NEXT IAS

(To be filled by candidate)

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Registration Number : NIAS2300020250 Date of Examination : 28/8

Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Online ☑

CSE (MAINS) TEST SERIES - 2023

Test-06	MATHEMATICS OPTIONAL
Dated : 20-08-2023	PAPER-II: FULL LENGTH

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are ***EIGHT*** *questions divided in* ***Two sections*** *and printed in* ***ENGLISH.***

Candidate has to attempt ***FIVE*** *questions in all.*

Question No. 1 *and* ***5*** *are compulsory and out of the remaining, any* ***THREE*** *are to be attempted choosing at least* ***ONE*** *question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA). Booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order. Unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA). Booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	68	15			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
			Grand Total		

Remarks

Observations:

SECTION-A

Q.1 (a) Find all the units of $Z[\sqrt{-5}]$. **(10)**

Let $Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in Z\}$

Let $a + b\sqrt{-5} \in Z[\sqrt{-5}]$ be a unit.

Then, $a + b\sqrt{-5} \mid 1$ and so

$\exists$ $c + d\sqrt{-5} \in Z[\sqrt{-5}]$ s.t.

$(a+b\sqrt{-5})(c+d\sqrt{-5}) = 1$ (Unity of $Z\sqrt{-5}$) — (1)

(where $a, b, c, d \in Z$)

$\Rightarrow$ $(ac - 5bd) + (bc + ad)\sqrt{-5} = 1$

Comparing both sides,

$ac - 5bd = 1$

and $bc + ad = 0$

From (1), taking conjugate,

$(a - b\sqrt{-5})(c - d\sqrt{-5}) = 1$ — (2)

Multiplying (1) and (2),

$(a^2 + 5b^2)(c^2 + 5d^2) = 1$

$\therefore$ $a^2 + 5b^2 = 1$, $\because$ $a^2, b^2, c^2, d^2 \geq 1$ $\forall \in Z$.

$\Rightarrow$ $a = \pm 1,\ b = 0$ is the only case.

Thus, $\boxed{1 \text{ and } -1}$ are the only two units of $Z[\sqrt{-5}]$

Q.1 (c) Show that every convergent sequence is bounded but the converse may not be true. (10)

(i) To show every convergent sequence is bounded.

Let (x_n) be a sequence which converges to L.

Then by Cauchy criterion of convergence, given $\epsilon > 0$, $\exists$ N s.t. $|x_n - L| < \epsilon \quad \forall \; n \geq N$.

~~Now consider~~

~~$A = Max$~~

$L-\epsilon < x_n < \epsilon + L$

$\forall \; n \geq N$

where ϵ is a constant.

Now consider

$A = Max\{|x_1|, |x_2|, \dots |x_{n-1}|, |\epsilon + L|, |L-\epsilon|\}$

Then, $x_k \leq A \quad \forall \; k \in Z$

i.e, (x_n) is bounded sequence

$\because \exists$ A s.t. $|x_n| \leq A \quad \forall \; n \in Z$

Thus every convergent sequence is bounded.

(ii) The converse may not be true.

consider (x_n) as
$x_n = (-1)^n$

Then (x_n) is $\{-1, 1, -1, 1, \dots\}$

clearly $|x_n| = 1 \ \forall\ n \in \mathbb{Z}$
$\Rightarrow$ (x_n) is a bounded sequence.

However (x_n) is an oscillating sequence as

$$x_n = \begin{cases} -1 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$$

and so (x_n) is NOT a convergent sequence.

Hence, converse may not be true: A bounded sequence may or may not be convgt

($(y_n) = 1$ is a sequence which is bounded and also convgt)

Q.1 (d) Prove that the converse of Lagrange's theorem is true in case of cyclic group. (10)

Lagrange's Theorem : If $(G,*)$ is a group and $(H,*)$ is a subgroup of G, then $o(H) / o(G)$.

Converse of Lagrange's Theorem :

If $(G,*)$ is a group and $\exists\ m \in \mathbb{Z}$ s.t. $m / o(G)$, then G has a subgroup of order m.

Let $(G,*)$ be a cyclic group. Say $o(G)=n$, and $G=\langle g \rangle$. Then $o(g)=n$.

Given $k>0$, $k \in \mathbb{Z}$ s.t. k/n.

~~Let $\gcd(k,n)=d$. Then we claim that $\langle g^{n/d} \rangle$ is a subgroup of G of order k.~~

~~Let $H=\langle g^{n/d} \rangle = \langle g^{n/(n,k)} \rangle$, $(g^{n/(n,k)})$~~

Then $n=kr$ (say), $r \in \mathbb{Z}$.

OR, $k=\frac{n}{r}$. Now let $p \in \mathbb{Z}$, $p<n$ be such that $(p,n)=r$

Now let $H = \langle g^{p} \rangle$

Clearly H is a subgroup of G.

Now $O(H) = \frac{n}{(p,n)}$

$= \frac{n}{r} = \boxed{k}$

$\therefore$ H is a subgrp of G with order k.

$\therefore$ Converse of Lagrange theorem holds in case of cyclic group.

