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# Majority Relations and Tournament Solutions

## A Computational Study

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# MAJORITY RELATIONS AND TOURNAMENT SOLUTIONS

A COMPUTATIONAL STUDY

HANS GEORG SEEDIG

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# ABSTRACT


Many methods to aggregate voters' preferences into a collective choice are based on the majority relation derived from the original preferences by taking pairwise majority comparisons. Whenever there are no majority ties, this induces a complete directed asymmetric graph, i.e., a tournament. This thesis deals with various aspects of majoritarian social choice and is divided into two parts. The first part focuses on structural features of majority relations and lays ground for subsequent studies on tournament solutions in the second part.

A common assumption in the area of computational social choice is that the number of voters may be arbitrarily large. In this work, effects of restricting the electorate size to a small constant are examined. Results include a strong expressive power of a small set of voters and computational intractability of several well-known concepts even for small electorates. On the other hand, winner determination may become easier when there is more homogeneity on the side of the alternatives. It is shown that a recursive procedure, coupled with an efficient decomposition method, gives theoretical and computational benefits. In a next step, the winner determination problem is extended to ask for possible and necessary winners in partially specified tournaments. In contrast to earlier work on partial preferences, it was found that most of the variants are computationally tractable.

The thesis contributes to a better understanding of the choice sets returned by the numerous tournament solutions considered here, complementing earlier theoretical work on inclusion relations. It is observed that the theoretical results on the lack of discriminative power of these set-valued concepts are far more negative than empirical and experimental results. In this context, illustrative and minimal examples where concepts differ are provided.

As a follow-up on a recent counterexample to a long-standing conjecture, several open questions in Social Choice Theory are addressed by settling the axiomatic properties of the solution concept *ME*. Lastly, we are concerned with several properties of tournament solutions that center around stability and identify the bipartisan set as a, from our perspective, most desirable tournament solution.

# PUBLICATIONS

This thesis is based on the following publications and working papers.

[1] On the fixed-parameter tractability of composition-consistent tournament solutions. In T. Walsh, editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 85–90. AAAI Press, 2011 (with F. Brandt and M. Brill).

[2] Possible and necessary winners of partial tournaments. In V. Conitzer and M. Winikoff, editors, *Proceedings of the 11th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 585–592. IFAAMAS, 2012 (with H. Aziz, M. Brill, F. Fischer, P. Harrenstein, and J. Lang)

[3] It only takes a few: On the hardness of voting with a constant number of agents. In *Proceedings of the 12th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 375–382. IFAAMAS, 2013 (with F. Brandt, P. Harrenstein, and K. Kardel).

[4] Bounds on the disparity and separation of tournament solutions. Submitted for publication to Discrete Applied Mathematics, 2013 (with F. Brandt and A. Dau).

[5] A tournament of order 24 with two disjoint TEQ-retentive sets. Technical report, `http://arxiv.org/abs/1302.5592`, 2013 (with F. Brandt).

[6] Minimal extending sets in tournaments. In *Proceedings of the 13th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 1539–1540. IFAAMAS, 2014 (with F. Brandt and P. Harrenstein).

[7] Identifying k-majority digraphs via SAT solving. In *Proceedings of the 1st AAMAS Workshop on Exploring Beyond the Worst Case in Computational Social Choice (EXPLORE)*, 2014 (with F. Brandt and C. Geist).

[8] On the discriminative power of tournament solutions. In *Proceedings of the 1st AAMAS Workshop on Exploring Beyond the Worst Case in Computational Social Choice (EXPLORE)*, 2014 (with F. Brandt).

[9] $k$-majority digraphs and the hardness of voting with a constant number of voters. Working paper (with G. Bachmeier, F. Brandt, C. Geist, P. Harrenstein, and K. Kardel).

[10] On the structure of stable tournament solutions. Working paper (with F. Brandt, M. Brill, and W. Suksompong).

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# CONTENTS

# 1 INTRODUCTION

Situations of collective choice where individual preferences of multiple agents have to be aggregated to make a decision of the group arise in many different fields. Some people are motivated by the idea of having a group of autonomous robots, each equipped with limited resources and programmed to try to fulfill certain objectives, automatically determine their joint next moves. Others are concerned with the need for decisions of several interdependent pieces of software. In contrast, the work presented in this thesis was not driven by possible applications but rather by theoretical curiosity. Following the usual terminology in social choice, we speak of *voters* having preferences over *alternatives* and call the method of aggregating the preferences into a decision a *social choice function*.

Obviously, there are infinitely many possibilities to define such a social choice function and it is immediate that some are more appealing than others. For example, it is generally accepted that, in a democracy, a function that always only takes the preferences of a distinguished single voter into account is not very desirable. The reason is that we feel that such a function should be impartial towards the voters, i.e., it should be *anonymous*. Similarly, we would not want a social choice function that does not treat all alternatives equally, it would not be *neutral*. Still, there is a universe of possible social choice functions and we will look at more involved properties later on.

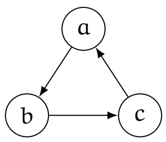
*two alternatives*

When the number of alternatives to vote on is limited to two, the most natural social choice function to think of is *majority rule* where an alternative that is preferred over the other by a majority of the voters is declared the group's choice.[1] In fact, in symmetric settings where there is no bias towards an alternative, e.g., by a *status quo*, there is overwhelming academic consensus that majority rule should be employed for two alternatives as it has many desirable properties.[2] For example, it is anonymous because it just counts the number of voters in favor of each alternative without making any distinctions between the voters and it also satisfies neutrality as the names of the alternatives do not affect the outcome of the rule.

1 For mathematical convenience, it is usually assumed that the number of voters is odd and that the voters have strict preferences to guarantee the existence of a strict majority in favor of one of the alternatives.

2 May (1952)

## MAJORITY RELATIONS

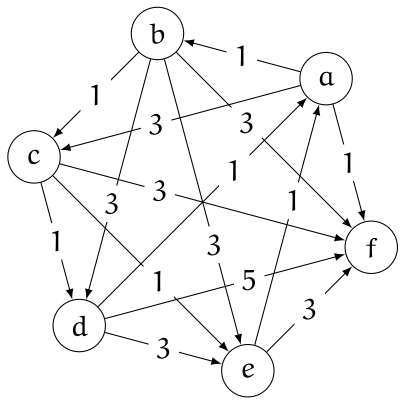



*Condorcet's Paradox*

The idea of taking pairwise majority comparisons first and use the resulting binary *majority relation* as the base for the final decision—in case of majority rule, take the maximal element—has been extended to any number of alternatives. Under the assumption of an odd number of voters and strict preferences, the resulting majority relation has to be asymmetric and complete, making the corresponding digraph a *tournament*. The interesting change when moving from two to more alternatives is that the majority relation does no longer necessarily have maximal elements. This was already observed in the 18th century and is now known as the Condorcet paradox.[3] Actually, it was shown that *every* tournament can represent the majority relation of voters' preferences—given that there are enough voters.

| 2 | 1 | 1 | 1 |
|---|---|---|---|
| a | b | d | c |
| b | d | e | b |
| c | f | a | e |
| d | e | c | d |
| e | a | f | f |
| f | c | b | a |

Preference profile R




Weighted majority graph induced by R

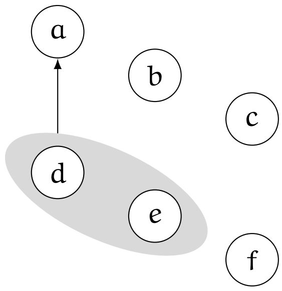
**Figure 1.1:** An example of a preference profile and the corresponding weighted majority graph.

