NEXT IAS

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Test - Test (06) - Full syllabus -(P2)

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1a) To find units of $z[\sqrt{-5}]$

$$z[\sqrt{-5}] = \{a + b\sqrt{-5} \text{ where } a, b \in z\}$$

a unit in an integral domain is that element whose multiplicative inverse exist

if u is unit then $\exists\, u'$ s.t. $uu' = 1$

To find units of $z[\sqrt{-5}]$

if $a+b\sqrt{-5}$ is unit than $\exists\, c+d\sqrt{-5}$

$$(a+b\sqrt{-5})(c+d\sqrt{-5}) = 1$$

$$\Rightarrow ac - 5bd + \sqrt{-5}[bc + ad] = 1$$

$$\Rightarrow \boxed{ac - 5bd = 1}$$

$$\text{and } \boxed{bc + ad = 0}$$

where a, b, c, d are $\in Z$

$$\Rightarrow ac = 1 \ \& \ bd = 0$$

$$\boxed{\text{units are } 1 \text{ and } -1}$$

1b) Define winding number.

1c) To show every convergent sequence is bounded

Let $\{x_n\}$ be cgt

$\therefore \underset{n\to\infty}{lt} \{x_n\} = l$ (finite)

$\Rightarrow |x_n - l| < \epsilon$ for $n \geq m$

for given $\epsilon > 0$ there exists values of $m \in Z$ such that $|x_n - l| < \epsilon$

Hence $\{x_n\}$ is a bounded sequence

Conversely, let $y_n = (-1)^n$

clearly $y_n = \{-1$ or 1 dependending on $n\}$

$\therefore |y_n| < 2$ (say)

$\because |y_n| = 1$

Hence $\{y_n\}$ is bounded

but clearly $\{y_n\}$ is not convergent

as $\underset{n\to\infty}{lt} \{y_n\}$ does not exist

1d) To prove converse of Lagrange theorem is true in cyclic group.

<u>Lagrange Theorem</u>: If H is subgroup of G then $o(H)$ divides $o(G)$

we show that in cyclic group if $o(H) \mid o(G)$ then $H \subseteq G$ and subgroup of G

Since G is cyclic

$$G = \langle a \rangle \quad \text{where } a^n = e$$

let H be subset of elements of G such that $o(H) \mid o(G)$

1(c)