(We proved for multiplicative group, it can be shown similarly for additive group)

Q.1 (e) Using graphical method, solve the linear programming problem.

$$Maximise\ Z = 3x_1 + 2x_2$$

Subject to
$$x_1 - x_2 \geq 1,$$
$$x_1 + x_3 \geq 3 \quad .$$
$$x_1, x_2, x_3 \geq 0$$

(10)

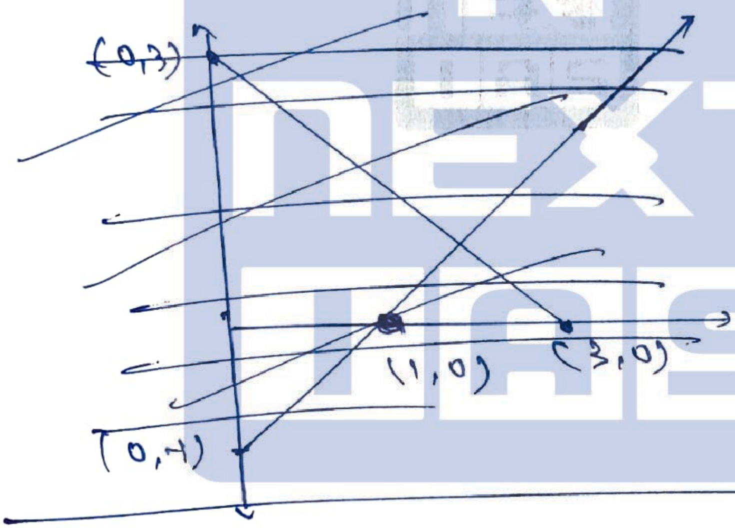
Max. $Z = 3x_1 + 2x_2$

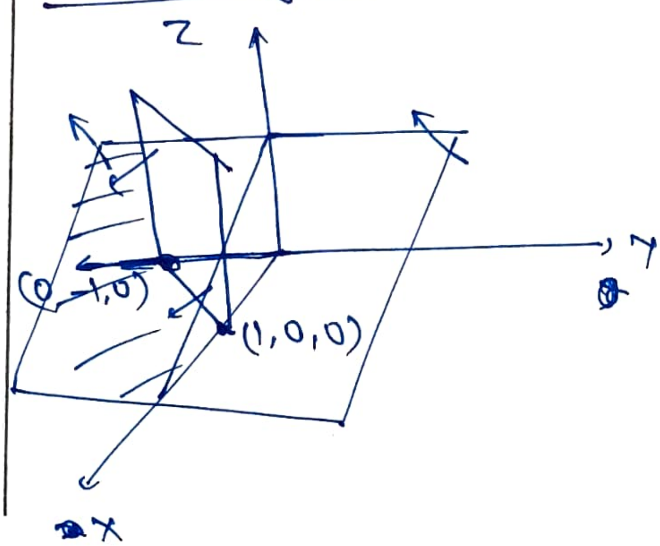
sub to $\quad x_1 - x_2 \geq 1$

$\quad x_1 + x_3 \geq 3$

$\quad x_1, x_2, x_3 \geq 0$

Let us take x_1 along x-axis and x_2 along y-axis, and x_3 along z-axis.





From the figure we can see that the feasible region is UNBOUNDED.

Thus, there is no solution to given LPP ($\because$ It is a maximizn type) (it is unbounded)

Q.2 (a) If G is a group s.t. $\frac{G}{Z(G)}$ is cyclic, where $Z(G)$ is centre of G then show that G is abelian. (15)

$Z(G) = \{x \mid x \in G,\ xa = ax\ \forall\ a \in G\}$

Given $\frac{G}{Z(G)}$ is cyclic, let it be generated by $gZ(G)$

$\left(\frac{G}{Z(G)} = \{gZ(G) \mid g \in G\}\right)$

Note: we have assumed G as a multiplicative group ; similar proof holds for additive group.

$\frac{G}{Z(G)} = \langle gZ(G) \rangle$

Let $x, y \in G$

Then $xZ(G) \in \frac{G}{Z(G)}$

$\Rightarrow$ $xZ(G) = g^i Z(G)$ for $i \in \mathbb{Z}$
($\because$ cyclic)

$\Rightarrow$ $\boxed{x = g^i z_1}$ for some $z_1 \in Z(G)$ — (1)

Similarly $yZ(G) = g^j Z(G)$
for $j \in \mathbb{Z}$

$\Rightarrow \boxed{y = g^j z_2}$ for $z_2 \in Z(G)$ — (2)

Then

$xy = (g^i z_1)(g^j z_2)$ (from (1), (2))

$= g^i (z_1 g^j) z_2$ ($\because$ G is associative)

$= g^i g^j z_1 z_2$ ($\because z_1 \in Z(G)$)

$\Rightarrow \boxed{xy = g^{i+j} z_1 z_2}$ — (3)

Similarly,

$yx = (g^j z_2)(g^i z_1)$

$= g^j (z_2 g^i) z_1$

$= g^j g^i z_2 z_1$

$= g^{j+i} (z_2 z_1)$ ($\because z_1, z_2 \in Z(G)$)

$\boxed{yx = g^{i+j} (z_1 z_2)}$ — (4)

($\because g^i g^j = g^j g^i$ as cyclic group is abelian)

From (3), (4) we get

$xy = yx$

Since x, y were arbitrary elements in G,
thus we get,

$$xy = yx \quad \forall\, x, y \in G$$

$\therefore$ G is abelian proved.

