ON THE COMPLETENESS OF S4
AND THE ALGEBRA OF POSSIBLE WORLDS

Thesis by

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May 7, 2012

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to express the deepest gratitude to my advisor, John Burgess, whose consistent guidance facilitated my research tremendously. I also appreciate his providing me with a copy of his excellent book, *Philosophical Logic*.

Thanks also to Edward Nelson for his support, and for his course in mathematical logic which greatly furthered my interest in the subject.

I also thank Edwin Williams. His thought-provoking course in semantics introduced many of the puzzles that inspired me to pursue this topic. He also suggested my advisor to me, for which I am very grateful.

I would like to give the utmost thanks to my wonderful family for all their encouragement throughout my years at Princeton, and for their constant provision of emotional, pecuniary, and alimentary assistance.

I thank my good comrades Matthew Weiss and Andrew Eron for tirelessly engaging me in Socratic dialogue and having the patience to lend an ear to my discursive musings.

I offer a heartfelt thanks to James Cole. We were an unlikely pair, but while we differed in comportment, we harmonized in our introspective and analytical nature. He has helped me through all of my tribulations these four years, and I will always consider him my roommate and a true friend. I am also thankful for the numerous diversions to which he has introduced me, such as "slinging the rock," "blasting pecs," and other bro activities.

Finally, I give my sincerest harn to Drs. Head, Bonbon, Waldo, Sticky, Harry, W.C., Sauce, Harnold, Bolt, Dude, Dang, Woody, Juicy, Twang, Stogie, Hooty, Cocoa, East, and Cardinal, to Rooks S-Joal and Archdylan, and to my sons, Beefer and Handy. May the melody linger on.

ON THE COMPLETENESS OF S4
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JEREMY SILVER

Abstract. This thesis will discuss two methods of proving completeness of the S4 modal system with respect to the class of reflexive, transitive frames. The first is based on the standard "modern proof" of Makinson (1966), which uses the concept of a maximal consistent set to construct a canonical model. The other approach is algebraic. In 1948, McKinsey and Tarski showed that S4 can be characterized by the class of closure algebras. Then in 1951, Tarski and Jónsson extended the Stone Representation Theorem (1936) to show that every closure algebra is representable as an algebra of sets with an additional operation arising from a reflexive, transitive relation. Using this representation, Tarski could have obtained the completeness result; however, he overlooked the connection between modal logic and algebra, and it was not until 1963 that Saul Kripke proved the completeness of S4. In this thesis I will explore both the modern and algebraic proofs, demonstrating that they are essentially identical.

1. Preliminaries

Modal Logic. In this paper we are concerned with the **S4** modal system. In defining this system, we take as our language the set of *modal formulas* constructed from a denumerable set of proposition letters $\{p_0, p_1, p_2, \dots\}$ closed under finite application of the logical operators $\neg$ (negation) and $\wedge$ (conjunction), and the unary modal operator $\Box$ (necessity). The following are some useful abbreviations:

(α and β are propositional variables which stand for arbitrary formulas)

$$\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta) \qquad (Disjunction)$$

$$\alpha \to \beta := \neg\alpha \vee \beta \qquad (Material\,Implication)$$

$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha) \qquad (Material\,Equivalence)$$

$$\alpha \twoheadrightarrow \beta := \Box(\alpha \to \beta) \qquad (Strict\,Implication)$$

$$\alpha \equiv \beta := (\alpha \twoheadrightarrow \beta) \wedge (\beta \twoheadrightarrow \alpha) \qquad (Strict\,Equivalence)$$

$$\Diamond\alpha := \neg\Box\neg\alpha \qquad (Possibility)$$

Definition 1.1. A *modal logic* is a set of modal formulas that includes all classical tautologies and is closed under *modus ponens* (from $\alpha \to \beta$ and α, infer β) and *uniform substitution* (substitution by any formula of all occurrences of a given proposition letter).

A *normal modal logic* is a modal logic that additionally includes formulas of the form

(K) $\qquad \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

and is closed under:

(Nec) from α, infer $\Box\alpha$

For a modal logic $\mathbf{L}$, we say that a formula α is a *theorem* of $\mathbf{L}$ (abbreviated $\vdash_{\mathbf{L}} \alpha$) if $\alpha \in \mathbf{L}$.

A subset of formulas S of a modal logic $\mathbf{L}$ is *derivationally consistent* if, for any subset $\{\alpha_1, \ldots, \alpha_n\}$ of S, the formula $\neg(\alpha_1 \wedge \cdots \wedge \alpha_n)$ is not a theorem of $\mathbf{L}$. A single formula α is derivationally consistent if $\neg\alpha$ is not a theorem of $\mathbf{L}$.

The *minimal modal logic* $\mathbf{L_0}$ is the normal modal logic that has only the tautologies and instances of (K) as its axioms.[1]

Definition 1.2. The **S4** modal system is the extension of $\mathbf{L_0}$ obtained by adding to it the following two axiom schemas:

(T) $\Box\alpha \to \alpha$

(4) $\Box\alpha \to \Box\Box\alpha$

Semantics for Modal Logic. The ordinary propositional calculus is interpreted via *truth valuations*, in which each sentence letter is assigned a truth value (0 or 1) and the logical connectives are interpreted as their corresponding Boolean truth functions. In this way every formula receives a truth value. A modal operator, on the other hand, is not truth functional (otherwise it could be reduced to a composition of the existing truth functions). These operators capture the "modality" of a proposition, i.e., the circumstances determining its truth conditions. The simplistic notion of a truth valuation therefore needs to be enriched to incorporate these differing circumstances.

Definition 1.3. A *Kripke frame* is a structure $\mathcal{F} = \langle W, R \rangle$ such that

(1) W is a non-empty set (the set of "worlds").
(2) R is a binary relation on W (the "accessibility relation").

Definition 1.4. A *Kripke model* is a structure $\mathcal{M} = \langle \mathcal{F}, v \rangle$ such that

(1) $\mathcal{F}$ is a Kripke frame.
(2) ν is a *valuation function* that assigns a truth value to each pair (p, u), where p is a proposition letter and $u \in W$.

The domain of ν can be extended from the set of proposition letters to the set of all formulas inductively as follows:

$$\nu(\neg\alpha, u) = 1 \text{ iff } \nu(\alpha, u) = 0$$
$$\nu(\alpha \wedge \beta, u) = 1 \text{ iff } \nu(\alpha, u) = \nu(\beta, u) = 1$$
$$\nu(\Box\alpha, u) = 1 \text{ iff } \nu(\alpha, v) = 1 \text{ for all } v \in W \text{ such that } uRv.$$ [2]

[1]By "axioms" I mean a subset of the theorems for which the complete set can be obtained by finite application of the rules of inference, starting with only this subset. An "axiom schema" is a generalization that describes a set of axioms having a certain form.

[2]In general, for binary relations I will use xRy to abbreviate the condition $(x, y) \in R$.

The expected results will arise for the other truth-conditional connectives $\vee$, $\to$, and $\leftrightarrow$, and we also have

Theorem 1.5. $\nu(\Diamond\alpha, u) = 1$ *iff* $\nu(\alpha, v) = 1$ *for some* $v \in W$ *such that* uRv.

Proof.

$$\begin{aligned}\nu(\Diamond\alpha, u) = 1 &\Leftrightarrow \nu(\neg\Box\neg\alpha, u) = 1 \Leftrightarrow \nu(\Box\neg\alpha, u) = 0\\ &\Leftrightarrow \exists v \in W,\ uRv,\ \nu(\neg\alpha, v) = 0\\ &\Leftrightarrow \exists v \in W,\ uRv,\ \nu(\alpha, v) = 1\end{aligned}$$

□

Definition 1.6. A formula α is said to be

- *Valid in a model*[3] $\mathcal{M} = \langle W, R, \nu\rangle$ if for all $u \in W$, $\nu(\alpha, u) = 1$.
- *Valid in a frame* $\mathcal{F} = \langle W, R\rangle$ if for every valuation function ν, α is valid in the model $\langle \mathcal{F}, \nu\rangle$.
- *Valid in a class of frames* F if α is valid in every frame in F.

We use '$\Vdash_S \alpha$' to denote that α is valid in S, where S is a model, frame, or class of frames.

Definition 1.7. A formula α is said to be *satisfied* in a model $\mathcal{M} = \langle W, R, \nu\rangle$ if there exists some $u \in W$ for which $\nu(\alpha, u) = 1$. A set of formulas Γ is satisfied in $\mathcal{M}$ if there exists some $u \in W$ for which $\nu(\gamma, u) = 1$ for every $\gamma \in S$.

Let **L** be a modal logic and F be a class of frames.

- **L** is *sound* with respect to F if $\vdash_{\mathbf{L}} \alpha$ implies $\Vdash_{\mathrm{F}} \alpha$ for every formula α.
- **L** is *complete* with respect to F if $\Vdash_{\mathrm{F}} \alpha$ implies $\vdash_{\mathbf{L}} \alpha$ for every formula α.

Theorem 1.8. *The following are two conditions equivalent to completeness of* **L** *with respect to a class of frames* F*:*

(1) For every formula α that is not a theorem of **L**, *there exists a model $\mathcal{M} = \langle \mathcal{F}, \nu\rangle$ for which $\mathcal{F}$ belongs to* F *and α is not valid in $\mathcal{M}$.*

(2) For every formula α that is derivationally consistent in **L**, *there exists a model $\mathcal{M} = \langle \mathcal{F}, \nu\rangle$ for which $\mathcal{F}$ belongs to* F *and α is satisfied in $\mathcal{M}$.*

Proof. Completeness is the condition that for any formula α, $\Vdash_{\mathrm{F}} \alpha$ implies $\vdash_{\mathbf{L}} \alpha$. Taking the contrapositive, if α is not a theorem of **L** it is not valid with respect to F. This latter condition is equivalent to the existence of a frame $\mathcal{F}$ belonging to F for which α is not valid, which is equivalent to the existence of a model $\mathcal{M} = \langle \mathcal{F}, \nu\rangle$ for which α is not valid. Hence completeness is equivalent to condition (1).

We now show (1) and (2) are equivalent. Suppose the former holds. Let α be a **L**-consistent formula, i.e. $\neg\alpha$ is not a theorem of **L**. Then by (1), there exists a model $\mathcal{M} = \langle \mathcal{F}, \nu\rangle$ for which

[3] Henceforth I may omit "Kripke" when using the terms "model" and "frame."

$\mathcal{F}$ belongs to F and $\neg\alpha$ is not valid, which means there is a $u \in W$ such that $\nu(\neg\alpha, u) = 0$, which implies $\nu(\alpha, u) = 1$, and hence α is satisfied in $\mathcal{M}$. Now suppose (2) holds. If α is not a theorem of **L**, then $\neg\neg\alpha$ is also not a theorem (or else α would be derivable by tautological consequence). This implies the consistency of $\neg\alpha$, which by (2) entails the existence of a model $\mathcal{M} = \langle \mathcal{F}, \nu \rangle$ for which $\mathcal{F}$ belongs to F and $\nu(\neg\alpha, u) = 1$ for some $u \in W$. Consequently $\nu(\alpha, u) = 0$, and so α is not valid in $\mathcal{M}$. □

The subject of this paper is to the completeness of **S4** with respect to the class of Kripke frames whose accessibility relation is reflexive and transitive. **S4** is also sound with respect to this class of frames, but the proof is routine and left to the reader. Kripke showed completeness in 1963 using a somewhat complicated proof using his "method of tableaux,"[4] but in the next section we will roughly follow a revised proof that uses the notion of maximal consistent sets to construct a canonical model.

[4]See Kripke [8].

2. The "Modern" Proof: Completeness via maximal consistent sets

A standard proof of the completeness of **S4** is found in Makinson [10], and in this section we will closely follow his method.

Definition 2.1. A *quasi-ordering* is a reflexive, transitive binary relation.

A frame $\langle W, R\rangle$ is *quasi-ordered* if R is a quasi-ordering on the elements of W.

Theorem 2.2. *The modal system* **S4** *is complete with respect to the class of quasi-ordered frames.*

By Theorem 1.8 it suffices to show that every derivationally consistent formula α has a Kripke model with a quasi-ordered frame that satisfies α. Makinson actually proves a *strong* completeness result, that in fact every consistent *set* of formulas has such a Kripke model satisfying that set. Since this result takes hardly more work to prove than the weaker completeness result, we will do so here.

Definition 2.3. A set of formulas Γ of a modal logic $\mathbf{L}$ is a *maximal consistent set* (MCS) if: (1) it is derivationally consistent, and (2) for every formula α, at least one of α and $\neg\alpha$ is in Γ.

Lemma 2.4. (Properties of MCS's) *Let* $\mathbf{L}$ *be a modal logic and* Γ *be an MCS of* $\mathbf{L}$*. Then the following properties hold:*

(1) For every formula α*, precisely one of* α *and* $\neg\alpha$ *is in* Γ.

(2) Every theorem of $\mathbf{L}$ *belongs to* Γ.

(3) If $\alpha \in \Gamma$ *and* $\alpha \to \beta \in \Gamma$*, then* $\beta \in \Gamma$.

(4) $\alpha \wedge \beta \in \Gamma$ *iff both* α *and* β *are in* Γ.

Proof.

(1) By definition of MCS, at least one of α and $\neg\alpha$ is in Γ. Suppose both are. Then since $\vdash_{\mathbf{L}} \neg(\alpha \wedge \neg\alpha)$ by tautology, it follows that Γ is inconsistent, which is a contradiction.

(2) Suppose $\vdash_{\mathbf{L}} \alpha$ and $\alpha \notin \Gamma$. Then by (1), $\neg\alpha \in \Gamma$. But it follows from $\vdash_{\mathbf{L}} \neg\neg\alpha$ that Γ is inconsistent, a contradiction. Hence $\alpha \in \Gamma$.

(3) Suppose $\alpha \in \Gamma$ and $\alpha \to \beta \in \Gamma$. If $\beta \notin \Gamma$, then by (1), $\neg\beta \in \Gamma$. But $\vdash_{\mathbf{L}} \neg(\alpha \wedge (\alpha \to \beta) \wedge \neg\beta)$ by tautology, and this leads to the inconsistency of Γ, a contradiction. Hence $\beta \in \Gamma$.

(4) Suppose $\alpha \wedge \beta \in \Gamma$ and (without loss of generality) $\alpha \notin \Gamma$. Then $\neg\alpha \in \Gamma$, which yields inconsistency of Γ in light of $\vdash_{\mathbf{L}} \neg((\alpha \wedge \beta) \wedge \neg\alpha)$. Conversely, suppose both α and β are in Γ, but $\alpha \wedge \beta \notin \Gamma$, i.e. $\neg(\alpha \wedge \beta) \in \Gamma$. Then $\vdash_{\mathbf{L}} \neg(\alpha \wedge \beta \wedge \neg(\alpha \wedge \beta))$ again leads to a contradiction. □

Lemma 2.5. (Lindenbaum's Lemma) *If* Σ *is an* $\mathbf{L}$*-consistent set of formulas, then there is a maximal consistent set* Σ^+ *of* $\mathbf{L}$ *such that* $\Sigma \subseteq \Sigma^+$.

Proof. First note that it is possible to enumerate the formulas of the modal language $\mathbf{L}$. For instance, we can write the proposition letters as $p, p', p'', \ldots,$ and if we use prefix notation for the operators, we may dispense with parentheses for disambiguation. We assign a unique "code digit" to each symbol p, $'$, $\neg$, $\wedge$, and $\Box$. To each formula we then assign a code number that is the concatenation of the code digits of the symbols comprising it. From there we can list the formulas in order of increasing code number, in order to get a denumerable sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ of formulas.

