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Maths Test Series - Test 7 & Test 8
(optional Next IAS)

Test- 7

1a) $M = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

we first find characteristic polynomial

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(\lambda-2)(\lambda-3)\right] = 0$$

$$(\lambda-2)^2(\lambda-3) = 0$$

minimal polynomial is polynomial of lowest degree $f(x)$ that satisfies $f(A) = 0$

check if $f(x) = (x-3)(x-2)$

$$= x^2 - 5x + 6$$

$$A^2 = \begin{bmatrix} 9 & -5 & 0 \\ 0 & 4 & 0 \\ 5 & -5 & 4 \end{bmatrix}$$

now

$$A^2 - 5A = \begin{bmatrix} 9 & -5 & 0 \\ 0 & 4 & 0 \\ 5 & -5 & 4 \end{bmatrix} - 5\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\therefore \boxed{A^2 - 5A + 6I = 0}$$

Hence minimal polynomial is

$(x-2)(x-3)$

1 b) $f(x)$ & $g(x)$ are continuous functions

$$f(x) = g(x) \quad \forall x \in \mathbb{Q}$$

To show $f(x) = g(x) \quad \forall x \in \mathbb{R}$

$\because \; f(x) = g(x)$

consider $h(x) = f(x) - g(x)$

$\therefore$ $h(x)$ is always continuous as $f(x)$ & $g(x)$ are continuous

$\therefore \; \lim_{x \to x_0} h(x) = h(x_0)$

if $x_0 \in \mathbb{Q}$

$\lim_{x \to x_0} h(x) = h(x_0) = 0 \quad [\because f(x_0) = g(x_0)]$

$\therefore \; \lim_{h \to 0} h(x_0 \pm h) = 0$

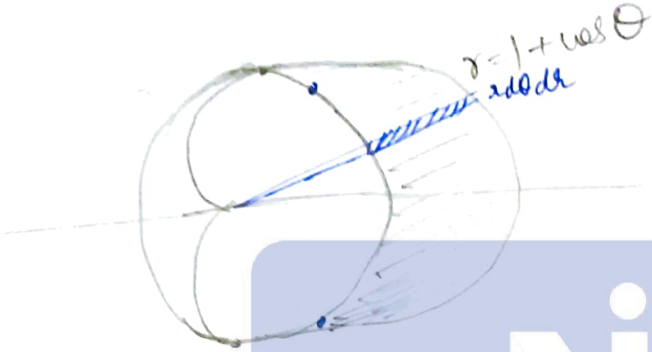
$\therefore$ for δ neighbourhood of each x_0

$h(x_0 + h) = 0 \Rightarrow f(x_0 + h) = g(x_0 + h)$

$\therefore \; \boxed{f(x) = g(x) \quad \forall x \in \mathbb{R}}$

$$I = \iint_R \sin\theta \, dA$$

$r = \cos\theta \Rightarrow$



$$I = \iint_R \sin\theta \, r\,dr\,d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=1}^{r=1+\cos\theta} \sin\theta \, r\,d\theta\,dr$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sin\theta}{2} \cdot \left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sin\theta}{2} \left[\cos^2\theta + 2\cos\theta\right] d\theta$$

$$= \int^{\pi/2} \frac{\sin\theta \cdot \cos^2\theta}{2} + \int_{-\pi/2}^{\pi/2} \frac{\sin 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin 2\theta \, d\theta = \frac{1}{2} \left. \frac{\cos 2\theta}{-2} \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{-1}{4}\left[\cos \pi - \cos -\pi\right] = 0$$

Ans $= 0$

1d) Generators are $\perp$ to plane

$$x + y - 3z = 5$$

$\therefore$ DRs of generator are $(1, 1, -3)$

DCs are $\left(\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}\right)$

Eq^n of cylinder is

$$\boxed{SS_1 = t^2}$$

$S_1 : l^2 + m^2 + n^2$

$t : lx + my + nz$

$\therefore$ Eq^n is

$$(x^2 + y^2 + z^2 - 9)(l^2 + m^2 + n^2) = (lx + my + nz)^2$$

$$11(x^2 + y^2 + z^2 - 9) = (x + y - 3z)^2$$

$$11x^2 + 11y^2 + 11z^2 - 99 = x^2 + y^2 + 9z^2 + 2xy - 6yz - 6xz$$

$$\boxed{10x^2 + 10y^2 + 2z^2 - 2xy + 6yz + 6xz - 99 = 0}$$

Answer

1e) Use Gauss - elimination

$$0.003x_1 + 59.14x_2 = 59.17$$

$$5.291x_1 - 6.13x_2 = 46.78$$

$$AX = B$$

$$[A|B] = \left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.13 & 46.78 \end{array}\right]$$

