(To be filled by candidate)

Name of Candidate : Shohani Tebeniwal

Roll No. : IMT23MATLI004

Registration Number : NIAS2300018169 Date of Examination : 23/07/23

Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Online ☑

CSE (MAINS) TEST SERIES - 2023

Test-03	MATHEMATICS OPTIONAL
Dated : 16-07-2023	

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA). Booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order. Unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA). Booklet must be clearly struck off.

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	21	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	41	20			
5. (a)	44	15			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	10			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	10			
8.(c)	79	10			
8. (d)	81	15			

Remarks

Observations:

SECTION-A

Q.1 (a) Examine the nature of the function $f(z)=\begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z\neq 0\\ 0 & z=0\end{cases}$ in a region including origin. (10)

Given

$$f(z) = \frac{x^3y^5 + i\,x^2y^6}{x^4+y^{10}} \qquad z \neq 0$$

(i) check for continuity

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^5(x+iy)}{x^4+y^{10}}$$

$$\lim_{x\to y^{5/2}} \frac{(y^{5/2})^2 y^5 (y^{5/2}+iy)}{2y^{10}} = 0$$

continuous

$$\lim_{z\to 0} f(z) = 0$$

(ii) check for derivability

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0)}{h+ik} = \frac{h^2k^5(h+ik)}{(h^4+k^{10})(h+ik)} = \frac{h^2k^5}{h^4+k^{10}}$$

$$\lim_{h\to k^{5/2}} \frac{f(h,k)}{h+ik} = \frac{k^{10}}{2k^{10}} = \frac{1}{2}$$

$$\lim_{h\to 0} f(h,k) = 0$$

$\therefore$ limit is not unique

$\Rightarrow$ $f(z)$ is not derivable

Q.1 (b) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$. (10)

To show
$f(x) = \sin x^2$ is not uniformly continuous
on $[0, \infty)$

let $\{x_n\}$ be sequence such that

$$\{x_n\} = \sqrt{\frac{n\pi}{2}}$$

& $\{y_n\}$ be another another sequence

$$\{y_n\} = \sqrt{\frac{(n+1)\pi}{2}}$$

$$\lim_{n\to\infty} |x_n - y_n| = \left|\sqrt{\frac{n\pi}{2}} - \sqrt{\frac{(n+1)\pi}{2}}\right|$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{\frac{n\pi}{2}} - \sqrt{\frac{(n+1)\pi}{2}}\right)\left(\sqrt{\frac{n\pi}{2}} + \sqrt{\frac{(n+1)\pi}{2}}\right)}{\sqrt{\frac{n\pi}{2}} + \sqrt{\frac{(n+1)\pi}{2}}}$$

$$= \lim_{n\to\infty} \left(\frac{\frac{n\pi}{2} - (n+1)\frac{\pi}{2}}{\sqrt{\frac{n\pi}{2}} + \sqrt{\frac{(n+1)\pi}{2}}}\right) = 0$$

$$\therefore \lim_{n\to\infty} |x_n - y_n| = 0$$

$f(x) = \sin x^2$

However

$$\lim_{n\to\infty} |f(x_n) - f(y_n)|$$

$$= \left|\sin\left(\frac{n\pi}{2}\right) - \sin\frac{(n+1)\pi}{2}\right|$$

for n to be even or odd

$$\left|\sin\frac{n\pi}{2} - \sin\frac{(n+1)\pi}{2}\right| = 1$$

$$\therefore \lim_{n\to\infty} |f(x_n) - f(y_n)| \neq 0$$

Hence $f(x)$ is not U.C. on $[0, \infty)$

(c) Suppose $f = (123456)$. Show that we can write $f = gh$, where $o(g) = 2$, $o(h) = 3$. (10)

Given

$f = (123456)$

To show

$f = gh$

where $o(g) = 2$ & $o(h) = 3$

Proof:

$f = (123456)$

$= (16)(15)(14)(13)(12)$

$= (156)(1234)$

$o(156) = 3$

alitet

$f = (123456)$

$o(f) = 6 \Rightarrow f^6 = (1)$

$\Rightarrow f = (123456)(1)$

$= (123456)[\underbrace{(123456)(123456)(123456)(123456)(123456)(123456)}_{6\ times}]$

$= [\underbrace{(123456)(123456)(123456)}_{(3\ times)}][\underbrace{(123456)(123456)(123456)(123456)}_{(4\ times)}]$

$g = (123456)^3$

clearly $o(g) = 2$

& $h = (123456)^4$

$o(h) = 3$

Answer

$\therefore f = gh$

where $o(g) = 2$ & $o(h) = 3$

d) Find the minimum value of $Z = -x_1 + 2x_2$,

Subject to :
$$-x_1 + 3x_2 \le 10,$$
$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

