$\Rightarrow a \in M$ or $b \in M$

Hence M is prime ideal

[M is maximal $\Rightarrow$ $\frac{R}{M}$ is field $\Rightarrow$ integral domain $\Rightarrow$ M is prime]

1(b) $f(z)$ is integral function

$|f(z)| \leq M$ for all values of z

$\because$ $f(z)$ is analytic for all values of z

$\Rightarrow f(z) = \sum a_n z^n$

$|\sum a_n z^n| \leq M$ for all values

if z takes very large value

i.e $z \to \infty$

then it will be contradiction

$\therefore \sum a_n z^n = a_0 + a_1 z + a_2 z^2 \ldots$

$\Rightarrow a_1 \ldots a_n$ are all zero

$$\Rightarrow \boxed{f(z) = a_0}$$

c) $\langle f_n \rangle = \tan^{-1}(nx)$

To show it's not uniformly convergent on $[0,1]$

$$\lim_{n\to\infty} \langle f_n \rangle = \begin{cases} 0 & x = 0 \\ \frac{\pi}{2} & x \neq 0 \end{cases}$$

Now, using M_n test

$$M_n = \sup_{x\neq 0} \left| \tan^{-1}(nx) - \frac{\pi}{2} \right| \text{ or } \sup_{x=0} \left| \tan^{-1}(nx) - 0 \right|$$

let $f(x) = \tan^{-1}(nx)$

$$f'(x) = \frac{n}{1+n^2x^2}$$

which is monotonically increasing

max. $f(x) = \tan^{-1}(n)$ at $x = 0$

$$M_n = \sup \left| \tan^{-1}(n) - 0 \right| = \frac{\pi}{2} \neq 0$$

$\therefore$ by M_n test $\langle f_n \rangle$ is not U.C. on $[0,1]$

on contrary if $x \in [a, 1]$
$a > 0$

then $\langle f_n \rangle$ is U.C.

$$M_n = \sup \left| \tan^{-1}(nx) - \frac{\pi}{2} \right| = 0$$

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d) If $\sum a_n$ cgs then $\lim a_n = 0$

let $S_n = \sum a_n$

we use cauchy's sequence theorem

given $\epsilon > 0$ $\exists\, m \in \mathbb{Z}$ such that

$|S_{n+p} - S_n| < \epsilon$ $\forall\, n \geq m$
$p \in \mathbb{Z}^+$

$\Rightarrow$ $|S_{n+1} - S_n| < \epsilon$ $\forall\, n \geq m$

$|(a_1 + a_2 \ldots a_{n+1}) - (a_1 \ldots + a_n)| < \epsilon$
$\forall\, n \geq m$

$\Rightarrow$ $|a_{n+1}| < \epsilon$ $\forall\, n \geq m$

for any given ϵ $|a_{n+1}| < \epsilon$ (consider $\epsilon \to 0$)
$\exists\, M$ such that $n \geq m$

$\Rightarrow \lim_{n \to \infty} a_{n+1} = 0$ $\Rightarrow \lim_{n \to \infty} a_n = 0$

e) Find ring with unity and subring of different unity

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

$a, b, c, d \in \mathbb{Z}$

$\therefore$ R is a ring

$(R, +^n)$ is abelian group

$(R, \times^n)$ is quasi-group

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in R$ is unity in R

$$R' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

R' is a subring of R

unity of R' is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3a) Show $R[x]$ is I.D iff R is I.D.

(i) we show that $R[x]$ is ID when R is I.D.

if R is ID

$\Rightarrow a \neq 0, \; b \neq 0 \in R$

$ab \neq 0 \in R$

or $ab = 0$ then $a = 0$ or $b = 0$

$R[x] = \{a_0 + a_1 x \, \ldots \mid a_0, a_1 \, \ldots \in R\}$

suppose $r_1(x) \cdot r_2(x) = 0$

$\Rightarrow (a_0 + a_1 x \, \ldots \, a_n x^n)(b_0 + b_1 \, \ldots \, b_n x^n) = 0$

$\Rightarrow a_0 b_0 + (a_1 b_0 + a_0 b_1) x \, \ldots = 0$

$\Rightarrow a_0 b_0 = 0 \quad \Rightarrow a_0 = 0$ or $b_0 = 0$ $\because a_0, b_0 \in R$

$\Rightarrow a_1 b_0 + a_0 b_1 = 0$

if $a_0 = 0$ & $b_0 \neq 0$ $\Rightarrow a_1 b_0 = 0 \Rightarrow a_1 = 0$

Hence $a_0, a_1 \, \ldots \, a_n = 0$

i.e. $r_1(x) = 0$

Hence $R[x]$ is I.D.