$$R_2 \to R_2 - \frac{5.291 R_1}{0.003}$$

$$\therefore \left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 0 & -104309.3767 & -104309.3767 \end{array}\right]$$

$$-104309.3767\, x_2 = -104309.3767$$

$$\therefore \boxed{x_2 = 1}$$

$$\therefore 0.003x_1 + 59.14 = 59.17$$

$$0.003x_1 = 0.03$$

$$\boxed{x_1 = 10}$$

Ans: $\boxed{x_1 = 10}$ & $\boxed{x_2 = 1}$

2a) T: linear operator on V

$\lambda_1 \dots \lambda_k$ are distinct eigen values

$v_1 \dots v_k$ are eigen vectors

To show $\{v_1 \dots v_k\}$ are independent (linearly)

Let's suppose, by contradiction they are not linearly independent

$\Rightarrow v_1 \dots v_k$ are linearly dependent

suppose $v_1 \dots v_n\ v_{n+1} \dots v_k$ are vectors $n < k$

and $v_{n+1}, v_{n+2} \dots v_k$ are L.D on $v_1 \dots v_n$

$\Rightarrow v_1 \dots v_n$ are L.I

$\therefore v_1 \dots v_n, v_{n+1}$ are L.D.

$\Rightarrow a_1 v_1 + a_2 v_2 \dots a_{n+1} v_{n+1} = 0$ — (1)

and $a_1, a_2 \dots a_{n+1} \neq 0$

$\therefore$ they are L.D.

by property of eigen values & eigen vectors

$$Tv_1 = \lambda_1 v_1$$

pre-multiply eq ① with T

$$\Rightarrow a_1(Tv_1) + a_2(Tv_2) \text{ --- } + a_{n+1}(Tv_{n+1}) = 0$$

$$\Rightarrow a_1\lambda_1 v_1 + a_2\lambda_2 v_2 \text{ --- } + a_{n+1}\lambda_{n+1} v_{n+1} = 0 \quad \text{—②}$$

now multiply eq ① with λ_{n+1}

$$\Rightarrow a_1\lambda_{n+1} v_1 + a_2\lambda_{n+1} v_2 \text{ --- } a_{n+1}\lambda_{n+1} v_{n+1} = 0 \quad \text{—③}$$

eq ③ − eq ②

$$\Rightarrow a_1(\lambda_{n+1} - \lambda_1)v_1 + a_2(\lambda_{n+1} - \lambda_2)v_2 \text{ -- } + a_n(\lambda_{n+1} - \lambda_n)v_n = 0$$

$\because$ v_1 -- v_n are L.I. (assumption) $n < k$

and $a_i(\lambda_{n+1} - \lambda_i)$ are [illegible]-zero

$\because \lambda_i \neq \lambda_j$

$\Rightarrow$ $\Rightarrow a_i$ -- $a_n = 0$

$\Rightarrow$ in eq ① $a_{n+1} v_{n+1} = 0$ $\Rightarrow a_{n+1} = 0$

we ....

Hence our assumption that these vectors are all linearly dependent is wrong

as $\alpha_1 - - \alpha_{n+1}$ are L.I.

Similarly we show

$\alpha_1 - - \alpha_{n+1}, \alpha_{n+2}$ are L.I.

$\vdots$

(on continuing)

$\vdots$

$\alpha_1 - - \alpha_n, \alpha_{n+1}, \alpha_{n+2}, - - - - - \alpha_k$

are are L.I.