Q.2 (b) Let $a<b$ be real numbers and $0<\lambda<1$. Put $x_1=a, x_2=b$ and $x_{n+2}=\lambda x_n+(1-\lambda)x_{n+1}$. Show that the sequence $\{x_n\}$ converges and find its limit. (15)

$x_1 = a, \quad x_2 = b \quad (b > a)$

$x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1}$

~~First we will show that (x_n) is a monotonically increasing sequence.~~

~~$x_2 > x_1$~~

~~Let $x_n > x_{n-1}$~~

$x_3 = \lambda x_1 + (1-\lambda)x_2$

$= \lambda a + (1-\lambda)b$

$= b + \lambda(a-b) < b$

and $x_3 > \lambda a + (1-\lambda)a = a$

$\therefore \; a < x_3 < b$

$x_{n+2} - x_{n+1} = \lambda x_n + (1-\lambda)x_{n+1} - x_{n+1}$

$= \lambda(x_n - x_{n+1})$

$\Rightarrow |x_{n+2} - x_{n+1}| = \lambda|x_{n+1} - x_n|$

$= \lambda^2 |x_n - x_{n-1}|$

$\vdots$

$\Rightarrow \; = \lambda^n |x_2 - x_1|$

$\Rightarrow |x_{n+2} - x_{n+1}| = \lambda^n |x_2 - x_1|$

$= \lambda^n (b - a)$

$\because$ $0 < \lambda < 1$, so as $n \to \infty$,

$|x_{n+2} - x_{n+1}| \to 0$.

Thus, the sequence converges.

Let $(x_n) \to L$ (ie let L be the limit)

~~Using~~ $\cancel{x_{n+2} =}$ $x_1 = a$, $x_2 = b$

$x_3 = \lambda a + (1-\lambda) b$

$x_4 = \lambda b + (1-\lambda)(\lambda a + (1-\lambda) b)$

$= (1-\lambda)(\lambda a) + b(\lambda + (1-\lambda)^2)$

$= (\lambda - \lambda^2) a + b(\lambda^2 + 1 - \lambda)$

$x_4 - a = (b - a) \times (\lambda^2 + 1 - \lambda)$

$(x_3 - a) = (1-\lambda)(b-a)$

$x_5 = \lambda(\lambda a + (1-\lambda) b) + (1-\lambda)\left((\lambda - \lambda^2) a + b(\lambda^2 + 1 - \lambda)\right)$

$x_5 - a = a(\lambda - 2\lambda^2 + \lambda^3 + \lambda^2 - 1) + b(\lambda - \lambda^2 + \lambda^2 + 1 - \lambda - \lambda^3 - \lambda + \lambda^2)$

$= (b-a)(1 - \lambda + \lambda^2 - \lambda^3)$

$\Rightarrow$ We get $x_n - a = (b-a)(1 - \lambda + \lambda^2 - \lambda^3 + \cdots (-1)^n \lambda^n)$

Taking $n \to \infty$

$\Rightarrow \boxed{L = \dfrac{b-a}{1+\lambda} + a}$ (Writing GP series)

$\Rightarrow \boxed{L = \dfrac{b + a\lambda}{1+\lambda}}$ is limit of sequence.

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Q.2 (c) Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m iff $f(z)$ can be written in the form $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non- zero at z_0. Moreover, $Res_{z=z_0} f(z)=\frac{\phi^{m-1}(z_0)}{(m-1)!} if\ m\geq 1.$ (20)

Given $f(z)$ and z_0 is an isolated singular point of $f(z)$.

(i) ~~To Prove~~ Given z_0 is a pole of order m

Then, we can write Laurent series of $f(z)$ around z_0 as

$$\cancel{f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots}$$

$$\cancel{f(z) = \sum_{k=-m}^{\infty} f^{(k)}(z_0)\cdot(z-z_0)^k}$$

$$f(z) = \sum_{r=-m}^{\infty} a_r (z-z_0)^r$$

$$= \frac{a_{-m}}{(z-z_0)^m} + \cdots + a_0 + a_1(z-z_0) + \cdots \quad \boxed{\text{where } a_{-m} \neq 0}$$

($\because$ Pole of order m, hence the Laurent series has exactly 'm' terms in Principal part).

Then, taking den^r as $(z-z_0)^m$, we get

$$f(z) = \frac{a_{-m} + a_{-m+1}(z-z_0) + \dots + a_0(z-z_0)^m + \dots}{(z-z_0)^m}$$

$$= \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots + a_0(z-z_0)^m + \dots$

Clearly $\phi(z)$ is analytic at z_0, and $\phi(z_0) = a_{-m} \neq 0$.

$\therefore$ $f(z)$ can be written in the form $\frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

(ii) Let $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0.

Then, $\lim_{z \to z_0} (z-z_0)^m . f(z)$

$$= \lim_{z \to z_0} \phi(z)$$

$$= a_{-m} \neq 0, \infty$$

i.e, $\lim_{z \to z_0} (z-z_0)^m f(z)$ is a non-zero, finite number.