(10)

To find min. value of $Z = -x_1 + 2x_2$

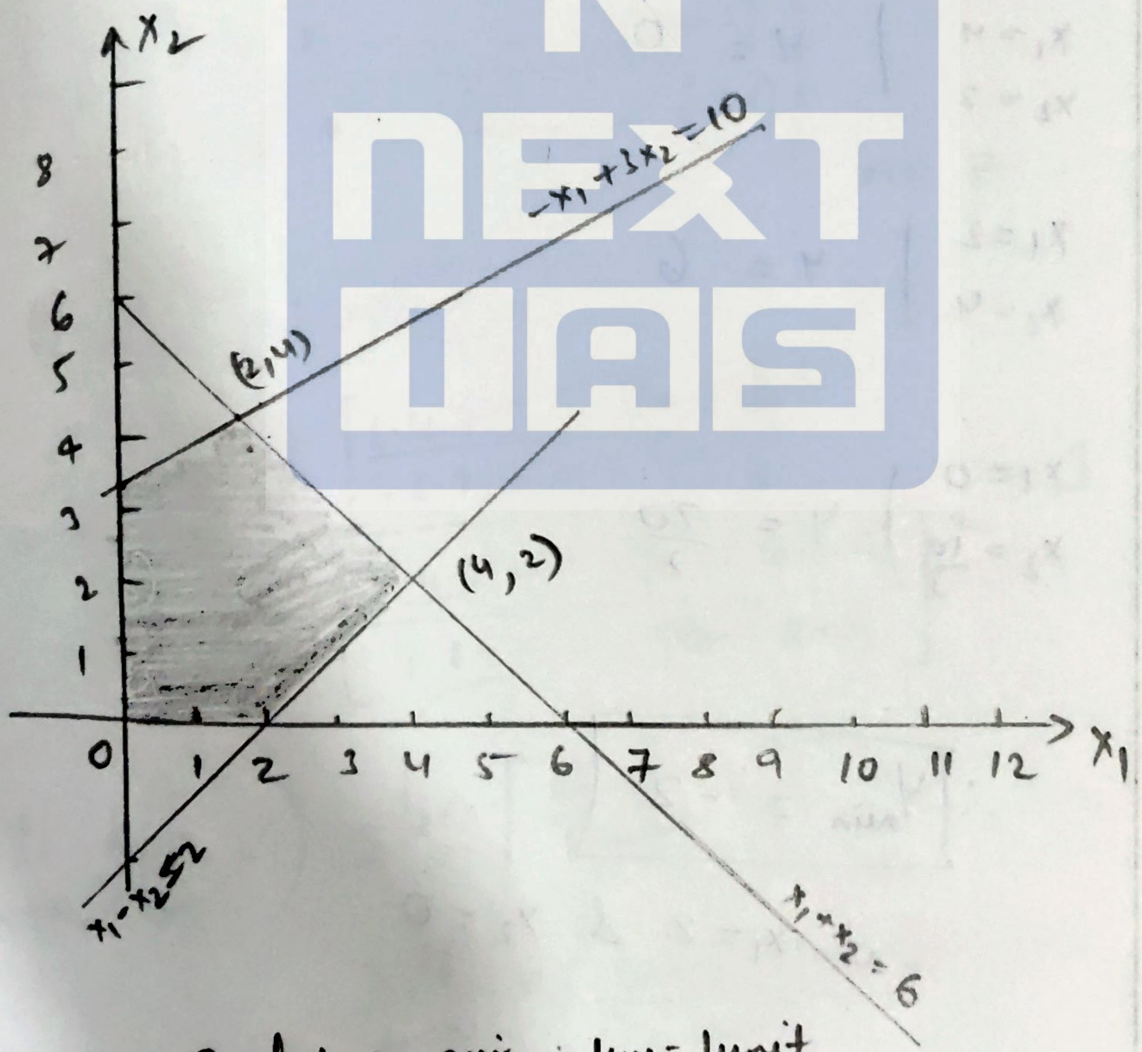
subject to $-x_1 + 3x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1, x_2 \ge 0$

We solve this using graphical method,



Scale: x_1 axis : 1cm = 1unit

x_2 axis : 1cm = 1unit

by convex region, we have following boundary points

$Y = -x_1 + 2x_2$

$x_1 = 0$, $x_2 = 0$ | $y = 0$

$x_1 = 2$, $x_2 = 0$ | $y = -2$

$x_1 = 4$, $x_2 = 2$ | $y = 0$

$x_1 = 2$, $x_2 = 4$ | $y = 6$

$x_1 = 0$, $x_2 = \frac{10}{3}$ | $y = \frac{20}{3}$

$\therefore \boxed{y_{min} = -2}$

at $x_1 = 2$ & $x_2 = 0$

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(e) Show that the series $\left(\frac{1}{3}\right)^2+\left(\frac{1.4}{3.6}\right)^2+\left(\frac{1.4.7}{3.6.9}\right)^2+\ldots\ldots$ converges. (10)

