(To be filled by candidate)

Name of Candidate : SHOHAM TEBERIWAL

Roll No. : MT 23MATL1004

Registration Number : NIAS2300018169 Date of Examination : 19/06/23

Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Online ☑

# CSE (MAINS) TEST SERIES - 2023

| Test-01 | MATHEMATICS OPTIONAL |
|---|---|
| Dated : 18-06-2023 | |

Time Allowed : Three Hours

Maximum Marks: 250

## QUESTION PAPER SPECIFIC INSTRUCTIONS

*(Please read each of the following instructions carefully before attempting questions)*

*There are EIGHT questions divided in Two sections and printed in ENGLISH.*

*Candidate has to attempt FIVE questions in all.*

*Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.*

*The number of marks carried by a question/part is indicated against it.*

*Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA). Booklet in the space provided.*

*No marks will be given for answers written in a medium other than the authorized one.*

*Word limit in questions, wherever specified, should be adhered to.*

*Attempts of question shall be counted in sequential order. Unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA). Booklet must be clearly struck off.*

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*(For filling by Examiners only)*

| Q. No. | Page No. | Max. Marks | Marks | Total | Signature |
|---|---|---|---|---|---|
| 1. (a) | 4 | 10 | | | |
| 1. (b) | 6 | 10 | | | |
| 1. (c) | 8 | 10 | | | |
| 1. (d) | 10 | 10 | | | |
| 1. (e) | 12 | 10 | | | |
| 2. (a) | 14 | 20 | | | |
| 2. (b) | 18 | 15 | | | |
| 2. (c) | 21 | 15 | | | |
| 3. (a) | 24 | 15 | | | |
| 3. (b) | 27 | 20 | | | |
| 3. (c) | 32 | 15 | | | |
| 4. (a) | 35 | 10 | | | |
| 4. (b) | 37 | 20 | | | |
| 4. (c) | 41 | 20 | | | |
| 5. (a) | 45 | 10 | | | |
| 5. (b) | 47 | 10 | | | |
| 5. (c) | 49 | 10 | | | |
| 5. (d) | 51 | 10 | | | |
| 5. (e) | 53 | 10 | | | |
| 6. (a) | 55 | 15 | | | |
| 6. (b) | 58 | 20 | | | |
| 6. (c) | 62 | 15 | | | |
| 7. (a) | 65 | 15 | | | |
| 7. (b) | 68 | 15 | | | |
| 7. (c) | 71 | 20 | | | |
| 8. (a) | 75 | 20 | | | |
| 8. (b) | 78 | 15 | | | |
| 8. (c) | 81 | 15 | | | |
| | | | Grand Total | | |

**Remarks**

**Observations:**

Candidate must not write on this margin

## SECTION-A

**Q.1 (a)** Let $u_1 = \begin{pmatrix}1\\1\\2\\4\end{pmatrix}, u_2 = \begin{pmatrix}2\\-1\\-5\\2\end{pmatrix}, u_3 = \begin{pmatrix}1\\-1\\-4\\0\end{pmatrix}$. Find the value of $h$ so that $b = \begin{pmatrix}2\\1\\1\\h\end{pmatrix}$ lie in span $\{u_1, u_2, u_3\}$. **(10)**

Given

$u_1 = (1\,1\,2\,4)^T$, $u_2 = (2,-1,-5,2)^T$, $u_3 = (1,-1,-4,0)^T$

Given that $b = (2,1\,1\,h)^T$ lie in span of $\{u_1\ u_2\ u_3\}$

To find value of $h$

we first check for linear independence of $u_1, u_2, u_3$

consider the row space and reduce it to row echelon form

$$\therefore \begin{bmatrix}1&1&2&4\\2&-1&-5&2\\1&-1&-4&0\end{bmatrix} \sim \begin{bmatrix}1&1&2&4\\0&-3&-9&-6\\0&-2&-6&-4\end{bmatrix} \sim \begin{bmatrix}1&1&2&4\\0&1&3&2\\0&1&3&2\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ ; $R_2 \to R_2/-3$, $R_3 \to R_3/-2$

$\therefore$ $u_1$ & $u_2$ is linearly independent

for $b$ to be span of $u_1\, u_2$ & $u_3$

consider matrix formed as row space of $u_1$ & $u_2$

$$\begin{bmatrix}1&1&2&4\\0&1&3&2\\2&1&1&h\end{bmatrix} \sim \begin{bmatrix}1&1&2&4\\0&1&3&2\\0&-1&-3&h-2(4)\end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

if $b$ is dependent on $a_1$ & $a_2$

$\Rightarrow$ the last row of matrix should be similar to row 2

$\Rightarrow$ $h - 8 = -2$

$\Rightarrow$ $\boxed{h = 6}$

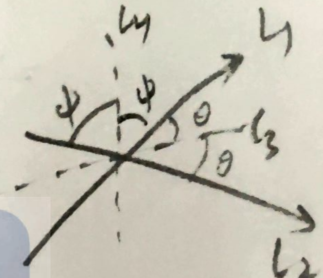
Answer

**Q.1 (b)** $(l_1, m_1, n_1)$ and $(l_2, m_2, n_2)$ are the dc's of two concurrent lines, show that the dc's of the two lines bisecting the angles between them are proportional to $(l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2)$. **(10)**

<u>Given</u>

$(l_1, m_1, n_1)$ & $(l_2\ m_2\ n_2)$ are DCs of 2 lines

<u>To find</u> DCs of lines bisecting angle between the 2 lines :-




if $L_1$ has DCs $l_1\ m_1\ n_1$

& $L_2$ has DCs $l_2\ m_2\ n_2$

let DCs of $L_3$ be $l_3\ m_3\ n_3$

$\because$ $L_3$ is angle bisector between $L_1$ & $L_2$

$\Rightarrow$ $L_3$ is equally inclined to $L_1$ & $L_2$

$\Rightarrow$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = l_2 l_3 + m_2 m_3 + n_2 n_3$$

$$\because \cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{\Sigma l_1^2}\sqrt{\Sigma l_2^2}}$$

$$\Rightarrow (l_1 - l_2) l_3 + (m_1 - m_2) m_3 + (n_1 - n_2) n_3 = 0$$

$\Rightarrow$ $(l_3, m_3, n_3)$ are proportional to

$$l_1 + l_2,\ m_1 + m_2,\ n_1 + n_2 \quad \text{--- (1)}$$

$\because \sum(l_1-l_2)(l_1+l_2) = \sum(l_1^2 - l_2^2)$

$\Rightarrow \sum l_1^2 - \sum l_2^2 = 1-1=0$

for line four or L4

$\cos\phi = l_1 l_4 + m_1 m_4 + n_1 n_4$

& $\cos(\pi-\phi) = l_2 l_4 + m_2 m_4 + n_2 n_4$

$\Rightarrow -\cos\phi = l_2 l_4 + m_2 m_4 + n_2 n_4$

$\Rightarrow \sum l_1 l_4 = -\sum l_2 l_4 \Rightarrow \sum l_4(l_1+l_2) = 0$

$\therefore (l_4, m_4, n_4)$ are proportional to

$(l_1-l_2,\ m_1-m_2,\ n_1-n_2)$ — (2)