*Example*

In order to illustrate and give a bit of intuition for a good part of the concepts that feature prominent roles in this thesis, we start with an example. Consider the preference profile given on the left in Figure 1.1. It shows the preference rankings of five agents over a set of alternatives $\{a, b, c, d, e, f\}$. The numbers indicate how many agents are having a particular preference ranking, e.g., there are two voters who have lexicographic preferences. From such a profile, we get a weighted majority graph by making pairwise comparisons between every two alternatives. For instance, the first three voters prefer $b$ over $c$ and the other two $c$ over $b$. This gives a (net) weighted majority of 1 in favor of $b$ over $c$. The full weighted majority graph for the profile is depicted on the right in Figure 1.1. In most of this thesis, we are concerned with *unweighted* majority graphs. Those graphs stem from the strict majority relation that indicates which of two alternatives is preferred by a majority (or *dominates*) the other. Since the number of voters in the example is odd, the majority relation is

3 Condorcet (1785)

complete and the corresponding tournament is depicted on the left in Figure 1.2.




Tournament T induced by profile R

| 1 | 1 | 1 |
|---|---|---|
| a | d | f |
| b | e | c |
| c | a | b |
| d | b | e |
| e | c | d |
| f | f | a |

A minimal profile inducing T.

Figure 1.2: The unweighted majority graph from the profile R in Figure 1.1. Since the majority relation is complete in this case, the graph is a tournament and omitted edges point downwards. The grey ellipse indicates that both d and e dominate a. On the right, a minimal profile inducing the same majority graph.

Generally, we are interested in the existence or non-existence of structure in majority relations. To this end, we investigated which relations can be induced by small preference profiles. For the example given here, there is a three-voter profile, shown in right of Figure 1.2 that gives the very same tournament. As it cannot be induced by less than three voters, we call it a 3-*majority digraph*.

When the number of voters is limited to a small constant, the space of possible majority relations is a little less rich.

**CONTRIBUTION 1**

> We address the significance of a restriction on the number of voters with respect to the possible majority relations. In particular, we define the *majority dimension* of a directed graph to be the smallest number of voters that can induce it through a majority relation. We also examine real-world preference profiles with respect to the complexity of their induced majority relations.

*majority dimension*

Coming back to the tournament on the left in Figure 1.2, we see that alternatives d and e, drawn together in a grey ellipse just to indicate that they both dominate alternative a, do in fact have identical relations to all other alternatives. They are indistinguishable from the perspective of the other alternatives and we say that, together, they form a *component* in the tournament. A closer inspection reveals that this tournament contains two additional non-trivial components, namely $\{a, b, c, d, e\}$ which all dominate f as well as $\{b, c\}$. All components of a graph can nicely be represented in a tree. For this tournament, the decomposition tree is depicted in Figure 1.3.

*component*

**CONTRIBUTION 2**

> We examine the decomposability of tournaments and define the

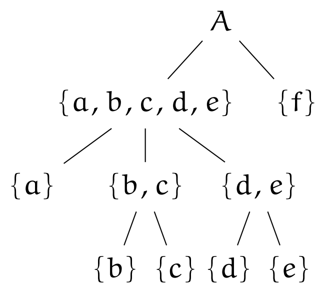



**Figure 1.3:** The decomposition tree of the example tournament.

*decomposition degree* as the maximum number of children of any node in the decomposition tree. We find that while random tournaments rarely exhibit any components, more natural tournaments (from stochastic simulations) very often are decomposable.

## TOURNAMENT SOLUTIONS

*choice from a tournament*

The idea of abstracting away from individual preferences and work on the derived (weighted) majority relation instead has spurred a number of interesting results in the past. Effectively, the new question has become how to choose from a tournament. Many solution concepts of this type have been proposed. Common examples are the top cycle, the uncovered set, or the Banks set for unweighted tournament solutions and maximin, Borda, and ranked pairs for weighted concepts. In this thesis, we are mainly but not exclusively concerned with unweighted concepts.

While there is no clear-cut "best" concept—neither among the unweighted nor the weighted concepts—there are still plenty of criteria to assess them with.

*axiomatic method*

A popular approach among social choice theorists is the *axiomatic method* that classifies solution concepts by the properties they satisfy. Ideally, concepts are even uniquely characterized by a set of appealing (and somewhat natural) properties. Practitioners on the other hand are mainly interested in the applicability of concepts. At this point, *computational social choice* has formed a new and thriving discipline over the last two decades, bringing methods and perspectives from computer science to the academic world of social choice.

*computational social choice*

Among the problems that have attracted the interest of computer scientists are complexity-theoretic questions. In particular, it is relevant for all concepts how they are actually computed.[4] For most of the common concepts, polynomial-time algorithms or some variant

4 Other popular questions for theoretical computer scientists are in the area of manipulation or compact representation.

of computational hardness has been shown. Since many of these results rely on the assumption of having an arbitrary number of voters, it was open whether the hardness results still hold for the restricted case of a small constant number of voters.

**CONTRIBUTION 3**

> We use our insights on the majority relations of few voters to adapt existing hardness constructions for scenarios with only few voters. This way, we are able to show that hardness of several winner determination problems prevails even when the number of voters is limited to a small constant.

In other words, seeking to exploit structural limitations on the majority relations due to small electorates offers no remedy for hard winner determination problems. Complementing these rather negative results, we seek to utilize the existence of components in tournaments to effectively shrink instances size for tournament solutions that treat components in a well-defined consistent way. For this approach, it is helpful that all components of a tournament can be recognized efficiently.

*tournament decomposition*

**CONTRIBUTION 4**

> We show how decomposing a tournament can be exploited to speed up the computation of winning sets for concepts that satisfy the property of composition-consistency and provide supplementary simulation results.

When preferences are not fully available yet, one may be interested in which alternatives still can possibly win and which alternatives will be winners for sure. These PossibleWinner and NecessaryWinner problems have been studied in the past for partial preference profiles.

**CONTRIBUTION 5**

> We extend this study to partial tournaments assuming that not all pairwise comparisons have yet been made. In addition to the classical possible and necessary winner problems, we also consider the problem of determining possible winning *sets* and give complexity results for the most common weighted and unweighted tournament solutions.

Any social choice function that satisfies very basic symmetry criteria (neutrality and anonymity) cannot be resolute, i.e., it has to be *set-valued*. But obviously, choice sets are of limited use if they are very large. After all, if only few alternatives remain unchosen, not much of a choice has been made. On the other hand, it is easier to define tournament solutions that make a consistent choice across different situations when the solution is less discriminative and only excludes alternatives from the choice set under rare circumstances.[5] In conclusion, a trade-off has to be made between discriminative power and axiomatic appeal.