Then by definition,
z_0 is a pole of order 'm' of $f(z)$.

To find residue $\text{Res}_{z=z_0} f(z)$

$f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic.

As $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + \dots$

hence $\boxed{\text{Res}_{z=z_0} f(z) = a_{-1}}$ — (1)

Now,
$\phi(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots$

$\Rightarrow \phi^{(m-1)}(z) = (m-1)!\, a_{-1} + m!\, a_0 (z-z_0) + \dots$

Putting $z=z_0$, we get
$\phi^{(m-1)}(z_0) = (m-1)!\, a_{-1} + 0$

$\Rightarrow a_{-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

Then from (1) we get,

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

if $m \geq 1$.

SECTION-B

Q.5 (a) Show that the equations $xp = yq$, $z(xp + yq) = 2xy$ are compatible and solve them. (10)

$xp = yq$ & $z(xp+yq) = 2xy$

Consider $f(x,y,z,p,q) = xp - yq$

Writing Charpit's eqn,

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x} = \frac{dq}{f_y}$$

$$\Rightarrow \frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{-px - qy} = \frac{dp}{p} = \frac{dq}{-q}$$

~~First~~ $\Rightarrow \boxed{pq = a}$ where 'a' is a constant.

And $xp = yq \Rightarrow xp = y \cdot a/p$

$\Rightarrow p^2 = ay/x$

$\Rightarrow p = \sqrt{ay/x}$, $q = \frac{a}{p} = \sqrt{ax/y}$

$dz = p\,dx + q\,dy = \sqrt{a}\left(\sqrt{\frac{y}{x}}\,dx + \sqrt{\frac{x}{y}}\,dy\right)$

$dz = \sqrt{a} \cdot \frac{(y\,dx + x\,dy)}{\sqrt{xy}}$

$$\Rightarrow \frac{dz}{\sqrt{a}} = \frac{d(xy)}{\sqrt{xy}}$$

$\Rightarrow \boxed{z/\sqrt{a} = 2\sqrt{xy} + c}$ (1) where c is an arbitrary constant.

We will show that there exist constants a, c s.t. (1) also satisfies $z(xp+yq) = 2xy$

Using $p = \sqrt{ay/x}$, $q = \sqrt{ax/y}$,

$$\sqrt{a}\left(2\sqrt{xy}+c\right)\left\{2\sqrt{axy}\right\} = 2xy$$

$$\Rightarrow \quad 4xy(\sqrt{a}) + 2ac\sqrt{xy} = 2xy.$$

which is always true if we take

$\boxed{c=0}$, $\boxed{a = 1/4}$

∴ The 2 systems / PDEs are compatible, and their sol^n is

$$\boxed{z = \sqrt{xy}}$$

Q.5 (c) Find the principal (or canonical) disjunctive normal form in three variables p,q,r for the Boolean expression $((p\wedge q)\rightarrow r)\vee((p\wedge q)\rightarrow\neg r)$. Is the given Boolean expression a contradiction or a tautology? (10)

We know that

$$p \rightarrow q \equiv \sim p \vee q$$

$$((p\wedge q)\rightarrow r)\vee((p\wedge q)\rightarrow \sim r)$$

$$= (\sim(p\wedge q) \vee r) \vee (\sim(p\wedge q) \vee \sim r)$$

$$= ((\sim p \vee \sim q) \vee r) \vee ((\sim p \vee \sim q) \vee \sim r)$$

$$\cancel{= (\sim p \vee \sim q \vee \sim r)}$$

$$\cancel{= (\sim(p \wedge q \wedge \sim r)) \vee (\sim(p \wedge q \wedge r))}$$

$$= \sim p \vee \sim q \vee \sim p \vee \sim q \vee r \vee \sim r$$

$$= \boxed{\sim p \vee \sim q \vee r \vee (\sim r)}$$

$$= \cancel{(\sim p \wedge (q \vee \sim q) \wedge (r \vee \sim r)) \vee (\sim q \wedge (p \vee \sim p) \wedge (r \vee \sim r)) \vee (r \wedge (p \vee \sim p) \wedge (q \vee \sim q)) \vee (\sim r \wedge (p \vee \sim p) \wedge (q \vee \sim q))}$$

$$= \boxed{1} \quad (\because r \vee \sim r = 1)$$

$$= 1 \wedge (p \vee \sim p) \wedge (q \vee \sim q) \wedge (r \vee \sim r)$$

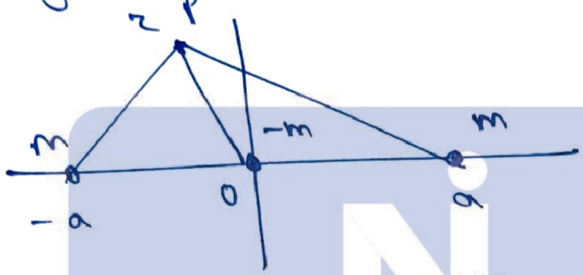
$$\boxed{\begin{aligned} &= (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee \\ &\quad (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee \\ &(\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee \\ &\quad (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r) \end{aligned}}$$

This is the reqd PDNF form.