$\because \sum(l_1-l_2)(l_1+l_2) = \sum(l_1^2 - l_2^2)$

$= \sum l_1^2 - \sum l_2^2 = 1-1=0$

$\because l^2+m^2+n^2 = 1$ for DCs

from (1) & (2)

DCs of 2 lines bisecting angles between them are proportional to

$(l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2)$

— Hence Proved

**Q.1 (c)** If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$, find $A^7-7A^6+15A^5-12A^4-A^3+8A^2-15A-4I$. **(10)**

Given

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

we find its characteristic polynomial

$$|A-\lambda I|=0$$

$$\Rightarrow \lambda^3-6\lambda^2+9\lambda-4=0$$

by cayley's hamilton theorem

$$\boxed{A^3-6A^2+9A-4I=0} \quad (1)$$

To find:

$$A^7-7A^6+15A^5-12A^4-A^3+8A^2-15A-4I$$

$$= A^4\cancelto{0}{(A^3-6A^2+9A-4I)} - A^6+6A^5-8A^4 - \cancelto{0}{(A^3-6A^2+9A-4I)} + 2A^2-6A-8I = 0$$

$$\Rightarrow -A^6+6A^5-8A^4+2A^2-6A-8I$$

$$= -A^3\cancelto{0}{(A^3-6A^2+9A-4I)} + A^4-4A^3 + 2A^2-6A-8I$$

$$= A(A^3-6A^2+9A-4I) + 2A^3-7A^2-2A-8I$$

$$= 2A^3 - 7A^2 - 2A - 8I$$

$$= 2\cancel{(A^3 - 6A^2 + 9A - 4I)}^{\,0}$$
$$+5A^2 - 20A$$

$$= 5A^2 - 20A = 5[A][A] - 20A$$

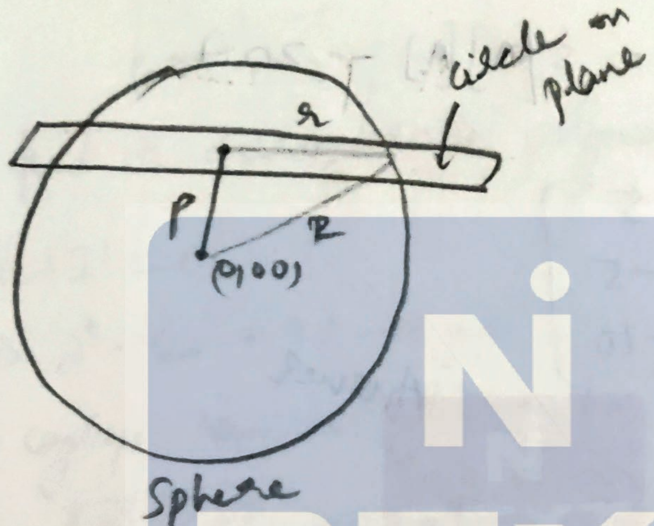
$$= \begin{bmatrix} -10 & -5 & 5 \\ -5 & -10 & -5 \\ 5 & -5 & -10 \end{bmatrix}$$

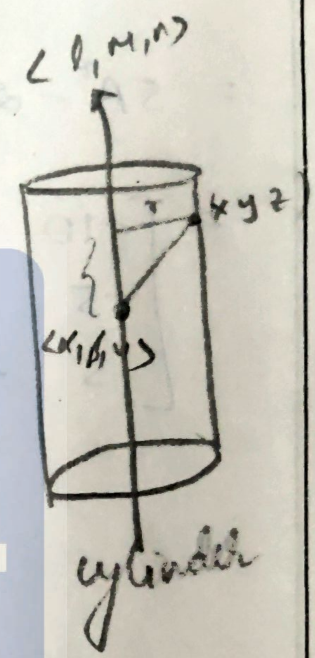
Answer

**Q.1 (d)** Find the equation of the right circular cylinder which passes through the circle $x^2+y^2+z^2=9, x-y+z=3.$ **(10)**

Given circle

$x^2+y^2+z^2=9$ & $x-y+z=3$




$\perp$ distance $= p = \dfrac{3}{\sqrt{1^2+1^2+1^2}} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

$R = 3$

$\therefore r_c^2 = R^2 - p^2 = 9-3 = 6$

$(l, m, n)$ are proportional to $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

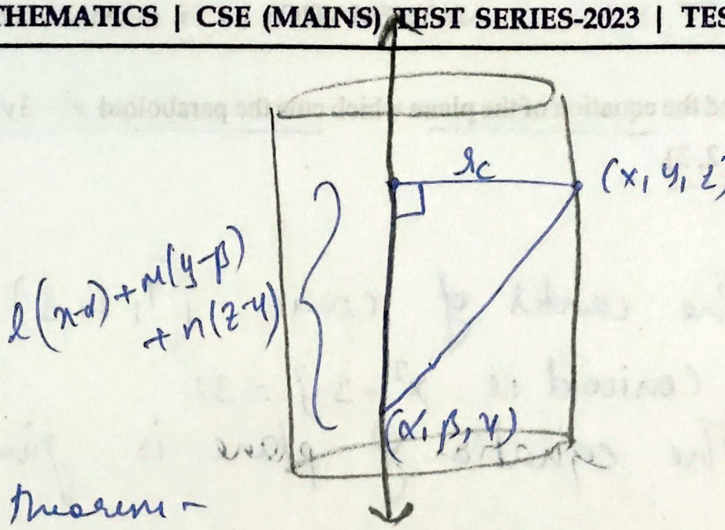
locus of center of circle is:

$(x_s + lr,\ y_s + mr,\ z_s + nr) = \left(0 + \frac{1}{\sqrt{3}}\sqrt{3},\ -1,\ 1\right)$

$\Rightarrow (1, -1, 1)$

$x_s, y_s, z_s$ is center of sphere

$\therefore (\alpha, \beta, \gamma) \equiv (1, -1, 1)$




using pythagoras theorem –

$$\therefore (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r_c^2 + \left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2$$

$$\Rightarrow (x-1)^2 + (y+1)^2 + (z-1)^2 = 6 + \frac{1}{3}\left[(x-1)-(y+1)+(z-1)\right]^2$$

or

$$(x-1)^2 + (y+1)^2 + (z-1)^2 = 6 + \frac{1}{3}\left[x-y+z-3\right]^2$$

— which is required equation of cylinder

**Q.1 (e)** Find the equation of the plane which cuts the paraboloid $x^2 - 3y^2 = 3z$ in the conic with centre (1, 2, 3). **(10)**

The center of conic $(1, 2, 3)$

Conicoid is $x^2 - 3y^2 = 3z$

The equation of plane is given by

$$T = S_1$$

$$\Rightarrow \quad x x_1 - 3yy_1 - \frac{3}{2}(z + z_1) = x_1^2 - 3y_1^2 - 3z_1$$

$$x - 6y - \frac{3}{2}(z+3) = 1 - 3(4) - 3(3)$$

$$x - 6y - \frac{3z}{2} = \frac{9}{2} - 20$$

$$\boxed{2x - 12y - 3z = -31}$$

Answer

Candidates must not write on this margin

(a) Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1,1,-1)$ and passes through the origin. (10)

sphere : $x^2+y^2+z^2-x+3y+2z-3=0$

let T.P @ $(1,1,-1)$ be

TP is $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

$\Rightarrow x+y-z-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+\frac{2}{2}(z-1)-3=0$

$x+y-z-\frac{x}{2}+\frac{3y}{2}+z=3$

$\frac{x}{2}+\frac{5y}{2}=3$ ; $\boxed{x+5y=6}$ required plane

any sphere is

$\boxed{S+\lambda P=0}$

$x^2+y^2+z^2-x+3y+2z-3+\lambda(x+5y-6)=0$

it passes through $(0,0,0)$

$\Rightarrow -3+\lambda(-6)=0 \Rightarrow 3+6\lambda=0$

$\lambda=\frac{-1}{2}$

$\therefore$ eq$^n$ is

$\boxed{x^2+y^2+z^2-\frac{3x}{2}+\frac{y}{2}+2z=0}$ Answer

Find the equation of the cone with vertex at $(2a, b, c)$ and passing through the curve $x^2+y^2=4a^2$ and $z=0$. Find $b$ and $c$ if the cone also passes through the curve $y^2=4a(z+a), x=0$. Also show that the cone is cut by the plane $y=0$ in two straight lines and the angle è between them is given by $\tan\theta=2$.

**(20)**

To find cone with vertex @ $(2a, b, c)$ passing through $x^2+y^2=4a^2$, $z=0$

let cone be

$$\frac{x-2a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \lambda \text{(say)} \quad (1)$$

it passes through $z=0 \Rightarrow \lambda = \frac{-c}{n}$

$$\Rightarrow x = \frac{-lc}{n} + 2a, \quad y = \frac{-mc}{n} + b$$

Cone is $x^2+y^2=4a^2$ from (1)

$$\Rightarrow \left[-\frac{l}{n}c + 2a\right]^2 + \left[-\frac{m}{c}n + b\right]^2 = 4a^2$$

$$\Rightarrow \boxed{\left[\frac{x-2a}{z-c} - 2a\right]^2 + \left[\frac{y-b}{z-c} - b\right]^2 = 4a^2} \quad (2)$$

— which is required equation

Cone also passes through

$y^2 = 4a(z+a)$, $x=0$ [from (1)]

$$\Rightarrow \lambda = \frac{-2a}{l} \quad \Rightarrow \quad y = \frac{-2a}{l}(m) + b$$

$$z = \frac{-2a}{l}(n) + c$$

put in $y^2 = 4a(z+a)$ (from ①)

$$\Rightarrow \left[-2a\left(\frac{m}{l}\right)+b\right]^2 = 4a\left[-a\left(\frac{n}{l}\right)+a+c\right]$$

$$\Rightarrow \boxed{\left(-2a\left(\frac{y-b}{x-2a}\right)+b\right)^2 = 4a\left(-2a\left(\frac{z-c}{x-2a}\right)+a+c\right)} \quad ③$$

which is required eq$^n$

② & ③ represent same cone

Equation ② can be written as

$$[x-2a-2a(z-c)]^2 + [(y-b)-b(z-c)]^2 = 4a^2(z-c)^2$$

$$[x-2az+2ac-2a]^2 + [y-bz+bc-b]^2 = 4a^2(z-c)^2$$

$$(x-2az)^2 + (2ac-2a)^2 + 2(2ac-2a)(x-2az) + (y-bz)^2 + (bc-b)^2 + 2(bc-b)(y-bz) = 4a^2(z-c)^2$$

$$x^2+y^2+z^2[4a^2+b^2-4a^2] + 4a(c-1)(x-2az) + 2b(c-1)(y-bz) = 4a^2(z-c)^2$$

from ③ $\quad x^2+y^2+4a(c-1)x+2b(c-1)y = 4a^2c^2$

put $z=0$ in eq ③

$$[-2a(y-b)+b(x-2a)]^2 = 4a(2ac+a(x-2a)+c(x-2a))(x-2a)$$

$$4a^2y^2+b^2x^2+4a^2b^2+4a^2b^2-2ab(x-2a)(y-b) = 8a^2c(x-2a)+4a^2x^2+4ac(x-2a)^2+4a^2(4a^2-4ax)$$

on comparing

$\Rightarrow \boxed{b=0}$

$$\Rightarrow 4a^2y^2 = 8a^2cx + 4a^2x^2 + 16a^4 - 16a^4x - 16a^3c + 4acx^2$$

$$4(a^2+ac) = -1$$

$$4a(a+c) = -1$$

$$a+c = \frac{-1}{4a} \quad \therefore \boxed{c = \frac{-1}{4a} - a}$$

$\boxed{4a^2=1}$

$$c = \frac{-1}{4\cdot\frac{1}{2}} - \frac{1}{2} = -\frac{1}{2} - \frac{1}{2} = -1$$