*top cycle*

To illustrate, we look at the choices of three common tournament solutions on the tournament in Figure 1.2. The *top cycle* of a tournament consists of all alternatives that can reach all other alternatives on some path. In case of the example tournament, the top cycle equals $\{a, b, c, d, e\}$. In fact, the only unchosen alternative $f$ is dominated by every other alternative[6] and we feel that no reasonable tournament solution should ever choose it. The *uncovered set* is a refinement of the top cycle and chooses all alternatives that reach every other alternative on a path of length at most 2. It is easy to verify that $\{a, b, d\}$ is the uncovered set of this tournament, i.e., alternatives $c$ and $e$ have been ruled out in comparison to the top cycle. An even more discriminative concept is the *Copeland set* that returns only those alternatives that win a maximum number of pairwise comparisons. For the tournament in question, this is alternative $b$ only as it is the only one that dominates four other alternatives. Therefore, the Copeland set is the singleton $\{b\}$. It can be shown in full generality, that the Copeland set always is contained in the uncovered set which in turn always chooses alternatives that are also in the top cycle. Several of such inclusion relations were already known but it was open from which tournament sizes on which tournament solutions may start to differ or even be disjoint.

*uncovered set*

*Copeland set*

**CONTRIBUTION 6**

We present our findings on the smallest tournaments for which choice sets of tournament solutions actually differ. These results were achieved by means of exhaustively examining all tournaments of increasing size and computing each tournament solution which we have implemented for these kinds of questions. We also add new theoretical results and show that two well-known tournament solutions do not always have to be contained in each other.

Regarding the actual size of choice sets, theoretical results indicated that even supposedly "small" tournament solutions have a strong tendency to not discriminate at all. This is in strong contrast to empirical results which showed that in real-world instances, the top cycle (which contains all other tournament solutions we are interested in) very rarely contains more than three alternatives.

**CONTRIBUTION 7**

We fill this gap by running simulations with more realistic dis-

5 Without going into details, it is obvious that a solution concept easily satisfies, e.g., *independence of unchosen alternatives* if there "never" are unchosen alternatives.

6 We say that $f$ is a *Condorcet loser*.

tributions than those used for the theoretical findings. Our results include a nice classifications of tournament solutions into groups of similar discriminative power.

Knowing about the trade-off between discriminativity and fulfillment of good properties, appealing tournament solutions are often characterized as being the smallest concept fulfilling a set of desirable properties. The tournament equilibrium set (or short $TEQ$) was conjectured to have such a characterization but the problem whether this was actually the case or whether $TEQ$ was severely flawed was open for more than two decades. In the meantime, Brandt[7] proposed a related tournament solution called $ME$ which would also have been a new smallest desirable refinement of existing concepts—but the corresponding conjecture remained unproven as well. In 2011, both of the conjectures were proven to be incorrect by non-constructively showing the existence of a counterexample of enormous size.[8] While the devastating consequences of this on $TEQ$ were immediate, many questions regarding $ME$ were now open again.

**CONTRIBUTION** *8*

We give a concrete and much, much smaller counterexample to the $TEQ$ conjecture. Also, we address the consequences for $ME$ which, unfortunately, are mostly negative. We also take the opportunity to engage in a discussion on the validness of the axiomatic method in cases where violations are very sparsely distributed.

Among the desirable properties a solution concept in general or a tournament solution specifically could satisfy, we focus on *stability*. The underlying idea is that there needs to be a reason for every chosen alternative why it cannot be excluded from the choice set as well as a justification for every unchosen alternative why it should not be added to the choice set.[9] Stability was shown to be satisfied by a number of common tournament solutions and to have nice implications regarding satisfaction of desirable basic properties.

*stability*

**CONTRIBUTION** *9*

We explore the connection of stability to other properties and identify the *bipartisan set* as a, from our perspective, most desirable tournament solution.

*bipartisan set*

## OVERVIEW OF THIS THESIS

This thesis is divided into two parts. In the first part, we cover majority relations and discuss structural aspects of induced majo-

7 Brandt (2011b)
8 Brandt et al. (2013a)
9 Wilson (1970) considered this property as natural as calling it *the solution property*.

rity graphs. Chapter 2 contains basic definitions and a treatment of various stochastic models that will be employed for numerous experiments throughout this thesis. In Chapter 3, we examine the decomposability of tournaments whereas Chapter 4 is concerned with majority relations under a restriction to only few voters.

The second part generally deals with various computational aspects of tournament solutions. After properly defining all concepts considered in this thesis in Chapter 5, we first turn to the computational complexity of winner determination problems. In Chapter 6, we show that several well-known tournament solutions remain computationally intractable even when the number of voters is a small constant. In contrast, Chapter 7 explores possibilities to speed up the computation of composition-consistent tournament solutions by theoretical insight and extensive simulations. Possible and necessary winner problems for partially specified tournaments are the topic of Chapter 8. In Chapter 9, we take two different looks at the differences of choice sets returned by different tournament solutions. Lastly, in Chapter 10 we deliberate on the implications of a recently found counter-example to a long-standing graph-theoretic conjecture, solving a number of open questions regarding the tournament solution $ME$ and also touch on several properties centered around stability.

## UNDERLYING PUBLICATIONS

This thesis is based on a number of joint publications and working papers, some of which have been presented at conferences and workshops. A full list was already given on page vii. Referring to this list, both Chapter 4 and Chapter 6 are based on [7], [3], and [9] whereas Chapter 3 and Chapter 7 are both based on [1]. Chapter 8, parts of which also appeared in the thesis of Brill (2012, Chapter 9) is based on [2]. Chapter 9 is based on material from [4] and [8]. Lastly, Chapter 10 is based on [5], [6], and [10].

## EXCLUDED WORK

In addition, my work contributed to several other publications and working papers that did not fit the theme of this thesis and whose results are therefore omitted. They are listed here for completeness.

- Optimal partitions in additively separable hedonic games. In T. Walsh, editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 43–48. AAAI Press, 2011 (with H. Aziz and F. Brandt).

- Stable partitions in additively separable hedonic games. In P. Yolum and K. Tumer, editors, *Proceedings of the 10th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 183–190. IFAAMAS, 2011 (with H. Aziz and F. Brandt).
- Computing Desirable Partitions in Additively Separable Hedonic Games. *Artificial Intelligence*, 195:316–334, 2013 (with H. Aziz and F. Brandt).
- Consistent Probabilistic Social Choice. 2014 (with F. Brandl and F. Brandt), Working paper.
- On the Susceptibility of the Deferred Acceptance Algorithm. 2014 (with H. Aziz and J. von Wedel), Working paper.

# Part I

# MAJORITY RELATIONS

# 2 FUNDAMENTALS

> An understanding of majority rule, of democracy, of liberalism which does without utilitarianism, and which does more than assert that rights are right, must travel a more mysterious space, must walk up odder stairs, and must employ a more intricate altimeter than transitive consistency.
>
> Douglas W. Rae, 1980

In this chapter, we introduce the main objects of our study in this thesis, i.e., majority relations, majority graphs, and tournaments as well as related notation with a summarizing list at the end of the chapter. We also describe and compare the different stochastic preference models that we will use throughout the thesis.

## 2.1 MAJORITY RELATIONS

### 2.1.1 Preferences

As a basic assumption, we presume individual voters to have linear preferences over a set of alternatives. Formally, let $A$ be a set of $m$ *alternatives* and $N = \{1, \dots, n\}$ a set of *voters*, also called an *electorate*. The preferences of voter $i \in N$ are represented by a linear (i.e., reflexive, complete, transitive, and antisymmetric) *preference relation* $\succsim_i \subseteq A \times A$. The interpretation of $(a, b) \in \succsim_i$, usually denoted by $a \succsim_i b$, is that voter $i$ values alternative $a$ at least as much as alternative $b$. Occasionally, we will also use $R_i$ synonymously with $\succsim_i$ in cases when the interpretation as a *preference ranking* is more natural.

