NEXT IAS

Name — SHOHAM

Roll No — IMT23MATL1004

Reg No — NIAS2300018169

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Test-0~~3~~4 — Paper-II [section B]

Maths Test

Section-A

1(a) $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

$\Rightarrow p\left[x(z-2y^2)\right] + q\left[y(z-y^2-2x^3)\right] = z(z-y^2-2x^3)$

$Pp + Qq = R$

$\therefore$ Lagrange auxilliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z} \Rightarrow \ln y = \ln z + \ln c$$

$$\boxed{y = cz}$$

b $\quad \dfrac{dx}{x(z-2y^2)} = \dfrac{dy}{y(z-y^2-2x^3)}$ $\qquad$ put $z = by$

$$\frac{dx}{xy(b-2y)} = \frac{dy}{y(by-y^2-2x^3)}$$

$$(by - y^2 - 2x^3)\,dx + x(2y - b)\,dy = 0$$

$$\frac{\partial M}{\partial y} = b - 2y$$

$$\frac{\partial N}{\partial x} = 2y - b$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\cancel{(2y-b)}}{x\cancel{(2y-b)}} = \frac{-2}{x}$$

$$If = e^{\int -\frac{2}{x}\,dx} = \frac{1}{x^2}$$

$$\frac{by - y^2 - 2x^3}{x^2}\,dx + \frac{1}{x}(2y-b)\,dy = 0$$

on integration

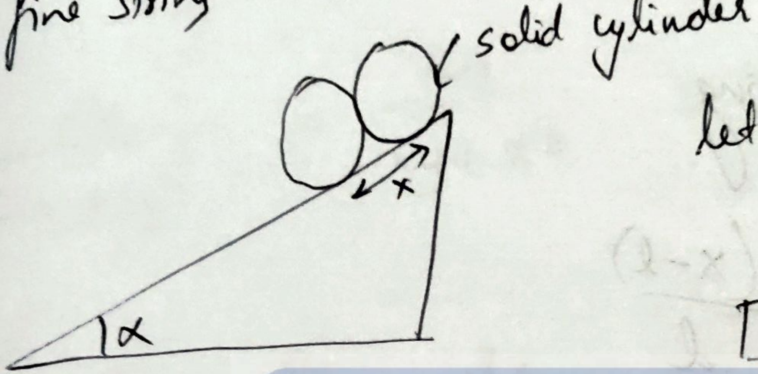
$$\Rightarrow (by - y^2)\left(\frac{-1}{x}\right) - \frac{2x^2}{2} = C_2$$

$$\frac{by - y^2}{x} + x^2 = C_2$$

$$\frac{z - y^2}{x} + x^2 = C_2$$

$$\phi\left(\frac{z}{y}, \frac{z - y^2}{x} + x^2\right) = 0$$

1b) cylinder rolls down smooth plane, inclination to horizon is α, unwrapping fine string



let a = radius

$$\boxed{x = a\theta}$$

$$T = K.E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

for solid cylinder $I = \frac{1}{2}Ma^2$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}M\right)\dot{x}^2 = \frac{3}{4}M\dot{x}^2$$

$$V = -Mgx\sin\alpha$$

$$L = T - V = \frac{3}{4}M\dot{x}^2 + Mgx\sin\alpha$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\Rightarrow \cancel{M}g\sin\alpha = \frac{3}{4}\cdot 2\cdot \cancel{M}\ddot{x}$$

$$\boxed{\ddot{x} = \frac{2}{3}g\sin\alpha}$$

(i) acceleration $= \frac{2g}{3} \sin\alpha$

for solid cylinder

(ii) tension of string

$$T = \frac{K(x-l)}{l}$$

where l is natural length

$$(g\sin\alpha)\, x = \frac{3}{4}\dot{x}^2$$

$$\therefore x = \frac{3}{4}\frac{\dot{x}^2}{g\sin\alpha}$$

depend on velocity at end point

$$\boxed{T = \frac{k\left(\frac{3}{4}\frac{\dot{x}^2}{g\sin\alpha} - l\right)}{l}}$$

1) $(z^2-2yz-y^2)p + (xy+zx)q = xy - zx$

its auxilliary eqn is

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+xz} = \frac{dz}{xy-zx}$$

$$\Rightarrow \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\frac{dy}{dz} = \frac{y+z}{y-z}$$

$$(y-z)dy - (y+z)dz = 0$$

$$ydy - zdz - (zdy + ydz) = 0$$

$$\frac{y^2-z^2}{2} - zy = C_1$$

or $\boxed{y^2 - z^2 - 2zy = c_1}$ ①

$$-2zy = c_1 - y^2 + z^2$$

$$\frac{dx}{z^2-y^2+c_1-y^2+z^2} = \frac{dy}{x(y+z)} = \frac{dz}{xy-zx}$$

$$\frac{xdx + ydy + zdz}{xz^2 - 2xyz - xy^2 + xy^2 + 2xyz - xz^2} = \frac{xdx + ydy + zdz}{0}$$