Given a consistent set of formulas Σ, we can form an increasing chain of consistent sets as follows:

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\gamma_n\}, \text{ if this is consistent} \\ \Sigma_n \cup \{\neg\gamma_n\}, \text{ otherwise} \end{cases}$$

$$\Sigma^+ = \bigcup_{n \geq 0} \Sigma_n$$

It is obvious that $\Sigma \subseteq \Sigma^+$. Also, it follows by construction that for every formula, either it or its negation is in Σ^+. Next, we show Σ_n is consistent for all n. The argument is from induction. Σ_0 is consistent by assumption. Now suppose Σ_n is consistent. We check to see if Σ_{n+1} is consistent. By construction, it could only be inconsistent if both $\Sigma_n \cup \{\gamma_n\}$ and $\Sigma_n \cup \{\neg\gamma_n\}$ are inconsistent. Suppose this is the case. Then there exist formulas α_1 and α_2 which are conjunctions of finite subsets of Σ_n such that $\vdash_{\mathbf{L}} \neg(\alpha_1 \wedge \gamma_n)$ and $\vdash_{\mathbf{L}} \neg(\alpha_2 \wedge \neg\gamma_n)$. Letting α abbreviate $\alpha_1 \wedge \alpha_2$, it follows that $\vdash_{\mathbf{L}} \neg(\alpha \wedge \gamma_n)$ and $\vdash_{\mathbf{L}} \neg(\alpha \wedge \neg\gamma_n)$. Then $\vdash_{\mathbf{L}} \alpha \to \neg\gamma_n$ and $\vdash_{\mathbf{L}} \alpha \to \gamma_n$, from which follows $\vdash_{\mathbf{L}} \alpha \to (\gamma_n \wedge \neg\gamma_n)$. From tautological consequence, $\vdash_{\mathbf{L}} \neg\alpha$, but since α is a conjunction of a finite subset of Σ_n, this contradicts that Σ_n is consistent, which we have by induction. Therefore Σ_n is consistent for all n. From this it follows that Σ^+ must be consistent. For suppose otherwise. Then there exists a finite set of formulas in Σ^+, $\gamma_{k_0}, \ldots, \gamma_{k_m}$ $(k_0 < \cdots < k_m)$, such that $\vdash_{\mathbf{L}} \neg(\gamma_{k_0} \wedge \cdots \wedge \gamma_{k_m})$. But from the construction, it must be the case that Σ_{k_m+1} also contains $\gamma_{k_0}, \ldots, \gamma_{k_m}$, and therefore Σ_{k_m+1} is inconsistent, a contradiction. Hence Σ^+ is consistent. These conditions together imply that Σ^+ is an MCS satisfying $\Sigma \subseteq \Sigma^+$. □

For a normal modal logic $\mathbf{L}$, we construct the *canonical Kripke model* $\mathcal{M}_{\mathbf{L}} = \langle W, R, \nu \rangle$ as follows:

- W is the set of all MCS's of $\mathbf{L}$.
- If $u, v \in W$, then uRv iff for every $\mathbf{L}$-formula α, $\alpha \in v$ whenever $\Box\alpha \in u$.
- For any proposition letter p and $u \in W$, $\nu(p, u) = 1$ iff $p \in u$.

Lemma 2.6. (Existence Lemma) *If u is an MCS and $\Diamond\alpha \in u$, then there is an MCS v such that uRv and $\alpha \in v$.*

Proof. Let Σ be the set of formulas consisting of α together with all β such that $\Box\beta \in u$. This set is consistent. For suppose otherwise; then there is a finite subset $\beta_0, \ldots, \beta_n$ of the β's such that $\neg(\beta_0 \wedge \cdots \wedge \beta_n \wedge \alpha)$ is a theorem of $\mathbf{L}$. By (Nec) this would imply $\Box\neg(\beta_0 \wedge \cdots \wedge \beta_n \wedge \alpha)$ is also a theorem, and this is equivalent to $\vdash_{\mathbf{L}} \neg\Diamond(\beta_0 \wedge \ldots \beta_n \wedge \alpha)$. It is possible to derive from this, using tautology, substitution, detachment, and the (K) axiom for normal modal logics, that $\vdash_{\mathbf{L}} \neg(\Box\beta_0 \wedge \cdots \wedge \Box\beta_n \wedge \Diamond\alpha)$. But this contradicts that u is an MCS. Therefore Σ is consistent, and by Lindenbaum's Lemma can be extended to an MCS $v = \Sigma^+$. v contains α since Σ did, and uRv holds because we constructed Σ to contain all β such that $\Box\beta \in u$, and $\Sigma \subseteq v$. ☐

Theorem 2.7. *For any consistent set Σ of formulas of* **L**, *the canonical model $\mathcal{M}_{\mathbf{L}}$ satisfies Σ.*

It will suffice to prove the following:

Lemma 2.8. (Truth Lemma) *For any* **L**-*formula α, $\nu(\alpha, u) = 1$ iff $\alpha \in u$.*

For if this results holds, suppose Σ is an **L**-consistent set. Then by Lindenbaum's Lemma, there is an MCS $u = \Sigma^+$ that contains every formula in Σ, and by the above lemma, $\nu(\alpha, u) = 1$ for all $\alpha \in \Sigma$, i.e. $\mathcal{M}_{\mathbf{L}}$ satisfies Σ.

The proof is by induction on the length of α.

- If α is a proposition letter, the result follows immediately from the definition of ν.
- $\nu(\neg\beta, u) = 1$ iff $\nu(\beta, u) = 0$ iff $\beta \notin u$ iff $\neg\beta \in u$. The final equivalence holds by property (1) of Lemma 2.4.
- $\nu(\beta \wedge \gamma, u) = 1$ iff $\nu(\beta, u) = \nu(\gamma, u) = 1$ iff $\beta \in u$ and $\gamma \in u$ iff $\beta \wedge \gamma \in u$. The final equivalence holds by property (4) of the same lemma.
- $\nu(\Box\beta, u) = 1$ iff $\forall v, uRv, \nu(\beta, v) = 1$ iff $\forall v, uRv, \beta \in v$ iff $\Box\beta \in u$. The final equivalence requires some explanation. The right-to-left direction follows directly from the definition of R. For the other direction, suppose $\Box\beta \notin u$. Then by property (1) of Lemma 2.4, $\neg\Box\beta \in u$. By property (3), MCS's are closed under entailment, and so $\Diamond\neg\beta \in u$. By Lemma 2.6, there is an MCS v such that uRv and $\neg\beta \in v$, which implies $\beta \notin v$, thus falsifying the condition $\forall v, uRv, \beta \in v$.

We now show that **S4** is complete with respect to the class of quasi-ordered frames. As indicated earlier, Theorem 1.8 tells us that we only have to show that for each **S4**-consistent formula α, there is a model with a quasi-ordered frame that satisfies α. Since we want to show *strong* completeness, we will show that for every set of **S4**-consistent formulas, there is a model with a quasi-ordered frame that satisfies that set. How do we find such a model? Simple. We now know from Theorem 2.7 that the canonical model $\mathcal{M}_{\mathbf{S4}}$ satisfies *any* consistent set of **S4** formulas. Therefore, we need only prove that $\mathcal{M}_{\mathbf{S4}}$ has a quasi-ordered frame.

Theorem 2.9. *If $\mathcal{M}_{\mathbf{S4}} = \langle W, R, \nu \rangle$ is the canonical model of* **S4**, *then R is a quasi-ordering on W.*

Proof. For reflexivity, let u be an MCS and $\Box\alpha \in u$. We know from property (2) of Lemma 2.4 that the (T) axiom of **S4**, $\Box\alpha \to \alpha$, is also in u. From property (3) of the same lemma, α must be in u, and so uRu holds.

For transitivity, suppose u, v, and w are MCS's such that uRv and vRw. Now suppose $\Box\alpha \in u$. The (4) axiom of **S4**, $\Box\alpha \to \Box\Box\alpha$, must also be in u, so it follows that $\Box\Box\alpha \in u$. Since uRv, we know $\Box\alpha \in v$, and since vRw, it follows that $\alpha \in w$. Thus uRw. ☐

We have shown the strong completeness of **S4** with respect to quasi-ordered frames, and Theorem 2.2 follows *a fortiori.*

The Finite Model Property. We know that for a normal modal logic, any consistent set of formulas has a model satisfying it. This was proven by showing that there exists a single model, the canonical model, that represents the entire logic. However, this model is not only infinite, but uncountable.

Theorem 2.10. *If* $\mathcal{M}_{\mathbf{L}} = \langle W, R, \nu \rangle$ *is the canonical model of a normal modal logic* $\mathbf{L}$*,* W *must be an uncountable set.*

Proof. First, if $\mathbf{L}$ has a canonical model, $\mathbf{L}$ cannot be the *inconsistent logic* (the logic in which every formula is a theorem). For then there would be no consistent sets, and so W would be the empty set, which is not allowed for a Kripke model. So we assume $\mathbf{L}$ is not inconsistent. Consider the collection $\mathscr{D}$ of sets $\{(\neg)p_0, (\neg)p_1, (\neg)p_2, \dots\}$ that contain every proposition letter or its negation (but not both), and no other propositions. There are $2^{\aleph_0}$ such sets, uncountably many. Moreover, each of these sets is consistent. For suppose otherwise. Then there would be an $\mathbf{L}$-theorem of the form $\neg((\neg p_{i_0} \wedge \cdots \wedge \neg p_{i_m}) \wedge (p_{j_0} \wedge \cdots \wedge p_{j_n}))$. By uniform subsitution, we could replace all the p_i's with some arbitrary formula α and replace all the p_j's with $\neg\alpha$, from which we would obtain α as a theorem. Therefore every formula is a theorem, and so $\mathbf{L}$ is the inconsistent logic, which contradicts our assumption. We conclude that each of the sets in $\mathscr{D}$ is consistent, and by Lindenbaum's Lemma can be extended to an MCS. It is easily verified that these MCS's must all be distinct, and hence the set W of all MCS's is uncountable. □

While the existence of a canonical model assures us that **S4** is complete, it does not give us a practical way of testing for the validity of arbitrary formulas. In order to do this, the previous claim tells us we would have to check the truth of a formula at uncountably many points u in W. Thus the canonical model is of little use toward the practical task of *decidability.* However, it is possible to establish the *finite model property* for **S4**, which does in fact provide a procedure for efficiently determining the validity of a formula.

Theorem 2.11. (Finite Model Property) *If* α *is a consistent formula of* **S4***, then there exists a finite model that satisfies* α*.*

The proof is very similar to that of the non-finite case; the difference is that we will essentially "miniaturize" the canonical model to be only concerned with a finite set of formulas.

Definition 2.12. A *subformula* of a formula α is any formula which occurs as a part of α. Any formula is a subformula of itself.

An α*-formula* is any formula that is either a subformula of α or a negation of a subformula of α.

A set of α-formulas Γ is an α*-maximal consistent set* (α-MCS) if: (1) it is derivationally consistent, and (2) for every α-formula β, at least one of β and $\neg\beta$ is in Γ.

α-MCS's naturally have the same properties as given in Lemma 2.4 for regular MCS's, except that the formulas under consideration are restricted to α-formulas. The proof is identical. Similarly, an analog of Lindenbaum's Lemma can be proven: every consistent set of α-formulas can be extended to an α-MCS. The proof is basically the same, but it is made slightly simpler by the fact that the chain of increasingly large consistent sets will terminate, and the largest one is the desired α-MCS, so it is unnecessary to take the union over the whole chain.

We are now ready to construct the *canonical finite model* $\mathcal{M}_\alpha = \langle W, R, \nu \rangle$ that will satisfy α:

- W is the set of all α-MCS's.
- If $u, v \in W$, then uRv iff for every α-formula β, whenever $\Box\beta$ is in u, both β and $\Box\beta$ are in v.
- For any proposition letter p occurring in α and $u \in W$, $\nu(p, u) = 1$ iff $p \in u$. For all other proposition letters q not occurring in α, $\nu(q, u) = 0$ for all $u \in W$.

Since α contains a finite number of subformulas, there can only be a finite number of α-formulas. Suppose there are n distinct α-formulas. Then there are at most 2^n sets of α-formulas, so there can only be a finite number of α-MCS's, and hence W is finite. We now have to show that $\mathcal{M}_\alpha$ satisfies α.

Lemma 2.13. *If u is an α-MCS and $\Diamond\beta \in u$, then there is an α-MCS v such that uRv and $\beta \in v$.*

Proof. Let Σ be the set of α-formulas consisting of β together with all $\Box\gamma$ such that $\Box\gamma \in u$. This set is consistent. For if not, then there is a finite subset $\gamma_0, \ldots, \gamma_n$ of the γ's such that $\neg(\Box\gamma_0 \wedge \cdots \wedge \Box\gamma_n \wedge \beta)$ is a theorem of **S4**. From this it is possible to derive that $\vdash_{\mathbf{S4}} \neg(\Box\Box\gamma_0 \wedge \cdots \wedge \Box\Box\gamma_n \wedge \Diamond\beta)$, and by the (4) axiom this implies $\vdash_{\mathbf{S4}} \neg(\Box\gamma_0 \wedge \cdots \wedge \Box\gamma_n \wedge \Diamond\beta)$, which contradicts that u is an α-MCS. Therefore Σ is consistent, and by the modified version of Lindenbaum's Lemma, it can be extended to an α-MCS v. v contains β since Σ did, and it likewise contains all $\Box\gamma$ that were in u. But by the (T) axiom and the fact that α-MCS's are closed under detachment (provided the entailed formula is an α-formula, which γ is in this case), it follows that all γ belong to v as well, and therefore uRv holds. $\Box$

Theorem 2.11 follows from a proof identical to that of Theorem 2.7, and it makes use of the above lemma. The induction on length still works because any subformula of an α-formula is still an α-formula. Also note that the model's valuation of non-α-formulas is completely inconsequential, because the goal is simply to show that $\mathcal{M}_\alpha$ satisfies α, which is itself an α-formula. It therefore does not matter what ν assigns to proposition letters not occurring in α.

Why did we impose the stronger condition on uRv that both $\beta \in v$ and $\Box\beta \in v$ must follow from $\Box\beta \in u$? Our aim was to "miniaturize" the canonical model, so in the case of **S4** we should hope that it has the same property as $\mathcal{M}_{\mathbf{S4}}$ of having a quasi-ordered frame. But suppose R were the same relation as in $\mathcal{M}_{\mathbf{S4}}$. In trying to prove transitivity, we would encounter a problem. For assume we have uRv, vRw, and $\Box\beta \in u$. In the infinite case, we were guaranteed that $\Box\Box\beta \in u$. But in the finite case, this is not necessarily so, since is it possible for $\Box\beta$ to be an α-formula without $\Box\Box\beta$ also being one. Without this step we cannot ensure that $\beta \in w$. But the property holds if we assume the stronger relation:

Theorem 2.14. *If $\mathcal{M}_\alpha = \langle W, R, \nu \rangle$ is the canonical finite model satisfying α, then R is a quasi-ordering on W.*

Proof. For reflexivity, let u be an α-MCS and $\Box\beta \in u$. Since β is an α-formula entailed by $\Box\beta$ via the (T) axiom, $\beta \in u$, and so uRu holds.

