AYAN JAIN Test 5 PTTP220063

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1(a) Given A & B are orthogonal matrices

$\Rightarrow AA^T = I = BB^T$

Given $\det A + \det B = 0$

To show $A+B$ is a singular matrix,

i.e, $|A+B| = 0$

Consider

$|A+B| = \cancel{AAA^T}\ |(A+B)^T|$

$= |B^T + A^T|$

$= |B^T(AA^T) + (B^TB)A^T|$

$= |B^T(A+B)A^T|$

$\Rightarrow |A+B| = |B|\,|A+B|\,|A|$

$\Rightarrow |A+B|(1 - |B||A|) = 0$ ——— (1)

But $AA^T = I$, $BB^T = I$

$\Rightarrow |A|, |B| = \pm 1$

& $\det A + \det B = 0$

$\Rightarrow |A||B| = |B||A| = -1$

$\Rightarrow 1 - |B||A| = 2$

$\therefore$ from (1), $\boxed{|A+B| = 0}$

OR, $\boxed{A+B \text{ is a singular matrix}}$

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(1b) Let $f: V \rightarrow V$ be an ONTO homomorphism.

To show : f is one-one.

Let $w \in V$, then $\exists\, u \in V$ s.t. $f(u) = w$.

Given V is finite dimensional, say dim $(V) = n$ = finite.

~~$\Rightarrow$ Any $(n+1)$ elements must be linearly dependent.~~

~~i.e. if $\exists\ a_1, a_2, \ldots a_{n+1} \in V$ and $\alpha_1, \alpha_2, \ldots \alpha_{n+1} \in F$~~

~~if~~ ~~Suppose~~

~~And since dim$(V) = n$, there can be upto n LI elements.~~

~~Let $a_1, a_2, \ldots a_n$ be LI.~~

~~$\Rightarrow$ If $\exists\ \alpha_1, \alpha_2, \ldots \alpha_n$ s.t.~~

~~$\sum a_i \alpha_i = 0$, $\Rightarrow \alpha_i = 0\ \forall i$~~

~~Now assume f is not one-one.~~

~~Then say for a_1~~

~~Let $f(b_1) = a_1 = f(b_2)$~~

~~($\because f$ is onto)~~

bungee

Let $\dim(V) = n$ = finite, and let $\{e_1, e_2, \dots e_n\}$ be the ~~standard~~ basis of V.

Suppose f is not one-one.
Then $\exists\ a \in V$ s.t. $a = f(b) = f(c)$
for $b, c \in V$ ($\because$ Onto).

$\Rightarrow f(b) - f(c) = a - a = 0$
$\Rightarrow f(b-c) = 0$ ($\because$ Homomorphism)

Now $b = \sum_{k=1}^{n} \alpha_k e_k$ & $c = \sum_{r=1}^{n} \beta_r e_r$

$\Rightarrow f(b-c) = 0 \Rightarrow f\left(\sum_{k=1}^{n} (\alpha_k - \beta_k) e_k\right) = 0$

$\Rightarrow \sum_{k=1}^{n} (\alpha_k - \beta_k) f(e_k) = 0$

But since $e_1, e_2 \dots e_n$ form the basis, hence $f(e_1), f(e_2)$ etc are also linearly independent.

Then, $\alpha_k - \beta_k = 0 \quad \forall\ k = 1, \dots n$
$\Rightarrow \alpha_k = \beta_k \quad \forall\ k = 1, \dots n$.

That means $\boxed{b = c}$

Thus, if $\boxed{f(b) = f(c) \Rightarrow b = c}$.

$\therefore$ f is one-one and hence an ISOMORPHISM.

1(c) Given cone $5yz - 8zx - 3xy = 0$

It is homogenous order 2 hence ~~passes through~~ has $(0,0,0)$ as vertex.

Given $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ as a generator.

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ represent a generator $\perp$ to given generator.

Then,

$l\cdot 1 + m\cdot 2 + n\cdot 3 = 0$ — (1)

$5mn - 8nl - 3lm = 0$ — (2)

From (1), $l = -2m - 3n$

Putting in (2),

$5mn - 8n(-2m-3n) - 3m(-2m-3n) = 0$

$\Rightarrow 5mn + 16nm + 24n^2 + 6m^2 + 9mn = 0$

$\Rightarrow 6m^2 + 30nm + 24n^2 = 0$

$\Rightarrow \left(\frac{m}{n}\right)^2 + 5\left(\frac{m}{n}\right) + 4 = 0$

$\Rightarrow \frac{m}{n} = -1, -4$

If $\frac{m}{n} = -1$, $n = -m$.