$\because$ The expression equals 1, hence it is a TAUTOLOGY.

Q.5 (d) Find the stream function of the two-dimensional motion due to two equal sources and an equal sink situated midway between them. (10)

Let 2 sources of strength 'm' be located at $z=a$ and $z=-a$, ($a>0$, $a\in\mathbb{R}$) and a sink of strength '$-m$' be at $z=0$.



(i.e. 2 sources at $(a,0)$, $(-a,0)$ and a sink at $(0,0)$)

complex potential at a point P ($z=z$) is given by

$$W = -m\log(z-a) - m\log(z+a) + m\log z$$

$$= m\log\left(\frac{z}{z^2-a^2}\right)$$

$$= m\log\left(\frac{x+iy}{(x^2+y^2-a^2)+2ixy}\right)$$

$W = \phi + i\psi$ where ϕ is potential function and ψ is stream fn.

Using $\boxed{\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + \tan^{-1}(b/a)}$,

$$\psi = m\tan^{-1}(y/x) - m\tan^{-1}\left(\frac{2xy}{x^2+y^2-a^2}\right)$$

$$= m\left\{\tan^{-1}(y/x) - \tan^{-1}\left(\frac{2xy}{x^2+y^2-a^2}\right)\right\}$$

$$= m\left\{\tan^{-1}\left(\frac{y/x - 2xy/x^2+y^2-a^2}{1+(y/x)(2xy/x^2+y^2-a^2)}\right)\right\}$$

Streamlines are $\psi = c$, or

$$\frac{\frac{y}{x} - \frac{2xy}{x^2+y^2-a^2}}{1+\frac{y}{x}\cdot\frac{2y}{x^2+y^2-a^2}} = c_2$$

$$\Rightarrow \boxed{y^3 - ya^2 - x^2y = xc_2(x^2+3y^2-a^2)}$$

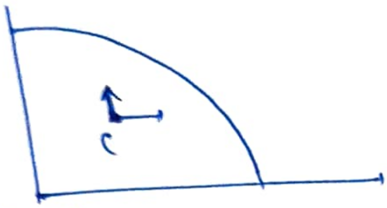
These represent the streamlines,

Stream function is

$$\psi = m\tan^{-1}\left\{\frac{y/x - \frac{2xy}{x^2+y^2-a^2}}{1+2y^2/x^2+y^2-a^2}\right\}$$

Q.5 (e) Show that at the centre of a quadrant of an ellipse, the principal axes in its plane are inclined at an angle $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$. (10)

consider a quadrant of the ellipse in first quadrant.



O(0,0)

c be the centre of the quadrant of ellipse.

Angle of inclination of principal axis be θ to horizontal, then

$$\theta = \cancel{\tfrac{1}{2}} \tan^{-1}\left(\frac{2F}{B-A}\right)$$

where B = moment of inertia along axis through c parallel to Y-axis,

A = MoI along axis through c parallel to X-axis.

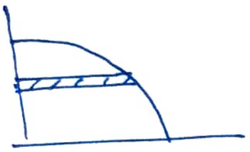
F = Product of inertia at C.

Let equation of ellipse be

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

$$\Rightarrow y = b\sqrt{1-x^2/a^2}$$

Let us first find coordinates of the centre of ellipse.



$$y_c = \frac{\int_0^b x\,dy}{\pi ab/4}$$

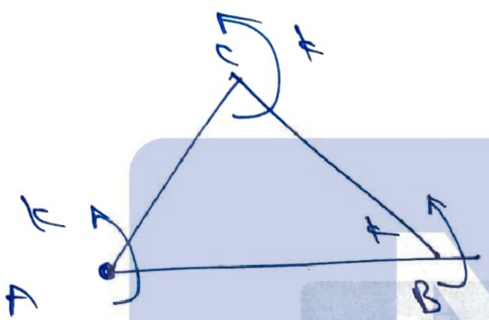
$$= \frac{4}{\pi ab}(b/a)\int_0^b \sqrt{b^2 - y^2}\,dy$$

$$= \left(\frac{4}{\pi b}\right)\left(y\sqrt{b^2-y^2} + \frac{b^2}{2}\sin^{-1}(y/b)\right)_0^b$$

$$= \frac{4}{\pi}\left(\frac{b^2}{2}\cdot\frac{\pi}{2}\right) = b^2$$

Q.6 (a) Three parallel rectilinear vortices of the same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. show that the vortices all move round the same cylinder with uniform speed in time $(4\pi^2a^2)/3k$. (20)

complex potential due to a vortex is $W = \frac{ik}{2\pi}\log(z-z_0)$
(at z_0)

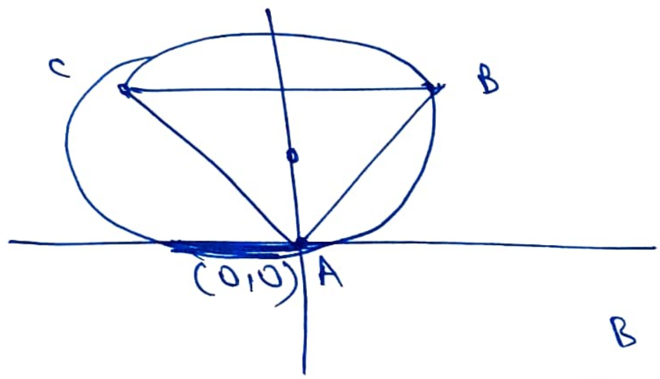


Let the 3 vortices be at
A (0, 0)
B $\left(a e^{i\pi/3}\right)$
C $\left(a e^{i2\pi/3}\right)$