For transitivity, suppose uRv and vRw for α-MCS's u, v, and w, and $\Box\beta \in u$. Then by definition of R, $\Box\beta \in v$, and hence $\Box\beta \in w$. Then $\beta \in w$ by detachment, so uRw holds. □

The finite model property gives a method for determining the validity (and by completeness, theoremhood) of an arbitrary formula α. Suppose we know an upper bound on the size of the canonical finite model satisfying $\neg\alpha$ (this number can be computed as a function of the length of α). We consider all models whose size is less than or equal to this upper bound, but only the ones that assign zero at all worlds to all proposition letters not appearing in α. Up to isomorphism, there are only finitely many models to consider. We check this set of models, and if $\neg\alpha$ is satisfied on any one of them, then α is not valid, and therefore not a theorem of **S4**. If on the other hand, $\neg\alpha$ is not satisfied on any of them, then $\neg\alpha$ is not consistent, and hence α is a theorem of **S4**. In this way, we have a means of deciding theoremhood for **S4**, albeit a cumbersome one. (It may also save time to search for *proofs* of α while simultaneously searching for models satisfying $\neg\alpha$).

3. McKinsey: S4 matrices and the algebraic finite model property

The proof of completeness in the previous section is rather straightforward. In this section, we will instead look at modal logic through the lens of algebra, as J.C.C. McKinsey did in his 1941 paper.[5] As the first step to obtaining the completeness of **S4** algebraically, we will reinterpret formulas as elements of "modal algebras" whose operations correspond to the logical operators. From this we will get an intermediate completeness result: for every non-theorem α of **S4**, a special kind of algebra can be constructed to "falsify" α.

Definition 3.1. A *matrix* is an a structure $\mathfrak{M} = \{K, D, -, \times, {}^*\}$, where K is a set, D a non-empty proper subset of K (the set of "designated" elements), $-$ and $*$ are unary functions on K, $\times$ is a binary function on K, and K is closed under $-$, $\times$, and $*$. Moreover, the symbols '$+$', '$\Rightarrow$', and '$\Leftrightarrow$' will abbreviate the following binary functions:

$$
\begin{aligned}
a + b &:= \; -(-a \times -b) \\
a \Rightarrow b &:= \; -{}^*(a \times -b) \\
a \Leftrightarrow b &:= (a \Rightarrow b) \times (b \Rightarrow a)
\end{aligned}
$$

A matrix interprets a modal formula in the following way. Every proposition letter is assigned an element of K, and each logical operator corresponds to an algebraic operation: $-$ for negation, $\times$ for conjunction, and $*$ for possibility. It is easily seen that $+$, $\Rightarrow$, and $\Leftrightarrow$ correspond to disjunction, strict implication, and strict equivalence respectively, and the box can be interpreted by the sequence of operations $-{}^*-$. In this way, every formula is interpreted as an element of K.

A matrix is said to *satisfy* a modal formula α if for every assignment of elements of K to the proposition letters of α, the evaluation results in an element belonging to D.

An *S4-matrix* is a matrix that satisfies all the theorems of **S4**.

An *S4-characteristic matrix* is an S4-matrix such that the only formulas it satisfies are the theorems of **S4**.

Definition 3.2. A matrix is *normal*[6] if for any $a, b \in K$ the following three conditions are met:

(1) if $a \in D$ and $b \in D$, then $a \times b \in D$
(2) if $a \Rightarrow b \in D$ and $a \in D$, then $b \in D$
(3) if $a \Leftrightarrow b \in D$, then $a = b$

Theorem 3.3. *There exists a normal S4-characteristic matrix,* $\mathfrak{M} = \{K, D, -, \times, {}^*\}$.

Proof. To begin, we define an equivalence relation on the set of modal formulas. For formulas α and β, we will say $\alpha \sim \beta$ if $\vdash_{\mathbf{S4}} \alpha \equiv \beta$.[7] It is easily seen that $\sim$ is an equivalence relation.

[5]See McKinsey [11].

[6]The properties of normality actually correspond to the rules of inference given for Lewis's original system of "strict implication," which is entirely equivalent to our system **S4** (see Lewis & Langford [9] and Gödel [4]). However, McKinsey was not yet aware of this equivalence.

[7]Note the *strict* equivalence. However, the (Nec) rule ensures that $\vdash_{\mathbf{S4}} \alpha \equiv \beta$ whenever $\vdash_{\mathbf{S4}} \alpha \leftrightarrow \beta$.

We denote the equivalence class of α (the set of formulas equivalent to α) by $[\alpha]$. We will now construct the matrix $\mathfrak{M}$.

Let K be the set of equivalence classes of formulas, and let D be the set of equivalence classes of **S4** theorems (by detachment, if α is a theorem of **S4**, all elements in its equivalence class are theorems of **S4**). If $[\alpha]$ is an element of K, then $-[\alpha] = [\neg\alpha]$, and $^*[\alpha] = [\Diamond\alpha]$. If $[\alpha]$ and $[\beta]$ are elements of K, then $[\alpha] \times [\beta] = [\alpha \wedge \beta]$.

We first show that $\mathfrak{M}$ is an S4-matrix. Suppose α is a theorem of **S4**. Let b be the result of some evaluation of α in $\mathfrak{M}$, that is, assigning elements of K to each proposition letter in α and applying the matrix operations to obtain some element b in K. $\mathfrak{M}$ will satisfy α if $b \in D$ for all such assignments. But it is obvious from the definitions of the matrix operations that b is just an equivalence class of some formula β that is a result of uniform substitution on α, and by the rule of uniform substitution, it is an equivalence class of an **S4** theorem, and thus belongs to D. Hence $\mathfrak{M}$ is an S4-matrix.

To show that it is S4-characteristic, suppose α is a formula satisfied by $\mathfrak{M}$. Then every assignment of elements of K to proposition letters of α evaluates to an element in D. In particular, if we assign to each proposition letter the equivalence class to which it belongs, then the resulting equivalence class (which is just the equivalence class of α itself) will be an element of D, and hence α is a theorem of **S4**.

We move on to the three conditions for normality:

(1) Suppose $[\alpha]$ and $[\beta]$ are elements of D, and hence α and β are theorems of **S4**. A simple **S4** proof shows that $\vdash_{\mathbf{S4}} \alpha \wedge \beta$, and so $[\alpha] \times [\beta] \in D$.

(2) Suppose $[\alpha] \Rightarrow [\beta]$ and $[\alpha]$ are elements of D. Then $\vdash_{\mathbf{S4}} \alpha \twoheadrightarrow \beta$ and $\vdash_{\mathbf{S4}} \alpha$. $\vdash_{\mathbf{S4}} \alpha \twoheadrightarrow \beta$ implies $\vdash_{\mathbf{S4}} \alpha \rightarrow \beta$, from which *modus ponens* yields $\vdash_{\mathbf{S4}} \beta$, and therefore $[\beta] \in D$.

(3) Suppose $[\alpha] \Leftrightarrow [\beta]$ is in D. Then $\vdash_{\mathbf{S4}} \alpha \equiv \beta$, which means $[\alpha] = [\beta]$.

Hence $\mathfrak{M}$ is a normal S4-characteristic matrix. □

The $\mathfrak{M}$ we have constructed is called the "Lindenbaum-Tarski matrix" of the logic. The fact that it is S4-characteristic yields a "soundness and completeness" result of sorts; $\mathfrak{M}$ is an algebraic structure that fails to satisfy all and only the non-theorems of **S4** on the given interpretation. It will take some work, however, to show how to obtain a Kripke model from this matrix.

Definition 3.4. Let $\mathfrak{B} = \langle K, -, \times \rangle$ be a structure in which K is a set closed under a unary operation $-$ and a binary operation $\times$. $\mathfrak{B}$ is a *Boolean algebra* if the following postulates hold:

(1) K contains at least two elements.

(2) $a \times b = b \times a$ *(Commutativity)*

(3) $(a \times b) \times c = a \times (b \times c)$ *(Associativity)*

(4) $a \times a = a$ *(Idempotence)*

(5) $-(-(a \times b) \times -(a \times -b)) = a$

We can define the binary operation $+$ as: $a+b=-(-a\times -b)$, with which the last postulate can be tidied up as:

(5a) $(a\times b)+(a\times -b)=a$

K is obviously closed under this addition operation, and from postulates 2–4, $+$ is commutative, associative, and idempotent.

Additionally we can define a binary relation $\leq$ by: $a\leq b$ iff $a=a\times b$. From the postulates it is easily verified that it is a (partial) ordering relation.

Finally, in a Boolean algebra it can be shown that K contains a unique element 0 such that $a+0=a$ for all a in K, and a unique element 1 such that $a\times 1=a$ for all a in K (i.e. 0 is an additive identity and 1 is a multiplicative identity). Simply let $0=a\times -a$ and $1=a+-a$ for some element $a\in K$. I leave the proof of their uniqueness to the reader, as well as the proofs of the following *distributivity* properties:

$$a+(b\times c)=(a+b)\times(a+c)$$
$$a\times(b+c)=(a\times b)+(a\times c)$$

It will be noted that there are many alternative axiomatizations of a Boolean algebra, with different operations primary and the others defined, but they are all equivalent to the system given here.[8]

Definition 3.5. Let $\mathfrak{C}=\langle K,-,\times,{}^*\rangle$ be a structure where $\langle K,-,\times\rangle$ is a Boolean algebra and K is closed under a unary operation *. Then $\mathfrak{C}$ is a *closure algebra* if the following additional postulates hold:

(1) $a\leq {}^*a$
(2) ${}^*0=0$
(3) ${}^{**}a={}^*a$
(4) ${}^*(a+b)={}^*a+{}^*b$

(where $\leq$ and $+$ are the same abbreviations as used above).

The unary operation satisfying these four postulates is called the closure algebra's *closure operation.* We can now extend the notion of a closure algebra to a matrix by simply including with it a set of distinguished elements:

Definition 3.6. A matrix $\mathfrak{M}=\langle K,D,-,\times,{}^*\rangle$ is called a *generalized closure matrix* if $\langle K,-,\times,{}^*\rangle$ is a closure algebra and D is a proper subset of K that contains the unit element 1. $\mathfrak{M}$ is called a *closure matrix* if, moreover, D is the singleton set containing only the unit element, i.e. $D=\{1\}$.

[8] See Huntington [6].

Theorem 3.7. *If $\mathfrak{M} = \langle K, D, -, \times, {}^*\rangle$ is a normal S4-matrix, then it is a generalized closure matrix.*

Proof. Suppose $\mathfrak{M}$ is a normal S4-matrix. We first show that K is a Boolean algebra with respect to $-$ and $\times$. Since D is a non-empty proper subset of K, we know K must contain at least two elements. For commutativity, we know that $\vdash_{\mathbf{S4}} (\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$. Since $\mathfrak{M}$ is an S4-matrix, $(a \times b) \Leftrightarrow (b \times a) \in D$ for any a, b in K, and since $\mathfrak{M}$ is normal, this implies $a \times b = b \times a$. We can similarly verify postulates 3-5 by acknowledging that $(\alpha \wedge \beta) \wedge \gamma \equiv \alpha \wedge (\beta \wedge \gamma)$, $(\alpha \wedge \alpha) \equiv \alpha$, and $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \equiv \alpha$ are all theorems (tautologies) of **S4**. Furthermore, since $\vdash_{\mathbf{S4}} (\alpha \vee \neg\alpha)$, we know that $(a + -a) = 1 \in D$.

It remains to show that the system also satisfies the four postulates for closure algebra. For the first, observe that $\vdash_{\mathbf{S4}} \alpha \equiv \alpha \wedge \Diamond\alpha$ is deducible from the (T) axiom of **S4**, and therefore $a = a \times {}^*a$, or $a \leq {}^*a$. Next, it can be shown that $\Diamond(\alpha \wedge \neg\alpha) \equiv (\alpha \wedge \neg\alpha)$ is provable in **S4**, which yields the second postulate. The third results from a simple derivation of $\vdash_{\mathbf{S4}} \Diamond\Diamond\alpha \equiv \Diamond\alpha$ from the (T) and (4) axioms of **S4**. Finally, the last postulate is obtained from $\vdash_{\mathbf{S4}} \Diamond(\alpha \vee \beta) \equiv \Diamond\alpha \vee \Diamond\beta$. □

Theorem 3.8. *If α is a formula that is not a theorem of* **S4***, then there exists a closure matrix that fails to satisfy α.*

Proof. From Theorem 3.3 there is a normal S4-characteristic matrix $\mathfrak{M} = \langle K, D, -, \times, {}^*\rangle$, which by definition fails to satisfy all non-theorems of **S4**, including α. $\mathfrak{M}$ is also normal and is therefore a generalized closure matrix. But we can in fact show that $D = \{1\}$. Recall that the designated elements of $\mathfrak{M}$ are the equivalence classes of **S4** theorems. Suppose β and γ are theorems of **S4**. By tautological consequence, $\vdash_{\mathbf{S4}} \beta \rightarrow \gamma$ and $\vdash_{\mathbf{S4}} \gamma \rightarrow \beta$, from which the (Nec) rule provides $\vdash_{\mathbf{S4}} \beta \equiv \gamma$. This means all **S4** theorems are in the same equivalence class, and so D can only contain one element, which must be 1. Hence if α is a non-theorem, $\mathfrak{M}$ is a closure matrix that fails to satisfy it. □

We will ultimately show that every closure matrix is representable as a Kripke model. Before we set about this task, however, we first present an even stronger result of McKinsey (1941) that will afford us two ways of proving the completeness theorem. This is the *algebraic finite model property.*

Theorem 3.9. (Algebraic Finite Model Property) *Let α be a formula that is not a theorem of* **S4***, where α contains r subformulas. Then there exists a normal S4-matrix with at most 2^{2^r} elements that fails to satisfy α.*

We begin with a lemma.

Lemma 3.10. *Let $\mathfrak{M} = \langle K, D, -, \times, {}^*\rangle$ be a normal S4-matrix, and let $a_1, a_2, \ldots, a_r$ be a finite sequence of elements of K. Then there exists a normal S4-matrix $\mathfrak{M}_1 = \langle K_1, D_1, -_1, \times_1, {}^*{}_1\rangle$ with at most 2^{2^r} elements such that K_1 contains a sequence of elements $b_1, b_2, \ldots, b_r$ satisfying the following conditions:*

(1) $a_i \in D$ iff $b_i \in D_1$
(2) $-a_i = a_j$ iff $-_1 b_i = b_j$
(3) $a_i \times a_j = a_k$ iff $b_i \times_1 b_j = b_k$
(4) if $*a_i = a_j$, then $*_1 b_i = b_j$

The proof is quite tedious and can be found in McKinsey [11], so we will provide only an outline. We construct $\mathfrak{M}_1$ in the following way: K_1 is the Boolean subalgebra of K generated by $a_1, a_2, \ldots, a_r$. That is, K_1 is the set of all elements obtained from finite application of $-$ and $\times$ on those elements. D_1 is the intersection of D and K_1. $-_1$ and $\times_1$ are just $-$ and $\times$ with their domains restricted to the elements of K_1. Finally, $*_1$ is defined as follows: $*_1 x = *x_1 \times *x_2 \times \cdots \times *x_n$, where $x_1, x_2, \ldots, x_n$ are the elements of K_1 that *cover* x. If x and y are elements of K_1, we say that y covers x if $x \Rightarrow y = 1$ and $*y \in K_1$. Finally, we simply let $b_i = a_i$ for all i.

With these definitions it is possible to show that the conditions of the lemma hold. Clearly all the b_i belong to K_1. Condition 1 holds by the way D_1 was defined, since all a_i are in K_1. Conditions 2-4 follow from the definitions of $-_1$, $\times_1$, and $*_1$.

$\mathfrak{M}_1$ is also a normal S4-matrix. To prove this, it suffices to show that it is a closure matrix and then apply the claim that every closure matrix is a normal S4-matrix, which can be proven by showing that every closure matrix satisfies the **S4** axioms, satisfies valid inferences from **S4** theorems, and obeys the normality conditions.