$l + 2m + 3(-m) = 0$

$\Rightarrow l = m = -n$

$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ is one generator. — ③

If $\frac{m}{n} = -4$, $m = -4n$

$l + 2(-4n) + 3n = 0$

$\Rightarrow l = 5n$

$\Rightarrow \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$ — ④

is another generator.

Clearly, $5 \cdot 1 + (-4) \cdot 1 + 1(-1) = 0$

so ③ & ④ are $\perp$ to each other.

Hence, the other 2 generators are given by

$$\boxed{\begin{aligned} &\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \\ \text{and} \quad &\frac{x}{5} = \frac{y}{-4} = \frac{z}{1} \end{aligned}}$$

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1(d) Let $I = \int_0^1 \tan^{-1}(1 - 1/x)\, dx$ — ①

$x \to 0+1-x$

$I = \int_0^1 \tan^{-1}\left(1 - \frac{1}{1-x}\right) dx$ — ②

①+② : $2I = \int_0^1 \tan^{-1}\left(\frac{x-1}{x}\right) + \tan^{-1}\left(\frac{-x}{1-x}\right) dx$

Now in $[0,1]$, $0 < x < 1$

$\Rightarrow \frac{x-1}{x} < 0$ and $\frac{-x}{1-x} < 0$

$\tan^{-1}\left(\frac{-x}{1-x}\right) = \tan^{-1}\left(\frac{x}{x-1}\right)$

And $\tan^{-1}(\alpha) + \tan^{-1}(1/\alpha) = -\pi/2$
when $\alpha < 0$

$\Rightarrow 2I = \int_0^1 -\frac{\pi}{2}\, dx = -\frac{\pi}{2}$

$\Rightarrow \boxed{I = -\pi/4}$

(1e) Let (α, β, γ) be the ~~cent~~ centre of such a circle.

Consider a sphere S_1 as

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2 \quad (1)$$

And plane P_1 as

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \quad (2)$$

Then (1) & (2) together represent a circle with centre at (α, β, γ) and radius 'a'.

Given it always intersects the coordinate axes.

(i) Intersects x-axis ⇒ ~~there~~ say at point $(x_1, 0, 0)$.

Then
$$\left.\begin{aligned}(x_1-\alpha)^2 + \beta^2 + \gamma^2 &= a^2 \\ A(x_1-\alpha) - B\beta - C\gamma &= 0\end{aligned}\right] \quad (A)$$

$$\Rightarrow \left(\frac{B\beta + C\gamma}{A}\right)^2 + \beta^2 + \gamma^2 = a^2$$

Similarly if it intersects y-axis and z-axis at $(0, y_1, 0)$, $(0, 0, z_1)$ resp.,

$$\left.\begin{aligned}\alpha^2 + (y_1-\beta)^2 + \gamma^2 &= a^2 \\ -A\alpha + B(y_1-\beta) - C\gamma &= 0\end{aligned}\right] \quad (B)$$

and

$$\left.\begin{array}{l}\alpha^2+\beta^2+(z_1-\gamma)^2=a^2\\ -A\alpha-B\beta+C(z_1-\gamma)=0\end{array}\right]\ (C)$$

Eliminating A, B, C from (A), (B), (C),

$$\begin{vmatrix} x_1-\alpha & -\beta & -\gamma \\ -\alpha & y_1-\beta & -\gamma \\ -\alpha & -\beta & z_1-\gamma \end{vmatrix}=0$$

$$\Rightarrow x_1y_1z_1=\alpha y_1z_1+\beta x_1z_1+\gamma x_1y_1 \quad \text{— (D)}$$

Also, (A) − (B) : $Ax_1=By_1$

$$\Rightarrow \frac{x_1}{B}=\frac{y_1}{A}$$

Similarly (B) − (C) : $By_1=Cz_1$

$\therefore$ $\cancel{\frac{x_1}{1/A}=\frac{y_1}{1/B}=\frac{z_1}{1/C}}$ $Ax_1=By_1=Cz_1$

$\Rightarrow A(x_1-\alpha)-B\beta-C\gamma=0$
becomes
$$x_1-\alpha-\frac{x_1}{y_1}\beta-\frac{x_1\gamma}{z_1}=0$$

$$\Rightarrow x_1-\alpha=\frac{x_1\beta}{y_1}+\frac{x_1\gamma}{z_1}$$

$$\alpha(x_1-\alpha)=\alpha\beta\frac{x_1}{y_1}+\alpha\gamma\frac{x_1}{z_1} \quad \text{— (3)}$$

$$\beta(y_1-\beta)=\beta\alpha\frac{y_1}{x_1}+\beta\gamma\frac{y_1}{z_1} \quad \text{— (4)}$$

$$\gamma(z_1-\gamma)=\gamma\alpha\frac{z_1}{x_1}+\gamma\beta\left(\frac{z_1}{y_1}\right) \quad \text{— (5)}$$