To show convergence of

$$S_n = \left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot4}{3\cdot6}\right)^2 + \left(\frac{1\cdot4\cdot7}{3\cdot6\cdot9}\right)^2 \ldots$$

$$S_n = \Sigma f_n$$

$$f_n = \left[\frac{1\cdot4\cdot7 \ldots (1+3(n-1))}{3\cdot6\cdot9 \ldots (3n)}\right]^2$$

$$\{f_n\} = \left[\frac{1\cdot4\cdot7 \ldots (3n-2)}{3\cdot6\cdot9 \ldots (3n)}\right]^2 \quad n \in Z$$

clearly $\{f_n\} > 0$

$$\lim_{n\to\infty} \frac{f_n}{f_{n+1}} = \frac{\left[\frac{1\cdot4\cdot7 \ldots (3n-2)}{3\cdot6\cdot9 \ldots (3n)}\right]^2}{\left[\frac{1\cdot4\cdot7 \ldots (3n-2)[3(n+1)-2]}{3\cdot6 \ldots (3n)[3(n+1)]}\right]^2} = 1$$

$$\lim_{n\to\infty} n\left(\frac{f_n}{f_{n+1}} - 1\right) = \lim n\left[\frac{(3n+3)^2}{(3n+1)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left(\frac{(3n+3)^2 - (3n+1)^2}{(3n+1)^2}\right)$$

$$= \lim_{n\to\infty} \frac{n(6n+4)2}{(3n+1)^2} = \frac{12}{9} > 1$$

(a)

$\therefore \lim_{n\to\infty} n\left(\frac{p_n}{p_{n+1}} - 1\right) > 1$

$\Rightarrow \sum p_n$ converges

by Raabe's test

$\Rightarrow \{S_n\}$ converges

$\Rightarrow$ Given series converges

Find the Laurent's Series of the function $f(z)=\exp\left[\frac{\lambda}{2}\left(z-\frac{1}{z}\right)\right]$ as $\Sigma_{-\infty}^{\infty} c_n z^n$ for $0<|z|<\infty$ where $c_n=\frac{1}{\pi}\int_0^{\pi}\cos(n\phi-\lambda\sin\phi)\,d\phi, n=0,\pm1,\pm2.....$ with λ a given complex number and taking the unit circle C given by $z=e^{i\phi}(-\pi\le\phi\le\pi)$ as contour in this region. (20)

$$f(z) = e^{\frac{\lambda}{2}\left(z-\frac{1}{z}\right)}$$

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n \quad \text{— laurent series}$$

$$c_n = \frac{1}{\lfloor n} f^n(z_0)$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \text{— by cauchy's theorem}$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{\lambda/2\left(z-\frac{1}{z}\right)}}{z^{n+1}}\,dz$$

for $z = e^{i\phi}$ — unit circle C

$dz = i e^{i\phi}\,d\phi$

$$c_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{\lambda}{2}\left[e^{i\phi}-e^{-i\phi}\right]}}{(e^{i\phi})^{n+1}}\, i e^{i\phi}\,d\phi$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{\lambda/2[2i\sin\phi]}}{e^{in\phi}}\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{i\lambda\sin\phi - in\phi}\right) d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\lambda\sin\phi - n\phi) + i\sin(\lambda\sin\phi - n\phi)\, d\phi \quad (1)$$

Now

$$\int_{-\pi}^{\pi} \sin(\lambda\sin\phi - n\phi)\, d\phi = \int_{-\pi}^{0} \sin(\lambda\sin\phi - n\phi)\, d\phi + \int_{0}^{\pi} \sin(\lambda\sin\phi - n\phi)\, d\phi$$

$$= -\int_{\pi}^{0} \sin(-\lambda\sin\phi + n\phi)\, d\phi + \int_{0}^{\pi} \sin(\lambda\sin\phi - n\phi)\, d\phi$$

$\boxed{\because \sin(-\phi) = -(\sin\phi)}$

$$= \int_{\pi}^{0} \sin(\lambda\sin\phi \pm n\phi)\, d\phi + \int_{0}^{\pi} \sin(\lambda\sin\phi - n\phi)\, d\phi$$

$$= 0$$

& $\cos(-\theta) = \cos\theta$

$\therefore \int_{-\pi}^{\pi} \cos x \, dx = 2\int_{0}^{\pi} \cos x \, dx$

$\therefore C_n = \frac{2}{2\pi}\int_{0}^{\pi} \cos(n\phi - 1 \sin\phi)\, d\phi$

$C_n = \frac{1}{\pi}\int_{0}^{\pi} \cos(n\phi - 1 \sin\phi)\, d\phi$

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2 (b) Find all the homomorphism from $\frac{Z}{4Z}$ to $\frac{Z}{6Z}$. (15)