Rough

$ydy - zdy$
$= ydz + zdz$
$= ydz + \ldots$

$y = 2zy + z^2$

$z^2 - y^2 = c_1 - 2zy$

$z^2 = y^2 - 2zy - c_1$

$$\frac{dx}{c_1 - 4yz} = \frac{dy}{xy + zx}$$

$xz^2 - 2xyz$
$- xy^2$
$+ xy^2 + xyz$
$+ xyz$
$- xz^2$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_2$ — ②

from ① & ②

$\phi(u, v) = 0$

$u = y^2 - z^2 - 2zy$

$v = x^2 + y^2 + z^2$

1d) Find root of $\cos x - 3x + 5 = 0$

using <u>False position method</u>

1e) $(3798.3875)_{10}$

we first convert it to binary

$3798 = 111011010110$

$0.3875 = 0.0110001100$

0.3875×2	$\rightarrow 0.775$	0
0.775×2	$\rightarrow 1.55$	1
0.55×2	$\rightarrow 1.1$	1
0.1×2	$\rightarrow 0.2$	0
0.2×2	$\rightarrow 0.4$	0
0.4×2	$\rightarrow 0.8$	0
0.8×2	$\rightarrow 1.6$	1
0.6×2	$\rightarrow 1.2$	1

$(3798.3875)_{10} \rightarrow (111011010110.01100011000)_2$

for octal take pair of 3's

$(3798.3875)_{10} = (7326.3061)_8$

for hexadecimal pair of 4's

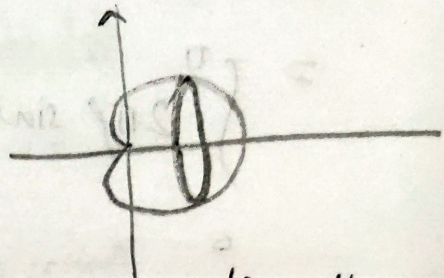
$(3798.3875)_{10} = (111011010110.011000110001)_2$

$= (ED6.631)_{16}$

3(a) solid body of density ρ

$$r = a(1+\cos\theta)$$

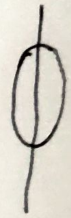
find M.I. about line $\perp$ to initial line



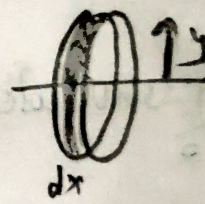
$$r = a(1+\cos\theta)$$

(Cardiod of revolution)

Consider a disc at x distance of radius $= y$



$$M.I = \underbrace{\iint dm\, x^2}_{\perp \text{ distance}} + \underbrace{\iint dm \left(\frac{y^2}{2}\right)}_{MI \text{ of a disc}}$$



$$dm = \rho(dx)(\pi y^2)$$

$$MI = \int_0^{2a} \rho \pi y^2 \left(x^2 dx + \frac{y^2}{2} dx\right)$$

$$= \int \rho \pi y^2 \left(x^2 + \frac{y^2}{2}\right) dx \quad = \iint 2\pi y \rho \left(x^2 + \frac{y^2}{2}\right) dx\, dy$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\iint 2\pi r \sin\theta \rho \left(r^2 \cos^2\theta + \frac{r^2 \sin^2\theta}{2}\right) r\, dr\, d\theta$$

$$MI = \int_0^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi\rho \sin\theta\left(\cos^2\theta + \frac{\sin^2\theta}{2}\right) r^4 \, dr \, d\theta$$

$$= \int_0^{\pi} 2\pi\rho \sin\theta\left(\cos^2\theta + \frac{\sin^2\theta}{2}\right) \frac{a^5(1+\cos\theta)^5}{5} \, d\theta$$

$$= 2\pi\rho \int_0^{\pi} \sin\theta\left(\cos^2\theta + \frac{\sin^2\theta}{2}\right)(1+\cos\theta)^5 \, d\theta \quad \text{— (1)}$$

$$\int_0^{\pi} \sin\theta(\cos^2\theta)(1+\cos\theta)^5 d\theta = -\int_1^{-1} t^2(1+t)^5 dt = \int_{-1}^{1} t^2(1+t)^5 dt$$

$$= \int_{-1}^{1} t^2(t^5+5t^4+10t^3+10t^2+5t+1) \, dt$$

$$= \int_{-1}^{1} 5t^6 + 10t^4 + t^2 \, dt = 2\left(\frac{5}{7}+\frac{10}{5}+\frac{1}{3}\right)$$

$$= \frac{128}{21} \quad \text{— (2)}$$

$$\int_0^{\pi} \sin\theta\left(\frac{1-\cos^2\theta}{2}\right)(1+\cos\theta)^5 dt = \int_{-1}^{1} \frac{1-t^2}{2}(1+t)^5 dt$$

$$= \int_{-1}^{1} \frac{1}{2}(1+t)^5 dt - \frac{1}{2}\int_{-1}^{1} t^2(1+t)^5 dt$$

$$= \frac{16}{3} - \frac{1}{2}\cdot\frac{128}{21} = \frac{16}{7} \quad \text{— (3)}$$

$$V = \iint 2\pi y \, dy \, dx$$

$$= \int_{r=}^{a(1+\cos\theta)} \int_{\theta=0}^{\pi} 2\pi r \sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi} 2\pi \sin\theta \, \frac{r^3}{3} \Big|_0^{a(1+\cos\theta)} d\theta$$

$$= \pi \int_0^{\pi} \frac{2}{3} \sin\theta \cdot a^3 (1+\cos\theta)^3 \, d\theta$$

$$= \frac{2\pi a^3}{3} \frac{(1+\cos\theta)^4}{-4} \Big|_0^{\pi} = \frac{2\pi a^3}{3} \cdot 4 = \frac{8\pi a^3}{3}$$