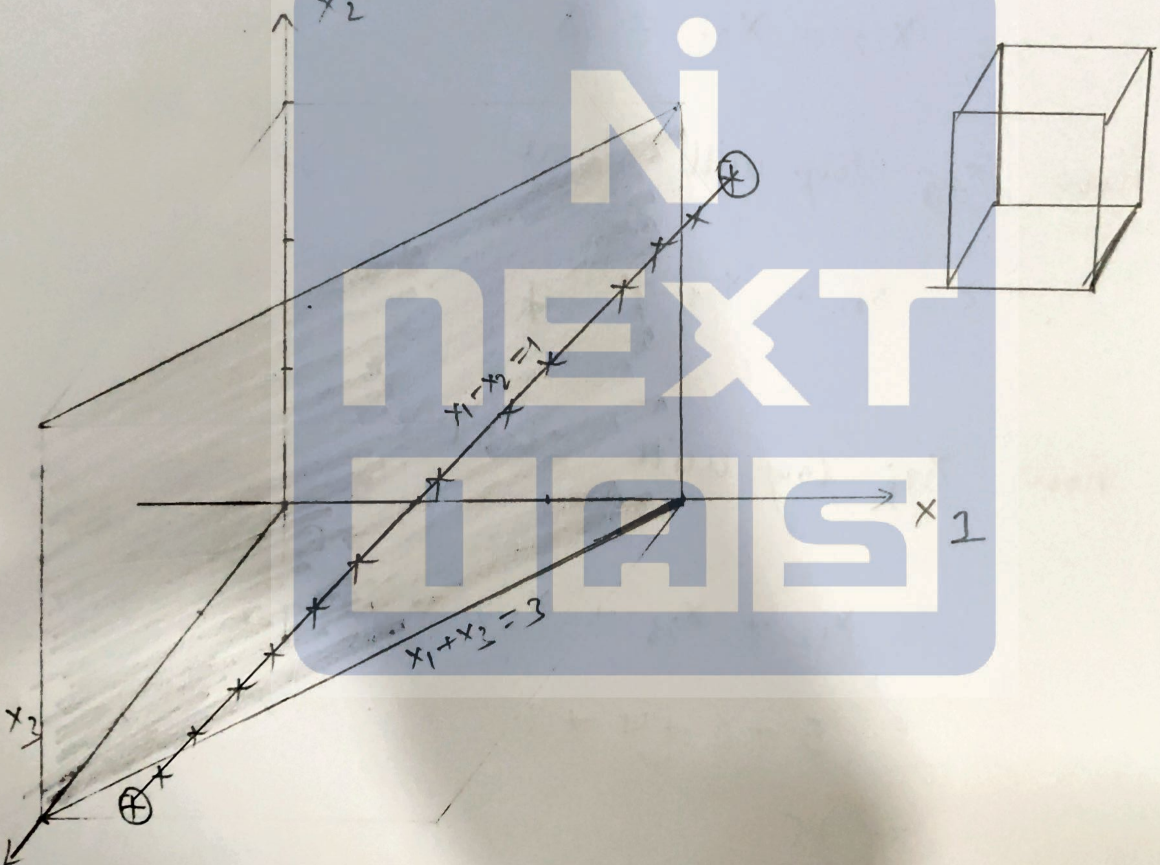
Max: $z = 3x_1 + 2x_2$

subject to

$x_1 - x_2 \geq 1$

$x_1 + x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$



for $x_1 + x_3 \geq 3$
cover all space beyond frame

& for $x_1 - x_2 \geq 1$ all points to right of line

for maximization, there is **no** bounded solution

3a) To show A_4 has no subgroup of order 6

A_4 is group of even permutations on S_4

S_4 is permutations on a, b, c, d

$o(S_4) = \lfloor 4 = 24$

$\therefore$ $o(A_4) = 12$

Elements of S_4 are

even [(a)

odd [(abcd), (abdc), (acbd), (acdb), (adbc), (adcb)

even [(abc), (acb), (acd), (adc), (abd), (adb), (bcd), (bdc),

even [(ab)(cd), (ac)·(bd), (ad)(bc)

odd [(ab), (ac), (ad), (bc), (bd), (cd)]

$\therefore$ A_4 = { (a), (abc), (acb), (acd), (adc), (abd), (adb), (bcd), (bdc), (ab)(cd), (ac)(bd), (ad)(bc) }

To show A_4 does not have subgroup of order 6

If suppose on contrary it has a subgroup of 6 then it will have (a) as identity element

Hence it must have 5 other elements

suppose

(abc) is one of element — 2^{nd} element

then (abc) (abc) = (acb) — 3^{rd} element

and (abc) (abc) (abc) = (a)

Let another element be (acd) — 4^{th} elem

now (acd) (acd) = adc — 5^{th} element

but (abc) (acd) = (abd) — which doesnot belong to set of above 5 elements

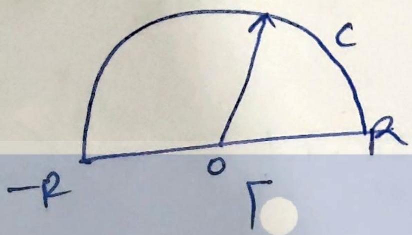
if another element was (ac) (bd) then also it will be a contradiction

Hence, A_4 cannot have subgroup of order 6

3b) $$\int_0^\infty \frac{dx}{x^4+a^4}$$

Consider

$$\int_{-\infty}^{\infty} \frac{dz}{z^4+a^4}$$ and curve Γ such that



$$\int_\Gamma \frac{dz}{z^4+a^4} = \int_C \frac{dz}{z^4+a^4} + \int_{-R}^{R} \frac{dz}{z^4+a^4}$$

$$\int_\Gamma \frac{dz}{z^4+a^4} = 2\pi i\,(\text{sum of residues})$$

for $z^4+a^4=0$ $\Rightarrow$ $z=(-1)^{\frac{1}{4}}a$

$\Rightarrow$ $z^4 = -a^4$ $\Rightarrow$

$$z = e^{\frac{(2k+1)\pi i}{4}}\cdot a \qquad k = 0,1,2,3$$

$$\underbrace{z = e^{i\pi/4}\cdot a,\ z = e^{i3\pi/4}\cdot a}_{\text{within contour}},\ z = e^{i\frac{5\pi}{4}},\ z = e^{i\frac{7\pi}{4}}$$

$$\therefore\ \text{Res}\,(z=\alpha) = \lim_{z\to\alpha} \frac{z-\alpha}{z^4+a^4} = \frac{1}{4z^3}$$