Hence distinct eigen vectors are L.I.

b) To find critical points of

$$f(x) = (x-1)^{2/3}$$

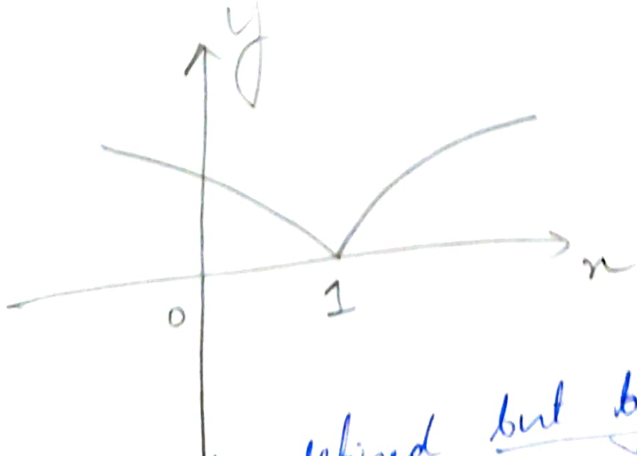
$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$$

for critical point

$$\left.\begin{array}{l} f'(x) = 0 \\ \text{i.e. } \dfrac{1}{(x-1)^{1/3}} = 0 \\ \text{which is not possible} \end{array}\right\} \quad f'(x) \text{ \underline{not} defined as } x \to 1$$

It's <u>neither</u> maxima, minima or inflection point

but when we draw the graph



Clearly $f'(x)$ is not defined <u>but by graph</u> we conclude that $x = 1$ is <u>local & global minima</u>

2 c) To show $(12, -18, 8)$ and $(-6, 18, -10)$ are feet of normal to ellipsoid $x^2+2y^2+3z^2=984$ that lie on $x+y+z=2$

let point be (α, β, γ)

feet of normal : (x_0, y_0, z_0)

DRs are $\langle x_0, 2y_0, 3z_0 \rangle$

$$\Rightarrow \frac{x_0}{\alpha - x_0} = \frac{2y_0}{\beta - y_0} = \frac{3z_0}{\gamma - z_0} = \lambda \text{ (say)}$$

if $x_0 + y_0 + z_0 = 2$ — (1)

$x_0 = \lambda\alpha - \lambda x_0$

$x_0(1+\lambda) = \lambda\alpha$

$x_0 = \frac{\lambda\alpha}{1+\lambda}$

$2y_0 = \beta\lambda - y_0\lambda$

$y_0(2+\lambda) = \beta\lambda$

$y_0 = \frac{\lambda\beta}{2+\lambda}$

$z_0 = \frac{\lambda\gamma}{3+\lambda}$

put in (1)

$$\lambda\left[\frac{\alpha}{1+\lambda} + \frac{\beta}{2+\lambda} + \frac{\gamma}{3+\lambda}\right] = 2 \quad \text{— (2)}$$

$\lambda = 1{:}2$

if one value is $(12, -18, 8)$

then $12 = \dfrac{r\alpha}{1+r}$, $-18 = \dfrac{r\beta}{2+r}$, $8 = \dfrac{r\gamma}{3+r}$

and $\dfrac{r^2\alpha^2}{(1+r)^2} + \dfrac{2r^2\beta^2}{(2+r)^2} + \dfrac{3r^2\gamma^2}{(3+r)^2} = 984$

$$\frac{\alpha^2}{(1+r)^2} + \frac{2\beta^2}{(2+r)^2} + \frac{3\gamma^2}{(3+r)^2} = \frac{984}{r^2}$$

& $\dfrac{\alpha}{1+r} + \dfrac{\beta}{2+r} + \dfrac{\gamma}{3+r} = \dfrac{2}{r}$

3a)

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

we find eigen values of A

$|A - \lambda I| = 0$

Rough

$\frac{-b}{a} = -3$

$\frac{c}{a} = 0$

$[-6-9+2-9+\cancel{12}+16]$

$\frac{-d}{a} = 4$

$\lambda^3 + 3\lambda^2 - 4 = 0$

$\lambda = 1, -2, -2$

Check eigen values and vectors at $\lambda = 1$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 4 & 7 & 3 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & -9 & -9 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = z$ & $y = -z$

eigen vector is $(1, -1, 1)$

(ii) if eigen value is -2

$(A + 2I)X = 0$

$$\begin{bmatrix} 4 & 6 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 3 \\ 4 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 6 & 3 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow x + y = 0$ & $z = 0$