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from 1, 2, 3

$$\therefore MI = \frac{2\pi \rho a^5}{5} \left[\frac{128}{21} + \frac{16}{7} \right]$$

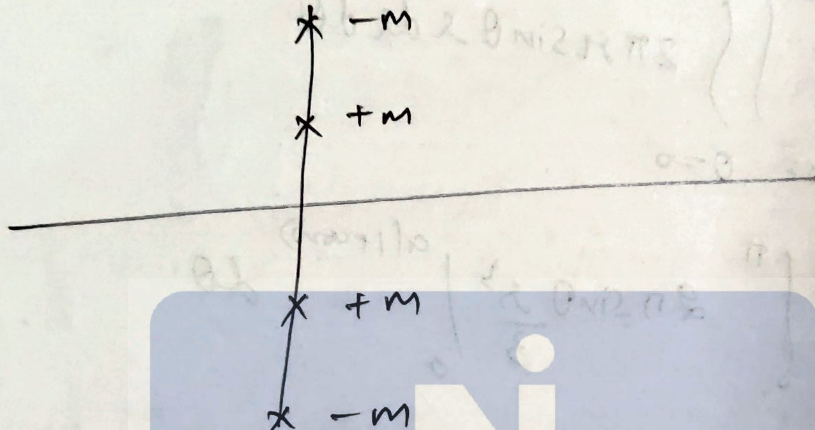
$$= \frac{2\pi \rho}{5} a^5 \times \frac{176}{21}$$

$$MI = \frac{352}{105} \pi \rho a^5$$

3b) Given source 'm' at $(0, a)$

& sink '$-m$' at $(0, b)$

and x-axis is rigid boundary



It's equivalent image system will be source at $(0,-a)$ & sink $(0,-b)$

$$\therefore \quad w = -m\log(z-ia) - m\log(z+ia) + m\log(z-ib) + m\log(z+ib)$$

$$w = -m\log(z^2+a^2) + m\log(z^2+b^2)$$

$$\frac{dw}{dz} = \frac{-m(2z)}{z^2+a^2} + \frac{m(2z)}{(z^2+b^2)}$$

$$= \frac{2mz(\cancel{-z^2}-b^2+\cancel{z^2}+a^2)}{(z^2+a^2)(z^2+b^2)} = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$\left|\frac{dw}{dz}\right| = |q| = \frac{|2mz(a^2-b^2)|}{|z^2+a^2||z^2+b^2|}$$

To find pressure, use bernollis eq^n

$$\frac{p}{\rho} + \frac{q^2}{2} + \Omega = \text{constant}$$

at infinity $q=0$

$$\frac{p_\infty}{\rho} = \frac{p}{\rho} + \frac{q^2}{2}$$

$$\frac{p-p_\infty}{\rho} = \frac{q^2}{2}$$ where p is pressure per unit length

on boundary $z = x$

resultant pressure

$$= \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx$$

$$= 2\rho m^2(a^2-b^2)^2 \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

Now,

$$\frac{x^2}{(x^2+a^2)^2(x^2+b^2)^2} = \frac{A}{x^2+a^2} + \frac{B}{x^2+b^2} + \frac{C}{(x^2+a^2)^2} + \frac{D}{(x^2+b^2)^2} \quad -(1)$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} = \frac{1}{a}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{a} \quad -(2)$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} \Rightarrow \int_{-\pi/2}^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta} = \frac{2}{a^3}\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= \frac{2}{a^3}\cdot\frac{\pi}{4} = \frac{\pi}{2a^3} \quad -(3)$$

$x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

in eq (1)

$$C = \frac{-a^2}{(b^2-a^2)^2} \quad \& \quad D = \frac{-b^2}{(b^2-a^2)^2}$$

$$\& \quad \frac{A}{a^2} + \frac{B}{b^2} + \frac{C}{a^4} + \frac{D}{b^4} = 0$$

$$\frac{A}{a^2} + \frac{B}{b^2} = \left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{1}{b^2-a^2}\right)^2$$

by symmetry $A = -B = \left(\frac{1}{b^2-a^2}\right)^2 \frac{a^2+b^2}{b^2-a^2}$

$$\frac{a^2}{4a^4(a^2+b^2)^2} = \frac{A}{2a^2} + \frac{B}{a^2+b^2} + \frac{C}{4a^4} + \frac{D}{(a^2+b^2)^2}$$

$$A = \frac{a^2+b^2}{(b^2-a^2)^3}$$

$$\therefore 2\rho\omega^2(a^2-b^2)^2\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{2\rho\omega^2}{(a^2-b^2)^2}\left\{\frac{A\cdot\pi}{a} + \frac{B\pi}{b} + \frac{C\pi}{2a^3} + \frac{D\pi}{2b^3}\right.$$