$\boxed{b=0}$ $\boxed{c=-1}$

cone cut by $y=0$

$$\Rightarrow \left(\frac{+2ab}{x-2a} + b\right)^2 = 4a\left(\frac{-2a(z-c)}{x-2a} + (a+c)\right)$$

$$\Rightarrow 0 = (a+c)(x-2a) - 2a(z-c)$$

or $$\left(\frac{(x-2a)}{z-c} - 2a\right)^2 = 4a^2$$

$$(x-2a-2a(z-c))^2 = 4a^2(z-c)^2$$

$$\Rightarrow [x-2az-2a(1-c)]^2 = 4a^2(z-c)^2$$

$$x^2 + \cancel{4a^2z^2} - 4azx + 4a^2(1-c)^2 - 4a(x-2az)(1-c) = \cancel{4a^2z^2} + 4a^2c^2 - 8a^2zc$$

$$x^2 + 4a^2 + \cancel{4a^2c^2} - 8a^2c + 4azx - 4ax(1-c) + 8a^2z = \cancel{4a^2c^2}$$

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \frac{2\,(2a)}{2a} = 2$$

$$\theta = \tan^{-1}(2)$$

State and prove Rank Nullity theorem for linear transformation. (20)

To state & prove Rank nullity theorem

statement:

For a linear transformation $T: U \to V$ (where U & V are vector spaces) with dimension of U (finite dimension) be dim(U) and rank of T be dimension of range space of V and nullity be dimension of $T(\alpha) = \hat{0}$

then

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(U)}$$

Proof:

consider the dimension of V as n (finite dimension)

∴ basis of U are $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$

supposity nullity of $T = m$

then $T(\alpha_i) = \hat{0} \in V$

if nullity = dimension of $T(\alpha) = \hat{0} = m$

⇒ there exists 'm' linearly independent vectors that span space $T(\alpha) = \hat{0}$

$\Rightarrow \alpha_1, \alpha_2 -- \alpha_m$ are the

basis of $T(\alpha) = \hat{0}$

where $m \leq n$

$\therefore$ we show dimension of range space of

V is ~~m~~-m

$\because \alpha_i$ is basis of V

$\therefore \alpha \in V$ can be written as

$\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 - - k_n\alpha_n$

where $k_1, -- k_n$ are non-zero

$V = T(\alpha)$ — since its linear transformation

$= T(k_1\alpha_1 + k_2\alpha_2 - - k_n\alpha_n)$

$= T(k_1\alpha_1) + T(k_2\alpha_2) - - T(k_n\alpha_n)$

$= k_1(T(\alpha_1)) + k_2 T(\alpha_2) - - k_m T(\alpha_m)$

$k_{m+1} T(\alpha_{m+1}) - - k_n(T(\alpha_n))$

$\because T(\alpha_1) = T(\alpha_2) = T(\alpha_m) = 0$

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$\Rightarrow$

$V = k_{m+1} T(\alpha_{m+1}) + k_{m+2} T(\alpha_{m+2}) \text{ --- } + k_n T(\alpha_n)$

$V = k_1' \beta_1 + k_2' \beta_2 \text{ -- } + k_{n-m}' \beta_{n-m}$

$\therefore$ V is linear span of
$\beta_1, \beta_2 \text{ --- } \beta_{n-m}$

we now show that $\beta_1 \text{ -- } \beta_{n-m}$ are linearly independent

if $k_1' \beta_1 + k_2' \beta_2 \text{ --- } + k_{n-m}' \beta_{n-m} = 0$

$\Rightarrow k_1' \beta_1 + k_2' \beta_2 \text{ -- } + k_{n-m}' \beta_{n-m} = T(\alpha)$

$\alpha \in$ null space

$\alpha = k_1 \alpha_1 \text{ --- } + k_m \alpha$
where $k_i \neq 0$

$\Rightarrow k_1' \beta_1 + k_2' \beta_2 + k_{n-m}' \beta_{n-m} =$
$T(k_1 \alpha_1 \text{ -- } k_m \alpha)$

$\because \beta_{n-m} = T(\alpha_n)$

$\Rightarrow k_1' \; k_2' \text{ -- } k_{n-m}' = 0$

$\therefore \alpha_1 \; \alpha_2 \text{ -- } \alpha_m \; \alpha_{m+1} \text{ -- } \alpha_n$ are all L.I.

therefore

$\text{rank}(T) = n-m$

$\therefore$ $(n-m)$ linearly independent vectors span $V$

$\therefore$ $\text{rank}(T) + \text{nullity}(T)$

$= (n-m) + m = n$

$= \dim(V)$

— Hence Proved

## SECTION-B

Q.5 (a) Find $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$ (10)

To find

$$\lim_{x\to 0}\frac{\left(1+\frac{1}{x}\right)^{\frac{1}{x}}-e}{x}$$

$\left[\text{this is } \frac{\to 0}{\to 0} \text{ form } \because \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{1/x}=e\right]$

$$\text{or } \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e^{1}}{x}$$

$$=\lim_{x\to 0}\frac{e\left[e^{\frac{1}{x}\ln(1+x)-1}-1\right]}{x}$$

$a^{b}-a^{c}$

$a^{c}\left[a^{b-c}-1\right]$

$$=\lim_{x\to 0}\frac{e\left[e^{\frac{\ln(1+x)-x}{x}}-1\right]}{x}$$

$\left[\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right]$

$$=\lim_{x\to 0}\frac{e\left[e^{\left(-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)\frac{1}{x}}-1\right]}{x}$$

$$=\lim_{x\to 0}\frac{e\left[e^{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}}-1\right]}{x}$$

$e^{y}=1+y+\frac{y^2}{2}\cdots$

$$=\lim_{x\to 0}\frac{e\left[\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}\cdots\right)+\frac{1}{2}\left(-\frac{x}{2}+\frac{x^2}{3}\cdots\right)^2-1\right)\right]}{x}$$

$$=\lim_{x\to 0}\frac{e}{x}\left[\left(-\frac{x}{2}+\frac{x^2}{3}\right)+\frac{1}{2}\left(-\frac{x}{2}+\frac{x^2}{3}\right)^2\cdots\right]=\frac{-e}{2}$$

$\boxed{\text{Ans: } -\frac{e}{2}}$

Check for the linear dependence the polynomials $i+x+x^2, -(1+i)-2x+2ix^2, x-x^2$ over

i C

ii R

(10)