From (D), (3), (4), (5) we get

$$\sum\alpha(x_1-\alpha)=\alpha(1-\alpha/x_1)$$

(3a) $S : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Enveloping cone with vertex $P(x_1, y_1, z_1)$ is given by

$SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)$$

$$= \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2 \quad —(1)$$

Section of (1) by $z = 0$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)$$

$$= \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\left(\Sigma\frac{x_1^2}{a^2}\right) - 1 - \frac{x_1^2}{a^2}\right)$$

$$+ \frac{y^2}{b^2}\left(\left(\Sigma\frac{x_1^2}{a^2}\right) - 1 - \frac{y_1^2}{b^2}\right)$$

$$+ \frac{2xx_1yy_1}{a^2b^2} + \frac{2xx_1}{a^2} + \frac{2yy_1}{b^2}$$

$$- \left(\Sigma\frac{x_1^2}{a^2}\right) = 0 \quad —(2)$$

For (2) to represent a

$\boxed{(i)}$ Parabola, we must have

$h^2 = ab$

$\Rightarrow \left(\frac{x_1 y_1}{a^2b^2}\right)^2 = \left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$

$\Rightarrow$ Locus of P :

$$\boxed{\frac{x^2y^2}{a^4b^4} = \left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)}$$

(ii) Rectangular hyperbola

$\Rightarrow$ coeff of x^2 + coeff of $y^2 = 0$

$\Rightarrow \left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right) = 0$

$\Rightarrow$ Locus of P:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2z^2}{c^2} = 2}$$

(iii) Circle

$\Rightarrow a = b$ & $h = 0$

$\Rightarrow x_1y_1 = 0$ & $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$

$\Rightarrow x_1 = y_1 = 0, \; z \in \mathbb{R}$

OR Locus of P : $\boxed{(0,0,z)}$

$\Rightarrow$ $\boxed{z\text{-axis}}$

(3b) $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

$$f_{xy}(0,0) = \lim_{y\to 0}\left(\frac{f_x(\cancel{y,0}\ 0,y) - f_x(0,0)}{y}\right)$$

$$= \lim_{y\to 0}\left\{\frac{1}{y}\left(\lim_{x\to 0}\frac{f(x,y) - f(0,0)}{x}\right)\right\}$$

$$= \lim_{y\to 0}\frac{1}{y}\left(\lim_{x\to 0}\frac{y^3}{x^2+y^2}\right)$$

$$= \lim_{y\to 0}\frac{1}{y}(y)$$

$$= \boxed{1} \quad \therefore \boxed{f_{xy}(0,0) = 1} \quad \text{Ans}$$

$$f_{yx}(0,0) = \lim_{x\to 0}\left(\frac{f_y(x,0) - f_y(0,0)}{x}\right)$$

$$= \lim_{x\to 0}\left\{\frac{1}{x}\left(\lim_{y\to 0}\frac{f(x,y) - f(0,0)}{y}\right)\right\}$$

$$= \lim_{x\to 0}\left\{\frac{1}{x}\lim_{y\to 0}\frac{xy^3}{x^2+y^2}\cdot\frac{1}{y}\right\}$$

$$= \lim_{x\to 0}\frac{1}{x}(0)$$

$$= \boxed{0} \quad \therefore \boxed{f_{yx}(0,0) = 0}$$

$$\therefore \boxed{f_{xy}(0,0) = 1,\ f_{yx}(0,0) = 0} \quad \text{Ans}$$

(3c) Given hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

$\Rightarrow \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right)$

system of generators is given by

$$\frac{x}{a} + \frac{z}{c} = \lambda\left(1+\frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\lambda}\left(1-\frac{y}{b}\right)$$

& $\frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \mu\left(1+\frac{y}{b}\right)$

Let $\frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \mu\left(1+\frac{y}{b}\right)$ — (1)

be any given generator.

We will show that $\exists$ 2 values of λ for which the generator

$$\frac{x}{a} + \frac{z}{c} = \lambda\left(1+\frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\lambda}\left(1-\frac{y}{b}\right) \quad \text{— (2)}$$

intersects (1) at right angles.