$$\frac{2\rho m^2}{(a^2-b^2)^2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = 2\rho m^2\cancel{(a^2-b^2)^2}\left[\left(\frac{1}{\cancel{a^2-b^2}}\right)^2\left(\frac{\pi}{a}-\frac{\pi}{b}\right)\left(\frac{a^4+b^4}{a^2-b^2}\right) - \frac{\pi a^2}{2a^3} - \frac{\pi b^2}{2b^3}\right)$$

$$= \frac{2\rho m^2\cancel{(a^2-b^2)^2}}{\cancel{(a^2-b^2)^2}}\left[\left(\frac{\pi}{a}-\frac{\pi}{b}\right]\left(\frac{a^2+b^2}{a^2-b^2}\right) - \frac{\pi}{2a} - \frac{\pi}{2b}\right]$$

on both sides

$$= 2\rho m^2\left[\frac{\pi}{a}\left(\frac{a^2+b^2}{a^2-b^2}-\frac{1}{2}\right)+\frac{\pi}{b}\left(\frac{-a^2-b^2}{a^2-b^2}-\frac{1}{2}\right)\right]$$

$$= 2\rho m^2\left\{\frac{\pi}{a}\left(\frac{2a^2+2b^2-a^2+b^2}{2(a^2-b^2)}\right)+\frac{\pi}{b}\left(\frac{-2a^2-2b^2-a^2+b^2}{2(a^2-b^2)}\right)\right\}$$

$$= 2\rho m^2\cdot\pi\left[\frac{-\cancel{b-a}}{ab}\cdot\frac{a^2+b^2}{\underset{(a+b)}{\cancel{a^2-b^2}}} - \frac{(a+b)}{2ab}\right]$$

$$= 2\rho m^2\pi\left[\frac{2(a^2+b^2)-(a+b)^2}{2ab(a+b)}\right] = \frac{\pi\rho m^2(a-b)^2}{ab(a+b)}$$

– on both sides of x-axis

Resultant pressure $= \dfrac{\pi\rho m^2(a-b)^2}{2ab(a+b)}$ (on +ve side of x-axis)

c) find family orthogonal to

$$\phi\left(z(x+y)^2,\ x^2-y^2\right)=0$$

$$\phi(u,v)=0$$

its auxilliary eqn is

$$Pp+Qq=R \quad \text{or} \quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$P=J\left(\frac{u,v}{y,z}\right)=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z}\end{vmatrix}=\begin{vmatrix}2z(x+y) & -2y\\ (x+y)^2 & 0\end{vmatrix}$$

$$=2y(x+y)^2$$

$$Q=J\left(\frac{u,v}{z,x}\right)=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x}\end{vmatrix}=\begin{vmatrix}(x+y)^2 & 2z(x+y)\\ 0 & 2x\end{vmatrix}$$

$$=2x(x+y)^2$$

$$R=J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2z(x+y) & 2z(x+y)\\ 2x & -2y\end{vmatrix}$$

$$=-2z(x+y)\left(2(x+y)\right)=-4z(x+y)$$

$\Rightarrow \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

its orthogonal eq^n is

$$P\,dx + Q\,dy + R\,dz = 0$$

$$2y(x+y)^2 dx + 2x(x+y)^2 dy - 4z(x+y)^2 dz = 0$$

$\because (x+y)^2 \neq 0$

$\Rightarrow$

$$y\,dx + x\,dy - 2z\,dz = 0$$

$\Rightarrow \boxed{xy - z^2 = c}$

where c is arbitrary constant

Section - B

5a) $2z + p^2 + qy + 2y^2 = 0$

$f = 2z + p^2 + qy + 2y^2$

by charpit method

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-(pf_p + qf_q)} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2+qy)} = \frac{dp}{2p} = \frac{dq}{q+4y+2q}$$

$$\frac{dx}{-2p} = \frac{dp}{2p} \Rightarrow dx + dp = 0$$

$$x + p = a$$

& $\frac{dy}{-y} = \frac{dp}{2p}$

(or) $\frac{dy}{-y} = \frac{dq}{3q+4y}$

$$\frac{3q+4y}{-y} = \frac{dq}{dy} \Rightarrow \frac{dq}{dy} + \frac{3q}{y} = -4$$

$$\frac{dq}{dy} + \frac{3q}{y} = -4$$

$$IF = e^{\int \frac{3}{y} dy} = e^{3\ln y} = y^3$$

$$\int -4y^3 dy = -y^4$$

$$q \cdot y^3 = -y^4 + b \Rightarrow q = -y + \frac{b}{y^3}$$

$$dz = p\,dx + q\,dy$$

$$dz = (a-x)\,dx + \left(-y + \frac{b}{y^3}\right) dy$$

on integrating

$$z = ax - \frac{x^2}{2} + \frac{-y^2}{2} - \frac{b}{2y^2} + C$$

or $z = -\frac{(a-x)^2 - y^2 - b/y^2}{2} + c$

where a, b, c are arbit constants

& q, b satisfies

$$2z + (a-x)^2 + y\left(-y + \frac{b}{y^3}\right) + 2y^2 = 0$$

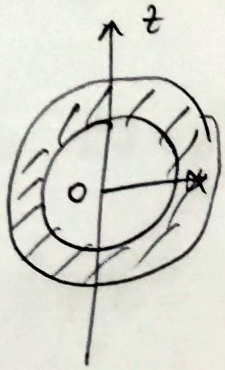
$$2z + (a-x)^2 + y^2 + \frac{b}{y^2} = 0$$