Total complex potential due to the 3 is

$$W_{tot} = \frac{ik}{2\pi}\log\left((z)(z-ae^{i\pi/3})(z-ae^{i2\pi/3})\right)$$

$$= \frac{ik}{2\pi}\log\left\{(z)\left(z-\left(\frac{a}{2}+ia\frac{\sqrt{3}}{2}\right)\right)\left(z-\left(\frac{-a}{2}+ia\frac{\sqrt{3}}{2}\right)\right)\right\}$$



Now let us consider W due to vortex at B and C, at A.

$$W_A = W_{tot} - \frac{ik}{2\pi}\log z$$

$$= \frac{ik}{2\pi} \log\left(\left(z - a/2 + i\sqrt{3}/2\, a\right) \left(z + \frac{a}{2} + i a\sqrt{3}/2\right) \right)$$

$$= \frac{ik}{2\pi} \log\left(\left(z^2 - \frac{a^2}{4} - \frac{3a^2}{4}\right) + i\left(z a\sqrt{3}\right) \right)$$

$$= \frac{ik}{2\pi} \log\left((z^2 - a^2) + i(z a\sqrt{3}) \right)$$

Velocity of vortex at A is then

$$-\frac{dW}{dz}\bigg|_{z=0}$$

$$= \frac{ik}{2\pi}\left(\frac{2z + i a\sqrt{3}}{(z^2 - a^2) + i(z a\sqrt{3})} \right)\bigg|_{z=0}$$

$$= \frac{ik}{2\pi} \cdot \left(\frac{i a\sqrt{3}}{-a^2} \right)$$

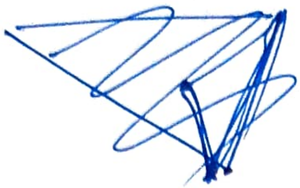
$$= \boxed{\frac{k\sqrt{3}}{2a\pi}}$$

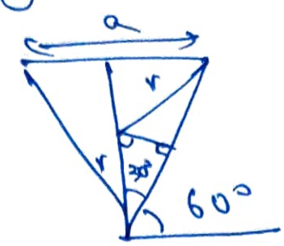
Clearly this velocity is along $+X$, ie along tangent to the circle / cylinder.

And this speed is constant.

$\therefore$ The vortices induce on each other a constant speed $k\sqrt{3}/2\pi a$ along tangent to cylinder.

Now radius of cylinder is





$r\cos 30 = a/2 \Rightarrow \boxed{r = a/\sqrt{3}}$

circumference $= 2\pi r = \boxed{\dfrac{2\pi a}{\sqrt{3}}}$

$\therefore$ Total time taken to move around cylinder is

$$\frac{2\pi a/\sqrt{3}}{k\sqrt{3}/2a\pi}$$

$$= \boxed{\frac{4\pi^2 a^2}{3k}}$$

Hence proved.

Q.6 (b) Find the integral surface of linear PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contain the straight line $x+y=0, z=1$. (15)

Given linear PDE,

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

comparing with $\underline{Pp+Qq=R}$,

$P = x(y^2+z)$, $Q = -y(x^2+z)$,

$R = x^2-y^2$

It's soln is by $\underline{\text{Lagrange eqn}}$,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2} \quad -(1)$$

Using multipliers $\underline{x, y, 0}$, each of (1) is equal to

$$\frac{x\,dx + y\,dy}{(x^2-y^2)z}$$

$$\Rightarrow \frac{x\,dx + y\,dy}{(x^2-y^2)z} = \frac{dz}{x^2-y^2}$$

$$\Rightarrow \boxed{x^2+y^2 = z^2+c_1}$$

Let $\underline{u = x^2+y^2-z^2}$.

$\underline{\text{Also}}$, taking Lagrange multipliers as $1/x, 1/y, 1$ we get

$$\frac{dx/x + dy/y + dz}{(y^2+z) - (x^2+z) + x^2 - y^2}$$

or $\dfrac{\frac{dx}{x} + \frac{dy}{y} + dz}{0}$

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + dz = 0$

$\Rightarrow \boxed{\log(|xy|) + z = c_2}$

Let $v = \log(xy) + z$.

Then, integral to PDE is given by $\boxed{\phi(u, v) = 0}$ or

$\phi(x^2 + y^2 - z^2, \log(|xy|) + z) = 0$

where ϕ is an arbitrary fn.

Integral containing line :

$x^2 + y^2 - z^2 = c_1,$

$\log(|xy|) + z = c_2$

Putting $z = 1$, $x + y = 0$ we get

$2x^2 - 1 = c_1, \quad \log(|-x^2|) = c_2 - 1$

$\Rightarrow x^2 = \frac{c_1 + 1}{2} = e^{c_2 - 1}$

$\Rightarrow c_1 + 1 = 2e^{c_2 - 1}$

$\Rightarrow x^2 + y^2 - z^2 + 1 = 2e^{\log(xy) + z - 1}$

$\Rightarrow \boxed{x^2 + y^2 - z^2 + 1 = 2xy \cdot e^{z-1}}$

is the required integral.