Finally, K_1 contains at most 2^{2^r} elements. Using properties of finite Boolean algebras, we know that since K_1 was generated from a finite set of a_i, every element can be written as a sum of products of the form

$$\prod_{i=1}^{r} \pm a_i$$

where the element 0 is taken to be the empty sum. There are 2^r products of the above form, and hence 2^{2^r} possible sums, giving a total of at most 2^{2^r} elements in K_1.

We now apply this lemma to prove Theorem 3.9. Suppose α is a non-theorem of **S4** with r subformulas $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then it fails to be satisfied by the S4-characteristic matrix $\mathfrak{M} = \langle K, \{1\}, -, \times, * \rangle$. That is, some assignment of K-elements to proposition letters produces a non-1 element. If $\alpha_1, \ldots, \alpha_n$ are the proposition letters occurring in α, let $a_1, \ldots, a_n$ be one such assignment of K-elements to $\alpha_1, \ldots, \alpha_n$. Now for the remaining subformulas $\alpha_{n+1}, \ldots, \alpha_r$, let $a_{n+1}, \ldots, a_r$ be the corresponding elements of K that result from the evaluation of each subformula under the given assignment. Without loss of generality, we let $\alpha_r = \alpha$, and so $a_r \neq 1$.

Using Lemma 3.10, there exists a normal S4-matrix $\mathfrak{M}_1 = \langle K_1, \{1\}, -_1, \times_1, *_1 \rangle$ with no more than 2^{2^r} elements, and a sequence $b_1, \ldots, b_r$ with all $b_i \in K_1$, satisfying the four specified conditions. Now suppose we assign $b_1, \ldots, b_r$ to the proposition letters of α. From conditions 2–4 of the lemma, it is seen from induction that each subformula α_i evaluates to b_i. In particular, α_r (which is just α) evaluates to b_r. By condition 1, since $a_r \neq 1$, $b_r \neq 1$, and hence $\mathfrak{M}_1$ fails to satisfy α. This proves the algebraic finite model property for **S4**.

The next corollary, which is a stronger version of Theorem 3.8, follows immediately:

Corollary 3.11. *If α is a formula that is not a theorem of* **S4***, then there exists a finite closure matrix that fails to satisfy α.*

Proof. A formula α can only have a finite number of subformulas, and so by Theorem 3.9, some finite normal S4-matrix $\mathfrak{M}_1$ fails to satisfy α. By Theorem 3.7, $\mathfrak{M}_1$ is a generalized closure matrix. But D_1 must contain the 1 element, so if we shrink D_1 to contain only the 1 element, the same assignment of K_1-elements to proposition letters of α will evaluate to an element not equal to 1, and hence this matrix (which is a closure matrix) fails to satisfy α. (Alternatively we can simply observe that $D_1 = D \cap K_1 = \{1\} \cap K_1 = 1$, since all Boolean subalgebras must contain 1). $\square$

From here there are two routes to obtain the representation of closure matrices as Kripke models, one using the algebraic finite model property, and the other based on the more general representation that can apply directly to the Lindenbaum-Tarski matrix, which is necessarily infinite (to see this, note that for instance, the equivalence classes of individual proposition letters are distinct, and there are infinitely many).

4. The Finite Case

From Algebra to Topology.

Definition 4.1. A *power set algebra* $\langle \mathcal{P}(I), ^{-}, \cap \rangle$ is a Boolean algebra whose elements are the subsets of some index set I, and $^{-}$ and $\cap$ are the operations of set complementation and set intersection, respectively. A simple set theoretic argument will show that the operation $\cup$ and the relation $\subseteq$ (set union and subset) are entirely analagous to the defined symbols $+$ and $\leq$.

A subalgebra of a power set algebra is called a *set algebra.*

It is easily seen that any set I induces a power set algebra, since the usual set-theoretic operations obey the postulates for a Boolean algebra. We can say that $\langle \mathcal{P}(I), ^{-}, \cap \rangle$ is *the* power set algebra of I.

Definition 4.2. A *topological matrix* $\langle \mathcal{P}(I), \{I\}, ^{-}, \cap, C \rangle$ is a closure matrix in which $\langle \mathcal{P}(I), ^{-}, \cap \rangle$ is the power set algebra of a set I, I is the sole distinguished element, and C is a unary operation on sets that obeys essentially the same postulates as * did in the previous section. The postulates for C are thus seen to correspond to the *Kuratowski closure axioms* for a topological space:

C1: $S \subseteq C(S)$
C2: $C(\emptyset) = \emptyset$
C3: $C(C(S)) = C(S)$
C4: $C(S_1 \cup S_2) = C(S_1) \cup C(S_2)$

Theorem 4.3. (Monotonicity) *If $S \subseteq S'$, then $C(S) \subseteq C(S')$.*

Proof. Suppose $S \subseteq S'$. Then $S' = S \cup (S' \backslash S)$. By axiom C5,
$C(S') = C(S) \cup C(S' \backslash S)$, and therefore $C(S) \subseteq C(S')$. □

Our next major objective is to show that finite closure matrices and finite topological matrices are in fact identical up to isomorphism.

Definition 4.4. If a and b are elements of a Boolean algebra, then a is a *proper part* of b if $a \leq b$, $a \neq 0$, and $a \neq b$.

An *atom* is a nonzero element that has no proper parts.

A Boolean algebra is *atomic* if for every nonzero element b, there is an atom a such that $a \leq b$.

A Boolean algebra is *complete* if every set of elements has a supremum (i.e. a sum).

Lemma 4.5. *Every finite Boolean algebra is complete and atomic.*

Proof. Completeness is guaranteed by the simple observation that every set of elements under consideration is a finite set, and therefore the sum is just a finite number of applications of binary sums, which are guaranteed to exist in any Boolean algebra.

For atomicity, consider a nonzero element b of a finite Boolean algebra $\mathfrak{B}$. If b is an atom, then certainly it has an atom below[9] it, since $b \leq b$. If b is not an atom, then it has a proper

[9]An element a is said to be "below" b if $a \leq b$. Note that any element is below itself.

part a. If a is an atom we are done, and if not then it must have a proper part. By continuing this process we see that b must have a proper part that is an atom, or else there is an infinite strictly descending chain of nonzero elements, contradicting that $\mathfrak{B}$ is finite. □

Lemma 4.6. *Let b be an element of a complete, atomic Boolean algebra $\mathfrak{B}$. Then*

$$b = \sum_i a_i$$

where $\{a_i\}$ is the set of atoms such that $a_i \leq b$. In other words, every element is the sum of atoms below it.

Proof. We take the empty sum to be the 0 element, so the result trivially holds for $b = 0$. Now suppose b is a nonzero element.

We note that $\leq$ is antisymmetric, for $c \leq d$ and $d \leq c$ both hold if and only if $c = c \times d = d$. So it suffices to show that the relation holds in both directions.

For the right-to-left direction, suppose $a_1 \leq b$ and $a_2 \leq b$. Then $a_1 = a_1 \times b$ and $a_2 = a_2 \times b$, so $a_1 + a_2 = (a_1 \times b) + (a_2 \times b) = (a_1 + a_2) \times b$. Therefore $a_1 + a_2 \leq b$. Extending this to arbitrary sum (via induction and completeness), we obtain $\sum a_i \leq b$.

For the other direction, note that $b \leq \sum a_i$ is equivalent to $b \times -(\sum a_i) = 0$, which is equivalent to $b \times \prod(-a_i) = 0$. Assume the contrary. Then since $\mathfrak{B}$ is atomic, there is an atom a such that $a \leq b \times \prod(-a_i) \leq b$. This means a must belong to the set of a_i. But then $a \leq \prod(-a_i) \leq -a$, which can only hold if $a = 0$, contradicting that a is an atom. Thus $b \leq \sum a_i$. □

Theorem 4.7. *Every complete, atomic Boolean algebra is isomorphic to the power set algebra of its set of atoms.*

Proof. Let $\mathfrak{B} = \langle K, -, \times \rangle$ be a complete, atomic Boolean algebra, with $At(K)$ the set of atoms in K. Let $\mathfrak{B}' = \langle \mathcal{P}(At(K)), ^-, \cap \rangle$ be the power set algebra of $At(K)$. Then the map $\phi : \mathfrak{B} \to \mathfrak{B}'$, $\phi(b) = \{a \in At(K) \,|\, a \leq b\}$ is an isomorphism.

Suppose $\phi(b_1) = \phi(b_2)$ for $b_1, b_2 \in K$. Then b_1 and b_2 have identical sets of atoms below them. $\mathfrak{B}$ is complete and atomic, so by Lemma 4.6, b_1 and b_2 are equal to the sum of the same set of atoms, and hence are equal to each other. ϕ is thus injective.

ϕ is also surjective. For suppose A is some set of atoms of K. Let b be the sum of elements in A. Then $\phi(b) = A$. For right-to-left inclusion, note that if $a \in A$, then $a \leq b$ and so $a \in \phi(b)$. For the other direction, suppose $a \in \phi(b)$, that is, a is an atom and $a \leq b$. Then $a = a \times b = a \times \sum\{a_i \in A\} = \sum\{a \times a_i \,|\, a_i \in A\}$ by distributivity. Since a is nonzero, there must be an a_i for which $a \times a_i \neq 0$. But in fact $a = a_i$, for otherwise their product would be a proper part of both of them, which contradicts that they are atoms. Hence $a \in A$.

The inverse map of ϕ is therefore seen to be the following: if A is a set of atoms, $\phi^{-1}(A) = \sum\{a_i \in A\}$.

It remains only to show ϕ is homomorphic with respect to the two operations of the Boolean algebra:

$$\phi(-b) = \{a \in At(K) \,|\, a \leq -b\} = \{a \in At(K) \,|\, a \not\leq b\} = \overline{\phi(b)}$$

The second equality holds by the following argument: Let a be an atom of K. For left-to-right inclusion, suppose both $a \leq -b$ and $a \leq b$. Then $a = a \times -b = a \times b$, which implies $a = 0$, a contradiction. Conversely, suppose $a \nleq b$, i.e. $a \neq a \times b$ We know that $a \leq 1 = b + -b$, so $a = a \times (b + -b) = (a \times b) + (a \times -b)$. Since a is an atom, either $a = a \times b$ or $a = a \times -b$, for otherwise a would have a proper part. Because $a \neq a \times b$, we deduce $a = a \times -b$, and so $a \leq -b$.

$$\begin{aligned}\phi(b_1 \times b_2) &= \{a \in At(K) \,|\, a \leq b_1 \times b_2\} = \{a \in At(K) \,|\, a \leq b_1 \text{ and } a \leq b_2\} \\ &= \{a \in At(K) \,|\, a \leq b_1\} \cap \{a \in At(K) \,|\, a \leq b_2\} = \phi(b_1) \cap \phi(b_2)\end{aligned}$$

Again the second equality holds as follows: Left-to-right inclusion is obvious from $a \leq b_1 \times b_2 \leq b_1$ and $a \leq b_1 \times b_2 \leq b_2$. For the other direction, suppose $a \leq b_1$ and $a \leq b_2$. Then $a = a \times b_1 = a \times b_2$. Multiplying on the right by b_2, we get $a \times b_1 \times b_2 = a \times b_2 \times b_2 = a \times b_2 = a$, whence $a \leq b_1 \times b_2$. □

It takes only a little more work to show the following:

Theorem 4.8. *Every finite closure matrix is isomorphic to a topological matrix.*

Proof. Let $\mathfrak{C} = \langle \mathfrak{B}, \{1\}, ^* \rangle$ be a finite closure matrix (we can write it as a "Boolean algebra part" with the two other entries added). Since $\mathfrak{B} = \langle K, -, \times \rangle$ is a finite Boolean algebra, we know by Lemma 4.5 that it is complete and atomic. The previous theorem provides an isomorphism ϕ between $\mathfrak{B}$ and the power set algebra of its set of atoms. Note that since K is a finite set of which $At(K)$ is a subset, $\mathcal{P}(At(K))$ must also be a finite set. To complete the proof, it only remains to specify the distinguished elements and the closure operation for the topological matrix, and to show the map ϕ preserves these. Let $\mathfrak{C}' = \langle \mathfrak{B}', \{At(K)\}, C \rangle$ be the topological matrix for which $\mathfrak{B}'$ is the power set algebra of $At(K)$ and C is defined as follows: if A is a set of atoms, $C(A) = \phi(^*\phi^{-1}(A))$, that is, $C(A)$ is the set of atoms below $^*b$, where b is the sum of atoms in A. Clearly $\mathcal{P}(At(K))$ is closed under C, and we now show that C satisfies the Kuratowski axioms:

C1: Suppose $a \in A$. Then $a \leq \phi^{-1}(A) \leq {}^*\phi^{-1}(A)$, and hence $a \in C(A)$.
C2: $C(\emptyset) = \phi(^*\phi^{-1}(\emptyset)) = \phi(^*0) = \phi(0) = \emptyset$
C3: $C(C(A)) = \phi(^*\phi^{-1}(\phi(^*\phi^{-1}(A))) = \phi(^{**}\phi^{-1}(A)) = \phi(^*\phi^{-1}(A)) = C(A)$
C4: $C(A_1 \cup A_2) = \phi(^*\phi^{-1}(A_1 \cup A_2)) = \phi(^*(\sum\{a_i \in A_1 \cup A_2\})) =$
$\phi(^*(\phi^{-1}(A_1) + \phi^{-1}(A_2))) = \phi(^*\phi^{-1}(A_1) + {}^*\phi^{-1}(A_2)) =$
$\phi(^*\phi^{-1}(A_1)) \cup \phi(^*\phi^{-1}(A_2)) = C(A_1) \cup C(A_2)$

Moreover, ϕ is homomorphic with respect to *, for we have

$$C(\phi(b)) = \phi(^*\phi^{-1}(\phi(b))) = \phi(^*b)$$

Finally, ϕ carries $\{1\}$ to $\{At(K)\}$ because all atoms of K are below the 1 element.
Hence ϕ is an isomorphism between $\mathfrak{C}$ and $\mathfrak{C}'$. □

This result provides the next step toward completeness:

Corollary 4.9. *If α is a formula that is not a theorem of* **S4**, *then there exists a finite topological matrix that fails to satisfy α.*

Proof. By Corollary 3.11, there is a finite closure matrix that fails to satisfy α, and by the previous theorem this closure matrix is isomorphic to a topological matrix. Hence there is a finite topological matrix that fails to satisfy α. □

Rings of subsets. We now proceed into what seems a digression from the matter at hand, but it will in fact be used to prove a crucial result.

Definition 4.10. A *completely distributive topology* on a set I is a unary operation $C : \mathcal{P}(I) \to \mathcal{P}(I)$ which satisfies the first three Kuratowski closure axioms and also the following axiom, which is a strengthening of C4 to arbitrary unions:

> **C5:** if $S = \cup S_i$, then $C(S) = \cup C(S_i)$

Note that in the case where I is a finite set, all unions are finite, so C5 is equivalent to C4.

Definition 4.11. A family of subsets $\mathfrak{R}$ of a set I forms a *ring of subsets* of I if for any two sets S and T in $\mathfrak{R}$, $S \cup T$ and $S \cap T$ are also contained in $\mathfrak{R}$.

A ring of subsets is *full* if it contains both the universal set I and the null set $\emptyset$.

A ring of subsets is *complete* if it contains, for every subfamily of sets S_i, the sum $\cup S_i$ and the product $\cap S_i$.

It is easily seen that if I is a finite set, all of its rings of subsets are complete.