$$\Rightarrow c = 0$$

$$\Rightarrow \boxed{2z + (a-x)^2 + y^2 + \frac{b}{y^2} = 0}$$

where a & b are arbit constants

b) find M.I of hollow sphere of internal & external radius a & b



let diameter be Oz

$$(MI)_z = \iiint dm\,(x^2+y^2)$$

$$= \rho \iiint dx\,dy\,dz\,(x^2+y^2)$$

in spherical coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$x^2+y^2 = r^2\sin^2\theta$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$(MI)_{Oz} = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{b} r^2\sin^2\theta\,(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$= \rho \int_a^b r^4 dr \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \rho \frac{(b^5 - a^5)}{5} \cdot 2\pi \int_0^{\pi} \sin^3\theta \, d\theta \quad \text{—} (1)$$

$$\int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{2}{3} \qquad \therefore \int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}$$

$$\therefore MI = \rho \frac{(b^5 - a^5)}{5} \cdot 2\pi \cdot \left(\frac{4}{3}\right) \quad \text{putting in } (1)$$

$$\rho = \frac{M}{\frac{4}{3}\pi (b^3 - a^3)}$$

$$\therefore MI = \frac{2M}{5} \frac{(b^5 - a^5)}{(b^3 - a^3)}$$

— Answer

c) use Newton Raphson method to show

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

for square root of N

let $x = \sqrt{N}$

(or) $x^2 - N = 0$

Let $f(x) = x^2 - N$ — to find value of x

as per Newton Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x_n) = x_n^2 - N$ & $f'(x_n) = 2x_n$

$$\therefore\ x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right)$$

— Hence Proved

d) To find principal disjunctive form

$\sim P \vee Q$

disjunctive form : sum of product

$\sim P \vee Q = \bar{P} + Q$

$\because 1 = P + \bar{P}$

$= \bar{P} \cdot 1 + Q \cdot 1$

$= \bar{P}(Q + \bar{Q}) + Q(P + \bar{P})$

$= \bar{P}Q + \bar{P}\bar{Q} + PQ + \bar{P}Q$

$= PQ + \bar{P}Q + \bar{P}\bar{Q}$

$\sim P \cup Q = (P \cap Q) \cup (\sim P \cap Q) \cup (\sim P \cap \sim Q)$

— which is in principal disjunctive form

e) find $y(1.2)$ use Runge-kutta with $h = 0.2$

from $\frac{dy}{dx} = xy$ & $y(1) = 2$

$$y_1 = y_0 + k'$$

$$k' = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

put $h = 0.2$ & $x_0 = 1$ & $y_0 = 2$

where $f(x,y) = xy$

$\therefore k_1 = 0.2 f(1, 2) = 0.4$

$k_2 = 0.2 f\left(1.1, 2 + \frac{0.4}{2}\right) = 0.484$

$k_3 = 0.2 f\left(1.1 ,\ 2 + \frac{0.484}{2}\right)$

$= 0.2 f(1.1,\ 2.242) = 0.49324$

$k_4 = 0.2 f(1.2,\ 2.49324)$

$= 0.5983776$

$k = \frac{1}{6}\left[0.4 + 2(0.484 + 0.49324) + 0.5983776\right]$

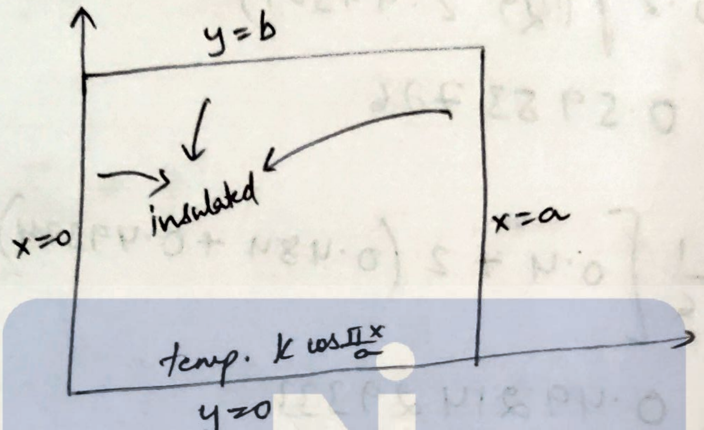
$k = 0.4921429333$

$k \sim 0.492143$

$\therefore y = 2.492143$ Answer

(correct to 6 decimal places)