To find homomorphisms from $\frac{Z}{4Z}$ to $\frac{Z}{6Z}$

$$\frac{Z}{4Z} \cong Z_4$$

& $\frac{Z}{6Z} \cong Z_6$

$\because$ question can be transformed to finding homomorphism Z_4 to Z_6

$$\phi : Z_4 \longrightarrow Z_6$$

$$\phi(1a) = \phi(1) \times_6 \phi(a)$$

$$\phi(a) = \phi(1) \cdot \phi(a)$$

let $\phi(1) = K$

$$\phi(a) = K\phi(a)$$

$\therefore o(K) | 4$ and $o(K) | 6$

$\Rightarrow o(K) = 1$

& $o(K) = 2$

$\because$ only 2 homomorphisms are possible if $o(k) = 1$

$\Rightarrow \phi(x) = 6x$

if $o(k) = 2$

$\Rightarrow \phi(x) = 3x$

$\therefore f : \frac{Z}{4Z} \to \frac{Z}{6Z}$

there are 2 homomorphism

$f(4Z+a) = 6a$

& $f(4Z+a) = 6Z+3a$

Q.2 (c) Examine the sequence $\langle f_n \rangle$, where $f_n(x) = nx(1-x)^n$ for uniform convergence on $[0,1]$. (15)

$f_n(x) = nx(1-x)^n$

$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x<1 \\ 0 & x=1 \quad \because f_n(1)=0 \end{cases}$

using M_n test

$M_n = |f_n(x) - 0| = |nx(1-x)^n|$

$f_n'(x) = n(1-x)^n - n^2x(1-x)^{n-1}$

$= n(1-x)^{n-1}[1-nx]$

$f_n'\left(\frac{1}{n}\right) = 0$

$f_n''(x) = n^2(1-x)^{n-1} - n^2(1-x)^{n-1} - n^2(n-1)x(1-x)^{n-2}$

$= -n^2(n-1)x(1-x)^{n-2}$

$f_n''\left(\frac{1}{n}\right) < 0$

$x = \frac{1}{n}$ is maximum point

$M_n = \sup|f_n(x) - 0| = \left|n\cdot\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^n\right|$

$= \sup\left|\left(1-\frac{1}{n}\right)^n\right|$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\log\left(1-\frac{1}{n}\right)}$$

$$= e^{n\left[-\frac{1}{n}-\frac{1}{n^2}\cdots\right]} = e^{-1}$$

$$M_n = \frac{1}{e} \neq 0$$

$\Rightarrow$ $f_n(x)$ is not convergent on $[0,1]$

consider $[a,1]$ $a>0$

$$f_n(x) = nx(1-x)^n$$

now, $f_n(x) - f(x) < \epsilon$

$\Rightarrow$ $nx(1-x)^n < \epsilon$

for $x \in S$

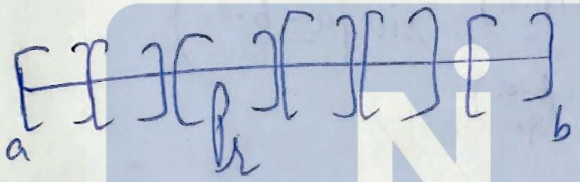
is uniformly convergent on $[a,1)$ $a>0$

(a) Show that $f(x)=\sin x$ is integrable on $\left[0,\frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}}\sin x\,dx=1$. (15)

To show $f(x)=\sin x$ is integrable on $\left[0,\frac{\pi}{2}\right]$

Let interval be $[a,b]$

consider 'n' Partitions



$n=\frac{b-a}{h}$

$P_r=\left[a+(r-1)h,\ a+rh\right]$ where $1\le r\le n$ } n partitions

$L(P,f)=\sum \delta_r m_r$

$\because \sin x$ is strictly increasing $\left[0,\frac{\pi}{2}\right]$

$\therefore$ m_r in P_r is $\sin(a+(r-1)h)$

$L(P,f)=\sum_{r=1}^{n} h\sin(a+(r-1)h)$

we know for AP of sin terms

$\sum \sin(x+dn)=\frac{\sin\frac{nd}{2}}{\sin d}\sin\left(\frac{x_1+x_n}{2}\right)$

$$L(P,f) = \sum_{r=1}^{n} h\sin(a+(r-1)h)$$

$$= \frac{h\sin\frac{nh}{2}}{\sin\frac{h}{2}} \cdot \sin\left(\frac{a+a+(n-1)h}{2}\right)$$