Given polynomials

$i+x+x^2$, $-(1+i)-2x+2ix^2$, $x-x^2$

considering $\{1, x, x^2\}$ as basis

coordinates are

$$\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -(1+i) \\ -2 \\ 2i \end{bmatrix} \& \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

consider three variables $k_1$ $k_2$ $k_3$

$$k_1\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} + k_2\begin{pmatrix} -1-i \\ -2 \\ 2i \end{pmatrix} + k_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$= \begin{bmatrix} -k_2 + i(k_1-k_2) \\ k_1 - 2k_2 + k_3 \\ k_1 - k_3 + i(2k_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

assuming $k_i \in \mathbb{R}$

$\Rightarrow$ $k_2 = 0$ ; $k_1 = k_2$ for row 1

& $k_1 + k_3 = 0$ for row 2

& $k_1 - k_3 = 0$ & $k_2 = 0$ for row = 3

$k_1 = k_2 = k_3 = 0$

if $k_1\ k_2\ k_3$ can take complex values

$\Rightarrow$

$-k_2 + i(k_1 - k_2) = 0$ $\quad \therefore \quad i k_1 = (1+i)k_2$

$k_1 - 2k_2 + k_3 = 0$ $\quad k_1 = -(i-1)k_2$

$k_1 - k_3 + i2k_2 = 0$ $\quad \boxed{k_1 = (1-i)k_2}$

$(-1-i)k_2 + k_3 = 0 \Rightarrow \boxed{k_3 = (1+i)k_2}$

$\Rightarrow k_1 - k_3 + i2k_2 = 0$

$\Rightarrow -2ik_2 + 2ik_2 = 0$ } $k_2$ can take any value

$\therefore$ they are linearly dependent on $\mathbb{C}$

$$\boxed{\begin{array}{l} \text{(i) } \mathbb{C} \rightarrow \text{dependent} \\ \text{(ii) } \mathbb{R} \rightarrow \text{independent} \end{array}}$$

Answer

c) Integrate the function over the domain $f(x,y) = xy(x^2+y^2)$ over the domain $R:\{-3 \le x^2 - y^2 \le 3, 1 \le xy \le 4\}$. **(10)**

To find

$$\iint_R xy(x^2+y^2)\,dx\,dy$$

let $u = xy$ & $v = x^2-y^2$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(y^2) - 2x^2$$

$$= -2(x^2+y^2)$$

$$dx\,dy = J\left(\frac{x,y}{u,v}\right) du\,dv$$

$$= \left[J\left(\frac{u,v}{x,y}\right)\right]^{-1} du\,dv = \frac{1}{-2(x^2+y^2)}\,du\,dv$$

$$\therefore \iint_R xy(x^2+y^2)\,dx\,dy = \iint_{R'} xy(x^2+y^2)\frac{-1}{2(x^2+y^2)}\,du\,dv$$

$$\Rightarrow \int_{u=1}^{4}\int_{v=-3}^{3} u\left(\frac{-1}{2}\right) du\,dv$$

$$= \frac{-1}{2}\int_1^4 u\,du \int_{-3}^{3} dv$$

$$= \frac{-1}{2}\cdot\frac{u^2}{2}\Big|_1^4 \cdot v\Big|_{-3}^{3}$$

$$= \frac{-1}{2}\times\frac{1}{2}\left[4^2-1\right]\left[6\right]$$

$$= \frac{-1}{4}\times 6\times 15 = \frac{-45}{2}$$

Answer

Candidates must not write on this margin

5 (d) Calculate approximately $(245)^{\frac{1}{5}}$ using LMV theorem. (10)

let $(245)^{\frac{1}{5}} = x$

$\Rightarrow 245 = x^5$

consider

$f(x) = x^5 - 245$

we find root of $f(x)$ using LMV theorem

$\int f(x)dx = \frac{x^6}{6} - 245x = g(x)$ say

$\exists\, c \in (a,b)$ such that (clearly $g(x)$ is continuous & derivable)

$g'(c) = \frac{g(a)-g(b)}{a-b} = f(c)$ — LMV theorem

we find $g(a) = g(b)$

$a^6 - 1470a = b^6 - 1470b$

$g(x) = \frac{x^6 - 1470x}{6}$

by hit & trial $g(2) = \frac{-1438}{3} = -479.3$

$g(2.5) = -571.8$

$g(2.8) = -605.684$

$g(2.9) = -611.36$

$g(3.1) = -611.58$

clearly $c \in (2.9, 3.1)$

$g(2.95) = -612.9048$

$g(3.05) = -613.08$

$c \in (2.95, 3.05)$

$g(2.98) = -613.3797$

$g(3.02) = -613.458$

$c \in (2.98, 3.02)$

$g(2.99) = -613.4598$

$g(3.01) = -613.499$

$\therefore \boxed{c \sim 3}$

Ans : $(245)^{1/5} \sim 3$

What 3 by 3 matrix represent the transformation that represent the transformation that rotate the *xy* plane through 90° leaving the z-axis alone? (10)



find transformation
xy plane through $90^\circ$
& z-axis as it is

$x' = l_1 x + m_1 y + n_1 z$

$y' = l_2 x + m_2 y + n_2 z$

$z' = l_3 x + m_3 y + n_3 z$

$\therefore$ matrix is given by

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

clearly $z' = z$

$\Rightarrow l_3 = 0, m_3 = 0$ & $\boxed{n_3 = 1}$

assuming rotation is anti-clockwise

$x' = 0 \cdot x + y$ & $y' = -x + 0 \cdot y$

$\therefore (l_1\ m_1\ n_1) \equiv (0, 1, 0)$

$(l_2\ m_2\ n_2) \equiv (-1, 0, 0)$

$\therefore$ matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

assuming anti-clockwise rotation

7(a) Show that the function $f(x, y)$ is differentiable at the origin

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } x^2+y^2\neq 0 \\ 0, & \text{if } x=y=0 \end{cases}$$ (15)