Birkhoff (1937) proves the following result:[10]

Theorem. *"The [full] complete rings of subsets of I can be identified with the different quasi-orderings of I or with the different completely distributive topologies on I."*

By "identified with" he means there is a bijective mapping between full complete rings of subsets and completely distributive topologies, and likewise with quasi-orderings. The composition of these mappings will then establish the crucial bijection between closure operations and quasi-orderings, with which topological matrices can be represented as "relational" matrices.

Let the classes of completely distributive topologies on I, full complete rings of subsets of I, and quasi-orderings on I be denoted by $\mathscr{T}_I$, $\mathscr{R}_I$, and $\mathscr{Q}_I$ respectively. We first show the bijection between $\mathscr{R}_I$ and $\mathscr{T}_I$.

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[10]See Birkhoff [1], Theorem 1.

Definition 4.12. Let ϕ_1 be the function that maps a full complete ring of subsets $\mathfrak{R}$ of I to the unary operation $P : \mathcal{P}(I) \to \mathcal{P}(I)$ defined by:

$$P(S) = \bigcap_{T_i \in \mathfrak{R}, T_i \supseteq S} T_i$$

Lemma 4.13. *For any $S \in I$, $P(S) \in \mathfrak{R}$.*

Proof. $P(S)$ is a product of sets in $\mathfrak{R}$, so by completeness of $\mathfrak{R}$ it is itself in $\mathfrak{R}$. □

Lemma 4.14. *$S \in \mathfrak{R}$ iff $P(S) = S$.*

Proof. If $S \in \mathfrak{R}$, then the intersection of sets in $\mathfrak{R}$ containing S is just S itself, hence $P(S) = S$. Conversely, assume that $P(S) = S$. Then S is the product of sets in $\mathfrak{R}$ containing it. But since $\mathfrak{R}$ is complete, this product is itself in $\mathfrak{R}$, so $S \in \mathfrak{R}$. □

Lemma 4.15. *If $S \subseteq S'$, then $P(S) \subseteq P(S')$.*

Proof. Suppose $S \subseteq S'$.

$$P(S) = \bigcap_{T_i \in \mathfrak{R}, T_i \supseteq S} T_i, \qquad P(S') = \bigcap_{T'_j \in \mathfrak{R}, T'_j \supseteq S'} T'_j$$

Since $T'_j \supseteq S' \supseteq S$ for all j, every T'_j is a T_i, and so the former intersection is a subset of the latter. Hence $P(S) \subseteq P(S')$. □

We now check that the operation $P = \phi_1(\mathfrak{R})$ satisfies the four axioms for a completely distributive topology.

- **C1:** $P(S)$ is the intersection of all sets in $\mathfrak{R}$ containing S; each of these sets contains S, so the intersection contains S. Hence $S \subseteq P(S)$.
- **C2:** $P(\emptyset)$ is the intersection of all sets in $\mathfrak{R}$ containing $\emptyset$, and since the empty set belongs to any full ring of subsets, $P(\emptyset) = \emptyset$.
- **C3:** Immediately from Lemmas 4.13 and 4.14.
- **C5:** Suppose $S = \cup S_i$. Then for all i, $S \supseteq S_i$, and so $P(S) \supseteq P(S_i)$ by Lemma 4.15; therefore $P(S) \supseteq \cup P(S_i)$.
  For the opposite inclusion, consider $\cup P(S_i)$. By Lemma 4.13, $P(S_i) \in \mathfrak{R}$ for all i, whence $\cup P(S_i) \in \mathfrak{R}$ by completeness of $\mathfrak{R}$. Now, axiom C1 provides $S_i \subseteq P(S_i)$, and so $\cup S_i = S \subseteq \cup P(S_i)$. Again using Lemma 4.15, $P(S) \subseteq P(\cup P(S_i))$, but $P(\cup P(S_i)) = \cup P(S_i)$ by Lemma 4.14, and therefore $P(S) \subseteq \cup P(S_i)$.

ϕ_1 is thus a well-defined function from $\mathscr{R}_I$ to $\mathscr{T}_I$. We now show that it is a bijection.

For injectivity, suppose $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are full complete rings of subsets of I, and $\mathfrak{R}_1 \neq \mathfrak{R}_2$. Then without loss of generality, there is a set S that belongs to $\mathfrak{R}_1$ but not $\mathfrak{R}_2$. Let $P_1 = \phi_1(\mathfrak{R}_1)$, $P_2 = \phi_1(\mathfrak{R}_2)$. By Lemma 4.14, $P_1(S) = S$ but $P_2(S) \neq S$. Hence P_1 and P_2 are not the same operation, i.e. $\phi_1(\mathfrak{R}_1) \neq \phi_1(\mathfrak{R}_2)$.

To show ϕ_1 is surjective, we will define a function ψ_1 that turns out to be the inverse of ϕ_1.

Definition 4.16. If C is a completely distributive topology on I, let $\psi_1(C) = \mathfrak{R}_C$, where $\mathfrak{R}_C = \{S \subseteq I \,|\, C(S) = S\}$. That is, $\mathfrak{R}_C$ is the family of *closed sets*, where a set S is "closed" if $C(S) = S$.

We first check that $\mathfrak{R}_C$ is a full complete ring of subsets of I.

- Suppose $\mathfrak{R}_C$ contains some collection of S_i. By definition of $\mathfrak{R}_C$, $C(S_i) = S_i$ for each i. By axiom C5, $C(\cup S_i) = \cup C(S_i) = \cup S_i$, and so $\cup S_i \in \mathfrak{R}_C$. Now consider the product. $\cap S_i = \cap C(S_i)$. Since $\cap S_i \subseteq S_i$ for all i, it follows from monotonicity that $C(\cap S_i) \subseteq C(S_i) = S_i$ for all i, and therefore $C(\cap S_i) \subseteq \cap S_i$. By C1, we also have $C(\cap S_i) \supseteq \cap S_i$, so $C(\cap S_i) = \cap S_i$, hence $\cap S_i \in \mathfrak{R}_C$. Thus $\mathfrak{R}_C$ meets the completeness condition, and is *a fortiori* a ring of subsets of I.
- $I \subseteq C(I)$ by C1, but since I is the largest set in $\mathcal{P}(I)$, $C(I) = I$, which means $I \in \mathfrak{R}_C$. Furthermore, $C(\emptyset) = \emptyset$ by C2, so $\emptyset \in \mathfrak{R}_C$. Therefore $\mathfrak{R}_C$ is full.

Hence ψ_1 is a well-defined function from $\mathscr{T}_I$ to $\mathscr{R}_I$. It remains only to show the following:

Claim. $\phi_1(\psi_1(C)) = C$ for all $C \in \mathscr{T}_I$.

Proof. Let $\mathfrak{R}_C$ and P_C abbreviate $\psi_1(C)$ and $\phi_1(\psi_1(C))$ respectively. We need to show that for all $S \subseteq I$, $P_C(S) = C(S)$. Fix such an S.

$$\begin{aligned} P_C(S) &= \bigcap_{T_i \in \mathfrak{R}_C, T_i \supseteq S} T_i \\ &= \bigcap_{T_i \supseteq S, C(T_i) = T_i} T_i \\ &= \bigcap_{T_i \supseteq S, C(T_i) = T_i} C(T_i) \end{aligned}$$

From axiom C3 and monotonicity, the condition $C(T_i) \supseteq S$ implies that $C(T_i) \supseteq C(S)$. Thus, because this condition holds for all $C(T_i)$, it holds for their product, and so $P_C(S) \supseteq C(S)$.

Now for the other direction. Again from axiom C3 we have that $C(S) \in \mathfrak{R}_C$, and by C1, $C(S) \supseteq S$. Therefore $C(S)$ belongs to the set of $T_i \supseteq S$ that belong to $\mathfrak{R}_C$. Thus,

$$C(S) \supseteq \bigcap_{T_i \in \mathfrak{R}_C, T_i \supseteq S} T_i = P_C(S)$$

□

Putting these results together, the following theorem has been proven:

Theorem 4.17. *ϕ_1 is a bijective map from $\mathscr{R}_I$ to $\mathscr{T}_I$ with inverse $\phi_1^{-1} = \psi_1$.*

We now show the bijection between $\mathscr{R}_I$ and $\mathscr{Q}_I$.

Definition 4.18. Let ϕ_2 be the function that maps a full complete ring of subsets $\mathfrak{R}$ of I to the relation on I given by $R = \{(x, y) \,|\, \forall S \in \mathfrak{R}, \text{ if } y \in S,\ x \in S\}$.

We check that $R = \phi_2(\mathfrak{R})$ is a quasi-ordering on I.

- *Reflexive.* Trivially, if $x \in S$, then $x \in S$. Hence xRx for all $x \in I$.
- *Transitive.* Suppose xRy and yRz. Then for all $S \in \mathfrak{R}$, $y \in S$ implies $x \in S$, and $z \in S$ implies $y \in S$. So $z \in S$ implies $x \in S$. Therefore xRz.

Hence ϕ_2 is a well-defined function from $\mathscr{R}_I$ to $\mathscr{Q}_I$.

Definition 4.19. Given a quasi-ordering Q on I, for every element $y \in I$, we let $f_Q(y) = \{x \in I \,|\, xQy\}$.

Lemma 4.20. *If* $R = \phi_2(\mathfrak{R})$,

$$f_R(y) = \bigcap_{T_i \in \mathfrak{R}, y \in T_i} T_i$$

Proof. Follows immediately from Definitions 4.18 and 4.19. □

Theorem 4.21. $S \in \mathfrak{R}$ *iff* $(y \in S \wedge xRy) \rightarrow x \in S$.

Proof. The left-to-right direction follows immediately from Definition 4.18. Now suppose the latter condition holds. Given an S, for every $y \in S$, S also contains all the x such that xRy. That is, $S \supseteq f_R(y)$. Furthermore,

$$S = \bigcup_{y \in S} f_R(y)$$

For suppose $z \in S$. By reflexivity of R, zRz, and hence $z \in f_R(z) \subseteq \bigcup_{y \in S} f_R(y)$. Conversely, suppose $z \in \bigcup_{y \in S} f_R(y)$. Then there exists a $y \in S$ for which $z \in f_R(y)$. But $f_R(y) \subseteq S$, so $z \in S$.

Using Lemma 4.20 and the fact that $\mathfrak{R}$ is complete, $f_R(y) \in \mathfrak{R}$ for all $y \in S$, and therefore (again by completeness of $\mathfrak{R}$), $S \in \mathfrak{R}$. □

We are now ready to show that ϕ_2 is injective. Suppose $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are full complete rings of subsets of I, and $\mathfrak{R}_1 \neq \mathfrak{R}_2$. Then without loss of generality, there is a set S that belongs to $\mathfrak{R}_1$ but not $\mathfrak{R}_2$. Let $R_1 = \phi_2(\mathfrak{R}_1)$, $R_2 = \phi_2(\mathfrak{R}_2)$. By Theorem 4.21, since $S \in \mathfrak{R}_1$, $\forall x, y$, $(y \in S \wedge xR_1y) \rightarrow x \in S$. And since $S \notin \mathfrak{R}_2$, $\exists x, y, (y \in S \wedge xR_2y \wedge x \notin S)$. But if $R_1 = R_2$, these statements would create a contradiction. Therefore $R_1 \neq R_2$, i.e. ϕ_2 is injective.

For surjectivity, we define a function ψ_2 that turns out to be the inverse of ϕ_2.

Definition 4.22. If Q is a quasi-ordering on I, let

$$\psi_2(Q) = \{S \subseteq I \,|\, (y \in S \wedge xQy) \rightarrow x \in S\}$$

Denoting $\psi_2(Q)$ by $\mathfrak{R}_Q$, we check that $\mathfrak{R}_Q$ is a full complete ring of subsets of I.

- Suppose $\mathfrak{R}_Q$ contains some collection of S_i. Then for each i, if $y \in S_i$ and xQy, then $x \in S_i$. Now suppose xQy and $y \in \cup S_i$. Then for some i, $y \in S_i$, whence $x \in S_i \subseteq \cup S_i$. Now consider the product. Suppose xQy and $y \in \cap S_i$. Then for all i, $y \in S_i$, whence $x \in S_i$. This implies $x \in \cap S_i$. With these results, it follows from Definition 4.22 that $\cup S_i \in \mathfrak{R}_Q$ and $\cap S_i \in \mathfrak{R}_Q$. Hence $\mathfrak{R}_Q$ is a complete ring of subsets of I.

- Since Q is a relation on I, all elements x and y such that xQy must be members of I, so $I \in \mathfrak{R}_Q$. As for the empty set, the condition for membership in $\mathfrak{R}_Q$ is vacuously met since there is no element $y \in \emptyset$ to satisfy its antecedent. Thus $\emptyset \in \mathfrak{R}_Q$, so $\mathfrak{R}_Q$ is full.

ψ_2 is therefore a well-defined function from $\mathscr{Q}_I$ to $\mathscr{R}_I$.

Claim. $\phi_2(\psi_2(Q)) = Q$ for all $Q \in \mathscr{Q}_I$.

Proof. Let $\mathfrak{R}_Q$ and R_Q abbreviate $\psi_2(Q)$ and $\phi_2(\psi_2(Q))$ respectively. We need to show that $R_Q = Q$.

Suppose xQy. Take any $S \in \mathfrak{R}_Q$, that is, an S for which $y \in S \wedge xQy \to x \in S$. Then if y is in S, x is in S. So by Definition 4.18, xR_Qy.

Conversely, suppose xR_Qy. For a given y, consider $f_Q(y)$. Suppose xQz and $z \in f_Q(y)$. Then zQy, so by transitivity of Q, xQy. Therefore $x \in f_Q(y)$, and so $f_Q(y) \in \mathfrak{R}_Q$ by Definition 4.22. Moreover, for every y, $y \in f_Q(y)$ by reflexivity. Now, since we assumed xR_Qy, this means that for all S in $\mathfrak{R}_Q$, $y \in S$ implies $x \in S$. Since $f_Q(y) \in \mathfrak{R}_Q$ and $y \in f_Q(y)$, consequently $x \in f_Q(y)$, i.e. xQy. □

We have proven the following theorem:

Theorem 4.23. *ϕ_2 is a bijective map from $\mathscr{R}_I$ to $\mathscr{Q}_I$ with inverse $\phi_2^{-1} = \psi_2$.*

It immediately follows from Theorems 4.17 and 4.23 that the composition $\phi_2 \circ \psi_1$ furnishes a bijection between completely distributive topologies and quasi-orderings on I. The next objective is to directly define this composition map and its inverse without invoking rings of subsets as an intermediary.

Let $\omega = \phi_2 \circ \psi_1$ be the bijective map from $\mathscr{T}_I$ to $\mathscr{Q}_I$ with inverse $\omega^{-1} = \psi_1^{-1} \circ \phi_2^{-1} = \phi_1 \circ \psi_2$.

Theorem 4.24. *If C is a completely distributive topology on I, then*

$$\omega(C) = \{(x, y) \mid \forall S \subseteq I, y \in S \to x \in C(S)\}$$

Proof. Let

$$\begin{aligned} R_1 &= \omega(C) = \phi_2(\psi_1(C)) = \phi_2(\{S \subseteq I \mid C(S) = S\}) \\ &= \{(x, y) \mid \forall S \subseteq I, C(S) = S, y \in S \to x \in S\} \end{aligned}$$

$$R_2 = \{(x, y) \mid \forall S \subseteq I, y \in S \to x \in C(S)\}$$

Suppose xR_1y. Now for any $S \subseteq I$, suppose $y \in S$. Let $T = C(S)$. By axiom C1, $S \subseteq C(S)$, so $y \in C(S)$. By axiom C3, $C(T) = T$, and because xR_1y, $x \in T = C(S)$. Therefore xR_2y.