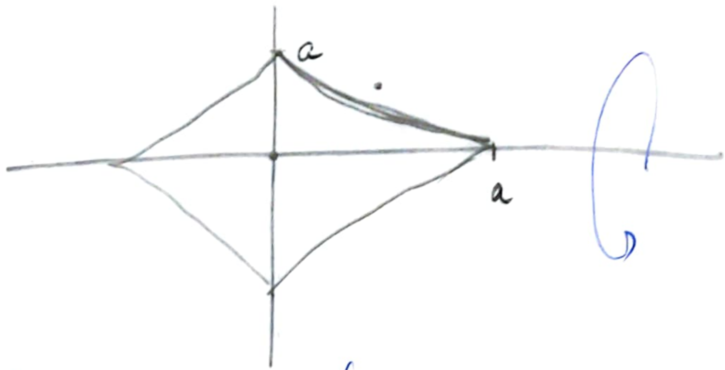
eigen vector is $(+1, -1, 0)$

$\therefore$ it has only 1 eigen vector

$\Rightarrow$ it's not diagnolizable

3b)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$



if it's rotated

$$V = \pi \int_0^a \int_0^{y=f(x)} 2y\,dy\,dx$$

$$= \pi \int_0^a \int_0^{(a^{2/3} - x^{2/3})^{3/2}} 2y\,dy\,dx$$

$$= \pi \int_0^a (a^{2/3} - x^{2/3})^3\,dx$$

$$= \pi \int_0^a (a^{2/3})^3 - x^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x \quad dx$$

Rough

~~Cancelled~~

$$= \pi \int_0^a a^2 - x^2 - 3a^{4/3} x^{2/3} + 3a^{2/3} x^{4/3} \, dx$$

$$= \pi \left[a^3 - \frac{\cancel{x}a^3}{3} - \frac{3a^{4/3} x^{2/3+1}}{\frac{2}{3}+1} + \frac{3a^{2/3} x^{4/3+1}}{\frac{4}{3}+1} \right]_0^a$$

$$= \pi a^3 \left[1 - \frac{1}{3} - \frac{3.3}{5} + \frac{3.3}{7} \right]$$

$$= \frac{16\pi a^3}{105}$$

c) To show $\perp$ from origin to generators of

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on cone

Let generator be

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c} \quad - (G)$$

Let $\perp$ from origin be — line L

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

$\therefore$ (G) $\perp$ (L)

$\Rightarrow$ $l(a\sin\theta)-(b\cos\theta)m+cn=0$

$$\boxed{al\sin\theta-bm\cos\theta+cn=0} \leftarrow (1)$$

and $\perp$ will intersect the generator G

$$\Rightarrow \begin{vmatrix} a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & c \\ l & m & n \end{vmatrix}=0$$

$$l\left[bc\sin\theta\right]+m\left[-ac\cos\theta\right]+n\left[-ab\right]=0$$

$$l\cdot\left(\frac{\sin\theta}{a}\right) - m\left(\frac{\cos\theta}{b}\right) - \frac{n}{c} = 0$$

$$l\; l(a\sin\theta) + m(-b\cos\theta) + n\cdot c = 0$$

$$\frac{l}{-\frac{c\cos\theta}{b} - \frac{b\cos\theta}{c}} = \frac{m}{-\frac{a\sin\theta}{c} - \frac{c\sin\theta}{a}} = \frac{n}{-\frac{b}{a}\sin\theta\cos\theta + \frac{a}{b}\sin\theta\cos\theta}$$

$$\frac{l}{\frac{(c^2+b^2)}{bc}(-\cos\theta)} = \frac{m}{-\sin\theta\left(\frac{a^2+c^2}{ac}\right)} = \frac{n}{\left(\frac{a^2-b^2}{ab}\right)\sin\theta\cos\theta} = r$$

$$l = -r\cos\theta\cdot\left(\frac{bc}{b^2+c^2}\right)$$

$$M = -r\sin\theta\left(\frac{ac}{a^2+c^2}\right)$$

$$n = r\sin\theta\cos\theta\left[\frac{(a^2-b^2)}{ab}\right]$$

elg.