Candidates must not write on this margin

Q.6 (c) Find a cubic polynomial in x which takes values $-3, 3, 11, 27, 57$ and 107 when $x = 0, 1, 2, 3, 4$ and 5 respectively. (15)

	x	$f(x)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	Δ^3	Δ^4
x_0	0	-3	6	2		
x_1	1	3	8	8	6	
x_2	2	11	16	14	6	0
x_3	3	27	30	20	6	0
x_4	4	57	50			
x_5	5	107				

(Difference Table)

Since 3rd difference $(\Delta^3 f(x))$ is constant hence it is a CUBIC polynomial.

Consider $\Delta^3 y_0 = 6$

We know: $\Delta = E - 1$,

So $(E-1)^3 y_0 = 6$

$\Rightarrow (E^3 - 3E^2 + 3E - 1) y_0 = 6$

$\Rightarrow y_3 - 3y_2 + 3y_1 - y_0 = 6$

Now let 'x' be any point.

Then $u = \frac{x-0}{1} = x$.

By $\boxed{\text{Newton's interpolation formula}}$

$$f(x) = f(0) + u \cdot \Delta f(0) + \frac{u(u-1)}{2!} \Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(0)$$

$$\Rightarrow f(x) = -3 + x \cdot 6 + \frac{(x)(x-1)}{2!} \cdot 2 + \frac{(x)(x-1)(x-2)}{3!} \cdot 6$$

$$= -3 + 6x + (x^2 - x) + (x(x^2 - 3x + 2))$$

$$= -3 + 6x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= \underline{x^3 - 2x^2 + 7x - 3}$$

$$\therefore \boxed{f(x) = x^3 - 2x^2 + 7x - 3}$$

is the required polynomial.

Q.8 (a) Obtain the Boolean Function $F(x,y,z)$ based on the table given below. Then simplify $F(x,y,z)$ and draw the corresponding Gate network: (15)

x	y	z	$f(x,y,z)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

taking the terms where $F(x,y,z)=1$,

$$\boxed{F(x,y,z) = xyz + xyz' + xy'z + x'yz}$$

Now simplifying it,

$$F(x,y,z) = xy(z+z') + xy'z + x'yz$$

$$= xy + x'yz + xy'z$$

$(\because z+z'=1 \text{ and } x+y=y+x)$

$$= (x+x'z)y + xy'z$$

$$= (x+z)y + xy'z$$

$(\because a+a'b = a+b)$

$$= xy + yz + xy'z$$

$$= x(y+y'z) + yz$$

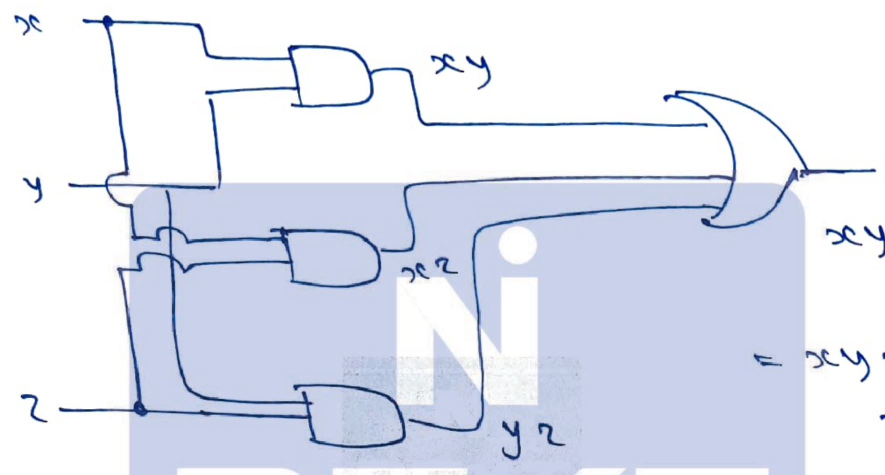
$$= x(y+z) + yz \quad (\because a+a'b = a+b)$$

$$= \boxed{xy + yz + xz}$$

Thus, simplified terms,

$$\boxed{F(x,y,z) = xy + yz + xz}$$

Corresponding gate network:



$xy + xz + yz$

$= xy + yz + xz$

Q.8 (b) A particle at a distance r from the centre of force moves under the influence of the central force $F=-\frac{k}{r^2}$, where k is constant. Obtain the Lagrangian and derive the equations of motion. (15)

Assuming planar (2D) motion.
Let position of the particle at any time be (r, θ).
Its velocity is given by

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$\therefore KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

Potential energy can be found as

$$PE = \int_{\infty}^{r} +\frac{k}{r^2}\,dr = \left.\frac{k}{r}\right|_{\infty}^{r}$$
$$= \frac{k}{r}$$

$\therefore$ Lagrangian $L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}}$$

Now Lagrangian eqn in a conservative force is

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{q}}\right) - \frac{\delta L}{\delta q} = 0$$

where $q = r, \theta$.