Now suppose xR_2y. Then for all $S \subseteq I$, $y \in S$ implies $x \in C(S)$. In particular, for all those S for which $C(S) = S$, $y \in S$ implies $x \in C(S) = S$, and hence xR_1y.

So $R_1 = R_2$, concluding the proof. □

Theorem 4.25. *If Q is a quasi-ordering on I, then for any $S \subseteq I$,*

$$\omega^{-1}(Q)(S) = \{x \mid \exists y \in S,\, xQy\}$$

Proof. Let

$$\begin{aligned} P_1(S) &= \omega^{-1}(Q)(S) = \phi_1(\psi_2(Q))(S) \\ &= \phi_1(\{T \subseteq I \mid (y \in T \wedge xQy) \to x \in T\})(S) \\ &= \bigcap T_i \mid T_i \supseteq S, (y \in T_i \wedge xQy) \to x \in T_i \end{aligned}$$

$$P_2(S) = \{x \mid \exists y \in S,\, xQy\}$$

To show $P_1(S) \supseteq P_2(S)$, suppose $x' \in P_2(S)$. Then there is a y' in S for which $x'Qy'$. Consider each $T_i \supseteq S$ for which the property $(y \in T_i \wedge xQy) \to x \in T_i$ holds. y' is in S, so it is in T_i, and since $x'Qy'$ holds, $x' \in T_i$ must hold. This is true for all such T_i, so it is true for their intersection, and hence $x' \in P_1(S)$.

For the opposite inclusion, it suffices to show that $S \subseteq P_2(S)$ and that $P_2(S)$ has the property $(y \in P_2(S) \wedge xQy) \to x \in P_2(S)$, for then obviously $P_1(S) \subseteq P_2(S)$.

Suppose $x \in S$. Then $x \in P_2(S)$ since xQx holds by reflexivity of Q. Hence $S \subseteq P_2(S)$.

Suppose $y \in P_2(S)$ and xQy. The first condition implies that there is a $z \in S$ for which yQz holds. By transitivity of Q, then, xQz, and so $x \in P_2(S)$.

Thus $P_1(S) = P_2(S)$, and the theorem is proven. □

Relational Matrices.

Definition 4.26. A *relational matrix* $\langle \mathcal{P}(I), \{I\}, ^{-}, \cap, C\rangle$ is a topological matrix for which there exists a quasi-ordering R on I such that for all $S \subseteq I$,

$$C(S) = \{x \in I \mid \exists y \in S,\, xRy\}$$

The bijection ω from the previous section provides a fundamental connection between finite topological matrices and relational matrices.

Theorem 4.27. *Every finite topological matrix is a relational matrix.*

Proof. Given a finite topological matrix $\mathfrak{T} = \langle \mathcal{P}(I), \{I\}, ^{-}, \cap, C\rangle$, it suffices to show the existence of a quasi-ordering R satisfying the condition in Definition 4.26.

We first observe that C is a completely distributive topology on I. By Definition 4.2, C satisfies the Kuratowski axioms C1-C4, but as remarked earlier, in the case of finite I, all unions taken over $\mathcal{P}(I)$ must be finite and can therefore be obtained from a finite number of binary unions. Axiom C5 therefore reduces to C4, so C is a completely distributive topology.

From the previous section, there is a one-to-one correspondence between completely distributive topologies on I and quasi-orderings on I, and from Theorem 4.24 the quasi-ordering corresponding to C is given by:

$$R := \omega(C) = \{(x, y) \mid \forall S \subseteq I, y \in S \to x \in C(S)\}$$

Then, since $C = \omega^{-1}(\omega(C)) = \omega^{-1}(R)$, it follows immediately from Theorem 4.25 that for all $S \subseteq I$,

$$C(S) = \{x \mid \exists y \in S,\, xRy\}$$

and hence $\mathfrak{T}$ is a relational matrix. □

Corollary 4.28. *If α is a formula that is not a theorem of* **S4***, then there exists a (finite) relational matrix that fails to satisfy α.*

Proof. Follows immediately from the previous theorem and Corollary 4.9. □

From Relational Matrices to Kripke Models. The final task is at hand. From the previous corollary, every non-theorem of **S4** has a relational matrix that fails to satisfy it. To prove the desired completeness result, it suffices by Theorem 1.8 to find for each non-theorem α a Kripke model $\mathcal{M}_\alpha$ with a quasi-ordered frame such that α is not valid in $\mathcal{M}_\alpha$.

Suppose a formula α is a non-theorem of **S4**. Then there exists a (finite) relational matrix $\mathfrak{T} = \langle \mathcal{P}(I), \{I\}, ^-, \cap, C\rangle$ and an assignment μ of subsets of I to the proposition letters of α such that when α is evaluated in $\mathfrak{T}$ under this assignment, it results in a proper subset of I. This evaluation can be made concrete by simply extending μ's domain to the set of all formulas built up from α's proposition letters, using the following inductive rules:

- $\mu(\neg\beta) = \overline{\mu(\beta)}$
- $\mu(\beta \wedge \gamma) = \mu(\beta) \cap \mu(\gamma)$
- $\mu(\Diamond\beta) = C(\mu(\beta))$

As indicated above, then, $\mu(\alpha) \subsetneq I$.

We now construct a Kripke model $\mathcal{M}_\alpha = \langle W, R, \nu\rangle$ as follows:

- $W = I$. Since $\mathfrak{T}$ is a matrix, $\mathcal{P}(I)$ contains at least 2 elements, so I cannot be empty. Therefore W is non-empty.
- $R = \omega(C)$. R is a quasi-ordering since it is in the image of ω.
- For any proposition letter p occurring in α and $u \in W$, $\nu(p, u) = 1$ iff $u \in \mu(p)$. For all other proposition letters q not occurring in α, $\nu(q, u) = 0$ for all $u \in W$.

Claim. α is not valid in $\mathcal{M}_\alpha$.

Since $\mu(\alpha) \subsetneq W$, there exists a $u' \in W$ for which $u' \notin \mu(\alpha)$. It suffices to show:

Lemma 4.29. $\nu(\alpha, u) = 1$ *iff* $u \in \mu(\alpha)$.

From this it will immediately follow that $\nu(\alpha, u') = 0$ and thus α is not valid. To prove the lemma, we use induction on the length of α.

- If α is a proposition letter, the result follows immediately from the definition of ν.
- $\nu(\neg\beta, u) = 1$ iff $\nu(\beta, u) = 0$ iff $u \notin \mu(\beta)$ iff $u \in \overline{\mu(\beta)} = \mu(\neg\beta)$
- $\nu(\beta \wedge \gamma, u) = 1$ iff $\nu(\beta, u) = \nu(\gamma, u) = 1$ iff $u \in \mu(\beta)$ and $u \in \mu(\gamma)$ iff $u \in \mu(\beta) \cap \mu(\gamma) = \mu(\beta \wedge \gamma)$
- $\nu(\Diamond\beta, u) = 1$ iff $\exists v, uRv, \nu(\beta, v) = 1$ iff $\exists v, uRv, v \in \mu(\beta)$ iff $u \in C(\mu(\beta)) = \mu(\Diamond\beta)$

(The $\alpha = \Box\beta$ case follows easily since $\Box$ can be defined in terms of $\Diamond$.)

Therefore α is not valid in $\mathcal{M}_\alpha$, and we have proven the completeness of **S4** with respect to the class of quasi-ordered frames. Not only this, but if we use the algebraic finite model property, it is assured that $\mathcal{M}_\alpha$ is always finite, and thus we have proven the same finite model property as in Section 2.

5. The Jónsson & Tarski Theorem

In the previous section we used the algebraic finite model property to represent finite closure matrices as relational matrices. We now show that a similar representation is possible even for infinite closure matrices.

Theorem 5.1. (Jónsson & Tarski Theorem) *Every closure algebra is isomorphic to a set algebra of some set W with an additional operation C_R arising from a quasi-ordering R on W by the stipulation $C_R(S) = \{u \mid \exists v \in S, uRv\}$.*

Definition 5.2. Let $\mathfrak{B} = \langle K, -, \times \rangle$ be a Boolean algebra. A *filter* of $\mathfrak{B}$ is a subset $F \subseteq K$ such that:

(1) $1 \in F$
(2) If $a, b \in F$, then $a \times b \in F$.
(3) If $a \in F$ and $a \leq b$, then $b \in F$.

F is a *proper filter* of $\mathfrak{B}$ if it is a filter and a proper subset of K. An *ultrafilter* is a proper filter that additionally satisfies:

(4) For every $a \in K$, either $a \in F$ or $-a \in F$.

This last condition actually implies an exclusive or. For suppose both a and $-a$ belong to a filter F. Then by (2), $a \times -a = 0 \in F$. 0 is below every element of K, so by (3), $F = K$. Thus F is not a proper filter and hence not an ultrafilter.

Definition 5.3. If a is an element of the Boolean algebra $\mathfrak{B}$, the *up-set* of a is defined as $\uparrow a = \{b \in K \mid a \leq b\}$, that is, the set of elements of K that a is below.

Claim. $\uparrow a$ is a filter of $\mathfrak{B}$.

Proof. $a \leq 1$ fulfills the first condition. For the second, suppose $b, c \in \uparrow a$. Then $a \leq b$ and $a \leq c$, which implies $a \leq b \times c$, so $b \times c \in \uparrow a$. The third property follows readily from the fact that $\leq$ is transitive. □

An up-set is in fact a special case of a filter that is *generated* by a set of elements.

Definition 5.4. Given a non-empty subset $A \subseteq K$, the filter generated by A is the set $F_A = \{b \in K \mid \exists a_1, \ldots, a_n \in A \text{ s.t. } a_1 \times \cdots \times a_n \leq b\}$.

We show this set is in fact a filter:

(1) Every element of A is below 1, so 1 belongs to F_A.
(2) Suppose $b_1, b_2 \in F_A$. Then there exist $a_1, \ldots, a_m$ and $a_1', \ldots, a_n'$ in A such that $\prod a_i \leq b_1$ and $\prod a_j' \leq b_2$. From Boolean algebra it is easily seen that $\prod a_i \times \prod a_j' \leq b_1 \times b_2$, and hence $b_1 \times b_2 \in F_A$.
(3) Suppose $b_1 \leq b_2$ and $b_1 \in F_A$. Then there exist $a_1, \ldots, a_m$ in A for which $\prod a_i \leq b_1 \leq b_2$, so $b_2 \in F_A$.

A subset $A \subseteq K$ is said to have the *finite meet property* if there exists no finite subset $\{a_1, \ldots, a_n\}$ of A such that $a_1 \times \cdots \times a_n = 0$.

Lemma 5.5. *If a set $A \subseteq K$ has the finite meet property, then F_A is a proper filter.*

Proof. Assume the contrary. Then $F_A = K$, and in particular $0 \in F_A$, which implies that there exists a finite set $\{a_1, \ldots, a_n\}$ of A such that $a_1 \times \cdots \times a_n \leq 0$, i.e. $a_1 \times \cdots \times a_n = 0$. This contradicts that A has the finite meet property. □

Lemma 5.6. (Ultrafilter Lemma) *Let F be a proper filter of $\mathfrak{B}$. Then F is contained in an ultrafilter of $\mathfrak{B}$.*

Proof. Consider the set P of all proper filters of $\mathfrak{B}$ containing F, with a partial order induced by $\subseteq$. Let Q be a chain in P, that is, a non-empty subset of P whose elements are pairwise ordered by $\subseteq$. We now show that $\cup Q \in P$:

(1) Every element of Q is a filter and therefore contains 1, hence $1 \in \cup Q$.

(2) Suppose $a, b \in \cup Q$. Then there exist filters $F_1, F_2 \in Q$ for which $a \in F_1, b \in F_2$. Since Q is a chain, we can say without loss of generality that $F_1 \subseteq F_2$. Therefore both a and b are in F_2, so $a \times b \in F_2 \subseteq \cup Q$.

(3) Suppose $a \leq b$ and $a \in \cup Q$. Then $a \in F_1$ for some filter F_1 in Q, and therefore b must also be in F_1, so $b \in \cup Q$.

$\cup Q$ is also proper. For suppose it contained 0. Then some $F_1 \in Q$ would contain 0, which contradicts that all elements of Q are proper filters.

Finally, $F \subseteq \cup Q$ since all filters $F_i \in Q \subseteq P$ contain F. Therefore $\cup Q \in P$.

Moreover, $\cup Q$ is an upper bound of Q since every element of Q is contained in it. By Zorn's Lemma, P contains a maximal element, which we will call u. We now show that u is an ultrafilter.

Assume the contrary. Then there is an element $a \in K$ such that neither a or $-a$ is in u. Consider the filters F_1 and F_2 generated by $u \cup \{a\}$ and $u \cup \{-a\}$, respectively. By maximality of u, neither of these filters belongs to P, but since they both contain F, they must be improper filters. So $0 \in F_1$ and $0 \in F_2$. By Lemma 5.5, then, these filters fail to have the finite meet property. Hence there are elements $b_1, \ldots, b_m \in u$, $b_1^{'}, \ldots, b_n^{'} \in u$ such that $b_1 \times \cdots \times b_m \times a = 0$ and $b_1^{'} \times \cdots \times b_n^{'} \times -a = 0$. By distributivity, $(b_1 \times \cdots \times b_m \times a) + (b_1 \times \cdots \times b_m \times -a) = b_1 \times \cdots \times b_m = 0 + (b_1 \times \cdots \times b_m \times -a) = b_1 \times \cdots \times b_m \times -a$. Hence $b_1 \times \cdots \times b_m \leq -a$. Similarly $b_1^{'} \times \cdots \times b_n^{'} \leq a$, and it follows that $b_1 \times \cdots \times b_m \times b_1^{'} \times \cdots \times b_n^{'} = 0$. Therefore u contains 0, contradicting that it is a proper filter. □

Theorem 5.7. *Let $\mathfrak{B} = \langle K, -, \times \rangle$ be a Boolean algebra, a an element of K, and F a proper filter of $\mathfrak{B}$ that does not contain a. Then there exists an ultrafilter containing F that does not contain a.*

Proof. Consider the set $G = F \cup \{-a\}$. This set has the finite meet property. For if it didn't, there would be a finite or empty set of elements $b_1, \ldots, b_n \in F$ such that $b_1 \times \cdots \times b_n \times -a = 0$.

If the set is empty, then $-a = 0$, which implies $a = 1$. But all filters contain 1, and therefore $a \in F$, a contradiction. So the set of b_i must be a finite, non-empty set. From distributivity it is seen that $b_1 \times \cdots \times b_n \leq a$. But since $b_1 \times \cdots \times b_n \in F$, this implies once again that $a \in F$, a contradiction. Hence G has the finite meet property, and by Lemma 5.5, F_G is a proper filter, which by the Ultrafilter Lemma is contained in an ultrafilter u. u contains F because $F \subseteq G \subseteq F_G \subseteq u$. Furthermore, u does not contain a because $-a \in G \subseteq u$ implies $a \notin u$. □

Definition 5.8. Given a closure algebra $\mathfrak{C} = \langle K, -, \times, {}^* \rangle$, we construct the *ultrafilter frame* of $\mathfrak{C}$, $\mathcal{F}_\mathfrak{C} = \langle Uf\mathfrak{C}, R \rangle$, where $Uf\mathfrak{C}$ denotes the set of all ultrafilters of $\mathfrak{C}$, and R is a relation on this set of ultrafilters given by the condition:

uRv iff $\forall x \in K$, if $x \in v$, then ${}^*x \in u$.