put $h = \frac{b-a}{n}$

$$= \frac{\frac{b-a}{n}\sin\left(\frac{b-a}{2}\right)}{\sin\left(\frac{b-a}{2n}\right)} \sin\left(\frac{a+b-h}{2}\right)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(P,f) = \lim_{h\to 0} \frac{h\sin\frac{b-a}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+b-h}{2}\right)$$

$$= 2\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{b+a}{2}\right)$$

$$= \cos a - \cos b \quad \text{— (1)}$$

$$U(P,f) = \sum \delta_r M_r$$

$$U(P,f) = \sum h\sin(a+rh)$$

$$= h\sum_{r=1}^{n}\sin(a+rh) = h\,\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+a+nh}{2}\right)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} U(P,f) = \lim_{h\to 0} \frac{h\sin\frac{b-a}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+b+h}{2}\right)$$

$$= 2\sin\frac{b-a}{2}\sin\frac{b+a}{2}$$

$$= \cos a - \cos b$$

$\therefore \int_{\bar{a}}^{b} f(x)dx = \int_{a}^{\bar{b}} f(x)\,dx$

$\therefore$ $f(x)$ is integrable on $[a, b]$

$\therefore$ $\sin x$ is integrable on $[0, \pi/2]$

$$\int_0^{\pi/2} \sin x\,dx = \cos 0 - \cos \pi/2 \quad \because \int_a^b \sin x\,dx = \cos a - \cos b$$

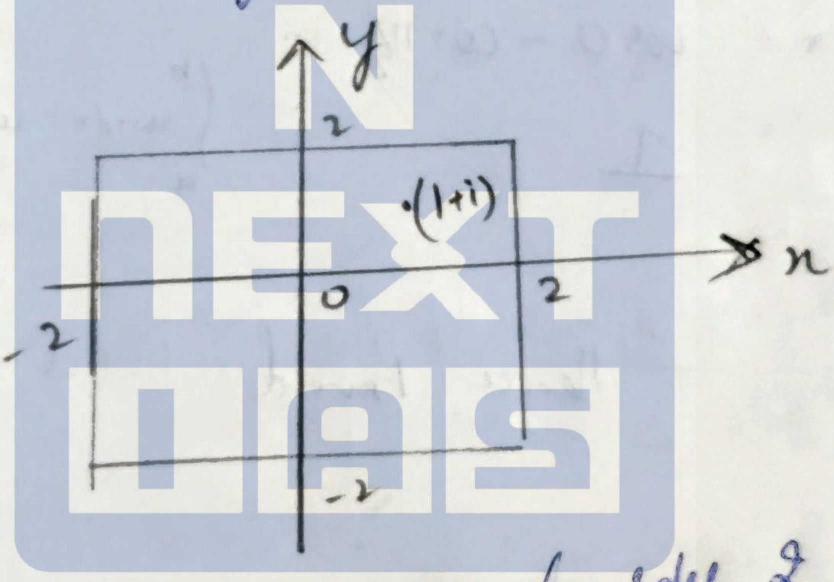
$$= \underline{1}$$

— Hence Proved

Q.4 (b) Evaluate $\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-1-i)^2}dz$, where C is the boundary of the square whose sides are the lines $x = \pm 2$ and $y = \pm 2$. (20)

$$\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-1-i)^2}dz$$

C : boundary of square with $x = \pm 2$, $y = \pm 2$



$z = 1+i$ is a pole of order 2

$z/2 < \pi/2$ ∴ $\tan z$ is defined

Let $f(z) = \frac{\tan z/2}{[z-(1+i)]^2}$

$$\int_C f(z)\,dz = 2\pi i \,\mathrm{Res}\,(z=(1+i))$$

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$$\text{Res}(z=\alpha) = \lim_{z \to 1} \frac{d}{dz} \cancel{(z-\alpha)^2} \frac{\tan z/2}{\cancel{(z-\alpha)^2}}$$

$$= \lim_{z \to \alpha} \frac{d}{dz} \tan\left(\frac{z}{2}\right)$$

$$= \lim_{z \to 1+i} \frac{1}{2} \sec^2\left(\frac{z}{2}\right)$$

$$= \frac{1}{2} \sec^2 \frac{(1+i)}{2}$$

$$\int_C f(z)\,dz = 2\pi i \left[\frac{1}{2} \sec^2\left(\frac{1+i}{2}\right)\right.$$

$$= \pi i \sec^2 \frac{(1+i)}{2}$$

Answer

$\because \cos i\theta = \cos\theta$

$\sin i\theta = i\sin\theta$

$$\sec^2\left(\frac{1+i}{2}\right) = \frac{2}{2\cos^2\left(\frac{1+i}{2}\right)} = \frac{2}{1+\cos(1+i)} = \frac{2}{1+\cos 1 \cos i - \sin 1 \sin i}$$

$$= \frac{2}{1+\cos^2 1 - i\sin^2 1}$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\pi i \sec^2\left(\frac{1+i}{2}\right) = \frac{\pi i \cdot 2}{1+\cos^2 1 - i(1-\cos^2 1)} = \frac{\pi i\, 2}{1-i+\cos^2 1\,(1+i)}$$