(1) can be re-written as:

$$\frac{x}{a} + \frac{1}{\mu}\left(\frac{y}{b}\right) + \frac{z}{c} = \frac{1}{\mu} \quad \equiv P_1$$

& $\frac{x}{a} - \mu\left(\frac{y}{b}\right) - \frac{z}{c} = \mu \quad \equiv P_2$

So the direction ratios of Line (1) (l, m, n) say can be found ($\because$ Line (1) is $\perp$ to P_1 & P_2)

$$\Rightarrow \frac{l}{a} + \frac{m}{b\mu} + \frac{n}{c} = 0$$

$$\& \quad \frac{l}{a} - \frac{\mu m}{b} - \frac{n}{c} = 0$$

$$\Rightarrow \frac{l}{-\frac{1}{bc\mu} + \frac{\mu}{bc}} = \frac{-m}{-\frac{1}{ac} - \frac{1}{ac}} = \frac{n}{-\frac{\mu}{ab} - \frac{1}{ab\mu}}$$

$$\Rightarrow \frac{l}{\left(\mu - \frac{1}{\mu}\right)\frac{1}{bc}} = \frac{m}{2/ac} = \frac{n}{\left(-\mu - \frac{1}{\mu}\right)\frac{1}{ab}}$$

OR, DRs are proportional to,

$$\left(\left(\mu - \frac{1}{\mu}\right)a,\ 2b,\ \left(-\mu - \frac{1}{\mu}\right)c\right)$$

Now repeating procedure for line (1),

(2) can be written as $P_3 = 0 = P_4$:

$$\frac{x}{a} - \frac{\lambda y}{b} + \frac{z}{c} = \lambda \equiv P_3$$

$$\frac{x}{a} + \frac{1}{\lambda}\frac{y}{b} - \frac{z}{c} = \frac{1}{\lambda} \equiv P_4$$

If (l_1, m_1, n_1) are DRs of (1)

then, $\frac{l_1}{a} - m_1\frac{\lambda}{b} + \frac{n_1}{c} = 0$

$$\frac{l_1}{a} + \frac{m_1}{\lambda b} - \frac{n_1}{c} = 0$$

$$\Rightarrow \frac{l_1}{\frac{\lambda}{bc} - \frac{1}{\lambda bc}} = \frac{-m_1}{-\frac{1}{ac} - \frac{1}{ac}} = \frac{n_1}{\frac{1}{a\lambda b} + \frac{\lambda}{ab}}$$

$\Rightarrow (l_1, m_1, n_1)$ are proportional to

$$\left(\left(\lambda - \frac{1}{\lambda}\right)a,\ 2b,\ \left(\lambda + 1/\lambda\right)c \right)$$

Then for ① & ② to be perpendicular we must have

$$ll_1 + mm_1 + nn_1 = 0$$

$$\Rightarrow \left(\mu - \frac{1}{\mu}\right)\left(\lambda - \frac{1}{\lambda}\right)a^2 + 4b^2 - \left(\mu + \frac{1}{\mu}\right)\left(\lambda + \frac{1}{\lambda}\right)c^2 = 0$$

$$\Rightarrow \lambda^2\left(a^2\left(\mu - \frac{1}{\mu}\right) - c^2\left(\mu + \frac{1}{\mu}\right)\right) + \lambda\left(4b^2\right) + 1\left(\left(\mu - \frac{1}{\mu}\right)a^2 - \left(\mu + \frac{1}{\mu}\right)c^2\right) = 0$$

which is a quadratic in λ and in general has 2 different real roots.

$\therefore$ In general, to any given generator of the hyperboloid, 2 generators can be drawn at right angles to it.

(4a) Given W_1 and W_2 are subspaces of $V(F)$.

(i) To show $W_1 + W_2$ is a subspace of $V(F)$

Let $u, v \in W_1 + W_2$, $\alpha, \beta \in F$.

Then $u = u_1 + u_2$, $v = v_1 + v_2$

where $u_1, v_1 \in W_1$; $u_2, v_2 \in W_2$.

Consider $\alpha u + \beta v$

$= (\alpha u_1 + \alpha u_2) + (\beta v_1 + \beta v_2)$

$= (\alpha u_1 + \beta v_1) + (\alpha u_2 + \beta v_2)$

($\because$ $V(F)$ is a vector space $\Rightarrow$ Associative)

Now since $u_1, v_1 \in W_1$ and $\alpha, \beta \in F$,

$\Rightarrow \alpha u_1 + \beta v_1 \in W_1$ ($\because W_1$ is a subspace)

Similarly $\alpha u_2 + \beta v_2 \in W_2$.

$\therefore (\alpha u_1 + \beta v_1) + (\alpha u_2 + \beta v_2) \in W_1 + W_2$

$\Rightarrow \alpha u + \beta v \in W_1 + W_2$

Thus for any $u, v \in W_1 + W_2$ and any $\alpha, \beta \in F$, $\alpha u + \beta v \in W_1 + W_2$

Thus $W_1 + W_2$ is a ~~sub~~ subspace of $V(F)$

(Note, $W_1 \subseteq V$ & $W_2 \subseteq V$ thus $W_1 + W_2 \subseteq V$ by closure)

(ii) To show $W_1 + W_2 = \{W_1 \cup W_2\}$

① To show $W_1 + W_2 \subseteq \{W_1 \cup W_2\}$

Consider $w \in W_1 + W_2$.