*alternatives*

*voters*

*preference relation*

A *preference profile* $R = (\succsim_1, \dots, \succsim_n)$ is an $n$-tuple containing a preference relation $\succsim_i$ for each agent $i \in N$. For a preference profile $R$ and two alternatives $a, b \in A$, the *majority margin* $g_R(a, b)$ is defined as the difference between the number of voters who prefer $a$ to $b$ and the number of voters who prefer $b$ to $a$, i.e.,

*preference profile*

*majority margin*

$$g_R(a, b) = |\{i \in N \mid a \succsim_i b\}| - |\{i \in N \mid b \succsim_i a\}|.$$

Thus, $g_R(b, a) = -g_R(a, b)$ for all $a, b \in A$.

The *majority relation* $\succsim_R$ of a given preference profile $R$ is defined as

*majority relation*

$$a \succsim_R b \quad \Leftrightarrow \quad g_R(a, b) \geqslant 0$$

where we write $\succsim$ if $R$ is clear from the context. We denote the strict part of $\succsim_R$ by $\succ_R$ and whenever $a \succ_R b$, we say that $a$ *dominates* $b$ or is *majority-preferred* over $b$.

*dominance*

One observation regarding strict majority relations is immediate: whenever the number of voters is odd, the majority margin between two alternatives can never be zero and the strict majority relation has to be complete.

*odd number of voters*

The majority relation can be extended to sets of alternatives by writing $A \succsim B$ when $a \succsim b$ for all $a \in A$ and $b \in B$. Moreover, for a subset of alternatives $B \subseteq A$, we will sometimes consider the restriction of the majority relation $\succsim_B = \succsim \cap (B \times B)$.

### 2.1.2 Majority Graphs and Tournaments

Every majority relation $\succsim_R$ is fully represented by an (asymmetric) digraph $G$ where a strict majority preference $a \succ_R b$ corresponnds to a edge from $a$ to $b$ in $G$, and *vice versa*. We say that $G$ is the *majority graph* of $R$ and that $R$ *induces* $G$. If $R$ has $k$ voters, we say that $G$ is $k$-*inducible*, or, equivalently, that $G$ is a $k$-*majority digraph*.

*majority graph*
$k$-*majority digraph*

If $\succ_R$ is complete, $G$ is as well and therefore a *tournament*, i.e., an asymmetric and complete digraph. We denote the set of all majority graphs by $\mathcal{G}$ and the set of all tournaments by $\mathcal{T}$ which we will use in case an argument is only made for tournaments. Often times throughout this thesis, we will argue over the majority graphs instead of the majority relations when statements are more intuitive or more easily formulated in graph-theoretic terms. In such cases, we also refer to alternatives as *vertices* and to majority preferences $(a, b) \in \succ_R$ as *edges*.

*tournament*

Occasionally, we will also come to consider *weighted graphs* $(A, w)$, where $w\colon A \times A \to \mathbb{Z}$ is a weight function associating edge $(a, b)$ with a weight. With a slight abuse of notation we also refer to weighted graphs as a pair $(A, \succ)$, where the weight function is subsumed and it is understood that

*weighted graphs*

$$\succ = \{(a, b) : w(a, b) > 0\}.$$

We say that a weighted graph $(A, w)$ *is induced by* $R$ if for all $a, b \in A$, $w(a, b) = g_R(a, b)$. In this case, $(A, w)$ is a *weighted* $k$-*majority digraph*.

Let $G = (A, \succsim)$ be a majority graph. The *order* $|G|$ of $G$ refers to the cardinality of $A$ and we let $\mathcal{G}_m$ (or $\mathcal{T}_m$) denote the set of all majority graphs (or tournaments) of order $m$. By $D(a, G)$ we denote the set of all alternatives that $a$ dominates in $G$, i.e.,

*order*

$$D(a, G) = \{b \in A : a \succ b\}$$

and call this set the *dominion* of $a$ in $G$. Similarly, let $\overline{D}(a, G)$ denote the *dominators* of $a$ in $G$, i.e.,

*dominion*
*dominators*

$$\overline{D}(a, G) = \{b \in A : b \succ a\}.$$

A non-empty subset $B \subseteq A$ of alternatives is *dominant* if $B \succ A \setminus B$. The size of the dominion of an alternative defines its *score* which is equivalent to its out-degree in $G$. Formally,

*dominant set*

*score*

$$s(a, G) = |D(a, G)|.$$

If the score of all vertices is identical, the graph is *regular*. If an alternative is majority-preferred over all other alternatives, the corresponding vertex has degree $m-1$ and we call such an alternative a *Condorcet winner* (Condorcet, 1785). We denote the set of Condorcet winners by $CW(G)$. Note that $CW(G)$ is either empty or a singleton.

*regular*

*Condorcet winner*

For a subset $B \subset A$, we write $G_B$ to denote the *subgraph* $(A, \succ_B)$ of $G$.

*subgraph*

Now let $G' = (A', \succsim')$ be a second majority graph. A bijective mapping $\pi : A \to A'$ is a digraph *isomorphism* if it holds that $a \succsim b$ if and only if $\pi(a) \succsim' \pi(b)$. In this case, $G$ and $G'$ are *isomorphic* to each other. Similarly, an *automorphism* $\pi$ on $G$ is an isomorphism from $A$ to itself. An *orbit* of a digraph contains all vertices that can be mapped to one another by an automorphism. Intuitively, two vertices are in the same orbit if they are indistinguishable in an unlabeled graph and we denote the set of all orbits of $G$ by $\mathcal{O}_G$.

*isomorphism*

*orbit*

Now, let $T = (A, \succ)$ be a tournament. The set of all linear orders on some set $A$ is denoted by $\mathcal{L}(A)$ and the maximal element of $A$ according to a linear order $L \in \mathcal{L}(A)$ is denoted by $\max(L)$. A set of vertices $B \subset A$ forms a *transitive subset* if $(B, \succ_B)$ is a linear order. Let $\mathcal{B}_T$ denote the set of all transitive subsets of $T$ and we will also write $\mathcal{B}_B$ for $\mathcal{B}_{T_B}$. Also, define

*transitive subset*

$$\mathcal{B}_T(a) = \{B \subseteq \mathcal{B}_T : \max(\succ_B) = a\}$$

as the set of all transitive subsets with maximal element $a$. For $B \in \mathcal{B}_T$, an alternative $a$ *extends* $B$ if $a \succ B$, implying $B \cup \{a\} \in \mathcal{B}_T(a)$.

A subset $B$ of $A$ is a *component* of $T$ if for all $a \in A \setminus B$ either $B \succ a$ or $a \succ B$. Components and the decomposition of a tournament will be introduced thoroughly in Chapter 3.

*component*

## 2.2 STOCHASTIC PREFERENCE MODELS

If certain phenomena in social choice—such as intransitivity of the majority relation, unintuitive outcomes of social choice functions, or opportunities for strategic manipulation—are known to occur *in theory*, a natural follow-up is to ask for their likelihood. Study of real-world data would be preferred but limited data availability is an almost unescapable problem[10] along with the fact that real-world data

10 The situation is currently improving due to the growing PrefLib library, established and maintained by Mattei and Walsh (2013), to which scholars can contribute their data sets.

may for the most part not exhibit the prerequisites for a meaningful study of the effect in question.