$$\frac{n}{l} = \frac{-\sin\theta\,(a^2-b^2)c}{(b^2+c^2)\,a} \quad \& \quad \frac{n}{m} = \frac{-(\cos\theta)(a^2-b^2)c}{(a^2+c^2)b}$$

put $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \frac{a^2(b^2+c^2)^2 n^2}{c^2(a^2-b^2)^2 l^2} + \frac{b^2(c^2+a^2)^2}{c^2(a^2-b^2)^2 m^2} = 1$$

$$\Rightarrow \frac{a^2(b^2+c^2)^2}{l^2} + \frac{b^2(c^2+a^2)^2}{m^2} = \frac{c^2(a^2-b^2)^2}{n^2}$$

for cone put $x = lr$, $y = mr$, $z = nr$

$\because$ cone passes through $(0,0,0)$

& (l, m, n) are DRs of generator

$\Rightarrow$ cone is

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}$$

— QED

Section - B

5a)

$$(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0$$

let $x^2 = u$ & $y^2 = v$

$$\therefore (2u+3v-7)du = (3u+2v-8)dv$$

$$\frac{dv}{du} = \frac{2u+3v-7}{3u+2v-8}$$

$u = p+h$ & $v = q+k$

$\therefore$
$$\left.\begin{array}{l} 2h+3k-7=0 \\ 3h+2k-8=0 \end{array}\right\} \begin{array}{l} h=2 \\ k=1 \end{array}$$

$$\frac{dq}{dp} = \frac{3q+2p}{2q+3p}$$

$$q = pt \Rightarrow \frac{dq}{dp} = t + p\frac{dt}{dp}$$

$$t + p\frac{dt}{dp} = \frac{3t+2}{2t+3} \Rightarrow p\frac{dt}{dp} = \frac{\cancel{3t}+2-2t^2-\cancel{3t}}{2t+3}$$

$$\frac{(2t+3)\,dt}{-1+t^2} = \frac{-2\,dp}{p}$$

$$\frac{2t+3}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1}$$

$$\Rightarrow At - A + Bt + B = 2t + 3$$

$$\left.\begin{array}{l} A+B=2 \\ -A+B=3 \end{array}\right\} \quad A = \frac{-1}{2}, \; B = \frac{5}{2}$$

$\therefore$ on integrating

$$A\ln(t+1) + B\ln(t-1) = -2\ln p + c$$

$$-\frac{1}{2}\ln(t+1) + \frac{5}{2}\ln(t-1) + 2\ln p = c$$

$$\frac{(t-1)^5 p^4}{(t+1)\cancel{p^4}} = c$$

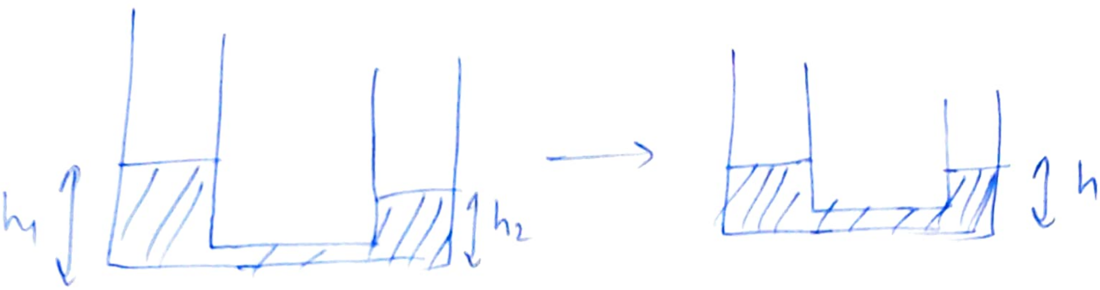
put $t = \frac{2}{p} = \frac{v-1}{u-2} = \frac{y^2-1}{x^2-2}$

$$\frac{(q-p)^5}{(q+p)} = c$$

$$\Rightarrow \boxed{(y^2-x^2+1)^5 = c(y^2+x^2-3)}$$

where c is arbitrary constant

3b)



Volume will be constant

$$Ah_1 + Ah_2 = 2Ah$$

$$h = \frac{h_1 + h_2}{2}$$

W_1(done by gravity) $= \rho A h_1^2 - \rho A h^2$

W_2 (work against gravity) $= \rho A h^2 - \rho A h_2^2$

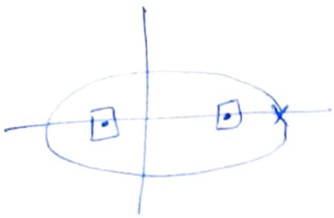
Total work done (by gravity) $= \rho A(h_1^2 + h_2^2 - 2h^2)$

$$= \rho A\left(\frac{2h_1^2 + 2h_2^2 - (h_1 + h_2)^2}{2}\right)$$

$$= \rho A\left[\frac{h_1^2 + h_2^2 - 2h_1h_2}{2}\right] = \rho A\frac{(h_1 - h_2)^2}{2}$$