(ii) conversely if $R[x]$ is I.D, we show R is I.D

if $r_1(x) \cdot r_2(x) = 0$

$\Rightarrow r_1(x) = 0$ or $r_2(x) = 0$

i.e. $a_0 b_0 = 0$

$a_1 b_0 + a_0 b_1 = 0$

$\vdots$

(coefficient) n^{th} term $= 0$

$a_0 = 0$

$a_1 = 0$

$\vdots$

$a_n = 0$

$\therefore r_1(x) = 0$

similarly if

$a_0 b_0 = 0 \quad a_0 \in R \quad b_0 \in R$

then $a_0 = 0$ from above

or $a_1 b_0 = 0 \Rightarrow a_1 = 0$

$\Rightarrow R$ is I.D.

Q.E.D.

3b) To maximize profits

operations	x	y	z	Total time
I	1	2	2	200
II	2	1	1	220
III	3	1	2	180
min sales	10	20	30	
Profit	10	15	8	

(→ time ←)

let # products of X → x

(no. of products) of Y → y

(—"—) of Z = z

Profit : $10x + 15y + 8z$

$x \geq 10$

$y \geq 20$

$z \geq 30$

} min. sales

time constraints : operation I : $x + 2y + 2z \leq 200$

operation II : $2x + y + z \leq 220$

operation III : $3x + y + 2z \leq 180$

$\because x \geq 10 \Rightarrow$ ~~x≥10~~

let $x = 10 + a$, $y = 20 + b$, $z = 30 + c$

$\Rightarrow$

Profit : $10(10+a) + 15(20+b) + 8(30+c)$

Profit $Y = 10a + 15b + 8c + 640$

time constraints :

(I) : $10 + a + 2(20+b) + 2(30+c) \leq 200$

$a + 2b + 2c \leq 90$

(II) : $2(10+a) + 20 + b + 30 + c \leq 220$

$2a + b + c \leq 150$

(III) : $3(10+a) + 20 + b + 2(30+c) \leq 180$

$3a + b + 2c \leq 70$

LPP : $Y : 10a + 15b + 8c$

$a + 2b + 2c \leq 90$

$2a + b + c \leq 150$

$3a + b + 2c \leq 70$

$a, b, c \geq 0$

		10	15	8	0	0	0	
		a	b	c	s_1	s_2	s_3	b
15	b	0	1	$\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{-1}{5}$	40
0	s_2	0	0	$\frac{-3}{5}$	$\frac{-1}{5}$	1	$\frac{-3}{5}$	90
10	a	1	0	$\frac{2}{5}$	$\frac{-1}{5}$	0	$\frac{2}{5}$	10
	Z_j	10	15	16	7	0	1	
	Δ	0	0	-8	-7	0	-1	

optimality is achieved

$\boxed{a = 10}$ $\boxed{b = 40}$

$\therefore x = 10 + a = 20$

$y = 20 + b = 60$

$z = 30 + c = 30$

Profit $= 10x + 15y + 8z$

$= 1340$

introduce surplus / slack variables

$$\left.\begin{aligned} a + 2b + 2c + s_1 &= 90 \\ 2a + b + c + s_2 &= 150 \\ 3a + b + 2c + s_3 &= 70 \end{aligned}\right\} \quad a, b, c, s_1, s_2, s_3 \geq 0$$

		10	15	8	0	0	0		
		a	b	c	s_1	s_2	s_3	b	θ
0	s_1	1	$\boxed{2}$	2	1	0	0	90	45 → outgoing
0	s_2	2	1	1	0	1	0	150	150
0	s_3	3	1	2	0	0	1	70	70
Z_j =		0	0	0	0	0	0		
Δ =		10	15	8	0	0	0		

Incoming

		a	b	c	s_1	s_2	s_3	b	θ
15	b	$1/2$	1	1	$1/2$	0	0	45	90
0	s_2	$3/2$	0	0	$-1/2$	1	0	105	70
0	s_3	$\boxed{5/2}$	0	1	$-1/2$	0	1	25	10 → outgoing
Z_j		$15/2$	15	15	$15/2$	0	0		
Δ		$5/2$	0	-ve	-ve	0	0		

Incoming

c) To find analytic function

$f(z) = u(r\theta) + iv(r\theta)$

$\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial\theta}$

& $\frac{\partial u}{r\partial\theta} = -\frac{\partial v}{\partial r}$

$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$

$\therefore \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta$ — (1)

$\frac{\partial v}{r\partial\theta} = \frac{1}{r}\left[-2\sin 2\theta\, r^2 + r\sin\theta\right]$ — (2)

$= -2r\sin 2\theta + \sin\theta$

$\therefore \frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$ from (2)

$u(r\theta) = -r^2\sin 2\theta + \frac{r}{1}\sin\theta + f(\theta)$

$= -r^2\sin 2\theta + r\sin\theta + f_1(\theta)$ — (3)