*analysis by simulation*

A remedy in such cases is to resort to stochastic analyses where stochastic models are used to create individual preferences in an electorate of a chosen size. Such simulations with stochastic preference models have been used for the analysis of several problems in (computational) social choice. For example, Laslier (2010) generated voting instances to derive estimates for the frequency of Condorcet winners and to compare the results of different voting rules such as plurality, Borda, approval voting, and Copeland's rule to each other. In his work, he has used a Rousseauist model, capturing the idea of a pre-existing truth, as well as spatial and redistributive models. Earlier, McCabe-Dansted and Slinko (2006) have used computational experiments to obtain a hierarchical clustering of voting rules. To this end, they considered the number of times two voting rules coincide on a sample set as a measure for their similarity. They used the same setting as Shah (2003) with 5 alternatives and 85 voters and employed the Pólya-Eggenberger urn model by Berg (1985) to generate preferences. Recent work employing stochastic preference models for comparison with empirical data include the papers by Tideman and Plassmann (2012) and Mattei et al. (2012).

*choice of stochastic models*

In this section, we will cover several stochastic models for linear preferences that have been proposed in the literature and that we will employ for our experiments in later chapters. Our choice of models was guided by our intent to use them for generating individual preferences and combine them into majority relations. Therefore, an efficient sampling procedure was necessary. Also, we favored models with few parameters over those with many parameters. The latter give more versatile models that are well-suited when it is asked whether given preferences can be modeled through a model. In our case, the huge number of parameters, e.g., in Thurstonian, Babington Smith, and multi-stage ranking models, is problematic as they need to be chosen in some reasonable manner for our sampling procedure.[11]

We refer to Critchlow et al. (1991) and Marden (1995) for a more in-depth treatment of stochastic models.

*culture*

For most of the models we consider, we sample preference profiles and work with the tournament induced by the majority relation of an odd number of voters. The term *culture* has been coined for probabilistic preference models where the draws for each voter are independent from each other. Cultures are defined by the probabilities they put on each possible preference ranking.

11 Also, sampling from a general Babington Smith model is a very tedious task. To our knowledge, there is no more efficient algorithm than to sample all $m(m-1)/2$ pairwise comparison with equally many non-identically distributed Bernoulli trials, return the resulting ranking if the outcome is transitive and start over if not.

### 2.2.1 Cultures of indifference

The most widely-studied culture is the *impartial culture* model (IC), where every possible ranking of the alternatives has the same probability of $1/m!$. IC is a member of the family of *dual cultures*, defined by the property that each ranking has the same probability as its inverse. Dual cultures have been criticized for being too unrealistic as they do not impose *any* structure on the preferences (see, e.g., Tsetlin et al., 2003; Regenwetter et al., 2006). Nevertheless, they are relevant for their susceptibility to analytical methods that helped to improve the understanding of voting phenomena (see, e.g., DeMeyer and Plott, 1970). If we add anonymity by having indistinguishable voters, the set of profiles is partitioned into equivalence classes. In the *impartial anonymous culture* (IAC), each of these equivalence classes is chosen with equal probability. Technically, this is not a culture in the static sense mentioned above.

*impartial culture*

*impartial anonymous culture*

### 2.2.2 Distance-based models

There are several models that assume a pre-existing truth in the form of reference rankings such that each agent reports a noisy estimate of said truth as his preferences. For these models, Laslier (2010) has introduced the term *Rousseauist cultures*. Such models are usually parameterized by a homogeneity parameter that scales the noisiness of individual perceptions. In its arguably simplest form, every agent $i$ provides possibly intransitive preferences $R_i$ where each pairwise preference $a\ R_i\ b$ is 'correct', i.e., coincides with the reference ranking $R_0$ with a probability $p$ where $0.5 \leqslant p \leqslant 1$. This model has been studied, for example, by Frank (1968), Nowicki (1989), and Łuczak et al. (1996) and since it is sometimes attributed to Condorcet (see, e.g., Young, 1988), we call it the *Condorcet noise model*.[12] This is the only model we consider in which individual preferences can be intransitive. For $p = 0.5$, the Condorcet noise model with any odd number of voters coincides with the model of uniform random tournaments.[13]

*Condorcet noise model*

In *Mallows-ϕ model* (Mallows, 1957), the distance to a reference ranking is measured by means of the *Kendall-tau distance* (Kendall, 1938) which counts the number of pairwise disagreements. Let $R_0$ be

*Mallows-ϕ model*

*Kendall-tau distance*

12 A practically useful aspect of this model is that all pairwise majority comparisons are independent of each other and can be computed directly by

$$\Pr(a \succ_R b \mid a\ R_0\ b) = \sum_{\nu=\frac{n}{2}+1}^{n} \binom{n}{\nu} p^{\nu}(1-p)^{n-\nu}.$$

13 A similar example for a Rousseauist culture would be the two-parameter model used in Drissi-Bakhkhat and Truchon (2004) where the probabilities of correct assessments may depend on the distance of the alternatives in the reference ranking.

the reference ranking. Then, the Kendall-tau distance of a preference ranking $R_i$ to $R_0$ is

$$\tau(R_i, R_0) = \binom{m}{2} - (|R_i \cap R_0| - m)\,.$$

According to the model, this induces the probability of a voter having $R_i$ as his preferences to be

$$\Pr_M(R_i, \phi, R_0) = \frac{\phi^{\tau(R_i,R_0)}}{C}$$

where $C$ is a normalization constant and $\phi \in (0, 1]$ is a dispersion parameter. Small values for $\phi$ put most of the probability on rankings very close to $R_0$ whereas for $\phi = 1$ the model coincides with IC.

Obviously, one can define a number of such distance-based models. Besides the Kendall-tau distance, Spearman's rho distance has been considered (resulting in Mallows-$\theta$ model), as well as the distance measures named after Cayley, Hammond, and Ulam. See Critchlow et al. (1991) for a discussion.

*unimodality*

A property that makes distance-based models less appealing for this particular study is their bias towards transitive majority relations which makes the issue of choosing trivial. In fact, Mallows-$\phi$ model even satisfies *strong unimodality* as defined in Critchlow et al. (1991) since a single preference ranking has maximum probability and ranking probabilities are non-increasing as we move along a path of rankings, where in each step two adjacent alternatives are swapped causing an increase in the Kendall-tau distance to the modal ranking.

*mixture models*

*Mallows k-mixtures*

To overcome this unimodality of the preference distribution to some extent, *mixtures* of models have been considered. A mixture model consists of several ordinary models with a probability distribution over them. While this idea could theoretically be applied to any set of models that may just differ in their parameterization or even belong to different model families, it has been considered the most with respect to the Mallows-$\phi$ model. For simplicity and to reduce the number of free parameters, we consider uniform mixtures over $k$ Mallows-$\phi$ models with a shared parameter $\phi$ and refer to this as *Mallows $k$-mixtures*. The probability of a preference ranking $\succsim_i$ to be chosen under a Mallows $k$-mixture is then

$$\Pr_{MM}\left(\succsim_i, \phi, (R_0^1, \ldots, R_0^k)\right) = \sum_{j=1}^{k} \frac{1}{k} \cdot \Pr_M(R_i, \phi, R_0^j).$$

Sampling from Mallows-$\phi$ (or Mallows mixtures) is conveniently possible by a repeated insertion model (Doignon et al., 2004; Lu and Boutilier, 2011).