7a) To find steady state temperature in rectangle plate $0<x<a$, $0<y<b$



eqⁿ of heat wave is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{(at steady state)}$$

$$u=X(x).Y(y)$$

$$\Rightarrow \boxed{X''Y+Y''X=0}$$

$$\text{or} \quad \frac{X''}{X}=-\frac{Y''}{Y}$$

boundary conditions

$\frac{\partial u}{\partial x}=0$ at $x=0,\ x=a$

$\frac{\partial u}{\partial y}=0$ at $y=b$

$u=K\cos\left(\frac{\pi x}{a}\right)$ at $y=0$

$\because \quad \frac{X''}{X} = \frac{-Y''}{Y} = \text{constant}$ or $\frac{X''}{X} + \frac{Y''}{Y} = \text{constant}$

case I

constant $= 0$

$Y'' = 0$

$X'' = 0$

$Y = c_1 y + c_2$

$X = c_3 x + c_4$

Rejected

since

$y = 0$

$y = k\cos\frac{\pi x}{a}$

case II

constant $=$ −ve

$= -\lambda_1^2 - \lambda_2^2$

$\frac{X''}{X} + \frac{Y''}{Y} = -(\lambda_1^2 + \lambda_2^2)$

$\frac{X''}{X} = -\lambda_1^2$

$\frac{Y''}{Y} = -\lambda_2^2$

Accepted

case III

constant $=$ +ve $= +\lambda^2$

$\frac{Y''}{Y} = \lambda^2$ & $\frac{X''}{X} = \lambda_2^2$

$Y = c_1 e^{\lambda t} + c_2 e^{-\lambda t}$

Rejected

$\because \; Y(0) \neq k\cos\frac{\pi x}{a}$

$X = c_1 \cos\lambda_1 x + c_2 \sin\lambda_1 x$ — ①

$Y = c_3 \cos\lambda_2 y + c_4 \sin\lambda_2 y$ — ②

put boundary conditions

$X' = -c_1 \lambda \sin\lambda_1 x + c_2 \lambda \cos\lambda_1 x$

$X'(x=0) \Rightarrow 0 = c_2 \lambda \Rightarrow c_2 = 0$

and $X'(x=a) \Rightarrow 0 = -c_1 \lambda \sin\lambda_1 a + c_2$

$\Rightarrow \sin\lambda_1 a = 0 \Rightarrow \lambda_1 a = n\pi$

$\lambda_1 = \frac{n\pi}{a}$ — ③

$\therefore \quad X(x) = \sum A \cos \frac{n\pi x}{a}$

$\& \quad Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y$

$Y'(y) = -c_3\lambda_2 \sin \lambda_2 y + c_4 \lambda_2 \cos \lambda_2 y$

put $Y'(b) = 0 \quad \Rightarrow \quad 0 = -c_3 \cancel{\lambda_2} \sin \lambda_2 b + c_4 \cancel{\lambda_2} \cos \lambda_2 b$ — (4)

$\& \quad Y(0) = k \cos \frac{\pi x}{a}$

$Y(0) = \sum A \cos \frac{n\pi x}{a} \cdot \left[c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y \right]$ — (5)

$k \cos \frac{\pi x}{a} = \sum A \cos \frac{n\pi x}{a} \left[c_3 + c_4 \cdot 0 \right]$ — (6)

from (4)

$c_3 \sin \lambda_2 b = c_4 \cos \lambda_2 b$

$c_4 = \frac{c_3 \sin \lambda_2 b}{\cos \lambda_2 b} = c_3 \tan \lambda_2 b$

from ⑥

$n=1$ & $Ar_3 = K$

$$\boxed{u(x,y) = K\cos\frac{\pi x}{a}\left[\cos\lambda_2 y + \tan\lambda_2 b\cdot\sin\lambda_2 y\right]}$$

Answer

where λ_2 is arbitrary +ve constant

7b) To show

$$\frac{x^2}{a^2k^2t^{2n}} + kt^n\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$$

is possible form of boundary surface

For boundary flow

$$\frac{df}{dt} = 0 \qquad \boxed{\because \frac{d}{dt} = \frac{\partial}{\partial t} + (q\cdot\vec{\nabla})}$$

$$\Rightarrow \frac{\partial f}{\partial t} + (\vec{q}\cdot\vec{\nabla})f = 0 \quad -(1)$$

$$(\vec{q}\cdot\vec{\nabla}) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad -(2)$$

$$\frac{\partial f}{\partial t} = -2nt^{-2n-1}\left(\frac{x^2}{a^2k^2}\right) + nkt^{n-1}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \quad -(3)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2k^2t^{2n}} \quad -(4)$$

$$\frac{\partial f}{\partial y} = \frac{2y}{b^2}kt^n \quad \& \quad \frac{\partial f}{\partial z} = \frac{2z}{c^2}kt^n \quad -(5)$$

from ①, ②, ③, ④, ⑤

$$\frac{dF}{dt}=0 \Rightarrow -\frac{2n}{t^{2n+1}}\left(\frac{x^2}{a^2k^2}\right) + nkt^{n-1}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$+ \frac{u\cdot 2x}{a^2k^2t^{2n}} + v\left(\frac{2y}{b^2}\right)kt^n + w\left(\frac{2z}{c^2}\right)kt^n = 0$$

on comparing terms

$$\frac{2n}{t^{2n+1}}\frac{x^2}{\cancel{a^2}\cancel{k^2}} = \frac{u\cdot\cancel{2}x}{\cancel{a^2}\cancel{k^2}t^{2n}}$$

we get $u = \frac{nx}{t}$

similarly, $v = \frac{-ny}{2t}$ & $w = \frac{-nz}{2t}$

now, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{n}{t} - \frac{n}{2t} - \frac{n}{2t}$

$$= \frac{n}{t} - \frac{n}{t} = 0$$

$\therefore$ motion is possible

$\because \vec{\nabla}\cdot\vec{q} = 0$

Hence it forms possible boundary surface

& at any time 't'

$$\underbrace{\vec{q} = \left(\frac{nx}{t}, \frac{-ny}{2t}, \frac{-nz}{2t}\right)}_{\text{velocity components}}$$