$\frac{\partial u}{r\partial\theta} = -2r\cos 2\theta + \cos\theta$ from (1)

$u_\theta = -2r^2\cos 2\theta + r\cos\theta \Rightarrow u = -r^2\sin 2\theta + r\sin\theta + f_2(r)$ — (4)

from ③ & ④

$$u(r, \theta) = -r^2 \sin 2\theta + r\sin\theta + C$$

$$\therefore f(z) = -r^2 \sin 2\theta + r\sin\theta + C + i\left(r^2 \cos 2\theta - r\cos\theta + 2\right)$$

$$f(z) = r^2\left(-\sin 2\theta + i\cos 2\theta\right) + r\left(\sin\theta - i\cos\theta\right) + 2i + C$$

$$\cos(2\theta + x) = -\sin 2\theta \qquad \left.\begin{array}{l}\cos(\theta + x) = \sin\theta \\ \sin(\theta + x) = -\cos\theta\end{array}\right] -i\left[\cos\theta + i\sin\theta\right]$$

$$\boxed{f(z) = r^2 e^{i(\pi/2 + 2\theta)} + r e^{i(\theta - \pi/2)} + C'}$$

where C' is constant

Section - B

5a) $(z - xp - yq)(p+q) = 1$

$\Rightarrow z - xp - yq = \frac{1}{p+q}$

$$\boxed{z = xp + yq + \frac{1}{p+q}} \quad \text{--- (1)}$$

by clairaut's form

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{-dx}{f_p} = \frac{-dy}{f_q} = \frac{-dz}{pf_p + qf_q}$$

$f : z - xp - yq - \frac{1}{p+q}$

$f_x = -p$ & $f_z = 1$ & $f_y = -q$

$\Rightarrow \frac{dp}{0} = \frac{dq}{0} \Rightarrow p = a, \; q = b$

put in (1)

$$\boxed{z = ax + by + \frac{1}{a+b}}$$

where a, b are arbitrary constants

5b) write algo to solve $xe^x - 1 = 0$ by bisection correct to 4 decimal places

Here correct to four decimal places means

max. error allowed = 0.00005 = e (say)

Algorithm is :—

1. Input $y = f(x)$ // $f(x) = xe^x - 1$

2. Input x_0, x_1

3. Find $y_0 = f(x_0)$ and $y_1 = f(x_1)$

4. If $y_0 y_1 > 0$, go to step 2
 Else proceed, go to step 5

5. If $|x_0 - x_1| < e$, go to step 9
 Else Go to step 6

6. $x_2 = \frac{x_0 + x_1}{2}$, find $y_2 = f(x_2)$

7. If $y_0 y_2 > 0$, then $y_0 = y_2$ and $x_1 = x_2$
 Else $y_0 = y_2$ and $x_0 = x_2$

8. Go to 5

9. Print x_1

10. End or stop

c) To find boolean expression

(i) Y is 1 only A is 1 and B is 1
or if A is 0 and B is 0

A	B	Y
0	0	1
0	1	
1	1	1
1	0	

$\therefore \boxed{Y = \bar{A}\bar{B} + AB}$

(ii)

A	B	C	Y
0	0	0	0
$\boxed{0}$	1	1	1
0	1	0	0
0	0	1	0
1	0	0	0
1	1	1	1
1	1	$\boxed{0}$	1
1	$\boxed{0}$	1	1

only A = 0

all 1

only C = 0

only B = 0

$\boxed{Y = \bar{A}BC + ABC + A\bar{B}C + AB\bar{C}}$

it can be simplified further
(P.T.O)

$$Y = \bar{A}BC + ABC + A\bar{B}C + AB\bar{C}$$

$$= BC + A\bar{B}C + AB\bar{C}$$

$$= C(B + \bar{B}A) + AB\bar{C}$$

$$= C(B+A) + AB\bar{C}$$

$A + \bar{A} = 1$

$X + \bar{X}Y = X + Y$

$$= AC + BC + AB\bar{C}$$

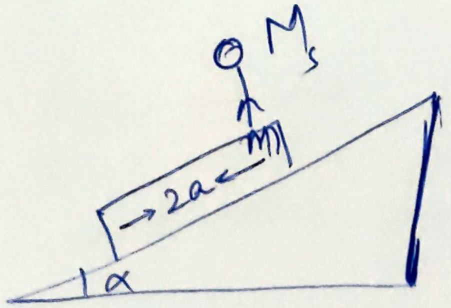
$$= A(C + B\bar{C}) + BC$$

$$= A(B+C) + BC$$

$$= AB + AC + BC$$

$$\boxed{Y = AB + BC + AC}$$

d)