### 2.2.3 Other models

*urn model*

In the Pólya-Eggenberger *urn model*, each possible preference ranking is thought to be represented by a ball in an urn from which individual preferences are drawn. After each draw, the chosen ball is put back and $\alpha \in \mathbb{N}_0$ new balls of the same kind are added to the urn (Berg, 1985). This models the effect of an interdependence of multiple voters' preferences as the next voter chooses from a modified distribution. Therefore, it does not fall under our definition of a culture. Still, the urn model subsumes both IC ($\alpha = 0$) and IAC ($\alpha = 1$).

A very different kind of model is the *spatial model*. Here, alternatives and voters are uniformly at random placed in a multi-dimensional space and the voters' preferences are determined by the (Euclidean) distances to the alternatives. The spatial model has played an important role in political and social choice theory where the dimensions are interpreted as different aspects or properties of the alternatives (see, e.g., Ordeshook, 1993; Austen-Smith and Banks, 2000). For a fixed natural number $d$ of issues, we assume that candidates as well as voters are located in the space $[0, 1]^d$. The position of candidates and voters can be thought of as their stance on the $d$ issues. Voters' preferences over candidates are given by the proximity to their own position according to the Euclidian distance. The one-dimensional case coincides with the well-studied model of single-peaked preferences. We generate tournaments by drawing the positions of candidates and voters uniformly at random from $[0, 1]^d$.

*spatial model*

The *uniform random tournament* model was used in previous analysis of the discriminativity of tournament solutions (Fisher and Reeves, 1995; Fey, 2008; Scott and Fey, 2012). It assigns the same probability to each *labeled* tournament $T$ of equal size, i.e.,

*uniform random tournament*

$$\Pr(T) = \frac{1}{2^{\binom{m}{2}}} \text{ for each } T \text{ with } |T| = m.$$

Note that it differs from all other models mentioned in the sense that it samples the tournament directly and does not construct it as a majority relation from a collection of sampled preference rankings.

## 2.3 COMPARISON OF STOCHASTIC MODELS

To get a better understanding of the majority relation typically produced by the stochastic models, we ran some experiments to assess their tendency towards majority relations that are transitive or exhibit a Condorcet winner.

### 2.3.1 Degree of Transitivity

A complete relation is transitive if and only if it does not contain any cycles. In fact, whenever a complete relation exhibits a cycle it does also contain a 3-cycle. We follow Kendall and Babington Smith (1940)

*3-cycle*

who consider $c_3(T)$, the number of 3-cycles in a tournament $T$ a valid measure of its transitivity:

> In discussing inconsistences, therefore, it seems best to confine attention to circular triads, which, so to speak, constitute the inconsistent elements of the configuration.

Computing $c_3(T)$ only requires the score sequence $(s_1, \ldots, s_{|T|})$ of $T$ (Moon, 1968, p. 11) since

$$c_3(T) = \binom{m}{3} - \sum_{i=1}^{m} \binom{s_i}{2}.$$

Kendall and Babington Smith (1940) observed that the maximum possible number of 3-cycles $c_3(T)$ in a tournament $T$ of order $m$ is

$$\frac{m^3 - m}{24} \qquad \text{if } m \text{ is odd}$$

and

$$\frac{m^3 - 4m}{24} \qquad \text{if } m \text{ is even}$$

and that both of these bounds are met by regular tournaments.[14] Consequently, they define

$$\zeta(T) = \begin{cases} 1 - c_3(T) \cdot \frac{24}{m^3 - m} & \text{if } |T| \text{ is odd} \\ 1 - c_3(T) \cdot \frac{24}{m^3 - 4m} & \text{if } |T| \text{ is even} \end{cases}$$

*measure of transitivity*

as a measure of consistence (or transitivity) of a tournament $T$.[15]

We generated tournaments as complete majority relations for scenarios with 5 to 305 voters and a varying number of alternatives up to 30. The resulting values of $\zeta$ for the different models are shown as a heat map in Figure 2.1. Higher values of $\zeta$ are displayed in green, corresponding to higher degree of transitivity.

We see that the number of voters does not seem to have a significant effect for the urn models (including IC and IAC), at least in this range of $n$. For the distance-based models, we see that an increase in the number of voters induces a higher degree of transitivity. The highest tendency towards rather transitive majority relations is observed for the spatial model.

14 For even $m$, a regular tournament has only scores $\frac{m}{2}$ and $\frac{m}{2} - 1$. These tournaments are also called *semi-regular*.

15 A second measure that comes to mind is the size of a minimum feedback arc set. One problem of this measure is that its computation is NP-hard (Alon, 2006; Charbit et al., 2007).

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.83 | 0.60 | 0.75 | 0.67 | 0.76 | 0.62 | 0.71 | 0.71 | 0.67 | 0.73 | 0.73 |
| 10 | 0.36 | 0.34 | 0.29 | 0.21 | 0.25 | 0.31 | 0.28 | 0.27 | 0.32 | 0.31 | 0.31 |
| | 0.17 | 0.14 | 0.11 | 0.11 | 0.12 | 0.14 | 0.14 | 0.13 | 0.09 | 0.11 | 0.11 |
| 20 | 0.12 | 0.05 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.06 | 0.06 | 0.06 | 0.08 |
| | 0.06 | 0.04 | 0.03 | 0.02 | 0.05 | 0.02 | 0.03 | 0.03 | 0.02 | 0.03 | 0.03 |
| 30 | 0.05 | 0.03 | 0.02 | 0.02 | 0.01 | 0.03 | 0.03 | 0.02 | 0.01 | 0.02 | 0.02 |

impartial culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.98 | 0.92 | 0.96 | 0.98 | 0.98 | 1.00 | 0.96 | 0.98 | 1.00 | 1.00 | 1.00 |
| 10 | 0.57 | 0.86 | 0.82 | 0.91 | 0.93 | 0.93 | 0.91 | 0.91 | 0.97 | 0.97 | 0.94 |
| | 0.72 | 0.66 | 0.83 | 0.88 | 0.85 | 0.87 | 0.86 | 0.91 | 0.86 | 0.88 | 0.90 |
| 20 | 0.54 | 0.69 | 0.73 | 0.78 | 0.81 | 0.84 | 0.84 | 0.81 | 0.87 | 0.83 | 0.85 |
| | 0.50 | 0.62 | 0.71 | 0.69 | 0.72 | 0.82 | 0.82 | 0.80 | 0.84 | 0.85 | 0.85 |
| 30 | 0.52 | 0.54 | 0.61 | 0.68 | 0.71 | 0.75 | 0.74 | 0.82 | 0.78 | 0.81 | 0.82 |

spatial model (dim $= 2$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.75 | 0.79 | 0.69 | 0.71 | 0.63 | 0.71 | 0.70 | 0.67 | 0.66 | 0.68 | 0.73 |
| 10 | 0.36 | 0.28 | 0.30 | 0.29 | 0.28 | 0.25 | 0.26 | 0.33 | 0.28 | 0.26 | 0.27 |
| | 0.17 | 0.10 | 0.13 | 0.12 | 0.11 | 0.12 | 0.14 | 0.08 | 0.10 | 0.10 | 0.10 |
| 20 | 0.11 | 0.05 | 0.06 | 0.04 | 0.07 | 0.05 | 0.04 | 0.06 | 0.07 | 0.05 | 0.05 |
| | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.04 | 0.03 | 0.02 | 0.04 | 0.02 | 0.05 |
| 30 | 0.04 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 | 0.03 | 0.01 | 0.02 | 0.02 | 0.01 |