7c) Use cylindrical coordinates to write Hamilton & its equations for mass m moving inside

$$x^2 + y^2 = z^2 \tan^2\alpha$$

cylindrical coordinates: r, θ, z

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\Rightarrow \quad r^2 = z^2\tan^2\alpha \quad \text{or} \quad \boxed{r = z\tan\alpha} \quad \text{—} \ (1)$$

$$K.E = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2)$$

Note:

$\because \dot{x}^2 + \dot{y}^2 = (\dot{r}\cos\theta - r\sin\theta\dot{\theta})^2 + (\dot{r}\sin\theta + r\cos\theta\dot{\theta})^2$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

from (1)

$$r = z\tan\alpha$$

$$\Rightarrow \dot{r} = \dot{z}\tan\alpha$$

$$KE = T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha)$$

$\because 1 + \cot^2\alpha = \operatorname{cosec}^2\alpha$

$$T = \frac{1}{2}m(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2)$$

$$V = mgz = mg(r\cot\alpha)$$

— due to external force gravity

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2) - mgr\cot\alpha$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}\operatorname{cosec}^2\alpha$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$H = \sum(p_r, \dot{r}) - L$$

$$H = \frac{1}{2}m\left[\frac{p_r^2}{m^2\operatorname{cosec}^2\alpha} + \frac{r^2 \cdot p_\theta^2}{m^2 r^4}\right] + mgr\cot\alpha$$

$$H = \frac{1}{2m}\left[\frac{p_r^2}{\operatorname{cosec}^2\alpha} + \frac{p_\theta^2}{r^2}\right] + mgr\cot\alpha$$

$$\boxed{H = \frac{1}{2m}\left(p_r^2\sin^2\alpha + \frac{p_\theta^2}{r^2}\right) + mgr\cot\alpha}$$

Hamiltonian

Hamilton eqⁿs are

$$\frac{\partial H}{\partial p_r} = \dot{r} \quad \& \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta}$$

$$\& \quad \frac{\partial H}{\partial r} = -\dot{p}_r \quad \& \quad \frac{\partial H}{\partial \theta} = -\dot{p}_\theta$$

$$\therefore \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r \sin^2\alpha}{m} \quad \text{— (1)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$-\dot{p}_r = \frac{\partial H}{\partial r} = \frac{-p_\theta^2}{mr^3} + mg\cot\alpha \quad \text{— (2)}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0$$

$$\Rightarrow p_\theta = \text{constant}$$

$\Rightarrow$ from (1) and (2)

$$\boxed{-m\,\mathrm{cosec}^2\alpha\,\ddot{r} = \frac{-c}{mr^3} + mg\cot\alpha}$$

Answer

8a) write algorithm for Lagrange interpolation for n values of x

$$f(x) = \sum \frac{(x-x_1)(x-x_2) \text{ --- } (x-x_n)}{(x_0-x_1)(x_0-x_2) \quad (x_0-x_n)} y_0$$

for n subintervals : x_0 --- x_n

(n+1) terms of x

1. Input x_i, y_i $(i = 0 \text{ to } n)$

2. Take $S=0$, $P=1$

3. Take $i=0$

4. Take $j=0$

5. If $j=i$, Go to 6
   Else $P = \frac{x - x_j}{x_i - x_j} \cdot P$

6. $j=j+1$

7. If $j<n$, go to 5, else go to 8

8. $S = S + P y_i$

9. $i = i + 1$

10. If $i < n$, go to 4, else go to 11

11. Print S

12. End

8b) $u = \frac{x}{1+t}$, $v = \frac{y}{2+t}$, $w = \frac{z}{3+t}$

i) to find streamlines

streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

$$\frac{dx}{x/1+t} = \frac{dy}{y/2+t} = \frac{dz}{z/3+t}$$

$$\Rightarrow (1+t)\frac{dx}{x} = \frac{(2+t)dy}{y}$$

$$\boxed{(1+t)\ln x = (2+t)\ln y + C_1} \quad (1)$$

& $$\boxed{(1+t)\ln x = (3+t)\ln z + C_2} \quad (2)$$

eq (1) & eq (2) together gives equation of streamlines

(i) For path lines

$$\frac{dx}{dt} = \frac{x}{1+t}$$

$$\ln x = \ln(1+t) + \ln a$$

$$x = a(1+t) \quad — (3)$$

$$\frac{dy}{dt} = \frac{y}{2+t} \Rightarrow \quad y = b(2+t) \quad — (4)$$

$$\frac{dz}{dt} = \frac{z}{3+t} \Rightarrow \quad z = c(3+t) \quad — (5)$$

path lines are

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\boxed{\vec{r} = a(1+t)\hat{i} + b(2+t)\hat{j} + c(3+t)\hat{k}}$$