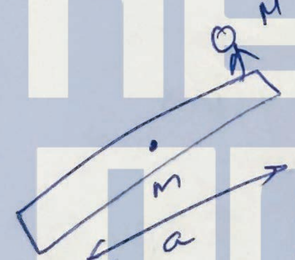


To show he will reach other end in 't'

$$t = \sqrt{\frac{4Ma}{(m+M)g\sin\alpha}}$$

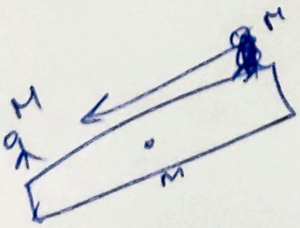
we solve using combined centre of mass of both bodies

initially :



$$(y_{cm})_i = \frac{\left(\frac{2a}{2}\right)m + (2a)M}{m+M} = \frac{2aM + am}{M+m}$$

finally



$$(y_{cm})_f = \frac{M(0) + \left(\frac{2a}{2}\right)m}{M+m} = \frac{am}{M+m}$$

$$\therefore \quad \Delta y_{cm} = \frac{2aM}{m+M}$$

$$y = ut + \frac{1}{2} a_y t^2$$

$$a_y = +g \sin\alpha \quad , \quad u = 0$$

$$\Delta y_{cm} = +\frac{1}{2} g \sin\alpha \, t^2$$

$$\frac{2aM}{m+M} = \frac{1}{2} g \sin\alpha \, t^2$$

$$t = \sqrt{\frac{4Ma}{(M+m) g \sin\alpha}}$$

— Hence Proved

e (i) flow where stream function exist
but velocity function doesn't exist

$\Rightarrow \psi(x,y)$ exist $\Rightarrow$ eq^n of continuity satisfied

& $\phi(x,y)$ doesn't exist $\Rightarrow$ irrotational flow

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ but } \frac{\partial v}{\partial x} \neq \frac{\partial u}{\partial y}$$

$$u(x,y) = -xy$$

$$v(x,y) = \frac{y^2}{2}$$

$$\boxed{\vec{q} = -xy\,\hat{i} + \frac{y^2}{2}\hat{j}}$$

Note: there can be infinitely such cases

(ii) $$W = \frac{k}{\pi}\tan^{-1}\left(\frac{z}{c}\right)$$

$$w = \phi + i\psi = \frac{k}{\pi}\tan^{-1}\left(\frac{z}{c}\right)$$

$$\psi(x,y) = \frac{W(z) - \overline{W}(\bar{z})}{2i}$$

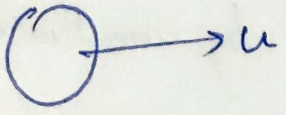
$$= \frac{k}{2\pi i}\left[\tan^{-1}\left(\frac{x+iy}{c}\right) - \tan^{-1}\left(\frac{x-iy}{c}\right)\right]$$

$$\psi(x,y) = \frac{k}{2i\pi}\left[\tan^{-1}\left(\frac{\frac{x+iy}{c} - \frac{x-iy}{c}}{1+\frac{x^2+y^2}{c^2}}\right)\right]$$

$$\psi(x,y) = \frac{k}{2\pi i}\tan^{-1}\left(\frac{2iyc}{x^2+y^2+c^2}\right)$$

or $$\boxed{\frac{y}{x^2+y^2+c^2} = k}$$

6a) Describble flow for circular cylinder of radius a moving with velocity u along x-axis



Clearly it's symmetric flow

(i) $\nabla^2 \phi = 0$: eqⁿ of continuity

(ii) boundary condⁿ: $(q_r)_{r=a} = +V\cos\theta$

$(q_r)_{r=\infty} = 0$

$\nabla^2 \phi = 0$

$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$ $\quad \because \boxed{\frac{\partial^2 \phi}{\partial z^2} = 0}$

$\phi(r,\theta) = R(r),\ \Theta(\theta)$

$\Rightarrow \quad \frac{\partial \phi}{r\,\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$

$\frac{R'\Theta}{r} + \frac{R''\Theta}{1} + \frac{R\Theta''}{r^2} = 0$

$\frac{rR' + r^2R''}{R} = -\frac{\Theta''}{\Theta} = \text{constant}$

for symmetrical flow
constant $= n^2$

$\Rightarrow \boxed{r^2 R'' + r R' - n^2 R = 0}$ and $\boxed{\Theta'' + n^2\Theta = 0}$

by cauchy's method

$[D(D-1) + D - n^2] R = 0$

$[D^2 - n^2] R = 0$

$R(r) = c_1 r^n + c_2 r^{-n}$ — ①

by auxilliary eq^n

$m^2 + n^2 = 0$

$\Theta(\theta) = c_1' \cos n\theta + c_2' \sin n\theta$ — ②

put $n=1$ for symmetry (or)