$\therefore$ Sum of residues

$$= \frac{1}{4}\left[z_1^{-3} + z_2^{-3}\right]$$

$$= \frac{1}{4a^3}\left[e^{-\frac{3i\pi}{4}} + e^{-\frac{i9\pi}{4}}\right]$$

$$= \frac{1}{4a^3}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4a^3}\left[\cancel{\frac{-1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}} \cancel{+\frac{1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}}\right]$$

$$= \frac{-i}{4a^3}\cdot\sqrt{2}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \cdot \frac{-i\sqrt{2}}{4a^3} = \frac{2\sqrt{2}}{4a^3}$$

$$\int_C \frac{dz}{z^4+a^4} = \int_0^{\pi} \frac{Re^{i\theta}\,d\theta}{R^4e^{i\theta}+a^4}$$

$$\int_0^{\pi}\left|\frac{Re^{i\theta}\,d\theta}{R^4e^{i\theta}+a^4}\right| \le \int_0^{\pi}\frac{R\,d\theta}{|R^4|-a^4} = \frac{R\pi}{R^4-a^4}$$

$$\lim_{R\to\infty} \frac{R\pi}{R^4-a^4} = 0$$

$$\therefore \lim_{R\to\infty}\left|\int_C \frac{dz}{z^4+a^4}\right| \le 0 \Rightarrow \lim_{R\to\infty}\int_C \frac{dz}{z^4+a^4} = 0$$

Hence

$$\int_{\Gamma} f(z)\,dz = \int_{C} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

where $f(z) = \frac{1}{z^4+a^4}$

taking limit $R \to \infty$

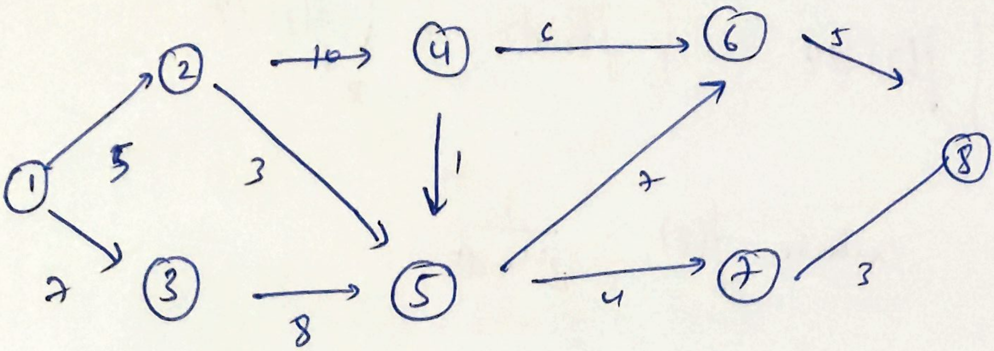
$$\therefore \quad \frac{2\sqrt{2}}{4a^3} = 0 + \int_{-\infty}^{\infty} \frac{dz}{z^4+a^4}$$

$$2\int_{0}^{\infty} \frac{dz}{z^4+a^4} = 2\,\frac{\sqrt{2}}{4a^3}$$

$$\boxed{\int_{0}^{\infty} \frac{dz}{z^4+a^4} = \frac{\sqrt{2}}{4a^3}}$$

Answer

3c) To formulate an LPP.



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Step I : Read all the nodes as $x_i$
Step II : Read all costs between nodes as $x_i x_j$ $(i \neq j)$
    [Eg: cost between ① & ② be $x_1 x_2 = 5$]
Step III : For other $x_i x_j$ value not provided consider it to be
    very large say 1000 & $S = 0$
Step IV : For $i = 1$, run loop till $i = 7$
Step V : For $j = 2$, run loop till $j = 8$
Step VI : If $i \geq j$, Go to step VIII
Step VII : If $x_i x_j < x_i x_{j+1}$    $S \to S + x_i x_j$
    Else Go to step VIII
Step VIII : $j \to j + 1$, Go to step 5
Step IX : $i \to j$, Go to step 4 ; Step X : Print S
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Therefore through the linear programming model