Ans

5(c)



$$u = \frac{1+e\cos\theta}{l}$$ } motion for ellipse

At apse $\frac{du}{d\theta} = 0$

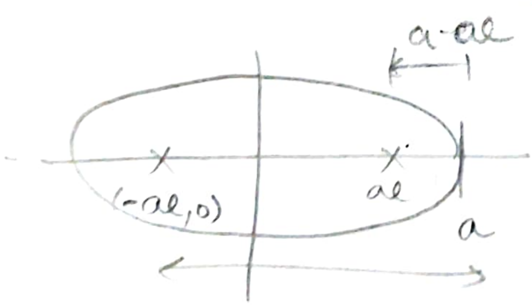
$$\frac{du}{d\theta} = \frac{-e\sin\theta}{l} = 0 \Rightarrow \boxed{\theta = 0^\circ}$$

$\therefore$ when $\theta = 0$ (or at apse)

$$u = \frac{1+e}{l} \qquad l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

$$u = \frac{1+e}{a(1-e^2)} = \frac{1}{a(1-e)} = \frac{1}{a-ae}$$

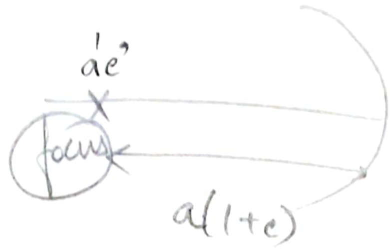
$$\therefore \boxed{r = a - ae}$$



center is transferred to $(-ae, 0)$

(+e)
e

$$\boxed{a + ae = a'(1-e')} \quad -①$$



conserving velocity

$$v_1 = v_2$$

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = h^2\left[u^2 + \left(\frac{e\sin\theta}{l}\right)^2\right]$$

$$= h^2\left[\frac{(1+e\cos\theta)^2 + (e\sin\theta)^2}{l^2}\right]$$

$$= \frac{h^2}{l^2}\left(1+e^2+2e\cos\theta\right)$$

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} \Rightarrow l^2 = a^2(1-e^2)^2$$

if $\theta = 0$

$$v^2 = \mu\frac{(1+e)^2}{a(1-e^2)} = \frac{(1+e)^2}{a(1+e)(1-e)} = \frac{(1+e)}{a(1-e)}$$

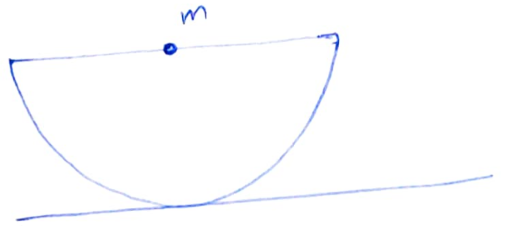
$$\frac{1}{a}\,\frac{1+e}{1-e} = \frac{1}{a'}\,\frac{1+e'}{1-e'}$$

$$\& \quad a' = \frac{a(1+e)}{1-e'} \quad \text{from ①}$$

$$\frac{1}{a}\cdot\frac{1+e}{1-e} = \frac{1-e'}{a(1+e)} \Rightarrow 1-e' = \frac{(1+e)^2}{1-e}$$

$$e' = 1 - \frac{(1+e)^2}{1-e} = \frac{1-e-1-e^2-2e}{1-e} = \frac{e(3+e)}{1-e}$$

5d)



$$h_{cm} = \frac{M\frac{5r}{8} + mr}{M+m}$$

$$h_{cm} \le \frac{r_1 r_2}{r_1+r_2}$$

or $\frac{1}{h_{cm}} \ge \frac{1}{r_1} + \frac{1}{r_2}$

for plane $r_2 \to 0$

& $r_1 \to r$ (for hemisphere

$\frac{1}{h_{cm}} \ge \frac{1}{r}$

or $h_{cm} \le r$

required condition

$$h_{cm} = r\left[\frac{\frac{5M}{8} + m}{M+m}\right] = r\left[\frac{M+m - \frac{3M}{8}}{M+M}\right]$$

$$= r\left[1 - \frac{\frac{3M}{8}}{M+m}\right] \le r$$

$\therefore$ Eqm is stable $\because$ $\boxed{h_{cm} < r}$

5e) Given

a, b, c & a', b', c' are reciprocal system

$\therefore \; a' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}\,\vec{c}]}$