Given

$$f(x,y) = \begin{cases} xy\dfrac{(x^2-y^2)}{x^2+y^2} & x^2+y^2\neq 0 \\ 0 & \end{cases}$$

we first show its continuous

$$|f(x,y)-f(0,0)| = \left|\frac{xy(x^2-y^2)}{x^2+y^2}-0\right|$$

if $x=r\cos\theta$, $y=r\sin\theta$

$$= \left|\frac{r^2\cos\theta\sin\theta\,(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$

$$= \left|\frac{r^4}{r^2}\frac{\sin 2\theta}{2}\cdot\cos 2\theta\right| = \frac{1}{4}r^2|\sin 4\theta| \leq \frac{r^2}{4} = \epsilon$$

clearly $\exists\ \delta$ such that $\sqrt{x^2+y^2} < \delta$

$r < \delta$ choose $\delta = \sqrt{4\epsilon}$

Now,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h(0)}{h}\frac{(h^2)}{h^2} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{f(0,k)}{k}$$

$$= \lim_{k\to 0} \frac{0\cdot(k)\left[-k^2\right]}{k \quad k^2} \Rightarrow = 0$$

Now, consider

$$f(0+h, 0+k) - f(0,0) = f_x \cdot h + f_y \cdot k + \sqrt{h^2+k^2}\,\phi(h,k) \quad \text{--- (1)}$$

$$\Rightarrow f(h,k) = 0\cdot h + 0\cdot k + \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\frac{hk(h^2-k^2)}{h^2+k^2} = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{hk(h^2-k^2)}{(h^2+k^2)^{3/2}}$$

to check existence of $\phi(h,k)$ at $(0,0)$

consider

$$|\phi(h,k) - 0| = \left|\frac{hk\,(h^2-k^2)}{(h^2+k^2)^{3/2}}\right|$$

if $h = R\cos\theta$, $k = R\sin\theta$

$$\Rightarrow |\phi(h,k) - 0| = \left|\frac{R^2\cos\theta\sin\theta \cdot R^2(\cos 2\theta)}{R^3}\right| = \left|R\,\frac{\sin 4\theta}{4}\right|$$

$\therefore \left|\phi(h,k)\right| = \frac{R}{4}|\sin 2\theta| \leq \frac{R}{4} \geq \epsilon$

choose $\delta = 4\epsilon$

$\therefore |\phi(h,k) - 0| < \epsilon$ when $\sqrt{h^2+k^2} < \delta$

$\therefore \lim\limits_{h\to 0, k\to 0} \phi(h,k) = 0$

$\therefore$ in eq (1)

we have

$f(x,y)$ as differentiable at $(0,0)$

— Hence Proved

**Q.7 (b)** Show that

$$\int_0^{\frac{\pi}{2}} \log\left(1-x^2\sin^2\theta\right)d\theta = \pi\log\left(1+\sqrt{1-x^2}\right)-\pi\log 2, if\ |x|<1.$$ (15)

Given to solve $\int_0^{\pi/2} \log(1-x^2\sin^2\theta)\,d\theta$

let

$$f(x) = \int_0^{\pi/2} \log(1-x^2\sin^2\theta)\,d\theta$$

$$\Rightarrow f'(x) = \int_0^{\pi/2} \frac{\partial}{\partial x}\left(\log(1-x^2\sin^2\theta)\right)d\theta$$

$$= \int_0^{\pi/2} \frac{-\sin^2\theta\ (2x)\ d\theta}{1-x^2\sin^2\theta}$$

$$= -2\int_0^{\pi/2} \frac{\sin^2\theta\ x\,d\theta}{1-x^2\sin^2\theta}$$

$$= -2x\int_0^{\pi/2} \frac{d\theta}{\operatorname{cosec}^2\theta - x^2}$$

$$f'(x) = -2x\int_0^{\pi/2} \frac{d\theta}{1-x^2+\cot^2\theta} = -2x\int_0^{\pi/2} \frac{d\theta}{1-x^2+\tan^2\theta} \quad \text{--- (1)}$$

let $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$$\Rightarrow \int_0^{\pi/2} \frac{d\theta}{(1-x^2)+t^2} = \int_0^{\infty} \frac{dt}{(1-x^2)+t^2}\frac{1}{(1+t^2)}$$

$$= \int_0^{\infty} \frac{A}{1-x^2+t^2} + \int_0^{\infty} \frac{B}{1+t^2}\, dt$$

$A(1+t^2) + B(1-x^2+t^2) = 1$

$A + B - Bx^2 = 1$

$A + B = 0$

$A = -B$

$A - A + Ax^2 = 1$

$A = \frac{1}{x^2}$

$\frac{1}{x^2}\left[1+t^2 - 1 + x^2 - t^2\right]$

$$= \frac{A}{\sqrt{1-x^2}}\tan^{-1}\left[\frac{t}{\sqrt{1-x^2}}\right]_0^{\infty} + B\tan^{-1}t\Big|_0^{\infty}$$

$$= \frac{A}{\sqrt{1-x^2}}\left(\frac{\pi}{2}\right) + B\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{1}{x^2}\right)\frac{\pi}{2}\left[\frac{1}{\sqrt{1-x^2}} - 1\right]$$

$$\therefore f'(x) = -2x \cdot \frac{\pi}{2(x^2)}\left[\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]$$ putting in (1)

$$f(x) = -\pi\int \frac{dx}{x\sqrt{1-x^2}} + \pi\int \frac{dx}{x} \quad — (2)$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta} = \int \operatorname{cosec}\theta\, d\theta = -\ln|\operatorname{cosec}\theta + \cot\theta| + C$$

$$= -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$$

$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore$

$$f(x) = -\pi\left[-\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right|\right] + \pi \ln x + C$$

$$\boxed{f(x) = \pi \ln\left(1+\sqrt{1-x^2}\right) + C}$$ 3

put $x=1 \Rightarrow f(1) = C \quad \because \ln(1)=0$

$$\int_0^{\pi/2} \log(1-x^2\sin^2\theta)\,d\theta = \int_0^{\pi/2} \log(1-\sin^2\theta)\,d\theta$$

if put $x=1$

$$f(1) = \int_0^{\pi/2} \log\cos^2\theta\,d\theta = 2\int_0^{\pi/2} \log\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \log\sin\theta\,d\theta$$

$$2f(1) = 2\int_0^{\pi/2}(\log\cos\theta + \log\sin\theta)\,d\theta = 2\int_0^{\pi/2} \log\frac{\sin 2\theta}{2}\,d\theta$$

$$f(1) = \int_0^{\pi/2} \log\sin 2\theta\,d\theta - \int_0^{\pi/2} \log 2\,d\theta$$