$\phi = R(r) \cdot \Theta(\theta) = \sum (c_1 r^n + c_2 r^{-n})(c_3 \cos n\theta + c_4 \sin n\theta)$

from ①, ②

if $r = a$, $-\frac{\partial \phi}{\partial r} = U\cos\theta$

$\Rightarrow$ clearly $n = 1$ and $c_4 = 0$

$\Rightarrow \boxed{\phi(r,\theta) = \left(a_0 r + \frac{b_0}{r}\right)\cos\theta}$

$$-\frac{\partial\phi}{\partial r} = q_r = \frac{-\partial}{\partial r}\left(a_0 r + \frac{b_0}{r}\right)\cos\theta$$

$$= -\left(a_0 - \frac{b_0}{r^2}\right)\cos\theta = \left(-a_0 + \frac{b_0}{r^2}\right)\cos\theta$$

at $r = R$, $q_r = U\cos\theta$

$$\Rightarrow -a_0 + \frac{b_0}{R^2} = U \quad \text{— (3)}$$

$a_0 = 0$

$b_0 = R^2 U$

at $r = \infty$, $q_r = 0$

$$\Rightarrow -a_0 = 0 \quad \text{— (4)}$$

$\Rightarrow$ from (3) & (4)

$$\phi(r,\theta) = \frac{R^2 U\cos\theta}{r} \quad \text{if } R = a$$

then $\boxed{\phi(r\theta) = \frac{Ua^2}{r}\cos\theta}$ — Answer

To find stream lines

$$\frac{dr}{-\frac{\partial\phi}{\partial r}} = \frac{r\,d\theta}{-\frac{\partial\phi}{r\,\partial\theta}} \Rightarrow \frac{dr}{-\frac{Ua^2\cos\theta}{r^2}} = \frac{r\,d\theta}{-\frac{Ua^2\sin\theta}{r\cdot r}}$$

$$\Rightarrow \frac{r^2\,dr}{\cos\theta} = \frac{r^3\,d\theta}{\sin\theta} \Rightarrow \frac{dr}{r} = \frac{\cos\theta\,d\theta}{\sin\theta}$$

$$\ln r = \ln\sin\theta + \ln c \Rightarrow \boxed{r = c\sin\theta}$$ — Answer

where c is arbit constant

6.b) To find orthogonal surface to

$$\frac{z(x+y)}{3z+1} = c$$

Let $f = \frac{z(x+y)}{3z+1}$

$$\frac{\partial f}{\partial x} = \frac{z}{3z+1}$$

$$\frac{\partial f}{\partial y} = \frac{z}{3z+1}$$

$$\frac{\partial f}{\partial z} = (x+y)\left(\frac{(3z+1) - 3z}{(3z+1)^2}\right) = \frac{x+y}{(3z+1)^2}$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

orthogonal surface is

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z}$$

$$\frac{dx}{z/3z+1} = \frac{dy}{z/3z+1} = \frac{dz}{x+y/(3z+1)^2}$$

$$\Rightarrow \quad dx = dy = \frac{z(3z+1)\,dz}{(x+y)}$$

$\Rightarrow dx = dy \Rightarrow \boxed{x = y + c_1}$ ①

& $\frac{x\,dx}{x} = \frac{y\,dy}{y} = \frac{z(3z+1)\,dz}{x+y}$

by lagrange multiplies $(x, y$ & $-1)$

$\Rightarrow \frac{x\,dx + y\,dy - z(3z+1)\,dz}{(x+y) - (x+y)}$

$\Rightarrow x\,dx + y\,dy - (3z^2 + z)\,dz = 0$

$\boxed{\frac{x^2}{2} + \frac{y^2}{2} - z^3 - \frac{z^2}{2} = c_2}$ ②

orthogonal surface is

$\phi\left(x-y, \frac{x^2+y^2-z^2}{2} - z^3\right) = 0$ Answer

if it passes through $z=1$ & $x^2+y^2=1$

then put in ②

$\frac{1}{2} - \frac{3}{2} = c_2 \Rightarrow \boxed{c_2 = -1}$

$\boxed{x^2+y^2 = 2z^3 + z^2 - 2}$ Answer

& $x = y + c_1 \Rightarrow x^2+y^2-2xy = c_1^2$

$1 - 2xy = c_1^2$

$1 - 2y(y+c_1) = c_1^2$ $\}$ $c_1^2 + 2yc_1 + 2y^2 - 1$

c_1 discriminant $= 0 \Rightarrow B^2 = 4AC \Rightarrow (2y)^2 = 4(2y^2-1) \Rightarrow \boxed{y = \pm 1}$

c) y_x is polynomial in 3^{rd} degree

$\Rightarrow$ it takes 4 values of x

x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^3 y_0$
x_0	y_0				
		$y_1 - y_0$			
x_1	y_1		$y_2 - 2y_1 + y_0$	$\Delta^2 y_1 - \Delta^2 y_0$	—
		$y_2 - y_1$			
x_2	y_2		$y_3 - 2y_2 + y_1$		
		$y_3 - y_2$			
x_3	y_3				