Then $w = w_1 + w_2$

where $w_1 \in W_1$, $w_2 \in W_2$.

And $w_1, w_2 \in W_1 \cup W_2$

Then, $w = 1w_1 + 1w_2$, and $1 \in F$ ($\because$ Field).

Thus w can be written as linear combination of element of $W_1 \cup W_2$.

Since w was any arbitrary element, thus we have

$$\boxed{W_1 + W_2 \subseteq \{W_1 \cup W_2\}} \quad -①$$

② To show $\{W_1 \cup W_2\} \subseteq W_1 + W_2$

Let $w \in \{W_1 \cup W_2\}$

Then $w = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$

where $\alpha_1, \alpha_2, \dots \alpha_n \in F$ and

$w_1, w_2, w_3, \dots w_n \in W_1 \cup W_2$.

Then each of $w_1, w_2, \dots w_n$ either

belong to W_1 or W_2.
Without any loss of generality we can assume
$w_1, w_2, \dots w_k \in W_1$
$w_{k+1}, w_{k+2}, \dots w_n \in W_2$

Then,
$w = (\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_k w_k)$ — (A)
$+ (\alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n)$

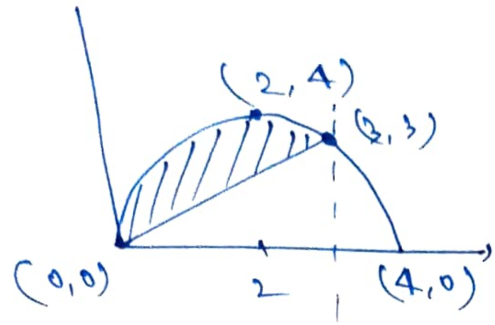
And, since W_1 is a subspace, so
$\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_k w_k \in W_1$
and similarly W_2 is subspace,
$\alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n \in W_2$.

Then from (A) we get,
$w \in W_1 + W_2$
(Say $\alpha_1 w_1 + \dots + \alpha_k w_k = u_1 \in W_1$
& $\alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n = u_2 \in W_2$
then $w = u_1 + u_2$)

Since w was arbitrary elmt of $\{W_1 \cup W_2\}$, thus $\boxed{\{W_1 \cup W_2\} \subseteq W_1 + W_2}$ — (2)

from (1) and (2),
$$\boxed{W_1 + W_2 = \{W_1 \cup W_2\}}$$

(4b)

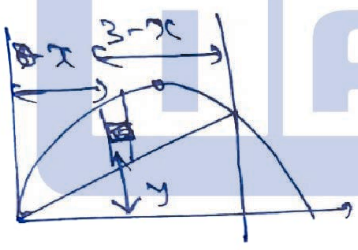


$y = 4x - x^2$
$y = x$
They meet at
$x = 4x - x^2$
$\Rightarrow 3x = x^2$
$\Rightarrow \boxed{x = 0, 3}$

The shaded region represents the area bound by $y = 4x - x^2$ and $x = y$.

Rotating about $x = 3$ to find volume.

Consider small element as (of dimensions dx, dy) At distance x from y-axis



~~⇒ Distance~~

(We integrate along vertical strings)

$\Rightarrow dV = 2\pi(3-x)\,dx\,dy$
(Ring element)

$$\Rightarrow V = \int_{x=0}^{3}\int_{y=x}^{y=4x-x^2} 2\pi(3-x)\,dy\,dx$$

$$= \int_0^3 \int_x^{4x-x^2} 2\pi(3-x)\cdot dy\, dx$$

$$= 2\pi \int_0^3 (3-x)(3x-x^2)\, dx$$

$$= 2\pi \int_0^3 (3-x)^2 x \, dx$$

$$= 2\pi \int_0^3 (x^3 + 9x - 6x^2)\, dx$$

$$= 2\pi \left(\frac{x^4}{4} + \frac{9x^2}{2} - 2x^3 \right)\Bigg|_0^3$$

$$= (2\pi)\left(\frac{81}{4} + \frac{81}{2} - 54 \right)$$

$$= (2\pi)\left(\frac{27}{4} \right) = \boxed{\frac{27\pi}{2}}$$

Thus, volume is $\boxed{27\pi/2}$

Note, we used concept that any small element $dx\,dy$ upon rotation will form a RING of radius $(3-x)$, and ~~dimension~~ thickness $dx \times dy$. Then we integrated all such elements.