x_{12} will be selected

$\therefore \; s = x_{12} = 5$

now x_{2j} loop will start

again x_{25} will be selected

$\therefore \; s = x_{12} + x_{25} = 8$

now x_{5j} loop will start

$\therefore \; s = x_{12} + x_{25} + x_{57} = 12$

now x_{7j} loop will start

$\therefore \; s = x_{12} + x_{25} + x_{57} + x_{78}$

$= 5 + 3 + 4 + 3$

$= \underline{\underline{15}}$ Answer

Section-B

5 a) $xp = yq$, $:\, z(xp+yq) = 2xy$

To show they are compatible

eq ①: $xp - yq = 0$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$$

$xy = c_1$

$z = c_2$

eq ②: $z(xp+yq) = 2xy$

$$\Rightarrow \frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{2xy}$$

$x = c_3 y$

& $\frac{dy}{z} = \frac{dz}{2x}$

$\frac{dy}{z} = \frac{dz}{2c_3 y}$

$c_3 y^2 = \frac{z^2}{2} + c_4$

$xy = \frac{z^2}{2} + c_4$

for equations to be compatible

$\boxed{c_1 = \frac{c_2^2}{2} + c_4}$ and $c_3 y^2 = c_1$

$\boxed{y^2 = \frac{c_1}{c_3}}$

Hence $z \cdot 2px = 2xy$

(or) $p = \frac{y}{z}$ $\quad q = \frac{x}{z}$

5

$p = \frac{y}{z}$, $q = \frac{x}{z}$

$z\frac{\partial z}{\partial x} = y$, $z\frac{\partial z}{\partial y} = x$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{z}$$

$$\boxed{DD'z = \frac{1}{z}}$$

$DD'z - \frac{1}{z} = 0$

$$\boxed{z^2 = xy + c}$$

where c is arbit constant

$zp = y$ | $zq = x$

$\frac{dx}{z} = \frac{dz}{y}$ | $\frac{dxy}{z} = \frac{dz}{x}$

$$\frac{y\,dx}{z} = \frac{x\,dy}{z}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\boxed{x = c'y}$$

$$\frac{\partial z}{\partial y} = \frac{c'y}{z}$$

$$\boxed{z^2 = c'y^2 + c''}$$

5b) $f(x) = \sin x$ is extrapolated by polynomial degree 9 with 10 points

To find Error

$$\text{Error} = \left| \prod_{i=1}^{10} \frac{(x-x_i)}{\lfloor 10} f^{10}(\xi) \right|$$

where $f(x) = \sin x$

$$\left| f^{10}(\varepsilon) \right| = |\sin x| \leq \sin 1 \quad x \in [0,1]$$

& $\prod_{i=1}^{10} (x-x_i) = (x-x_1)(x-x_2) - - (x-x_{10})$

To find max. value of $\Pi(x-x_i)$

$$\frac{d}{dx} \Pi(x-x_i) = 0$$

$\therefore$ Max. Error is $\Pi \frac{(x-x_i)}{\lfloor 10} \cdot \sin 1$

$x \in (0,1]$

5c)

$$((p \wedge q) \rightarrow r) \vee (p \wedge q \rightarrow \sim r)$$

if $a \rightarrow b$ then $\sim a \cup b$

so, $(p \wedge Q) \rightarrow R$ then $\sim (p \wedge q) \cup r$

$$= \overline{PQ} + R = \bar{P} + \bar{Q} + R$$

& $(p \wedge q) \rightarrow \sim r = \overline{PQ} + \bar{R} = \bar{P} + \bar{Q} + \bar{R}$

now,

$$(p \wedge q \rightarrow r) \cup (p \wedge q \rightarrow \sim r)$$

$$= \overline{PQ} + R + \overline{PQ} + \bar{R}$$

$$= \overline{PQ} = \sim p \cup \sim q \quad \text{or} \quad \sim (p \wedge q)$$

To write in normal disjunctive format

$$\bar{P} + \bar{Q} = \bar{P}(Q+Q)(R+\bar{R}) + (P+\bar{P})\bar{Q}(R+\bar{R})$$

$$= \bar{P}(QR + \bar{Q}R + Q\bar{R} + \bar{Q}\bar{R}) + (PR + \bar{P}R + P\bar{R} + \bar{P}\bar{R})\bar{Q}$$

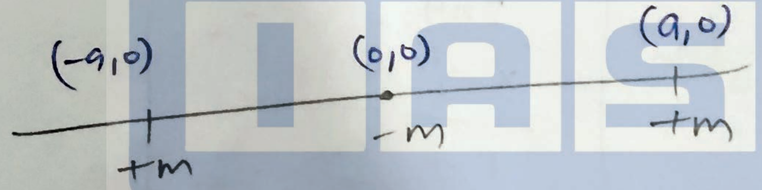
$$= \bar{P}QR + \bar{P}\bar{Q}R + \bar{P}Q\bar{R} + \bar{P}\bar{Q}\bar{R} + P\bar{Q}R + P\bar{Q}\bar{R}$$

so normal disjunctive form is

$$(\sim p \wedge q \wedge r) \cup (\sim p \wedge q \wedge \sim r) \cup (\sim p \wedge \sim q \sim r) \cup (\sim p \wedge \sim q \wedge \sim r)$$
$$\cup (p \wedge \sim q \wedge r) \cup (p \wedge \sim q \wedge \sim r)$$