Q.4 (c) Show that in a PID, every non zero prime ideal is maximal. (15)

To show in Principal Ideal Domain,
non zero prime ideal is maximal

Proof:

let R be principal ideal domain
& I be prime ideal of R $I \neq 0$

to show I is maximal
let $I = (x)$
on contrary, let's assume I is not maximal

$\exists$ an ideal U such that
$I \subset U \subset R$

$\because$ U belongs to P.I.D

$\Rightarrow U = (a) \quad a \in R$

$\Rightarrow (x) \subset (a) \subset R$

$\Rightarrow x \in (a)$

let $x = ar_1$ —(1)

$\Rightarrow ar_1 \in (x)$

now for prime ideal
$ar_1 \in (x)$

$\Rightarrow a \in (x)$ or $r_1 \in (x)$
by definition

if $a \in (x)$ then $(a) = (x)$
not possible $\because$ U is distinct ideal than I

$\Rightarrow r_1 \in (x)$
$\Rightarrow r_1 = x r_2 \quad r_2 \in R$

now in (1)
$x = ar_1 \Rightarrow x = axr_2$

$x(1 - ar_2) = 0$
$\because$ R is integral domain

& $x \neq 0 \Rightarrow ar_2 = 1$

$\Rightarrow \exists\ r_2 \in R$ such that $ar_2 = 1$

$\Rightarrow 1 \in (a)$
$\Rightarrow (a) = R \Rightarrow U = R$

Hence it's contradiction that I is not maximal

$\therefore \quad I \cup U = R$

Hence I is maximal

Hence Proved

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SECTION-B

a) Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (10)

To show $\lim_{n\to\infty} (n)^{1/n} = 1$

let $\lim_{n\to\infty} (n)^{1/n} = l$

$\Rightarrow \quad \log l = \log n^{1/n} = \frac{1}{n} \log n$

$\lim_{n\to\infty} \frac{1}{n} \log n = \frac{\to\infty}{\to\infty}$ form

using L'Hospital Rule

$\lim_{n\to\infty} \frac{\log n}{n} = \lim \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \log l = 0$

$\Rightarrow \boxed{l = 1}$

$\Rightarrow \lim_{n\to\infty} (n)^{1/n} = 1$

Let R be a ring with unity, such that R has no right ideals except $\{0\}$ *and* R. Show that R is a division ring. (10)

Given : R is ring with unity
: R has no right ideals except $\{0\}$ & R

To show : R is division ring

Proof:

for ring R let a be element & $1 \in R$

if $a = 0$ then $aR = 0$
i.e $\{0\}$ is ideal (given)

if $a \neq 0$
then aR must be equal to R
$\because$ aR is right ideal of R
[$\because$ $ar \in aR$ for $r \in R$]

$\Rightarrow aR = R$
$\Rightarrow ar_1 \in R$
$\Rightarrow ar_1 = 1$ $\quad \because (1 \in R$ given$)$

$\Rightarrow$ any given element a (non zero)
there exist element r_1 such that
$ar_1 = 1$ $\quad (r_1 \neq 0)$
$\Rightarrow$ a has multiplicative inverse

This is true for all non-zero elements a

$\Rightarrow a \neq 0 \in R$ has inverse in R

$\Rightarrow$ R is division ring

[$\because$ Division ring has unity element & every non-zero element has inverse]

— Hence Proved

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What kind of singularity the given function $\frac{\cot \pi z}{(z-a)^2}$ has at $z = 0,\infty$. (10)

$$f(z) = \frac{\cot \pi z}{(z-a)^2}$$

(i) singularity at $z = 0$

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z \cot(\pi z)}{(z-a)^2}$$

$$= \lim_{z\to 0} \frac{z \cos(\pi z)}{\sin(\pi z)(z-a)^2} = \frac{1}{\pi a^2}$$

$$\lim_{z\to 0} z f(z) \neq 0$$

$\Rightarrow$ $\boxed{f(z) \text{ is a pole of order } 1}$

(ii) singularity at $z \to \infty$

$$f(z) = \frac{\cot \pi z}{(z-a)^2}$$

$$f(1/z) = \frac{\cot(\pi/z)\, z^2}{(1-az)^2}$$

$$\lim g(z) = \frac{z^2 \cot(\pi/z)}{(1-az)^2}$$

we check singularity at $z=0$ for $g(z)$

clearly $g(z)$ is essential singularity at $z=0$

$\Rightarrow$ $\boxed{f(z) \text{ has essential singularity at } z=\infty}$

<u>Ans:</u>

(i) $z=0$: pole

(ii) $z\to\infty$: essential

Express the following LPP in the standard matrix form

Maximise $Z = 4x_1 + 2x_2 + 6x_3$

Subject to :
$$2x_1 + 3x_2 + 2x_3 \geq 6,$$
$$3x_1 + 4x_2 = 8$$
$$6x_1 - 4x_2 + x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