(ii) condition if streamlines identical to path lines

stream lines are $\vec{q} \times \vec{dr}$

path lines are $\vec{r}$

$$\vec{q} \times \vec{dr} = \vec{r}$$

Stream lines:

$$x^{(1+t)} = c_1\, y^{(2+t)}$$

$$\left.\begin{aligned} \& \quad x &= a(1+t) \\ y &= b(2+t) \end{aligned}\right\} \qquad [a(1+t)]^{1+t} = c_1 [b(2+t)]$$

$$\left(\frac{a(1+t)}{b(2+t)}\right)^{1+t} = c_1 b(2+t)$$

at $t=0$ $\qquad \frac{a}{2b} = c_1 2b$

$$\Rightarrow \boxed{c_1 = \frac{a}{4b^2}}$$

&

$$x^{1+t} = c_2 z^{3+t}$$

$$(a(1+t))^{1+t} = c_2 (c(3+t))^{3+t}$$

$$a = c_2 (3c)^3$$

$$c_3 = \left(\frac{27c^3}{a}\right)^{7} \Rightarrow \boxed{c_3 = \frac{a}{27c^3}}$$

$\therefore$ stream lines are

$$\boxed{x^{(1+t)} = \frac{a}{4b^2} y^{(2+t)}} \quad \& \quad \boxed{x^{(1+t)} = \frac{a}{27c^3} z^{(3+t)}}$$

8c) $x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

$KE = \frac{1}{2} M v^2 + \frac{1}{2} I \dot{\theta}^2$

$v^2 = \dot{x}^2 + \dot{y}^2$

$= a^2(\dot{\theta} + \cos\theta\,\dot{\theta})^2 + a^2(\sin\theta\,\dot{\theta})^2$

$v^2 = a^2\dot{\theta}^2 \{(1+\cos\theta)^2 + \sin^2\theta\}$

$= a^2\dot{\theta}^2[2 + 2\cos\theta] = 2a^2\dot{\theta}^2(1+\cos\theta)$

$I = \frac{2}{5} mb^2$ for sphere

at $\theta = 0$

$v^2 = 4a^2\dot{\theta}^2$

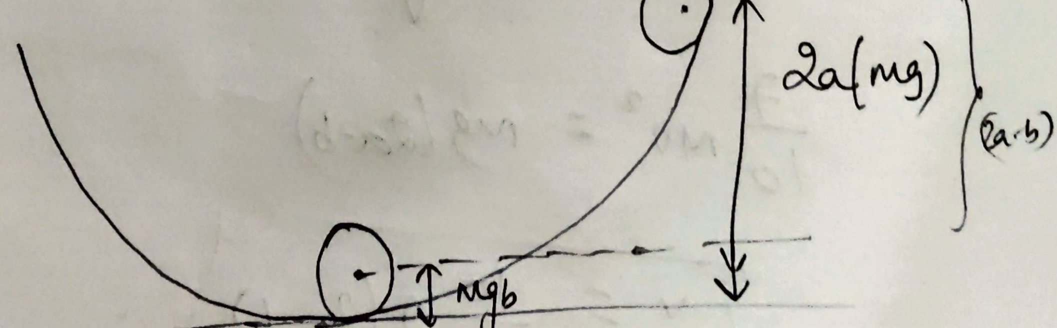
$\boxed{v = 2a\dot{\theta}}$

$KE = ma^2\dot{\theta}^2(1+\cos\theta) + \frac{1}{5} mb^2\dot{\theta}^2$

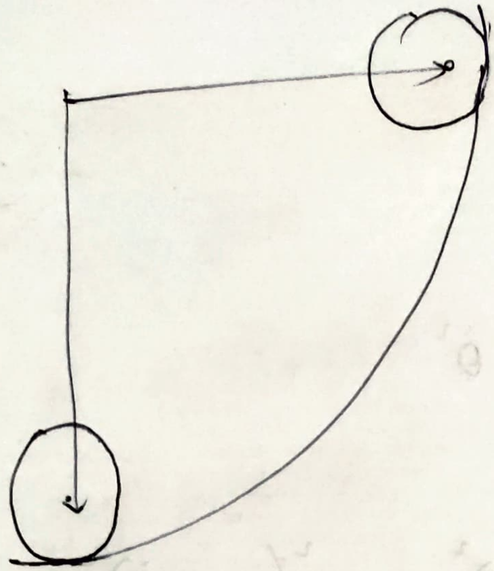
$V = mg(y - y_0) + mgb = mg(a(1-\cos\theta) - 2a)$

$= -mga(1+\cos\theta) + mg\,b$

at lowest point $\theta = 0$



Work by gravity $= 2mg(2a - b)$



$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}m(b\dot{\theta})^2 + \frac{1}{2}\frac{2}{5}mb\dot{\theta}^2$$

$$= mb^2\dot{\theta}^2\left(\frac{1}{2}+\frac{1}{5}\right)$$

$$= mb^2\dot{\theta}^2 \cdot \frac{7}{10}$$

$$KE = PE(\text{gained})$$

$$\frac{7}{10}mv^2 = mg(2a-b)$$

$$\boxed{v = \frac{10}{7}g(2a-b)}$$