which is neither tautology nor contradiction

5d) To find stream function of two equal sources and an equal sink midway



let them be place along x-axis & sink at origin & distance between two sources be 2a

$$\therefore w = -m\log(z+a) - m\log(z-a) + m\log z$$

complex potential

$$w = -m\log(z^2-a^2) + m\log z$$

$$\boxed{w = -m\log\left(\frac{z^2-a^2}{z}\right)}$$

$$W = \phi + i\psi$$

$$\therefore \ \psi = -m \tan^{-1} \frac{y}{x-a} - m \tan^{-1}\left(\frac{y}{x+a}\right) + m \tan^{-1}\left(\frac{y}{x}\right)$$

$$= -m\left[\tan^{-1}\left(\frac{y}{x-a}\right) + \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= -m\left[\tan^{-1}\left(\frac{\frac{y}{x-a} + \frac{y}{x+a}}{1 - \frac{y^2}{x^2-a^2}}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= -m\left[\tan^{-1}\left(\frac{2xy}{x^2-a^2-y^2}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= -m \tan^{-1}\left[\frac{\frac{2xy}{x^2-y^2-a^2} - \frac{y}{x}}{1 + \frac{2xy}{x^2-y^2-a^2} \cdot \frac{y}{x}}\right]$$

$$= -m \tan^{-1}\left[\frac{2x^2y - yx^2 + y^3 + a^2y}{x^3 + xy^2 - a^2x}\right]$$

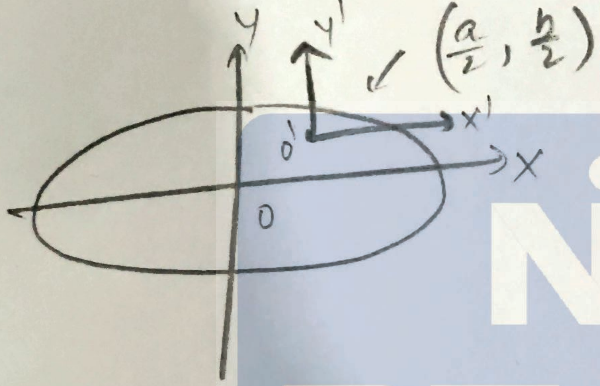
$$\psi = -m \tan^{-1}\left[\frac{x^2y + y^3 + a^2y}{x(x^2+y^2-a^2)}\right]$$

$$= -m \tan^{-1}\left(\frac{y}{x} \frac{(x^2+y^2+a^2)}{(x^2+y^2-a^2)}\right)$$

c) To show at center of quadrant, principal axes are inclined at $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$

We know angle of inclination is given by

$$2\theta = \tan^{-1}\left(\frac{2F}{B-A}\right)$$



$$I_{o'x'} = I_{ox} + M\left(\frac{b}{2}\right)^2 = \frac{Mb^2}{4} + \frac{Mb^2}{4} = \frac{Mb^2}{2}$$

$$I_{o'y'} = I_{oy} + M\left(\frac{a}{2}\right)^2 = \frac{Ma^2}{4} + \frac{Ma^2}{4} = \frac{Ma^2}{2}$$

$$I_{x'y'} = ?$$

$$I_{x'y'} = \int dm \left[x-\frac{a}{2}\right]\left[y-\frac{b}{2}\right]$$

$$\boxed{\sigma = \frac{M}{\pi ab}}$$

$$= \iint_R \sigma\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)dx\,dy$$

$$= \sigma\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)dy\;dx$$

$$I_{x'y'} = \sigma \int_{-a}^{a} \left(x - \frac{a}{2}\right) \left(\frac{y^2}{2} - \frac{by}{2}\right) \Bigg|_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \sigma \int_{-a}^{a} \left(x - \frac{a}{2}\right) \left(-b^2\right) \sqrt{1-\frac{x^2}{a^2}}\, dx$$

$$= 2\sigma \int_{0}^{a} \frac{a}{2} \cdot b^2 \sqrt{\frac{a^2-x^2}{a}}\, dx$$

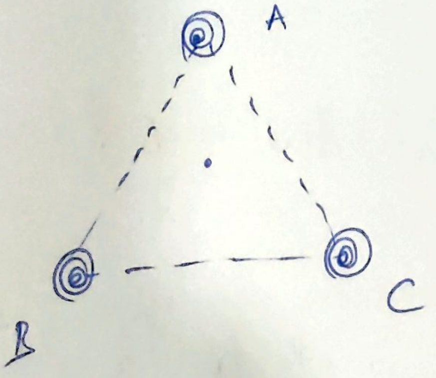
$$= \sigma b^2 \int_{0}^{a} \sqrt{a^2-x^2}\, dx$$

$$= \sigma b^2 \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

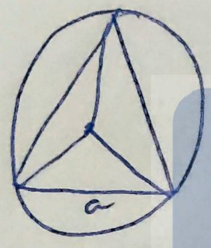
$$= \frac{\sigma a^2 b^2}{2} \cdot \frac{\pi}{2} = \frac{Mab}{4}$$