Theorem 5.9. *uRv iff $\forall x \in K$, if $-{}^*{-x} \in u$, then $x \in v$.*

Proof. From the definition of R, it suffices to show that $x \in v \rightarrow {}^*x \in u$ is equivalent to $-{}^*{-x} \in u \rightarrow x \in v$. Suppose the former condition holds. Now, say $-{}^*{-x} \in u$. Since u is an ultrafilter, ${}^*{-x} \notin u$, so by hypothesis $-x \notin v$, and likewise since v is an ultrafilter, $x \in v$. The opposite direction is proven analagously. □

Theorem 5.10. *R is a quasi-ordering on $Uf\mathfrak{C}$.*

Proof. For reflexivity, let x be some element of an ultrafilter u. Since $x \leq {}^*x$, it follows that ${}^*x \in u$. Hence uRu.

For transitivity, suppose uRv and vRw. If $x \in w$, then ${}^*x \in v$ and therefore ${}^{**}x \in u$. But in a closure algebra, ${}^{**}x = {}^*x$, so the condition for uRw is met. □

Definition 5.11. Let $\mathcal{F} = \langle W, R \rangle$ be a frame. Then the structure $\mathcal{F}^+ = \langle \mathcal{P}(W), {}^-, \cap, C_R \rangle$ is the *complex algebra* of $\mathcal{F}$, where $\langle \mathcal{P}(W), {}^-, \cap \rangle$ is the power set algebra of W, and C_R is the unary operation on $\mathcal{P}(W)$ given by $C_R(S) = \{u \mid \exists v \in S, uRv\}$.

If $\mathcal{F}_\mathfrak{C}$ is the ultrafilter frame of a closure algebra $\mathfrak{C}$, then $\mathcal{F}_\mathfrak{C}^+$ is called the *canonical embedding algebra* of $\mathfrak{C}$.

Letting $W = Uf\mathfrak{C}$, we see that since R is a quasi-ordering on W and $C_R(S) = \{u \mid \exists v \in S, uRv\}$, the Jónsson & Tarski Theorem can be proven directly by identifying an isomorphic map from $\mathfrak{C}$ to a subalgebra of $\mathcal{F}_\mathfrak{C}^+$. In other words, we must find an *embedding* (injective homomorphism) of $\mathfrak{C}$ into $\mathcal{F}_\mathfrak{C}^+$. This is precisely the *Stone embedding*:

$$\rho : K \rightarrow \mathcal{P}(Uf\mathfrak{C})$$

$$\rho(x) = \{u \in Uf\mathfrak{C} \mid x \in u\}$$

To show that ρ is injective, suppose a and b are distinct elements of K. Without loss of generality, we can say $b \nleq a$. The element a therefore does not belong to the up-set of b, which is a filter. By Theorem 5.7, there is an ultrafilter u containing $\uparrow b$ that also does not contain a. But then $u \in \rho(b)$ and $u \notin \rho(a)$, and hence $\rho(a) \neq \rho(b)$, proving injectivity.

We now show that ρ is homomorphic with respect to the operations of the closure algebra:

- $\rho(-a) = \overline{\rho(a)}$. All ultrafilters containing $-a$ do not contain a, and vice versa.
- $\rho(a \times b) = \rho(a) \cap \rho(b)$. An ultrafilter contains $a \times b$ iff it contains both a and b iff it belongs to $\rho(a) \cap \rho(b)$.
- $\rho(*a) = C_R(\rho(a))$. The right-to-left inclusion is easy to prove. Suppose $u \in C_R(\rho(a))$. Then there exists a $v \in \rho(a)$ such that uRv. Since $a \in v$ and uRv, it follows that $*a \in u$, and hence $u \in \rho(*a)$. The other direction is the tricky one. Suppose $u \in \rho(*a)$, i.e. $*a \in u$. We need to show the existence of an ultrafilter v such that $a \in v$ and uRv.

  Let $S = \{x \in K \,|\, -*-x \in u\}$. S is closed under $\times$. For suppose $-*-x_1 \in u$ and $-*-x_2 \in u$. Then $(-*-x_1) \times (-*-x_2) \in u$ since ultrafilters are closed under $\times$. But $(-*-x_1) \times (-*-x_2) = -((*-x_1) + (*-x_2)) = -*(-x_1 + -x_2) = -*-(x_1 \times x_2)$, and so $-*-(x_1 \times x_2) \in u$, giving the desired result.

  Now consider the set $T = \{a \times x \,|\, x \in S\}$. T has the finite meet property. For assume the contrary. Then there is a finite set $x_1, \ldots x_n \in S$ such that $(a \times x_1) \times \cdots \times (a \times x_n) = (a \times \cdots \times a) \times (x_1 \times \cdots \times x_n) = a \times (x_1 \times \cdots \times x_n) = 0$. Since S is closed under $\times$, a finite product of such x_i will also be in S, and so it suffices to show that for any $x \in S$, $a \times x = 0$ will lead to a contradiction. For if $a \times x = 0$, then $a \leq -x$, from which it follows that $*a \leq *-x$.[11] Recall that $*a \in u$. Therefore $*-x \in u$, which contradicts that $x \in S$. Hence T has the finite meet property.

  T is a subset of K with the finite meet property. By Lemmas 5.5 and 5.6, there is an ultrafilter v that contains T. Additionally, $S \subseteq v$. For suppose $x \in S$. Then $a \times x \in v$, and since $a \times x \leq x$, x is in v.

  Finally, we show that $a \in v$ and uRv. The first condition holds because $1 \in S$, and therefore $a \in T \subseteq v$. For the second condition, suppose that $-*-x \in u$. Then $x \in S \subseteq v$, which implies uRv by Theorem 5.9.

With this, the Jónsson & Tarski Theorem has been proven. The next corollary follows right away from the fact that the Stone embedding always maps the closure algebra's 1 element to the set of all ultrafilters, because all ultrafilters contain 1.

Corollary 5.12. *Every closure matrix is isomorphic to a relational matrix.*

As we have seen, the Jónsson & Tarski Theorem is a means of obtaining this result without resort to the algebraic finite model property. Combining it with Theorem 3.8, we have shown that every non-theorem of **S4** has a relational matrix that fails to satisfy it. Although we have not guaranteed that this matrix is finite, the same construction of a Kripke model that we discussed in Section 4 will yield the desired completeness result.

However, if we want to see how the algebraic method developed in this section precisely parallels the completeness proof of Section 2, we can directly construct a "canonical model" algebraically. This canonical model will be a single model for which all and only the theorems of **S4** are valid.

[11] This property is easily derivable from postulate (4) of a closure algebra. See Lemma 6.11.

Recall from Section 3 the Lindenbaum-Tarski matrix $\mathfrak{M}$ for **S4**, that is, the normal S4-characteristic matrix whose elements consist of the equivalence classes of **S4** formulas, and whose designated elements are the equivalence classes of **S4** theorems. In the proof of Theorem 3.8, it was shown that $\mathfrak{M}$ is a closure algebra with a single designated element, 1 (since all theorems are in the same equivalence class). Let $\mathfrak{C}$ be the closure algebra part of $\mathfrak{M}$, called the *Lindenbaum-Tarski algebra.* We construct the canonical model $\mathcal{M} = \langle W, R, \nu \rangle$ as follows:

- $\langle W, R \rangle$ is the ultrafilter frame $\mathcal{F}_{\mathfrak{C}}$. That is, $W = Uf\mathfrak{C}$, and uRv iff $x \in v$ implies $^*x \in u$ for all $x \in K$.
- For any proposition letter p and $u \in W$, $\nu(p, u) = 1$ iff $[p] \in u$.

To show completeness, we first prove the following:

Lemma 5.13. $\nu(\alpha, u) = 1$ *iff* $[\alpha] \in u$.

Proof by induction on the length of α.

- If α is a proposition letter, the result follows immediately from the definition of ν.
- $\nu(\neg\beta, u) = 1$ iff $\nu(\beta, u) = 0$ iff $[\beta] \notin u$ iff $-[\beta] = [\neg\beta] \in u$. The last equivalence holds since u is an ultrafilter.
- $\nu(\beta\wedge\gamma, u) = 1$ iff $\nu(\beta, u) = \nu(\gamma, u) = 1$ iff $[\beta] \in u$ and $[\gamma] \in u$ iff $[\beta]\times[\gamma] = [\beta\wedge\gamma] \in u$.
- $\nu(\Diamond\beta, u) = 1$ iff $\exists v, uRv, \nu(\beta, v) = 1$ iff $\exists v, uRv, [\beta] \in v$ iff $^*[\beta] = [\Diamond\beta] \in u$. To show the last equivalence, the left-to-right direction follows by definition of R. For the other direction, recall that the embedding ρ has the property $\rho(^*[\beta]) = C_R(\rho([\beta]))$. Therefore, any ultrafilter containing $^*[\beta]$ is such that $\exists v \in \rho([\beta]), uRv$, i.e. v contains $[\beta]$.

We can now prove completeness easily. Suppose α is valid in $\mathcal{M}$. Then for all $u \in W$, $\nu(\alpha, u) = 1$. By the above lemma, $[\alpha] \in u$ for all $u \in W = Uf\mathfrak{C}$. That is, $\rho([\alpha]) = Uf\mathfrak{C}$. Since ρ is injective and $\rho(1) = Uf\mathfrak{C}$, we see that $[\alpha]$ is the 1 element of the Lindenbaum-Tarski algebra. Therefore α is in the equivalence class of **S4** theorems, and hence a theorem itself. Also, R is a quasi-ordering on W by Theorem 5.10, so we are done.

Correspondence Between Modern and Algebraic Proofs. It should be clear to the reader by now just how similar are the two methods we used to construct canonical models for **S4**. In fact, the models are not just similar, but they are for practical purposes exactly the same! In Section 2, we built our Kripke frame out of maximal consistent sets of formulas. In the present section, we considered the equivalence classes of formulas as elements of a closure algebra, and then built the Kripke frame out of the ultrafilters of that closure algebra. But in fact, the ultrafilters *are* maximal consistent sets, once we "unpack" the equivalence classes into their constituent sets of formulas.

The following table should elucidate the correspondence between the two approaches we have used:

Logic	Algebra
Formulas	Equivalence classes of formulas
$\neg, \wedge, \Diamond$	$-, \times, ^*$
$\vdash_{\mathbf{S4}} \alpha \to \beta$	$[\alpha] \leq [\beta]$
$\vdash_{\mathbf{S4}} \alpha \leftrightarrow \beta$	$[\alpha] = [\beta]$
Set of **S4** theorems	1
Consistency	Finite Meet Property
Maximal consistent sets	Ultrafilters

This identification of ultrafilters with MCS's is apparent from the definition of each. An ultrafilter is merely a "condensed" way of viewing an MCS, because we know that if a formula belongs to an MCS, by consistency all provably equivalent formulas belong to that MCS, i.e. its equivalence class belongs to that MCS (ultrafilter).

With this correspondence in mind, observe how the algebraic completeness proof is essentially the same as the modern completeness proof. We will outline the proofs in parallel. The modern proof begins with a lemma enumerating four properties of MCS's. These properties have analogues in the properties of ultrafilters. As an example, for every formula α, precisely one of $\{\alpha, \neg\alpha\}$ is in any MCS. Likewise, precisely one of $\{[\alpha], [\neg\alpha]\}$ is in any ultrafilter.

The next step is to prove Lindenbaum's Lemma, that every consistent set can be extended to an MCS. In the algebraic proof, the Ultrafilter Lemma serves the same role: every proper filter can be extended to an ultrafilter, and hence any set with the finite meet property can be extended to one. Note that the proofs of these lemmas are slightly different. In proving Lindenbaum's Lemma, we can enumerate the formulas and add each formula or its negation successively to the consistent set to obtain an MCS. The Ultrafilter Lemma, on the other hand, is more general and does not assume countability of the algebraic elements. The proof therefore requires Zorn's Lemma.

If we were to prove the Ultrafilter Lemma more specifically with respect to the Lindenbaum-Tarski algebra, we could actually use the same method as for Lindenbaum's Lemma. That is, we could enumerate the equivalence classes of formulas in the following way: Using the Axiom of Choice, select from each equivalence class a formula of minimum length. That formula has an associated "code number," and since no two formulas of different equivalence classes can have the same code number, we can order the equivalence classes by code number of the selected formulas. From there the proof proceeds exactly as for Lindenbaum's Lemma.

The canonical models we construct are basically the same in both approaches: W is the set of MCS's, or ultrafilters. R is the relation for which uRv iff $\alpha \in v$ implies $\Diamond\alpha \in u$; equivalently, iff $[\alpha] \in v$ implies $^*[\alpha] = [\Diamond\alpha] \in u$. The valuation function ν is the function that makes p true at u precisely when p belongs to the MCS u, or when $[p]$ belongs to the ultrafilter u. The reflexivity and transitivity of R are proven easily in both cases, either by the (T) and (4) axioms of **S4**, or by postulates (1) and (3) of a closure algebra given in Definition 3.5.

The remainder of the two proofs vary in the details. The algebraic proof is more general and utilizes a deep algebraic result: the Jónsson & Tarski Theorem. But the basic idea is the same for both proofs. We show that the canonical model invalidates all non-theorems of **S4**. The proof of this involves "lifting" the "truth as membership" condition for proposition letters to the same condition for all formulas. This is done via induction, and it hinges on a crucial "existence lemma." In the logic proof, it is Lemma 2.6: If $\Diamond\alpha \in u$, there exists a v such that uRv and $\alpha \in v$. In the algebraic proof, this corresponds to the condition $\rho({*}[\alpha]) = C_R(\rho([\alpha]))$, which implies that if $[\Diamond\alpha] \in u$, there exists a v such that uRv and $[\alpha] \in v$.

The proofs of these are, unsurprisingly, almost identical. We assume that α (or $[\alpha]$) is in the MCS/ultrafilter u. We then form the set of propositions/elements of u that are "necessary" (i.e. $\beta \in u$ s.t. $\vdash_{\mathbf{S4}} \Box\beta$, or, $-{*}-[\beta] \in u$) and show that this set is consistent when α is adjoined (or has the finite model property when $[\alpha]$ is multiplied to each element). Then by Lindenbaum's/Ultrafilter Lemma the set can be extended to an MCS/ultrafilter v such that uRv and $\alpha \in v$ (or $[\alpha] \in v$).

Hence we see that the algebraic proof is an almost equivalent formulation of the modern proof. It invokes the heavier machinery of the Jónsson & Tarski Theorem, but the canonical model it constructs is essentially the same as Makinson's, once we realize that ultrafilters of equivalence classes can be identified with maximal consistent sets.

It is left to the reader to uncover a similar correspondence between the two proofs of the finite model property we have given in Sections 2 and 4.

6. THE EXTENSION THEOREM

In the previous section, we provided a proof of the Jónsson & Tarski Theorem by means of the Stone embedding map that took algebra elements to the set of ultrafilters containing them. In their actual 1951 paper, however, Jónsson and Tarski took a different route. Their aim was broader, as they were concerned not only with closure algebras, but in general with "BAOs" (Boolean algebras with operators), of which a closure algebra is just one particular instance.

Recall from Theorem 4.7 that every complete, atomic Boolean algebra is isomorphic to the power set algebra of its set of atoms. This was subsequently used to show that every finite closure matrix is isomorphic to a topological matrix. Moreover, since the operation on this topological matrix is a completely distributive topology, the matrix can in fact be represented as a relational matrix.

We can also generalize the same procedure to non-finite closure matrices of a certain kind.

Definition 6.1. On a Boolean algebra, a unary function f is *additive* if $f(a+b) = f(a) + f(b)$ for all a, b.