(4c)

$$\sin v = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$\Rightarrow v = \sin^{-1}\left(\frac{x+2y+3z}{(x^8+y^8+z^8)^{1/2}}\right)$$

Consider $f(x,y,z) = \dfrac{x+2y+3z}{(x^8+y^8+z^8)^{1/2}}$

Then $f(\lambda x, \lambda y, \lambda z) = \dfrac{1}{\lambda^3} f(x,y,z)$

$\Rightarrow f(x,y,z)$ is homogenous of order $\boxed{-3}$

Then by $\boxed{\text{EULER's}}$ identity,

$$\boxed{x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} + z\frac{\delta f}{\delta z} = -3f} \quad \text{—(1)}$$

Now consider $\sin v = \sin(f(x,y,z))$

partially Differentiating wrt x, y, z resp.

$$\frac{\delta(\sin v)}{\delta x} = \frac{\delta \sin f}{\delta f} \cdot \frac{\delta f}{\delta x}$$

$$= \cos(f) \cdot \frac{\delta f}{\delta x}$$

~~And~~ $x\dfrac{\delta(\sin v)}{\delta x} = x \cdot \cos v \dfrac{\delta v}{\delta x}$

So we get,

$x\frac{\delta(\sin v)}{\delta x} = x\cos v\frac{\delta v}{\delta x} = x\cos f\left(\frac{\delta f}{\delta x}\right)$ — (2)

Similarly

$y\cos v\cdot\frac{\delta v}{\delta y} = y\cos f\cdot\frac{\delta f}{\delta y}$ — (3)

$z\cos v\frac{\delta v}{\delta z} = z\cos f\cdot\frac{\delta f}{\delta z}$ — (4)

(2) + (3) + (4)

$(\cos v)\left(x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}+z\frac{\delta v}{\delta z}\right)$

$= \cos(f(x,y,z))\left(x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}\right)$

Using (1),

$(\cos v)\left(x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}+z\frac{\delta v}{\delta z}\right) = (\cos v)\times(-3v)$

$(\because v \equiv f(x,y,z))$

OR

$f(x,y,z) = \sin v$, so in (1),

$x\frac{\delta(\sin v)}{\delta x}+y\frac{\delta(\sin v)}{\delta y}+z\frac{\delta(\sin v)}{\delta z} = -3\sin v$

$\Rightarrow \cos v\left(x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}+z\frac{\delta v}{\delta z}\right) = -3\sin v$

Dividing by $\cos v$, we get

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} + 3\tan v = 0}$$

Hence proved.

(5a) $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ ——— (1)

Differentiating wrt x,

$$\frac{2x}{a^2+\lambda} + \frac{2y \cdot dy/dx}{b^2+\lambda} = 0$$

$$\Rightarrow x(b^2) + (y\,dy/dx)\,a^2 = -\lambda(x + y\,dy/dx)$$

$$\Rightarrow \lambda = \frac{-(b^2x + a^2y\,dy/dx)}{x + y\,dy/dx} \quad \text{——— (2)}$$

Putting λ in (1),

$$\frac{x^2}{\frac{(a^2-b^2)x}{x+y\,dy/dx}} + \frac{y^2}{\frac{(b^2-a^2)y\,dy/dx}{x+y\,dy/dx}} = 1$$

$$\Rightarrow \frac{x(x+y\,dy/dx)}{a^2-b^2} + \frac{\frac{y(x+y\frac{dy}{dx})}{dy/dx}}{b^2-a^2} = 1$$

$$\Rightarrow \boxed{\left(x-\frac{y}{dy/dx}\right)(x+y\,dy/dx) = a^2-b^2} \quad \text{(3)}$$

is the Differential equation of the family.

To get orthogonal trajectory, we put $\frac{dy}{dx} \longrightarrow \frac{-1}{dy/dx}$,

$$\Rightarrow \left(x + y\frac{dy}{dx}\right)\left(x + y\left(-1/dy/dx\right)\right) = a^2 - b^2$$

$$\Rightarrow \left(x + y\frac{dy}{dx}\right)\left(x - \frac{y}{dy/dx}\right) = a^2 - b^2 \quad - (4)$$

which is the same as (3).

$\therefore$ Given family is self-orthogonal.

$\therefore$ ~~family~~ orthogonal trajectories to $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is given by

$$\boxed{\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} = 1}$$

where k is a parameter.