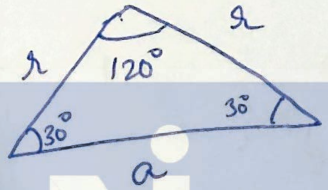
$$2\tan\theta = \frac{2\left(\frac{Mab}{4}\right)}{\frac{Ma^2}{2} - \frac{Mb^2}{2}} = \frac{ab}{a^2-b^2}$$

6a)



for Δ, orthocenter/circumcenter will be O



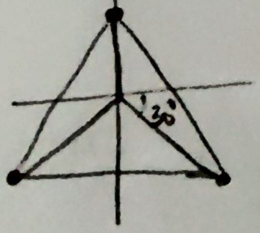


$$\cos 120^\circ = \frac{2r^2-a^2}{2r^2} \Rightarrow \frac{-1}{2} = \frac{2r^2-a^2}{2r^2}$$

$$\Rightarrow -r^2 = 2r^2 - a^2$$

$$a^2 = 3r^2$$

$$\boxed{r = \frac{a}{\sqrt{3}}}$$



complex potential will be given by

$$W = \frac{ik}{2\pi}\log\left(z - re^{i\pi/2}\right) + \frac{ik}{2\pi}\log\left(z - re^{i\pi+\frac{\pi}{6}}\right)$$

$$+ \frac{ik}{2\pi}\log\left(z - re^{\frac{-i\pi}{6}}\right)$$

to find velocity at A
we take complex potential due to B & C

$$w' = \frac{ik}{2\pi}\log\left(z - re^{\frac{i7\pi}{6}}\right) + \frac{ik}{2\pi}\log\left(z - re^{-i\pi/6}\right)$$

$$\frac{dw'}{dz} = \frac{ik/2\pi}{z - re^{i7\pi/6}} + \frac{ik/2\pi}{z - re^{-i\pi/6}}$$

at $z = re^{i\pi/2}$

$$q = \frac{ik/2\pi}{r\left(e^{i\pi/2} - e^{i7\pi/6}\right)} + \frac{ik/2\pi}{r\left(e^{i\pi/2} - e^{-i\pi/6}\right)}$$

$e^{i\pi/2} = i$

$e^{i7\pi/6} = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt3}{2} - \frac{i}{2}$

$e^{-i\pi/6} = \frac{\sqrt3}{2} - \frac{i}{2}$

$$q = \frac{ik/2\pi}{r\left(+\frac{\sqrt3}{2} + \frac{3i}{2}\right)} + \frac{\frac{ik/2\pi}{r}}{\left(\frac{-\sqrt3}{2} + \frac{3i}{2}\right)} = \frac{ik}{2\pi r}\left[\cancel{\frac{\sqrt3}{6}} - \frac{i}{2} + \cancel{\frac{-\sqrt3}{6}} - \frac{1}{2}i\right]$$

$$= \frac{ik}{2\pi r}\cdot -i = \frac{k}{2\pi r}$$

$$T = \frac{2\pi r}{q} = \frac{2\pi r}{k/2\pi r} = \frac{4\pi^2 r^2}{k} = \frac{4\pi^2 a^2}{3k}$$

put $r = \frac{a}{\sqrt3}$

6 b)

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

it's auxilliary Lagarange equation is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2}$$

$$\frac{\frac{dx}{x} + + \frac{dy}{y} + \frac{dz}{z}}{y^2+z-x^2-z+x^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\Rightarrow \boxed{xyz = c_1}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - x^2z + y^2z}$$

$$\Rightarrow \boxed{x^2+y^2 = z^2 + c_2}$$

$$\phi(xyz,\ x^2+y^2-z^2) = 0$$

For particular solution

along $x+y=0$ & $z=1$

$x=-y$ & $z=1$

$\Rightarrow \quad -x^2 = c_1 \qquad \because \; xyz = c_1$

& $\quad 2x^2 = 1 + c_2$

$$\Rightarrow \boxed{+2c_1 + 1 + c_2 = 0}$$

$$\Rightarrow \boxed{2xyz + 1 + x^2 + y^2 - z^2 = 0}$$

(or)

$$\underbrace{f(x,y,z) = x^2 + y^2 - z^2 + 2xyz + 1 = 0}_{\text{integral surface}}$$