impartial anonymous culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.83 | 0.70 | 0.68 | 0.88 | 0.81 | 0.64 | 0.65 | 0.71 | 0.77 | 0.65 | 0.70 |
| 10 | 0.34 | 0.33 | 0.29 | 0.27 | 0.28 | 0.26 | 0.33 | 0.24 | 0.31 | 0.28 | 0.30 |
| | 0.16 | 0.11 | 0.13 | 0.10 | 0.10 | 0.10 | 0.11 | 0.12 | 0.07 | 0.12 | 0.14 |
| 20 | 0.10 | 0.06 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.04 | 0.06 |
| | 0.06 | 0.04 | 0.04 | 0.04 | 0.03 | 0.02 | 0.02 | 0.04 | 0.04 | 0.02 | 0.04 |
| 30 | 0.05 | 0.02 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |

urn ($\alpha = 10$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.81 | 0.66 | 0.71 | 0.73 | 0.72 | 0.78 | 0.72 | 0.58 | 0.77 | 0.76 | 0.76 |
| 10 | 0.42 | 0.50 | 0.39 | 0.46 | 0.52 | 0.45 | 0.60 | 0.56 | 0.60 | 0.65 | 0.68 |
| | 0.23 | 0.20 | 0.29 | 0.33 | 0.44 | 0.53 | 0.53 | 0.65 | 0.56 | 0.67 | 0.69 |
| 20 | 0.10 | 0.15 | 0.23 | 0.34 | 0.39 | 0.57 | 0.58 | 0.65 | 0.65 | 0.68 | 0.67 |
| | 0.08 | 0.20 | 0.26 | 0.41 | 0.49 | 0.59 | 0.57 | 0.69 | 0.71 | 0.73 | 0.69 |
| 30 | 0.06 | 0.15 | 0.21 | 0.41 | 0.57 | 0.58 | 0.64 | 0.69 | 0.71 | 0.75 | 0.77 |

Mallows-$\phi$ ($\phi = 0.95$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.79 | 0.68 | 0.65 | 0.67 | 0.73 | 0.79 | 0.72 | 0.75 | 0.76 | 0.78 | 0.79 |
| 10 | 0.37 | 0.45 | 0.38 | 0.43 | 0.48 | 0.58 | 0.51 | 0.52 | 0.52 | 0.60 | 0.65 |
| | 0.22 | 0.23 | 0.29 | 0.34 | 0.40 | 0.51 | 0.38 | 0.52 | 0.51 | 0.52 | 0.56 |
| 20 | 0.13 | 0.21 | 0.22 | 0.34 | 0.34 | 0.41 | 0.44 | 0.50 | 0.46 | 0.52 | 0.55 |
| | 0.08 | 0.11 | 0.19 | 0.31 | 0.37 | 0.42 | 0.46 | 0.49 | 0.46 | 0.54 | 0.55 |
| 30 | 0.08 | 0.11 | 0.20 | 0.23 | 0.34 | 0.39 | 0.46 | 0.47 | 0.50 | 0.55 | 0.58 |

Mallows 4-mixture ($\phi = 0.9$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.49 | 0.88 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10 | 0.06 | 0.54 | 0.95 | 0.98 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | 0.01 | 0.49 | 0.82 | 0.96 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 20 | 0.00 | 0.23 | 0.65 | 0.91 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | 0.00 | 0.17 | 0.60 | 0.85 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30 | 0.00 | 0.14 | 0.42 | 0.75 | 0.96 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Condorcet noise ($p = 0.65$)

**Figure 2.1:** Degree of transitivity $\zeta$ for tournaments obtained from different stochastic models. The more green, the higher the value of $\zeta$, corresponding to a lower number of 3-cycles in the generated tournaments.

### 2.3.2 Frequency of Condorcet winners

In anticipation of Part II where most of the functions considered all coincide on tournaments with a Condorcet winner, the frequency of which Condorcet winners exist is a second meaningful criterion for comparing the different models. The results of our simulations (this time for up to 50 alternatives to show a curious non-monontonicity in Mallows' models) are depicted in Figure 2.2.

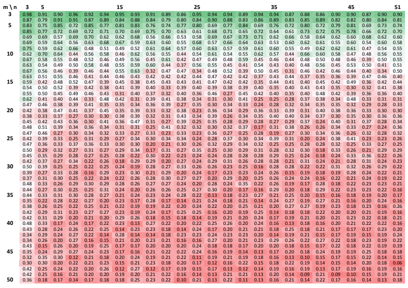
Again, we see that for urn-based based models, the number of voters does not have a noteworthy effect on the criterion in question. For the distance-based and the spatial model, larger number of voters more frequently induce tournaments with Condorcet winners. Perhaps unsurprisingly, there obviously is a strong correlation between the degree of transitivity and the frequency of Condorcet winners.

## 2.4 SUMMARY

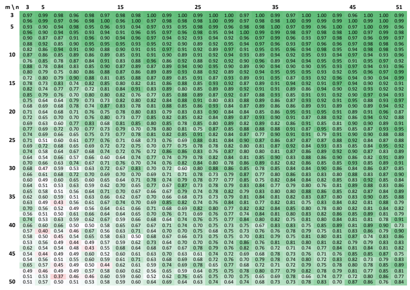
This chapter covered basic definitions and terminology, and discussed the stochastic preference models we will use for the experiments in the rest of the thesis. We compared the models by means of their tendency to produce transitive majority relations or majority graphs with Condorcet winners. The most notable insights are that the number of voters does have little effect on the majority graphs obtained from urn models, including the impartial and the impartial anonymous culture. Among the models considered, the spatial model tends the most towards transitivity in the majority relation and Condorcet winners, only matched by the distance-based models when the number of voters becomes very large.

A short comment on the effects of varying the chosen parameters in our models is in order. Changes that increase homogeneity in the the electorate—such as increasing $p$ in the Condorcet noise model, decreasing $\phi$ in Mallows-$\phi$ model, or increasing $\alpha$ in the urn model—increase the degree of transitivity and the frequency of Condorcet winners. For the spatial model, we have not found the dimension to have a large impact on the results as long as it is at least 2 (data not shown).

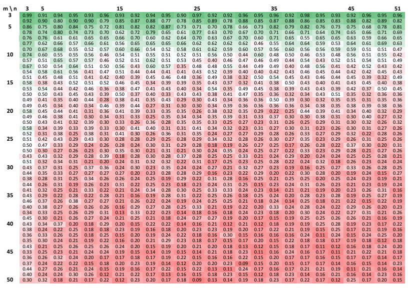
We summarize introduced notation on majority graphs in Table 2.1 for future reference.