(10)

P,Q

To express LPP in standard matrix form

$\max Z = 4x_1 + 2x_2 + 6x_3$

Subject to

$2x_1 + 3x_2 + 2x_3 \geq 6$

$3x_1 + 4x_2 = 8$

$6x_1 - 4x_2 + x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

$\Rightarrow$

$2x_1 + 3x_2 + 2x_3 - x_4 = 6$

$3x_1 + 4x_2 = 8$

$6x_1 - 4x_2 + x_3 + x_5 = 10$

introducing slack & surplus variable x_4 & x_5

$x_4 \geq 0 \quad x_5 \geq 0$

and given $x_1\ x_2\ x_3 \geq 0$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & -4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$$

$$\boxed{AX = B}$$

and $\boxed{\max : CX^T}$

where

$$A = \begin{bmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & -4 & 1 & 0 & 1 \end{bmatrix}$$

$B = [6 \;\; 8 \;\; 10]^T$

$C = [4 \;\; 2 \;\; 6 \;\; 0 \;\; 0]$

$X = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5]^T$

where $x_p \geq 0$

b) Suppose G is a finite group of order pq, where p, q are primes and $p > q$. Show that G has at most one subgroup of order p.

(10)

Given: G is finite group $o(G) = pq$

p & q are prime & $p > q$

To show: G has atmost one subgroup of order p

Proof:

Let $H \subseteq G$

$o(H) \mid o(G)$ — Lagrange theorem

$o(H) \mid pq$

$\Rightarrow o(H) = p$

$\Rightarrow$ a subgroup of order p is possible

To show atmost one subgroup is possible

on contrary, if possible

let $o(H_2) = p$ where $H_1 \neq H_2$

we know

$$o(H \cap K) = \frac{o(H) \cdot o(K)}{o(G)}$$

$$\therefore o(H \cap H_1) = \frac{o(H) \cdot o(H_1)}{o(G_1)} = \frac{p^2}{pq} = \frac{p}{q}$$

if p & q are prime
then p ∤ q & q ∤ p

$\therefore\ o(H_1 \cap H_2) = \frac{p}{q}$ not possible

Hence there does not exist another subgroup H_2 of order p

if $o(H) = p$

Hence, atmost one subgroup of order p exists

7 (a) Four new machines M_1, M_2, M_3 *and* M_4 are to be installed in a machine shop. There are five vacant places *A, B, C, D* and *E* available. Because of limited space, machine M_2 cannot be placed at *C* and M_3 cannot be placed at *A*.

C_{ij}, is the assignment cost of machine i to place j in rupees is shown below.

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	—	5	4
M_3	—	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule. (15)

Given 4 machines & 5 available places with given cost & M_2 cannot go to C & M_3 cannot go to A

to find ~~find~~ cost — Minimum

let cost for $M_2 - C$ be X & M_3 A be X where X is infinitely large (max M)

& introduce dummy machine M_5

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	M	5	4
M_3	M	6	9	6	2
M_4	9	3	7	2	3
M_5	0	0	0	0	0

For assignment problem, apply hungarian method

(i) create zero in all columns (✓)

(ii) create zero in all rows (subtract least value from other elements)

∴ we have

	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	M	1	0
M_3	M	4	7	4	0
M_4	7	1	5	0	1
	0	0	0	0	0

Min. no. of lines required = 5

∴ optimality achieved.

$$\left.\begin{array}{l} M_1 \longleftrightarrow A \\ M_2 \longleftrightarrow B \\ M_3 \longleftrightarrow E \\ M_4 \longleftrightarrow D \end{array}\right\}$$

optimal assignment

no machine is assigned at C.