$b' \Rightarrow \dfrac{\vec{c} \times \vec{a}}{[a, b, c]}$

$c' = \dfrac{\vec{a} \times \vec{b}}{[a\,b\,c]}$

$\vec{a} \cdot \vec{a}' = \dfrac{\vec{a} \cdot \vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$

$\vec{b} \cdot \vec{b}' = \dfrac{\vec{b} \cdot (\vec{c} \times \vec{a})}{[a\,b\,\vec{c}]} = \dfrac{[\vec{b}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$= 1$

similarly $\vec{c} \cdot \vec{c}' = 1$

$\therefore \boxed{\sum \vec{a} \cdot \vec{a}' = 3}$

8a)

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = x^4 - 4x^2y$$

or $$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$$

let $x = z$ (change of independent variable)

$\because Q = 4x^2 \quad \therefore \quad z = x^2$

$\therefore \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{constant}$

let $z = x^2$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot 2x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\cdot\frac{dy}{dz}\right) = 2\cdot\frac{dy}{dz} + 2x\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= 2\cdot\frac{dy}{dx} + (2x)^2\frac{d^2y}{dz^2}$$

$\Rightarrow y'' - \frac{1}{x}y' + 4x^2y = x^4$ becomes

$$\left[4x^2\frac{d^2y}{dz^2} + \cancel{2\frac{dy}{dz}}\right] - \cancel{\frac{1}{x}\cdot 2x\frac{dy}{dz}} + 4x^2y = x^4$$

$$\Rightarrow 4x^2 \frac{d^2y}{dz^2} + 4x^2 y = x^4$$

$$\text{or} \quad \frac{d^2y}{dz^2} + y = \frac{x^2}{4}$$

put $x^2 = z$

$$\boxed{\frac{d^2y}{dz^2} + y = \frac{z}{4}}$$

$$(D^2+1)y = \frac{z}{4}$$

$y_{C.F}$ = solution of homogeneous eq^n

AE : $m^2+1=0 \Rightarrow m = \pm i$

$$y_{CF} = C_1 \sin z + C_2 \cos z$$

$$y_{PI} = \frac{1}{D^2+1} \cdot \frac{z}{4} = (1+D^2)^{-1} \frac{z}{4}$$

$$= (1 - D^2 + D^4 \cdots) \frac{z}{4} = \frac{z}{4}$$

$$\therefore \boxed{y = C_1 \sin x^2 + C_2 \cos x^2 + \frac{x^2}{4}}$$

where C_1 & C_2 are arbitrary

b) To show $\vec{F}$ is irrotational

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

find $\vec{\nabla} \times \vec{F}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - xz & z^2 - xy \end{vmatrix}$$

$$= \sum \left\{ \frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - xz) \right\} \hat{i}$$

$$= \sum (-x + x)\hat{i} = \sum 0\hat{i} = \vec{0}$$

(by symmetry)

$\because \vec{\nabla} \times \vec{F} = 0 \implies \vec{F}$ is irrotational

To find scalar ϕ

$$\vec{F} = \vec{\nabla}\phi$$

$$\Rightarrow \vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x^2 - yz \quad \Rightarrow \phi = \frac{x^3}{3} - xyz + f(y,z) \quad — (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 - xz \quad \Rightarrow \phi = \frac{y^3}{3} - xyz + g(x,z) \quad — (2)$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \quad \Rightarrow \phi = \frac{z^3}{3} - xyz + h(x,y) \quad — (3)$$

from (1), (2), (3)

$$\boxed{\phi(x,y,z) = \frac{x^3+y^3+z^3}{3} - xyz + C}$$

where C is arbitrary constant

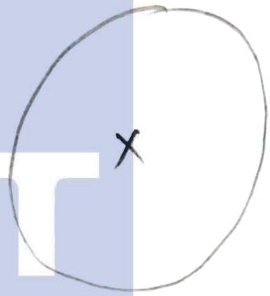
8c) $\vec{F} = (y^2+z^2-x^2)\hat{i} + (-y^2+z^2+x^2)\hat{j} + (y^2+x^2-z^2)\hat{k}$

to show stoke's theorem is verified

$$\iint_S \text{curl } F \cdot \hat{n}\, ds$$

$S: x^2+y^2+z^2-2ax+az=0$ above $z=0$ (plane)