$$f(1) = \frac{1}{2}\int_0^{\pi} \log\sin\theta\,d\theta - \frac{\pi}{2}\log 2$$

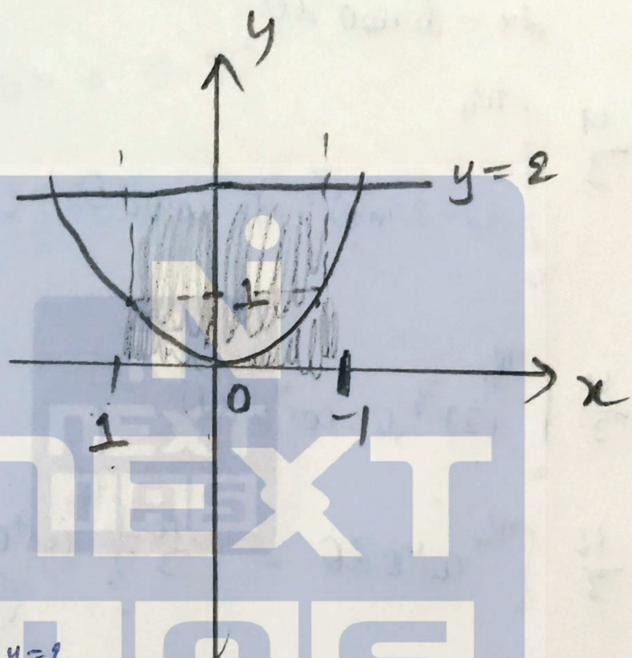
$$f(1) = \underbrace{\int_0^{\pi/2} \log\sin\theta\,d\theta}_{f(1)/2} - \frac{\pi}{2}\log 2 \Rightarrow f(1) = -\pi\log 2$$

$$\therefore \boxed{f(x) = \pi\ln\left(1+\sqrt{1-x^2}\right) - \pi\log 2}$$

7(c) Evaluate $\iint_R \sqrt{|y-x^2|}\, dxdy$ where $R=[-1,1;0,2]$. (20)

$$\iint_R \sqrt{|y-x^2|}\, dxdy$$

$R : [-1,1, 0,2]$



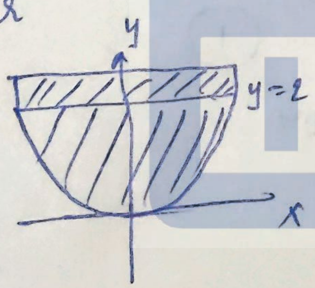

region R

$x : [-1,1]$

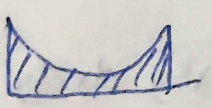
$y : [0,2]$

consider

$R_1$




$R_2$ :



for $R_1$ : $y > x^2$

$$\iint_{R_1} \sqrt{y-x^2}\, dx\, dy = \int_{-1}^{1}\int_{x^2}^{2} \sqrt{y-x^2}\, dx\, dy$$

$$= \int_{-1}^{1} \frac{(y-x^2)^{3/2}}{3/2}\Bigg|_{x^2}^{2} dx = \int_{-1}^{1} \frac{(2-x^2)^{3/2}}{3/2}\, dx$$

$$= \int_{-1}^{1} 2\frac{(2-x^2)^{3/2}}{3}\,dx = \frac{4}{3}\int_0^1 (2-x^2)^{3/2}\,dx$$

let $x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\,d\theta$

$$\frac{4}{3}\int_0^{\pi/4} (2-2\sin^2\theta)^{3/2}\cdot\sqrt{2}\cos\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/4} (2)^{3/2}\sqrt{2}\cdot\cos^4\theta\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/4}\cos^4\theta\,d\theta = \frac{16}{3}\cdot\int_0^{\pi/4}\cos^4\theta\,d\theta$$

1

$$= 2.904$$

$R_2 : x^2 > y$

$$\iint_{R_2}\sqrt{x^2-y}\,dx\,dy = \int_{-1}^{1}\int_0^{x^2}\sqrt{x^2-y}\,dx\,dy$$

$$= \int_{-1}^{1} \left.\frac{(x^2-y)^{3/2}}{-3/2}\right|_0^{x^2} dx = \frac{2}{3}\int_{-1}^{1}(x^2)^{3/2}\,dx$$

$$= \frac{4}{3}\int_0^1 x^3\,dx = \frac{4}{3}\cdot\frac{1}{4} = \frac{1}{3}$$

Ans: $\iint_R \sqrt{|y-x^2|}\, dx\,dy$

$= \iint_{R_1} \sqrt{y-x^2}\, dx\,dy + \iint_{R_2} \sqrt{x^2-y}\, dx\,dy$

$= 2.904 + 0.333$

$= \underline{3.237}$

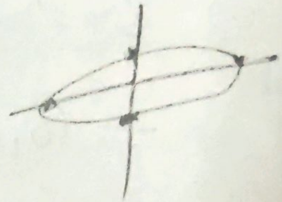
Candidates must not write on this margin

8 (a) Find the ends of the major and minor axes of the ellipse $3x^2-2xy+3y^2=4$. (20)

Given:

Ellipse:-

$3x^2-2xy+3y^2=4$



Compare with

$ax^2+2hxy+by^2-d=0$

$A=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$

$|A-\lambda I|=0 \Rightarrow \lambda^2-6\lambda+8=0$

$\lambda=4 \quad \lambda=2$

$\lambda=4$

$(A-\lambda I)X=0 \Rightarrow \begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$\Rightarrow x=-y \Rightarrow \begin{pmatrix}1\\ -1\end{pmatrix}$ is characteristic vector

$\langle l,m,n\rangle$ are $\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\rangle$

for $\lambda=2$

$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ $\quad$ $(l,m,n)$ are $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$

for center

$\therefore f_x = 0$ & $f_y = 0$

$\Rightarrow 6x - 2y = 0$ & $-2\cancel{x} + 6y = 0$

$\Rightarrow (0,0)$ is center

$d' = u\alpha + v\beta + d = -4$

$\therefore \quad x' = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$

$y' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$

Eqn of Ellipse with transformed axes are:

$\lambda_1 x^2 + \lambda_2 y^2 = 4$

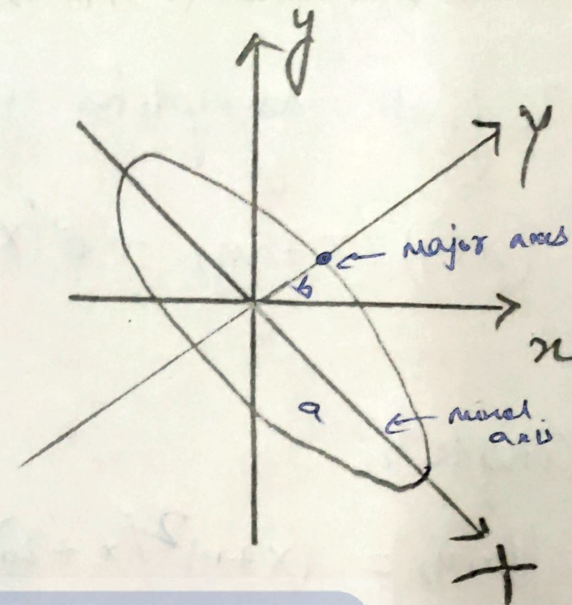
$\Rightarrow \boxed{4x^2 + 2y^2 = 4}$

$4\left(\frac{x-y}{\sqrt{2}}\right)^2 + 2\left(\frac{x+y}{\sqrt{2}}\right)^2 = 4$

Check: $2(x^2+y^2-2xy) + x^2+y^2+2xy = 4$

Eq^n is

$$\frac{X^2}{1} + \frac{Y^2}{2} = 1$$




Comparing with

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$\Rightarrow \quad a = 1, \quad b = \sqrt{2}$

Coordinates of ends Major axis $b = 2$

is $\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\right), \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\right) = (1, 1)$

& $\left(\sqrt{2}\left(\frac{-1}{\sqrt{2}}\right), \sqrt{2}\left(\frac{-1}{\sqrt{2}}\right)\right) = (-1, -1)$

Coordinates of end minor axis are

$\left(1\left(\frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

& $\left(-1\left(\frac{1}{\sqrt{2}}\right), -1\left(\frac{-1}{\sqrt{2}}\right)\right) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Answer

**Q.8 (b)** Find the asymptotes of the curve $(x+y)^2(x+2y)+2(x+y)^2-(x+9y)-2=0$.

(15)

To find asymptotes of

$(x+y)^2(x+2y) + 2(x+y)^2 - (x+9y) - 2 = 0$

Consider

$f(x,y) = (x+y)^2(x+2y)$

homogeneous part of degree 3

$\phi_3(m) = (m+1)^2(2m+1)$

$\phi_2(m) = 2(m+1)^2$

$\phi_1(m) = -(9m+1)$

for m calculation

$\phi_3(m) = 0 \Rightarrow (m+1)^2(2m+1) = 0$

$\Rightarrow m = -1, -1, \frac{-1}{2}$

$\phi_3'(m) = 2(m+1)(2m+1) + 2(m+1)^2$

for $m = \frac{-1}{2}$

$C = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-\phi_2(\frac{-1}{2})}{\phi_3'(\frac{-1}{2})} = \frac{-2(\frac{1}{2})^2}{2(-\frac{1}{2})^2} = -1$

$\therefore \boxed{y = \frac{-1}{2}(x) - 1}$ one of asymptote

for $m = -1$

$c = \frac{-\phi_2(-1)}{\phi_3'(-1)}$ $\frac{\to 0}{\to 0}$ form

$\therefore \frac{c^2}{2}\phi_3''(m) + \frac{c}{2}\phi_2'(m) + \phi_1(m) = 0$

$\phi_3''(m) = 2(2m+1) + 4(m+1) + 4(m+1)$
$= 2(2m+1) + 8(m+1) = 12m + 10$

$\phi_2'(m) = 4(m+1) = 4m+4$

$\phi_3(m) = -9m - 1$

$\phi_3''(-1) = -2$

$\phi_2'(-1) = 0$

$\phi_3(-1) = 8$

$\frac{c^2}{2}(-2) + c(0) + 8 = 0$

$\Rightarrow \boxed{-c^2 + 8 = 0}$

$\Rightarrow c = \pm 2\sqrt{2}$

$\therefore \boxed{y = -x \pm 2\sqrt{2}}$ other asymptotes

∴ asymptotes of curve are

$$\boxed{y = -\frac{x}{2} - 1}$$

$$\boxed{y = -x - 2\sqrt{2}}$$

$$\boxed{y = -x + 2\sqrt{2}}$$

8(c) Let $U$ and $V$ be finite dimensional vector spaces over $F$. Then prove that $U$ and $V$ are isomorphic if and only if $\dim U = \dim V$.

(15)

Given U & V as finite dimensional vector spaces over F

To show

U & V are isomorphic $\Longleftrightarrow \dim U = \dim V$

Proof:

let $\dim U = m$

$\dim V = n$

Two vector spaces are isomorphic if

(i) $f(x+y) = f(x) + f(y)$ $\quad x, y \in U$

$f(x), f(y) \in V$

(ii) it is one-one

(iii) it is onto

consider $f$ to be a linear transformation

$$T_f : U \to V$$

$$\therefore \; T_f(a\alpha + b\beta) = aT_f(\alpha) + bT_f(\beta)$$

by property of linear transformation

$\therefore \; f(x+y) = f(x) + f(y)$ — homomorphism

(i) To show its onto

$T(\alpha) \in V$ or if $\beta \in V$

then $\beta = k_1 T(\alpha_1) + k_2 T(\alpha_2) - - + k_n T(\alpha_n)$

where $\alpha_1 - - \alpha_n$ are linearly independependent "basis" of $U$

$\therefore$ there exists

$\alpha$ such that

$\alpha = k_1\alpha_1 + k_2 \alpha_2 - - k_n \alpha_n$

for $\beta = T(\alpha)$

hence its onto

(ii) to show its 1-1

for one-one

basis of $U$ should correspond to basis of $V$

such that $T(\alpha_i) = \beta_i$

only possible if $\dim V = \dim U$

$\Rightarrow$ $U$ & $V$ are isomorphic

Conversely,

If $U$ & $V$ are isomorphic we show

$\dim U = \dim V$

let $\alpha_1 \dots \alpha_m$ be basis of $U$

& $\beta_1$ — $\beta_n$ be basis of $V$

for one-one

$T(\alpha_i) = \beta_i$

for all $i$

$\Rightarrow \quad m = n$

$\Rightarrow \quad \dim U = \dim V$

$\Rightarrow$ nullity $= 0 \Rightarrow T(\alpha) = 0 \quad \alpha = 0$

which is also condition for isomorphism

Hence Proved