Total cost $= 4 + 4 + 2 + 2$
$= \underline{12\ Rs}$ (min. possible)

Q.7 (b) Show that $\int_0^\infty \frac{\sin x}{x} dx$ converges but not absolutely. (20)

To show

$\int_0^\infty \frac{\sin x}{x} dx$ converges but not absolutely

$$I = \int_0^\infty \frac{\sin x}{x} dx$$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

& $\lim_{x \to \infty} \frac{\sin x}{x}$ is defined $= 0$

Infinite integral

we use $\boxed{\text{Abel's Theorem}}$

$$\int_0^\infty \frac{\sin x}{x} = \int_0^\infty \sin x \cdot \frac{1}{x} dx$$

$$= \int_0^\infty f(x) \cdot g(x) \, dx$$

$f(x) = \sin x$ & $g(x) = \frac{1}{x}$

$$\int_0^t f(x) dx = \int_0^t \sin x \, dx = -\cos x \Big|_0^t$$

$$= 1 - \cos t$$

$1 - \cos t$ is bounded $\Rightarrow \int_0^t f(x) dx$ is bounded

Candidates must not write on this margin

& $g(x) = \frac{1}{x}$ $g(x) > 0$, monotonic decreasing function & $\lim_{x \to \infty} g(x) = 0$

$\therefore$ by Abel's theorem

$\int_0^\infty f(x) \cdot g(x) dx$ converges

$\Rightarrow \int_0^\infty \frac{\sin x}{x} dx$ converges —— 1

Now, consider $\left|\int_0^\infty \frac{\sin x}{x} dx\right|$ convergence criteria :-

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \le \int_0^\infty \frac{dx}{x}$$

$\because |\sin x| \le 1$

& $|x| > 0$ for $x \in (0, \infty) \Rightarrow x > 0$

$I_2 = \int_0^\infty \frac{dx}{x}$ which diverges

$$\Rightarrow \int_0^\infty \left|\frac{\sin x}{x}\right| dx \le \int_0^\infty \frac{dx}{x}$$

or $I_3 = \int_0^\infty \frac{dx}{\sqrt{x^3}} = \int_0^1 \frac{dx}{x^{3/2}} + \int_1^\infty \frac{dx}{x^{3/2}}$ } diverges

$\lim_{x \to \infty} \left|\frac{\sin x}{x}\right| \sqrt{x^{3/2}} \to \infty \Rightarrow$ It I_3 diverges

Alitee,

$$\int_0^\infty \frac{|\sin x|}{x}\,dx$$

$x>0$

& $\frac{1}{x} < k$

$|\sin x|$ is cyclic in $(0, \pi/2)$ cycles

$$\int_0^{\pi/2} \sin x\,dx = 1$$

$$\therefore \int_0^\infty \frac{|\sin x|}{x}\,dx = c\int_0^\infty \frac{dx}{x}$$

$\int_0^\infty \frac{dx}{x}$ diverges

$\therefore \int_0^\infty \frac{\sin x}{x}\,dx$ is convergent but not absolutely

Q.7 (c) Evaluate $\int_0^{2\pi} \frac{1}{1-2p\sin 2\theta + p^2} d\theta$, $|p| < 1$. (15)

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin 2\theta + p^2}$$

$$2\theta \longrightarrow \theta$$

$$I = \int_0^{4\pi} \frac{d\theta}{2(1-2p\sin\theta+p^2)}$$

$$= \int_0^{2\pi} \frac{d\theta}{2(1-2p\sin\theta+p^2)} + \int_{2\pi}^{4\pi} \frac{d\theta}{2(1-2p\sin\theta+p^2)}$$

$$= \cancel{2} \int_0^{2\pi} \frac{d\theta}{\cancel{2}(1-2p\sin\theta+p^2)}$$

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$$

Consider curve $C : |z| = 1$

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{zi}$$

$$\therefore \frac{z - \frac{1}{z}}{2i} = \sin\theta \Rightarrow 2\sin\theta = \frac{z^2-1}{zi}$$

$$I = \int_C \frac{dz}{iz\left[1 - p\left(\frac{z^2-1}{iz}\right) + p^2\right]} = \int_C \frac{dz}{iz(p^2+1) - p(z^2-1)}$$

$$= \int_C \frac{dz}{-pz^2 + izp^2 + iz + p}$$

$$= \int \frac{dz}{-pz(z-ip) + i(z-ip)} = \int \frac{dz}{(z-ip)(-pz+i)}$$

$$= \int_C \frac{dz}{(z-ip) -p\left(z-\frac{i}{p}\right)} = \int \frac{-1}{p} \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)}$$

for $g(z) = (z-ip)\left(z-\frac{i}{p}\right)$

roots are ip & i/p

if $|p|<1 \Rightarrow |ip|<1$ inside C

$$\int f(z)\,dz = 2\pi i \,\text{Res}\,(z = ip)$$

$$= 2\pi i \left[\frac{-1}{p} \cdot \frac{1}{\left(ip - \frac{i}{p}\right)}\right]$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}} \cdot \frac{-1}{\cancel{p}} \cdot \frac{1}{\frac{p^2-1}{\cancel{p}}} = \frac{2\pi}{1-p^2}$$

Ans: $\boxed{\frac{2\pi}{1-p^2}}$