It is *completely additive* if $f(\sum a) = \sum f(a)$ for arbitrary sums of elements.

An n-ary function f is additive if it is additive in each argument. That is, when f is viewed as a unary function of each argument, holding all other arguments constant, that function is additive. Likewise, f is completely additive if it is completely additive in each argument.

We can now make the following definition:

Definition 6.2. A closure algebra is a *complete, atomic closure algebra* if: 1) the Boolean algebra part is complete and atomic, and 2) the closure operation is completely additive.

In the proof of Theorem 4.7, we only used the finitude of the closure algebra to guarantee its Boolean algebra was complete and atomic. But if we assume these properties initially, we get the following:

Theorem 6.3. *Every complete, atomic closure matrix*[12] *is isomorphic to a topological matrix whose closure operation is a completely distributive topology.*

Proof. Let $\mathfrak{C} = \langle K, \{1\}, -, \times, {}^*\rangle$ be a complete, atomic closure matrix. The isomorphism is the same as before; it is the map ϕ which takes each algebra element to the set of atoms below it.

ϕ maps $\mathfrak{C}$ to the topological matrix $\mathfrak{C}' = \langle \mathcal{P}(At(K)), \{At(K)\}, {}^-, \cap, C\rangle$, where $C(A) = \phi({}^*\phi^{-1}(A))$. We now show that if * is completely additive, then C is a completely distributive topology on $At(K)$. We already know that C satisfies the Kuratowski axioms, so it suffices to show that if $S = \cup S_i$ is an arbitrary union of sets in $\mathcal{P}(At(K))$, then $C(S) = \cup C(S_i)$.

[12]Recall that a closure matrix is just a closure algebra with the 1 element designated, so they are only trivially distinct concepts.

$$C(S) = \phi({}^*\phi^{-1}(S)) = \phi({}^*\sum\{a \in S\}) = \phi(\sum\{{}^*a \,|\, a \in S\})$$
$$= \{b \,|\, b \leq \sum\{{}^*a \,|\, a \in S\}\} = \{b \,|\, b \leq \sum_i \sum\{{}^*a \,|\, a \in S_i\}\}$$

The crucial third equality holds by complete additivity of *. Similarly,

$$\bigcup_i C(S_i) = \bigcup_i \phi({}^*\phi^{-1}(S_i)) = \bigcup_i \phi({}^*\sum\{a \in S_i\}) = \bigcup_i \phi(\sum\{{}^*a \,|\, a \in S_i\})$$
$$= \bigcup_i \{b \,|\, b \leq \sum\{{}^*a \,|\, a \in S_i\}\} = \{b \,|\, b \leq \sum_i \sum\{{}^*a \,|\, a \in S_i\}\}$$

These two items are equal, so C is a completely distributive topology. □

The relational matrix representation of a complete, atomic closure matrix comes as a corollary of the following, whose proof is identical to that of Theorem 4.27 without the extra assumption of a finite matrix.

Theorem 6.4. *Every topological matrix whose closure operation is a completely distributive topology is a relational matrix.*

This is excellent progress, but still not as strong a result as the Jónsson & Tarski Theorem, which assures us that *any* closure matrix is representable as a relational matrix.

It suffices to prove the following theorem:

Theorem 6.5. (Extension Theorem) *Any closure algebra is a subalgebra of a complete, atomic closure algebra.*

We see that Theorem 5.1 directly follows from this. For suppose $\mathfrak{C}$ is a closure algebra that is a subalgebra of $\mathfrak{D}$, where $\mathfrak{D}$ is complete and atomic. Then $\mathfrak{D}$ is isomorphic, via ϕ, to a power set algebra $\mathfrak{D}'$ whose closure operation corresponds to a quasi-ordering on the atoms of $\mathfrak{D}$. An isomorphism carries subalgebras to subalgebras, and therefore $\mathfrak{C}$ is isomorphic to a subalgebra of $\mathfrak{D}'$.

Jónsson and Tarski proved a more general form of the Extension Theorem for Boolean algebras with arbitrarily many operators of any arity.[13] In this section, however, we will prove the theorem only in the case of closure algebras.

Suppose $\mathfrak{C} = \langle K, -, \times, f\rangle$ is a closure algebra (we have changed the notation from * to f for convenience). We seek to find a complete, atomic closure algebra $\mathfrak{D} = \langle K_1, -_1, \times_1, f_1\rangle$ of which $\mathfrak{C}$ is a subalgebra.

Theorem 6.6. (Stone Extension) *Any Boolean algebra is a subalgebra of a complete, atomic Boolean algebra.*

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[13] See Jónsson and Tarski [7].

Proof. This statement is a consequence of our earlier Stone Representation Theorem. The Stone mapping ρ embeds a Boolean algebra into a power set algebra, and power set algebras are always complete and atomic (the atoms are the singleton sets). Thus, a Boolean algebra $\mathfrak{B}$ is isomorphic to a set algebra $\mathfrak{B}'$ which is a subalgebra of a (complete and atomic) power set algebra $\mathfrak{A}'$. We can construct the complete atomic extension $\mathfrak{A}$ of $\mathfrak{B}$ by simply "replacing" each element of $\mathfrak{B}'$ in $\mathfrak{A}'$ by the corresponding element of $\mathfrak{B}$. □

Using this theorem, we can simply let the Boolean algebra part of $\mathfrak{D}$, $\langle K_1, -_1, \times_1 \rangle$, be the Stone extension of $\langle K, -, \times \rangle$. For ease of notation, we will write $\langle K_1, -_1, \times_1 \rangle$ as $\langle L, -, \times \rangle$, keeping in mind that these $-$ and $\times$ are technically defined over the larger domain L, but they are the same as $-$ and $\times$ for $\mathfrak{C}$ when their domain is restricted to K. To show that $\mathfrak{C}$ is a subalgebra of $\mathfrak{D}$, it only remains to find f_1 so that $f_1(x) = f(x)$ for all $x \in K$.

Definition 6.7. Let $\langle K, -, \times \rangle$ be a Boolean subalgebra of $\langle L, -, \times \rangle$. An element $x \in L$ is *closed* if

$$x = \prod_{x \le y \in K} y$$

Lemma 6.8. *If $x \in K$, then x is closed.*

Proof. The set of $y \in K$ such that $x \le y$ includes x itself, so by definition of $\le$, the product of this set is equal to x. □

Definition 6.9. Let C denote the set of closed elements in L. If $f : K \to K$, let $f^+ : L \to L$ be the function:

$$f^+(x) = \sum_{x \ge y \in C} \prod_{y \le z \in K} f(z)$$

Lemma 6.10. *If y is closed, then*

$$f^+(y) = \prod_{y \le z \in K} f(z)$$

Proof. Suppose y is closed. Then

$$f^+(y) = \prod_{y \le z \in K} f(z) + \sum_{y \gneq y' \in C} \prod_{y' \le z \in K} f(z)$$

For each $y' \in C$, $y' \lneq y$, consider

$$\prod_{y' \le z \in K} f(z)$$

Since $y' \le y$, the set of z occurring in this product is a superset of the set of $z \in K$, $y \le z$. Therefore,

$$\begin{aligned}\prod_{y' \leq z \in K} f(z) &\leq \prod_{y \leq z \in K} f(z) \\ \sum_{y \gneq y' \in C} \prod_{y' \leq z \in K} f(z) &\leq \prod_{y \leq z \in K} f(z) \\ f^+(y) &= \prod_{y \leq z \in K} f(z)\end{aligned}$$

□

Lemma 6.11. (Monotonicity) *If f is additive, then if $x \leq y$, $f(x) \leq f(y)$.*

Proof. Suppose $x \leq y$. Then $y = x + y$, so $f(y) = f(x + y) = f(x) + f(y)$, i.e. $f(x) \leq f(y)$. □

Theorem 6.12. *If f is additive, then $f = f^+ \backslash K$. That is, $\forall x \in K$, $f^+(x) = f(x)$.*

Proof. Using the previous three lemmas, if $x \in K$,

$$f^+(x) = \prod_{x \leq z \in K} f(z) = f(x) \times \prod_{x \lneq z \in K} f(z) = f(x)$$

The last equality holds because for each $z \gneq x$, $f(x) \leq f(z)$ holds by monotonicity, and hence $f(x) \leq \prod f(z)$. □

Since f is additive by closure algebra postulate (4), we can let $f_1 = f^+$, and this theorem guarantees $f = f^+ \backslash K$, which implies that $\mathfrak{C}$ is a subalgebra of $\mathfrak{D}$. It remains to show that $\mathfrak{D}$ is a complete, atomic closure algebra. The Stone extension assures us that $\mathfrak{D}$'s Boolean algebra is complete and atomic. Moreover, f^+ is completely additive by the following theorem:

Theorem 6.13. *If f is additive, then f^+ is completely additive.*

The proof is very tedious and can be found in Jónsson and Tarski [7], Theorem 2.4.
It only remains to show that $\mathfrak{D}$ satisfies the four postulates for a closure algebra.

(2) $f^+(0) = 0$

Since 0 is an element of K, by Theorem 6.12 we have $f^+(0) = f(0) = 0$.

(4) $f^+(a + b) = f^+(a) + f^+(b)$

f^+ is completely additive, and therefore additive.

(1) $a \leq f^+(a)$
(3) $f^+(f^+(a)) = f^+(a)$

These two conditions are much harder to prove by definition. Instead we will employ the following theorem:

Theorem 6.14. *If f is an additive function of arity m, and $g_1, g_2, \ldots, g_m$ are additive functions of arity n, then*

$$(f(g_1, g_2, \ldots, g_n))^+ = f^+(g_1^+, g_2^+, \ldots, g_n^+)$$

That is, composition of additive functions is preserved under $^+$.

Note that in Definition 6.9, $^+$ was only defined for unary functions. This was done for clarity, but the definition can be easily extended to functions of any arity. The proof of Theorem 6.14 is again very complicated, and the reader may refer to Jónsson and Tarski [7], Theorem 2.10 for the more generalized version.

Now, let Id_K and Id_L be the identity maps on K and L respectively.

Theorem 6.15. *$Id_K^+ = Id_L$.*

The proof will take some work. First, it should be disclosed that the actual extension theorem found in Stone appears in a stronger (and much more esoteric) form than the version we presented in Theorem 6.6. Jónsson and Tarski paraphrase it as:

Theorem. *"Every Boolean algebra is isomorphic to a set-field consisting of all open and closed sets in a totally-disconnected compact space."*

Although we will not delve into further details here, this result has an important consequence in that the Stone extension $\langle L, -, \times \rangle$ has an additional property:

- If u and v are distinct atoms of $\langle L, -, \times \rangle$, then there exists an element $k \in K$ such that $u \leq k$ and $v \times k = 0$.

To prove Theorem 6.15, we begin with a lemma:

Lemma 6.16. *If u is an atom of $\mathfrak{D}$, then u is closed.*

Proof. Let

$$z = \prod_{u \leq y \in K} y$$

Using the above property of the Stone extension,

$$z = k \times \prod_{u \leq y \in K} y$$

Since $\mathfrak{D}$'s Boolean algebra is complete and atomic, by Lemma 4.6 the latter product can be written as a sum of the atoms below it. Let this set of atoms be A. We know that u belongs to A. Therefore

$$z = k \times \sum_{v \in A} v = k \times (u + \sum_{v \in A, v \neq u} v) = k \times u = u$$

where we have again used the additional property of the Stone extension, and also distributivity. We now have

$$u = \prod_{u \leq y \in K} y$$

And hence u is closed. □

We can now prove Theorem 6.15.

Proof.

$$\begin{aligned} Id_K^+(x) &= \sum_{x \geq y \in C} \prod_{y \leq z \in K} Id_K(z) = \sum_{x \geq y \in C} \prod_{y \leq z \in K} z \\ &= \sum_{x \geq y \in C} y = \sum_{x \geq y \in At(L)} y + \sum_{x \geq y \in C, y \notin At(L)} y \\ &= x \end{aligned}$$

The second-to-last equality is a result of the previous lemma that all atoms are closed, and the last equality holds because the first term is just x and the second term is a sum of elements below x. Hence we have proven $Id_K^+ = Id_L$. □

In a similar fashion, it can be shown that the $-$ and $\times$ operations of the extension $\mathfrak{D}$ are really just $-^+$ and $\times^+$ for the corresponding operations of $\mathfrak{C}$.

We finally return to the task of proving closure algebra postulates (1) and (3) for $\mathfrak{D}$.

Let $h_K : K \to K$ be the function $h_K(x) = x \times f(x)$, and $h_L : L \to L$ be the corresponding function on domain L, $h_L(x) = x \times f^+(x)$. We see that h_K is the composition of $\times$ with Id_K and f, and h_L is the composition of $\times^+$ with Id_L and f^+. By Theorem 6.14,

$$h_K^+ = (\times(Id_K, f))^+ = \times^+(Id_K^+, f^+) = \times^+(Id_L, f^+) = h_L$$

Since $\mathfrak{C}$ satisfies postulate (1), we know that $a \leq f(a)$, or equivalently, $a = a \times f(a)$. This corresponds to the condition $Id_K = h_K$. Applying the $^+$ operator to both sides, we get $Id_K^+ = h_K^+$, i.e. $Id_L = h_L$, which means that $\mathfrak{D}$ also satisfies postulate (1).

For postulate (3), we know that $f(f(a)) = f(a)$. Applying $^+$ to the composition of f with itself, we get $(f(f))^+ = f^+(f^+)$, by Theorem 6.14. Hence $f^+(f^+) = f^+$, and $\mathfrak{D}$ satisfies postulate (3).

The proof of the extension theorem is done, for we have shown that $\mathfrak{C}$ is a subalgebra of a complete, atomic closure algebra, $\mathfrak{D}$.

7. Conclusion

Given the parallelism we have seen between the algebra and logic, it is clear that the work of McKinsey, Tarski and Jónsson could have been used to prove the completeness of **S4** (and several other modal systems) years before Kripke. The method of constructing an relational canonical model comes directly from applying the Jónsson & Tarski Theorem to McKinsey's "S4-characteristic matrix," which is just the Lindenbaum-Tarski algebra of equivalence classes of formulas. Alternatively, completeness could have been shown using McKinsey's algebraic finite model property in conjunction with the identification of finite topological matrices with relational matrices, which can be extracted from Birkhoff's paper on Rings of Sets. Hence the completeness of **S4** could have been proven as early as 1951 by the former method, or as early as 1941 by the latter. Alfred Tarski, having worked extensively on both modal logic and algebra, would have been the likely candidate to discover the link between the two.

Yet despite all the crucial components now floating around in the literature, Tarski somehow overlooked the connection. While working on representations for BAOs, he simply did not have modal logic on his mind. Instead he was focused on the algebra of binary relations; in fact, the second half of Jónsson and Tarski [7] dealt almost wholly with "relation algebras." Even as late as 1962, Tarski told Kripke that he could not see the connection between his representation theorem and Kripke's work. Kripke was the first to discover it, and he mentions in a footnote that the insight struck him just as he was finishing up the completeness proof.

Goldblatt [5] poses the question, "Could Tarski have invented Kripke semantics?" I would reply that, given the conspicuous parallels we have seen between logic and algebra, it was certainly within his capacity to do so. By historical accident, he failed to connect the dots before moving on to a different line of research. I concede, however, that without the preestablished conceptual framework of "possible worlds" that semanticists now take for granted, it would have been much more difficult to instinctively grasp what was going on. "Possible worlds" harmonize with our modal intuitions far more readily than do "ultrafilters." Neverthless, Tarski's obliviousness serves as a reminder that mathematical disciplines are not insular, and that we can make important discoveries by bridging the gaps between them.

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