by newton's forward interpolation

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0$$

where $u = \dfrac{x - x_0}{h}$

where $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$h = 1$

$\therefore \boxed{u = x}$

$$y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{\lfloor 2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{\lfloor 3}\Delta^3 y_0$$

$$\int_0^2 y_x\,dx = \int_0^2 y_0 + x\Delta y_0 + \frac{x(x-1)}{\lfloor 2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{\lfloor 3}\Delta^3 y_0 \, dx$$

$$x(x-1)(x-2) = x^3-3x^2+2x$$

$$= 2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{\lfloor 2}\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^2 + \frac{1}{6}\Delta^3 y_0\left[\frac{x^4}{4}-x^3+x^2\right]_0^2$$

$$= 2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0 + 0$$

$$= 2y_0 + 2(y_1-y_0) + \frac{1}{3}(y_2-2y_1+y_0)$$

$$= 2y_1 + \frac{y_2}{3} - \frac{2y_1}{3} + \frac{y_0}{3}$$

$$\boxed{\int_0^2 y_x\,dx = \frac{y_0}{3} + \frac{4}{3}y_1 + \frac{y_2}{3}}$$

Now

$$\int_1^2 y_x\,dx = y_0 x\Big|_1^2 + \Delta y_0 \frac{x^2}{2}\Big|_1^2 + \frac{\Delta^2 y_0}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 + \frac{1}{6}\Delta^3 y_0\left[\frac{x^4}{4}-x^3+x^2\right]_1^2$$

$$= \int_0^2 y_x\,dx - \int_0^1 y_x\,dx$$

$$= \int_0^2 y_x\,dx - \left[y_0 + \frac{\Delta y_0}{2} + \frac{\Delta^2 y_0}{2}\left(\frac{-1}{6}\right) + \frac{1}{6}\Delta^3 y_0\left(\frac{1}{4}\right)\right]$$

$$\int_0^1 y_x\,dx = y_0 + \frac{\Delta y_0}{2} - \frac{\Delta^2 y_0}{12} + \frac{\Delta^3 y_0}{24}$$

$$= y_0 + \frac{y_1 - y_0}{2} - \frac{(y_2 - 2y_1 + y_0)}{12} + \frac{1}{24}\left[y_3 - 2y_2 + y_1 - y_2 + 2y_1 - y_0\right]$$

$$= y_0\left(1 - \frac{1}{2} - \frac{1}{12} - \frac{1}{24}\right) + y_1\left(\frac{1}{2} + \frac{1}{6} + \frac{3}{24}\right)$$

$$+ y_2\left(\frac{-1}{12} - \frac{3}{24}\right) + \frac{y_3}{24}$$

$$= \frac{3}{8}y_0 + \frac{19}{24}y_1 - \frac{5y_2}{24} + \frac{y_3}{24}$$

$$\therefore \int_1^2 y_x\,dx = \int_0^2 y_x\,dx - \int_0^1 y_x\,dx$$

$$= \left[\frac{y_0}{3} + \frac{4}{3}y_1 + \frac{y_2}{3}\right] - \left[\frac{3}{8}y_0 + \frac{19}{24}y_1 - \frac{5y_2}{24} + \frac{y_3}{24}\right)$$

$$= \frac{-y_0}{24} + \frac{13y_1}{24} + \frac{13y_2}{24} - \frac{y_3}{24}$$

——— QED

8a) To reduce to cannonical form

$$y u_{xx} + (x+y) u_{xy} + x u_{yy} = 0$$

$$yr + (x+y)s + xt = 0$$

i.e. $Rr + Ss + Tt = 0$

$\hookrightarrow$ $\left.\begin{array}{l} R\lambda^2 + S\lambda + T = 0 \\ \\ y\lambda^2 + (x+y)\lambda + x = 0 \end{array}\right\}$ characteristic auxillary eqn

$$y\lambda(\lambda + x) + 1(y\lambda + x) = 0$$

$$(\lambda + 1)(y\lambda + x) = 0$$

$\Rightarrow \lambda = -1$ and $\lambda = \frac{-x}{y}$

Characteristic is $\frac{dy}{dx} + \lambda = 0$

$$\frac{dy}{dx} - 1 = 0 \quad \& \quad \frac{dy}{dx} - \frac{x}{y} = 0$$

$\Rightarrow y - x = c_1 \quad \& \quad \frac{y^2 - x^2}{2} = c_2$

Let $\left.\begin{array}{l} u(x,y) = x - y \\ v(x,y) = \frac{x^2 - y^2}{2} \end{array}\right\}$ they are L.I.