Writing the r eqns,

$$\frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 + \frac{k}{r^2}\right) = 0$$

$$\Rightarrow \boxed{m\ddot{r} = mr\dot{\theta}^2 + k/r^2} \quad \text{--- (1)}$$

Similarly the θ - eqn is

$$\frac{d}{dt}(mr^2\dot{\theta}) - (0) = 0$$

$$\Rightarrow \boxed{mr^2\ddot{\theta} + 2mr\dot{\theta}\dot{r} = 0} \quad \text{--- (2)}$$

(1) and (2) together are the equations of motion

Q.8 (c) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating by giving each point a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time 5ØaÜ. (20)

Let the string be along x-axis.

We know wave eqn in a tight string is guided by

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \quad \text{--- (1)}$$

where $y(x,t)$ is the vertical displacement of any point at a time 't' and distance 'x' from $x=0$.

Let $y = X(x)T(t)$ be a soln to (1), then

$$XT'' = c^2 X'' T$$

OR $$\frac{X''}{X} = \frac{T''}{c^2 T}$$

As they are indep of each other, so both can be equated to a constant μ.

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \mu \quad \text{--- (2)}$$

$\boxed{\text{Case I}}$: $\mu = 0$

Then (2) gives $X = Ax + B$,
$T = Ct + D$

But then using $X(0) = X(l) = 0$,
we get $A = B = 0$.
$\Rightarrow$ Reject this case.

$\boxed{\text{Case II}}$ $\mu > 0$ (say $\mu = \lambda^2$)

Then (2) gives
$X'' = \lambda^2 X$, $T'' = c^2\lambda^2 T$
~~$\Rightarrow X'X'' = \lambda^2 X X'$~~ which give
~~$\Rightarrow (X')^2 =$~~ $X = A\cosh(\lambda x) + B\sinh(\lambda x)$
& $T = C\cosh(\lambda ct) + D\sinh(\lambda ct)$

Then ~~as~~, using $X(0) = X(l) = 0$,
we get $A = B = 0$.
$\Rightarrow$ Reject this case.

Thus, $\boxed{\text{Case III}}$: $\mu < 0$, $= -\lambda^2$ (say)

from (2), $X = A\cos\lambda x + B\sin\lambda x$
$T = C\cos(\lambda ct) + D\sin(\lambda ct)$

Using $X(0) = X(l) = 0$, we get
$A = 0$, $\lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$, $n \in \mathbb{Z}$

Now using $y(x,t) = X(x)T(t)$

and superposing all possible values of λ, we get

$$y = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\left(C_n \cos(\lambda c t) + D_n \sin(\lambda c t)\right) \quad (3)$$

At $t=0$, string is at equilibrium, i.e. $y = 0 \ \forall x$

$\Rightarrow$ We get $C_n = 0 \ \forall n$

Taking $B_n \times D_n = E_n$, (3) becomes:

$$y(x,t) = \sum_{1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) \times \sin\left(\frac{n\pi c t}{l}\right)$$

Now differentiating wrt 't' and putting $t=0$, we get:

$$\lambda x(l-x) = \sum_{n=1}^{\infty}\left(E_n \cdot \frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

RHS is the fourier series of $\lambda x(l-x)$. Thus we get,

$$E_n \cdot \frac{n\pi c}{l} = \frac{2}{l}\int_0^l \lambda x(l-x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow E_n = \frac{2\lambda}{n\pi c}\int_0^l x(l-x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\lambda}{n\pi c}\left\{ x(l-x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right)\Bigg|_0^l + \int_0^l (l-2x)\left(\frac{+\cos(n\pi x/l)}{(n\pi/l)}\right)dx \right\}$$

$$= \frac{2\lambda}{n\pi c}\left\{ \frac{(l-2x)\cdot\sin(n\pi x/l)}{(n\pi/l)^2}\Bigg|_0^l - \int_0^l \frac{-2\cdot\sin\left(\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2}dx \right.$$

$$= \frac{2\lambda}{n\pi c}\left\{ \frac{2\cdot\cos(n\pi x/l)}{n^3\pi^3/l^3}\Bigg|_0^l \right\}$$

$$= \frac{4l^3}{n^4\pi^4 c}\left(1-\cos(n\pi)\right)$$

$$\Rightarrow E_n = \begin{cases} 0 & n = \text{even} \\ 8l^3/n^4\pi^4 c & n = \text{odd} \end{cases}$$

$\therefore$ We get

$$y(x,t) = \sum_{n=\text{odd}} \left(\frac{8l^3}{n^4\pi^4 c}\right)\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

$$= \sum_{m=1}^{\infty} \frac{8l^3}{(2m-1)^4\pi^4 c}\sin\left(\frac{(2m-1)\pi x}{l}\right)\sin\left(\frac{(2m-1)\pi ct}{l}\right)$$

$$\boxed{= \left(\frac{8l^3}{\pi^4 c}\right)\sum \frac{1}{(2m-1)^4}\cdot\sin\left(\frac{(2m-1)\pi x}{l}\right)\sin\left(\frac{(2m-1)\pi ct}{l}\right)}$$

is the displacement of the string at dist x from $x=0$, at time 't'.