6(i)

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
0	-3	6	2	6	0
1	3	8	8	6	0
2	11	16	14	6	
3	27	30	20		
4	57	50			
5	107				

Clearly $\Delta^4 y_0 = 0$

$\Rightarrow$ polynomial will be of degree 3

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0$$

$$u = \frac{x-x_0}{h} = \frac{x-0}{1} = x$$

$$\therefore f(x) = -3 + x(6) + \frac{x(x-1)}{\cancel{2}}\cdot\cancel{2} + \frac{x(x-1)(x-2)}{\cancel{6}}\cdot\cancel{6}$$

$$= -3 + 6x + x^2 - x + x(x^2-3x+2)$$

$$= x^2 + 5x - 3 + x^3 - 3x^2 + 2x$$

$$\boxed{f(x) = x^3 - 2x^2 + 7x - 3}$$ Answer

8a) To simplify F(x,y,z)

using sum of product form (conjunctive form)

use of Σ 1's

$$f(x,y,z) = xyz + xy\bar{z} + x\bar{y}z + \bar{x}yz$$

$$= x(yz + y\bar{z} + \bar{y}z) + \bar{x}yz$$

$$= x(\, y(z+\bar{z}) + \bar{y}z) + \bar{x}yz$$

$$= x(\, y + \bar{y}z) + \bar{x}yz$$

$\because \boxed{A+\bar{A}=1}$

& $\boxed{A+\bar{A}B = A+B}$

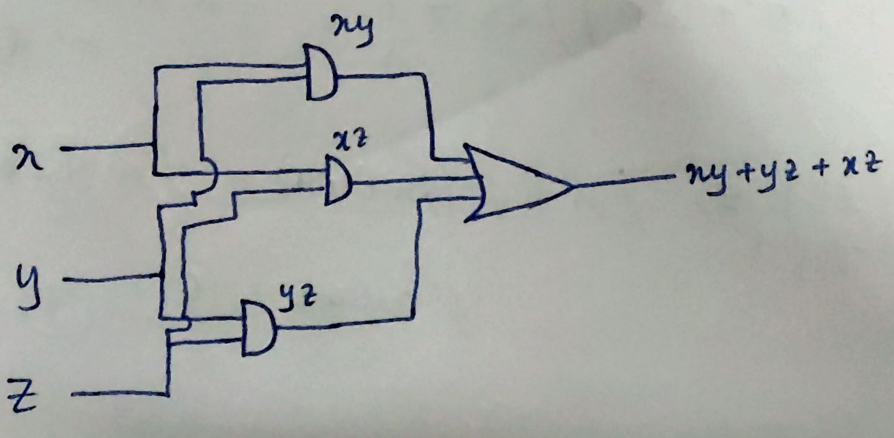
$$= x(y+z) + \bar{x}yz$$

$$= xy + xz + \bar{x}yz$$

$$= y(x+\bar{x}z) + xz$$

$$= y(x+z) + xz$$

$$= xy + yz + xz$$



8b) Given $F = \frac{-k}{r^2}$

$$V = \int F\,dr = \int \frac{-k}{r^2}\,dr = \frac{k}{r} + C$$

$$K.E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$K.E. = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\,\dot{\phi}^2 + r^2\dot{\theta}^2)$$

$$L = K.E. - V \quad (\text{or } T - v)$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \frac{k}{r} - C}$$

which is Lagrangian in spherical coordinates

Now, $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$

$$\therefore \frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$\Rightarrow \frac{1}{2}m(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,2r + \frac{k}{r^2} = \frac{1}{2}m\ddot{r}\cdot 2 \quad — ①$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow \cancel{\frac{1}{2}}\cancel{m}\,\cancel{r^2}\,\dot{\phi}^2(\cancel{2}\sin\theta\cos\theta) = \cancel{\frac{1}{2}}\cancel{m}\,\cancel{r^2}\,\ddot{\theta}$$

$$\dot{\phi}^2\sin\theta\cos\theta = \ddot{\theta} \quad — ②$$

and $\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) \Rightarrow 0 = \frac{1}{2} m r^2 \sin^2\theta \cdot 2\ddot{\phi}$