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial v}{\partial x} = x$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\cdot x$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\cdot x\right) = \frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} + x\left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right]$$

$$P \quad Q \quad R \quad S \quad T$$

$$r = R + xS + Q + x(S + xT)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= -\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial v}$$

$$t = \frac{\partial}{\partial y}\left(-\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial v}\right)$$

$$= -\underbrace{\frac{\partial^2 z}{\partial u^2}}_{R}\cdot\frac{\partial u}{\partial y} - \underbrace{\frac{\partial^2 z}{\partial u \partial v}}_{S}\cdot\left(\frac{\partial v}{\partial y}\right) - \underbrace{\frac{\partial z}{\partial v}}_{Q} - y\left[\underbrace{\frac{\partial^2 z}{\partial u \partial v}}_{S}\frac{\partial u}{\partial y} + \underbrace{\frac{\partial^2 z}{\partial v^2}}_{T}\cdot\frac{\partial v}{\partial y}\right]$$

$$= R + yS - Q + yS + y^2 T$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)$$

$$= \underbrace{\frac{\partial^2 z}{\partial u^2}}_{R}\cdot\frac{\partial u}{\partial y} + \underbrace{\frac{\partial^2 z}{\partial u \partial v}}_{S}\cdot\frac{\partial v}{\partial y} + x\left[\underbrace{\frac{\partial^2 z}{\partial u \partial v}}_{S}\frac{\partial u}{\partial y} + \underbrace{\frac{\partial^2 z}{\partial v^2}}_{T}\frac{\partial v}{\partial y}\right]$$

$$= -R - yS - xS - xyT$$

now, $yr + (x+y)s + xt = 0$

$$\Rightarrow y\left[\cancel{R}+2xS+Q+\cancel{x^2 T}\right] - (x+y)\left[\cancel{R}+(x+y)S+\cancel{xyT}\right]$$

$$+\ x\left[\cancel{R}+2yS+\cancel{y^2 T}-Q\right]=0$$

$$\Rightarrow S\left(2xy-(x+y)^2+2xy\right) = Q(x-y)$$

$$(x+y)^2 = x^2+y^2+2xy$$

$$S\left[2xy-x^2-y^2\right] = Q(x-y)$$

$$-S(x-y)^2 = Q(x-y)$$

$$S(x-y) = -Q$$

cannonical form is

$$\boxed{\frac{\partial^2 z}{\partial u\partial v}u = -\frac{\partial z}{\partial v}} ;\ \boxed{z_{uv}+\frac{z_v}{u}=0}$$

or $\quad u\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{\partial z}{\partial v}=0$

or $\quad uv\dfrac{\partial^2 z}{\partial u\partial v}+v\dfrac{\partial z}{\partial v}=0$ $\qquad \ln u = U \Rightarrow u = e^{U}$

$\qquad \ln v = V \Rightarrow v = e^{V}$

$$\Rightarrow (DD'+D')z=0$$

$$D'(D+1)z=0$$

$$Z = e^{-U}\phi_1(V)+\phi_2(U) \Rightarrow z = u\,\phi_1(\ln v) + \phi_2 \ln(u)$$

$$z = u\phi_1(v) + \phi_2(v)$$

$$z = (x-y)\phi_1\left(\frac{x^2-y^2}{2}\right) + \phi_2(x-y)$$

or $$\boxed{z = (x-y)\phi_1\left(\frac{x^2-y^2}{2}\right) + \phi_2(x-y)}$$

$$z = (x-y)\phi_1\left(\frac{x^2-y^2}{2}\right) + \phi_2(x-y)$$

where ϕ_1 & ϕ_2 are arbit functions

8b) $$V_r = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$V_\theta = -u\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For stream lines

$$\frac{dr}{V_r} = \frac{r\,d\theta}{V_\theta}$$

$$\Rightarrow \quad V_\theta\, dr - V_r\, r\, d\theta = 0$$

$$-u\left(1+\frac{a^2}{r^2}\right)\sin\theta\, dr \quad -u\left(1-\frac{a^2}{r^2}\right)\cos\theta\, r\, d\theta = 0$$

$$u\left(1+\frac{a^2}{r^2}\right)\sin\theta\, dr + u\left(r-\frac{a^2}{r}\right)\cos\theta\, d\theta = 0$$

$$M\,dr + N\,d\theta = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial \theta} &= u\left(1+\frac{a^2}{r^2}\right)\cos\theta \\ \frac{\partial N}{\partial r} &= u\left(1+\frac{a^2}{r^2}\right)\cos\theta\end{aligned}\right\}\quad \text{it's exact}$$

$$U\sin\theta\int\left(1+\frac{a^2}{r^2}\right)dr \;+\; \cancelto{0}{\int u\left(r-\frac{a^2}{r}\right)\cos\theta\, d\theta} = C$$