(5b) Let the cycloid's equation be

$$s = 4a\sin\psi \quad \text{— (1)}$$

~~Given~~ $\cancel{\frac{d\psi}{dt} = \text{const} = \omega \text{ (say)}}$

Differentiating (1) w.r.t. time,

$$\frac{ds}{dt} = 4a\cos\psi \cdot \left(\frac{d\psi}{dt}\right) = (4a\cos\psi)\,\omega$$

Direction of motion is along tangent, i.e. along ds/dt.

$$\frac{ds}{dt} = 4a\cos\psi \cdot \frac{d\psi}{dt}$$

Given ds/dt rotates with a constant angular velocity.

$$\Rightarrow \frac{d^2s}{dt^2} = -4a\sin\psi\,\frac{d\psi}{dt} + 4a\cos\psi\,\frac{d^2\psi}{dt^2}$$

$$= \text{constant.}$$

(5c) $xy - \frac{dy}{dx} = y^3 e^{-x^2}$ — (1)

Dividing by y^3,

$$\frac{x}{y^2} - \frac{1}{y^3}\frac{dy}{dx} = e^{-x^2}$$

Let $z = 1/y^2 \Rightarrow \frac{dz}{dx} = \frac{-2}{y^3} \cdot \frac{dy}{dx}$

$$\Rightarrow xz + \frac{1}{2}\frac{dz}{dx} = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + (2x)z = 2e^{-x^2} \quad \text{— (2)}$$

which is a linear DE.

Integrating factor is

$$I.F: e^{\int 2x\,dx} = e^{x^2}$$

$\Rightarrow$ soln to (2) is

$$z(e^{x^2}) = \int e^{x^2} \cdot 2e^{-x^2}\,dx$$

$$= 2x + c_1$$

$$\Rightarrow z = 2xe^{-x^2} + c_1 e^{-x^2}$$

$$\Rightarrow \boxed{\frac{1}{y^2} = 2xe^{-x^2} + c_1 e^{-x^2}}$$

OR $y = \left(\frac{1}{2xe^{-x^2} + c_1 e^{-x^2}}\right)^{1/2}$ where c_1 is an arbitrary constant.

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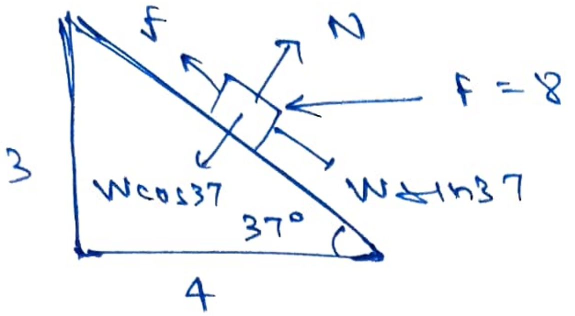
(5d)



Let N be the normal reaction and f be the friction.

At limiting point of motion,

$f = \mu N$ — (1)

Balancing forces along and parallel to plane,

$f + 8\,\cancel{\sin 37}\ \cos 37^\circ = W\sin 37$

$\Rightarrow \cancel{f = 12\cdot 3/5 = \frac{36}{5}}$

$\Rightarrow f = 20\cdot\frac{3}{5} - 8\cdot\frac{4}{5}$

$= 12 - \frac{32}{5} = \boxed{\frac{28}{5}}$ — (2)

And, $N = W\cos 37 + 8\sin 37$

$= 20\cdot\frac{4}{5} + 8\cdot\frac{3}{5} = \boxed{\frac{104}{5}}$ — (3)

From (1), (2), (3) we get

coeff of friction $\boxed{\mu = \frac{7}{26}}$

(5e) $u = u_1 i + u_2 j + u_3 k$ is unit vector.

$r(t) = (x_0 + t u_1) i + (y_0 + t u_2) j + (z_0 + t u_3) k$

$\frac{d\vec{r}}{dt} = u_1 i + u_2 j + u_3 k$

$\Rightarrow dx = u_1 dt, \; dy = u_2 dt, \; dz = u_3 dt$

Arc length $ds = \left((dx)^2 + (dy)^2 + (dz)^2\right)^{1/2}$

$\Rightarrow ds = (u_1^2 + u_2^2 + u_3^2)^{1/2} \cdot dt$

But u is a unit vector

$\Rightarrow |u| = 1 \Rightarrow |u_1^2 + u_2^2 + u_3^2|^{1/2} = 1$

$\therefore \boxed{ds = dt}$

$\Rightarrow$ Integrating,

$\boxed{s = t + c}$ where c is a constant

clearly, ~~...~~ the arc length parameter is t itself

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(7a) $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$ — (1)

Let $u = x+y$, $v = x^2+y^2$

$\Rightarrow \frac{du}{dx} = 1+p$, $\frac{dv}{dx} = 2(x+yp)$

(1) becomes:

$$v\left(\frac{du}{dx}\right)^2 - 2(u)\left(\frac{du}{dx}\right)\left(\frac{1}{2}\frac{dv}{dx}\right) + \frac{1}{4}\left(\frac{dv}{dx}\right)^2 = 0$$

$$\Rightarrow 4v - 4u\cdot\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 = 0$$

$$\Rightarrow \cancel{\left(\frac{dv}{du}\right)^2}\ v = u\cdot\frac{dv}{du} + \frac{1}{4}\left(\frac{dv}{du}\right)^2$$

which is in Clairaut's form of ~~the~~ ordinary differential equation.