$S: (x-a)^2+y^2+\left(z+\frac{a}{2}\right)^2 = \frac{5a^2}{4}$



$$\text{curl } f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= (2y-2z)\hat{i} + (2z-2x)\hat{j} + (2x-2y)\hat{k}$$

$$\hat{n} = \frac{(2x-2a)\hat{i} + 2y\hat{j} + (2z+a)\hat{k}}{\sqrt{(2x-2a)^2+(2y)^2+(2z+a)^2}}$$

$$\iint_S \text{curl } f \cdot \hat{n}\, ds = \iint_R \text{curl } f \cdot \hat{n} \cdot \frac{dx\,dy}{\hat{n}\cdot\hat{k}} =$$

$$= \iint_R \frac{2(y-z)\cdot 2(x-a) + 2(\cancel{z}-x)\cdot 2y + 2(x-y)(2z+a)}{2z+a}\, dx\, dy$$

$$R: (x-a)^2 + y^2 = a^2$$

$$= \frac{2}{a} \iint 2y(x-a) + -2xy + a(x-y)\, dx\, dy$$

$$= \frac{2}{a} \iint a(x-y-2y)\, dx\, dy$$

$$= \frac{2a}{a} \iint (x-3y)\, dx\, dy$$

$x = r\cos\theta$, $y = r\sin\theta$

$r^2 = 2ar\cos\theta \Rightarrow \boxed{r = 2a\cos\theta}$

$$= \frac{2a}{a} \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2(\cos\theta - 3\sin\theta)\, dr\, d\theta$$

$$= \frac{2a}{a} \int_{-\pi/2}^{\pi/2} (\cos\theta - 3\sin\theta)\frac{(2a\cos\theta)^3}{3}\, d\theta$$

$$= \frac{(2a)^4}{3a} \int_{-\pi/2}^{\pi/2} \cos^4\theta - 3\sin\theta \cos^3\theta \, d\theta$$

$$= 2 \cdot \frac{(2a)^4}{a\,3} \int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{2^5 a^3}{3} \cdot \frac{3\pi}{16} = 2a^3$$

— (1)

As per stoke's theorem

$$\iint_{curl} F \cdot n \, ds = \int \vec{F} \cdot d\vec{r}$$

put ; $z = 0$ & $x^2 + y^2 = 2ax$

$(x-a)^2 + y^2 = a^2$

$x = a + a\cos\theta$

$\int (y^2 - x^2)\,dx + (x^2 - y^2)\,dy$ $\quad y = a\sin\theta$

$y^2 - x^2 = a^2\sin^2\theta - (a + a\cos\theta)^2 = a^2[\sin^2\theta - \cos^2\theta - 1 - 2\cos\theta]$

$dx = -a\sin\theta, \quad dy = a\cos\theta \, d\theta$

$$a^2 \int_0^{2\pi} (\cos 2\theta + 1 + 2\cos\theta)(\cos\theta + \sin\theta) \, d\theta$$

$$= a^3 \int_0^{2\pi} (\cos^2\theta - \sin^2\theta)(\cos\theta + \sin\theta) + 2\cos^2\theta \, d\theta$$

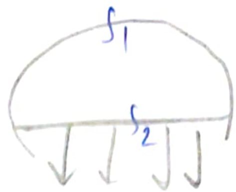
$$= a^3 \cdot 8 \int_0^{\pi/2} \cos^2\theta \, d\theta = a^3 \cdot 8 \cdot \frac{\pi}{4} = 2a^3$$

— (2)

Hence from (1) & (2) Ans = $2a^3$

Check:

take $\hat{n} = -\hat{k}$



$$\iint_{S_1} \text{curl} F \cdot \hat{n}\, dS = -\iint_{S_2} \text{curl} F \cdot \hat{n}\, dS$$

$$= \iint (2x - 2y)\, dS = 2\iint (x-y)\, dx\, dy$$

$$= 2\iint (x-y)\, dx\, dy$$

$$= \oint (y^2 - x^2)\, dx + (x^2 - y^2)\, dy$$

Green's theorem

$$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$