θ-constant

no term of r

$$\boxed{U\sin\theta\left(r-\frac{a^2}{r}\right) = C}$$ streamlines

Answer

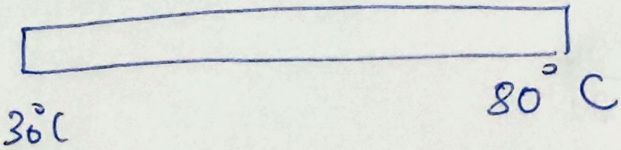
$$\boxed{\psi(r,\theta) = -U\sin\theta\left(r-\frac{a^2}{r}\right)}$$ stream function

Ans

Check:

$$\frac{\partial\psi}{r\,\partial\theta} = \frac{\partial\psi}{\partial r} = -u_r = -u\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

c)



for stready state

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$u = ax + b$$

at $x = 0 \Rightarrow b = 30$

at $x = l \Rightarrow al + 30 = 80 \Rightarrow a = \frac{50}{l}$

$$u(x) = \frac{50}{20}x + 30 = \frac{5x}{2} + 30$$

$\therefore$ at $t = 0 \Rightarrow u(x,0) = \frac{5x}{2} + 30$ — (1)

and later conditions

$$\left.\begin{aligned} u(0,t) &= 40°C \\ u(l,t) &= 60°C \end{aligned}\right\}$$

$u(x,t) = x + 40$

@ $x = 0, T = 40$

@ $x = 20, T = 60$

BC' Conditions

Let $y(x,t) = u(x,t) - (x + 40)$ — (2)

$\therefore y(0,t) = 0$ & $y(20,t) = 0$

Now heat eq^n is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{c^2 \partial t}$$

by variable separable $y = X(x) \cdot T(t)$

$$\frac{X''}{X} = \frac{T'}{c^2 T} = \text{constant}$$

(applying boundary conditions)

if constant $= 0$

$X''(x) = 0$

$ax + b = X(x)$

@ $x = 0$

$y(x) = 0$

@ $x = l$

$y(x) = 0$

$a = b = 0$

Rejected

if constant $= +ve$

constant $= n^2$

$X'' = n^2 X$

$X(x) = c_1 e^{nx} + c_2 e^{-nx}$

@ $x = 0$, $c_1 + c_2 = 0$

@ $x = l$ $c_1 e^{nl} + c_2 e^{-nl} = 0$

$c_1 = c_2 = 0$

Rejected

if constant $= -ve$

constant $= -n^2$

$X'' + n^2 X = 0$

$X(x) = c_1 \cos nx + c_2 \sin nx$

@ $x = 0$

$c_1 = 0$

@ $x = l$

$\sin nl = 0$

$nl = n'\pi$

$n = \frac{n'\pi}{l}$ $n' \in \mathbb{Z}$

$$\therefore y(x,t) = \sum a_n \sin \frac{\pi n x}{l} \, e^{-\frac{c^2 n^2 \pi^2}{l^2} t}$$

Now applying initial conditions

$$y(x,0) = u(x,0) - (x+40) \qquad \text{from } (1)$$

$$= \left(\frac{5x}{2}+30\right) - (x+40)$$

$$= \frac{3x}{2} - 10$$

$$\Rightarrow \quad \frac{3x}{2} - 10 = \sum a_n \sin \frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l \left(\frac{3x}{2}-10\right)\sin\frac{n\pi x}{l}\,dx \qquad \text{put } l=20$$

$$a_n = \frac{1}{10}\int_0^{20}\left(\frac{3x}{2}-10\right)\sin\frac{n\pi}{20}x\,dx$$

$$a_n = \frac{1}{10}\left[\frac{\left(\frac{3x}{2}-10\right)\cos\frac{n\pi x}{20}}{-\left(\frac{n\pi}{20}\right)}\Bigg|_0^{20} + \int_0^{20}\frac{\frac{3}{2}\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\,dx\right]$$

$$= \frac{1}{10}\left[\frac{20\cos n\pi + 10}{-\left(\frac{n\pi}{20}\right)} + \frac{3}{2}\cdot\frac{1}{\left(\frac{n\pi}{20}\right)^2}\underbrace{\sin\frac{n\pi x}{20}\Bigg|_0^{20}}_{0}\right]$$

$$a_n = \frac{-20}{n\pi}\left(1+2\cos n\pi\right)$$

$$\therefore \quad y(x,t) = \sum \frac{-20}{n\pi}(1+\cos 2n\pi) \sin \frac{n\pi x}{20}\, e^{-\frac{c^2 n^2 \pi^2}{400}t}$$

$$u(x,t) = y(x,t) + (x+40) \quad \text{from (2)}$$

$$u(x,t) = (x+40) + \sum \frac{-20}{n\pi}(1+\cos 2n\pi) \sin \frac{n\pi x}{20}\, e^{-\frac{c^2 n^2 \pi^2}{400}t}$$