($y = px + f(p)$.
Then solution is $p = c$, or
$y = cx + f(c)$)

Here, solution is $\frac{dv}{du} = c$

$$\Rightarrow \boxed{v = cu + \frac{1}{4}c^2}$$

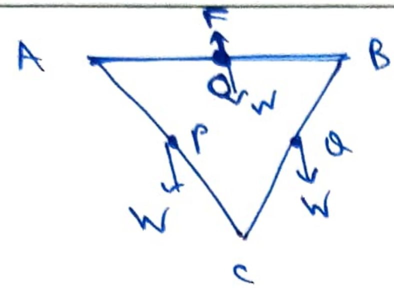
Or, substituting back values

of $u = x+y$, $v = x^2+y^2$, we get

$$\boxed{x^2+y^2 = c(x+y) + \frac{c^2}{4}}$$

is the required solution, where c is an arbitrary constant.

(7b)



Let F be the total support at O.
Then by force balance,

$\boxed{F = 3W}$ — (1)

(7c) $y'' - 3y' + 2y = 4t + e^{3t}$

$y(0)=1,\ y'(0)=-1$

Taking Laplace both sides,

$\mathcal{L}(y'') - 3\mathcal{L}(y') + 2\mathcal{L}(y) = 4\mathcal{L}(t) + \mathcal{L}(e^{3t})$

$\Rightarrow \left(s\mathcal{L}(y') - y'(0)\right) - 3\left(s\mathcal{L}(y) - y(0)\right) + 2\mathcal{L}(y) = \frac{4}{s^2} + \frac{1}{s-3}$

$\Rightarrow \left(s\left(s\mathcal{L}(y) - 1\right) + 1\right) - 3\left(s\mathcal{L}(y) - 1\right) + 2\mathcal{L}(y) = 4/s^2 + 1/s-3$

Denoting $\mathcal{L}(y)$ by $Y(s)$,

$\left(s^2Y(s) - s + 1\right) + \left(-3sY(s) + 3\right) + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$

$\Rightarrow Y(s)\left(s^2 - 3s + 2\right) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$

$\Rightarrow Y(s) = \frac{4}{s^2(s-1)(s-2)} + \frac{1}{(s-3)(s-1)(s-2)} + \frac{s-4}{(s-1)(s-2)}$ —— (1)

Now, $\frac{4}{s^2(s-1)(s-2)} = 2\left(\frac{1}{(s)(s-1)(s-2)} - \frac{1}{s^2(s-1)}\right)$

$$= \frac{1}{(s-1)(s-2)} - \frac{1}{(s)(s-1)} - 2\left(\frac{1}{(s)(s-1)} - \frac{1}{s^2}\right)$$

$$= \left(\frac{1}{s-2} - \frac{1}{s-1}\right) - \left(\frac{1}{s-1} - \frac{1}{s}\right) + \frac{2}{s^2} - 2\left(\frac{1}{s-1} - \frac{1}{s}\right)$$

$$= \frac{2}{s^2} + \frac{3}{s} + \frac{-4}{s-1} + \frac{1}{s-2} \quad \text{—} (2)$$

And

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{1}{2}\left(\frac{1}{(s-2)(s-3)} - \frac{1}{(s-1)(s-2)}\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{s-3} - \frac{1}{s-2}\right) - \left(\frac{1}{s-2} - \frac{1}{s-1}\right)\right)$$

$$= \frac{1/2}{s-1} + \frac{-1}{s-2} + \frac{1/2}{s-3} \quad \text{—} (3)$$

And,

$$\frac{s-4}{(s-1)(s-2)} = \frac{1}{s-2} - 3\left(\frac{1}{s-2} - \frac{1}{s-1}\right)$$

$$= \frac{3}{s-1} - \frac{2}{s-2} \quad \text{—} (4)$$

from (2), (3), (4), we get (1) as:

$$Y(s) = \frac{2}{s^2} + \frac{3}{s} + \frac{-1/2}{s-1} + \frac{-2}{s-2} + \frac{1/2}{s-3}$$

Taking Laplace inverse, we get,

$$\boxed{y(t) = 2t + 3 - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}}$$

is the solution to the given IVP.

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