$\Rightarrow \boxed{r^2 \sin^2\theta \, \ddot{\phi} = 0}$ —③

$\therefore \ddot{\phi} = 0 \Rightarrow \boxed{\dot{\phi} = c}$

in eq ②

T(a

$$\ddot{\theta} = c^2 \sin\theta \cos\theta$$

$$\frac{d^2\theta}{dt^2} = c^2 \sin\theta \cos\theta$$

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = c^2 \cdot \frac{\sin^2\theta}{2}$$

$$\boxed{\dot{\theta}^2 = c^2 \sin^2\theta + c'}$$ —④

in eq ①

$$\frac{1}{2} m \left(c^2 \sin^2\theta + c' + \sin^2\theta \cdot c^2\right) 2r + \frac{k}{r^2} = \cancel{\frac{1}{2}} m \ddot{r}$$

$$\boxed{\frac{m\dot{r}^2}{2} = \frac{-k}{r} + m\left(2c^2 \sin^2\theta + c'\right)\frac{r^2}{2} + c''}$$ ⑤

at t

eq ④ & ⑤ are required eqⁿ of motion

8c) string in eq^n position @ $x=0$, $x=l$

Initially in eq^n position

& Initial velocity is $\lambda x(l-x)$

∴ Boundary conditions:

$x=0$, $u(x,t)=0$

$x=l$, $u(x,t)=0$

Initial conditions:

$u_t(x,0) = \lambda x(l-x)$

wave eq^n is given by

$$\frac{c^2\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

let $u(x,t) = X(x)\,T(t)$ — variable separable method

$\Rightarrow c^2 X'' T = X T''$

or $\frac{X''}{X} = \frac{T''}{c^2 T} = \text{constant}$

Case I constant $=0$

$\Rightarrow X''=0$

$\Rightarrow X(x) = c_1 x + c_2$

but boundary conditions not satisfied

se II constant $= +ve = k^2$ (say)

$$\frac{X''}{X} = k^2$$

$$\Rightarrow X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

B.Cs are @ $x=0 \Rightarrow 0 = c_1 + c_2$

@ $x=l \Rightarrow 0 = c_1 e^{kl} + c_2 e^{-kl}$

$\Rightarrow c_1 = c_2 = 0$ Rejected

Case III constant $= -ve = -k^2$ (say)

$$\frac{X''}{X} = -k^2$$

$$X(x) = c_1 \sin kx + c_2 \cos kx$$

@ $x=0 \Rightarrow c_1(0) + c_2 = 0 \Rightarrow c_2 = 0$

@ $x=l \Rightarrow c_1 \sin kl = 0 \Rightarrow kl = n\pi$ or $\boxed{k = \frac{n\pi}{l}}$

$$\therefore \frac{T''}{c^2 T} = -k^2 \text{ (or) } \frac{T''}{T} = -c^2 k^2$$

$$T(t) = c_3 \sin ckt + c_4 \cos ckt$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{l} \left(A_n \sin \frac{cn\pi}{l} t + B_n \, \text{sos} \frac{cn\pi}{l} t \right)$$

at $t=0$, $u(x,t)=0 \Rightarrow A_n(0) + B_n = 0 \Rightarrow B_n = 0$

$$u_t = \sum \sin \frac{n\pi x}{l} \left[A_n \cdot \frac{cn\pi}{l} \cdot \cos \frac{cn\pi}{l} t \right]$$

$$\lambda x(l-x) = \sum \left(A_n \cdot \frac{cn\pi}{l}\right) \sin\frac{n\pi x}{l}$$

at $t=0$

$$\Rightarrow A_n\left(\frac{cn\pi}{l}\right) = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\lambda x(l-x)\Big|_0^l + \int_0^l \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\cdot\lambda[l-2x]\,dx\right]$$

$$= \frac{2}{l}\left[\frac{1}{\left(\frac{n\pi}{l}\right)^2}\sin\frac{n\pi x}{l}\,\lambda(l-2x)\Big|_0^l - \frac{1}{\left(\frac{n\pi}{l}\right)^2}\int_0^l \sin\frac{n\pi x}{l}(-2\lambda)\,dx\right]$$

$$= \frac{2}{l}\,\frac{2\lambda}{\left(\frac{n\pi}{l}\right)^2}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4\lambda}{l\cdot\left(\frac{n\pi}{l}\right)^3} - \cos\frac{n\pi x}{l}\Big|_0^l$$

$$= \frac{+4\lambda l^2}{n^3\pi^3}\left[1-\cos n\pi\right]$$

$$A_n\left(\frac{(2k+1)\pi}{l}\right) = \frac{8\lambda l^2}{(2k+1)^3\pi^3}$$ if n is odd, $\cos n\pi = -1$

$$A_n = \frac{8\lambda l^3}{(2k+1)^4 \pi^3}$$

Therefore

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8\lambda l^3}{(2k+1)^4 \pi^3} \sin \frac{(2k+1)\pi}{l} x \sin \frac{c(2k+1)\pi}{l} t$$

which is required eqn