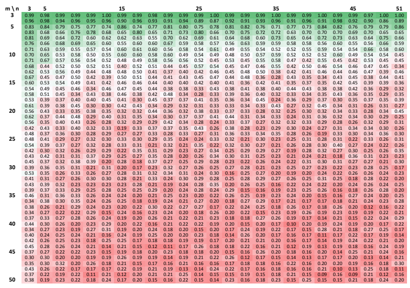
impartial culture



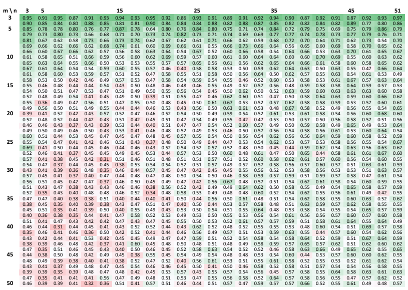
spatial model (dim = 2)



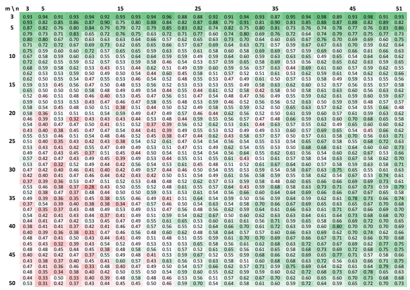
impartial anonymous culture



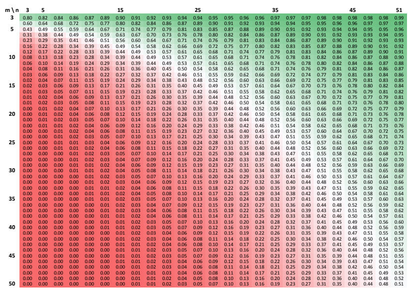
urn ($\alpha = 10$)



Mallows-$\phi$ ($\phi = 0.95$)



Mallows 4-mixture ($\phi = 0.95$)



Condorcet noise ($p = 0.65$)

**Figure 2.2:** Frequency of Condorcet winners in tournament generated from different stochastic preference models with 3 to 51 voters (left to right on horizontal axis) and 3 to 50 alternatives (top to bottom on vertical axis). Green entries indicate a high frequency of tournaments with Condorcet winners. Values for the first six models are taken over 100 samples whereas the probabilities for the Condorcet noise model were computed directly.

| | |
|---|---|
| $A = (a_1, \dots, a_m)$ | set of alternatives |
| $N$ | set of voters, electorate |
| $\succsim_i, R_i$ | individual preferences, preference ranking |
| $R = (\succsim_1, \dots, \succsim_n)$ | preference profiles |
| $\succsim$ | (weak) majority relation |
| $\succ$ | strict majority relation |
| $G = (A, \succsim)$ | a majority graph |
| $\mathcal{G}$ | set of all majority graphs |
| $\mathcal{G}_m$ | set of all majority graphs of size $m$ |
| $D(a)$ | dominion of $a$ |
| $\overline{D}(a)$ | dominators of $a$ |
| $\mathcal{O}_G$ | set of all orbits of $G$ |
| $G = (A, w)$ | a weighted majority graph |
| $T = (A, \succ)$ | a tournament |
| $\mathcal{T}$ | set of all tournaments |
| $\mathcal{T}_m$ | set of all tournaments of size $m$ |
| $\mathcal{B}_T$ | set of all transitive sets in $T$ |

**Table 2.1:** Notation for majority graphs

# 3 DECOMPOSITION OF TOURNAMENTS

> *Decomposition* allows the subdivision of the explanatory task so that the task becomes manageable and the system intelligible.
>
> William Bechtel and Robert C. Richardson, 2010

Decomposition techniques have seen numerous applications to reduce the complexity of a system at hand. They have been implemented in fields as diverse as sequence optimization (Sidney and Steiner, 1986), network reliability (Shogan, 1986), graph drawing (Papadopoulos and Voglis, 2007), and protein interaction (Gagneur et al., 2004). In our case, we are interested in identifying sets of similar alternatives in a tournament with the intention to exploit this knowledge about the tournament's structure later for computational problems. Section 3.1 gives the general definitions of components and decompositions. In Section 3.2, we discuss decomposition trees and introduce the decomposition degree of a tournament. Decomposition trees can be efficiently computed as reviewed in Section 3.3. Finally, Section 3.4 contains the results of simulations regarding the decomposability of tournaments generated by various stochastic preference models. Section 3.5 summarizes our findings.

## 3.1 COMPONENTS AND DECOMPOSITIONS

A natural structural concept in the context of tournaments is that of a component which is a subset of alternatives that bear the same relationship to all alternatives not in the set.

DEFINITION 3.1

Let $T = (A, \succ)$ be a tournament. A non-empty subset $B$ of $A$ is a *component* of $T$ if for all $a \in A \setminus B$ either $B \succ a$ or $a \succ B$. A *decomposition* of $T$ is a set of pairwise disjoint components $\{B_1, \dots, B_k\}$ of $T$ such that $A = \bigcup_{i=1}^{k} B_i$.

*component*
*decomposition*

The *null decomposition* of a tournament $T = (A, \succ)$ is $\{A\}$; the *trivial decomposition* consists of all singletons of $A$. Any other decomposition is called *proper*. A tournament is said to be *decomposable* if it admits a proper decomposition. Given a particular decomposition, the *summary* of a tournament is defined as the tournament on the individual components rather than the alternatives.

summary

**Definition 3.2**
Let $T = (A, \succ)$ be a tournament and $\widetilde{B} = \{B_1, \ldots, B_p\}$ a decomposition of $T$. The summary of $T$ with respect to $\widetilde{B}$ is defined as the tournament $T_B = (\{1, \ldots, p\}, \widetilde{\succ})$, where

$$i \mathrel{\widetilde{\succ}} j \quad \text{if and only if} \quad B_i \succ B_j.$$

inducible tournament

A tournament is called *reducible* if it admits a decomposition into *two* components. Otherwise, it is *irreducible*. Laslier (1997) has shown that there exist a natural unique way to decompose any tournament. Call a decomposition $\widetilde{B}$ *finer* than another decomposition $\widetilde{B}'$ if $\widetilde{B} \neq \widetilde{B}'$ and for each $B \in \widetilde{B}$ there exists $B' \in \widetilde{B}'$ such that $B \subseteq B'$. $\widetilde{B}'$ is said to be *coarser* than $\widetilde{B}$. A decomposition is *minimal* if its only coarser decomposition is the null decomposition.

minimal decomposition

**Proposition 3.3** *(Laslier, 1997, Thm. 1.3.11)*
Every irreducible tournament with more than one alternative admits a unique minimal decomposition.

This is obviously not true for *reducible* tournaments, as witnessed by the tournament $T = (\{a, b, c\}, \succ)$ with $a \succ b$, $a \succ c$, and $b \succ c$, which admits two minimal decompositions, namely $\{\{a\}, \{b, c\}\}$ and $\{\{a, b\}, \{c\}\}$. Nevertheless, there is a unique way to decompose any reducible tournament. A *scaling decomposition* is a decomposition with a transitive summary.

scaling decomposition

**Proposition 3.4** *(Laslier, 1997, Thm. 1.3.13)*
Every reducible tournament admits a unique scaling decomposition such that each component is irreducible.

This scaling decomposition into irreducible components is also the finest scaling decomposition. In graph-theoretic terms, this decomposition partitions the tournament into its strongly connected components.

## 3.2 DECOMPOSITION TREES AND DECOMPOSITION DEGREE

Propositions 3.3 and 3.4 offer a straightforward method to iteratively decompose tournaments. If the tournament is reducible, take the finest scaling decomposition. If it is irreducible, take the minimal decomposition. The repeated application of these decompositions leads to the decomposition tree of a tournament.

decomposition tree

**Definition 3.5**
The *decomposition tree* $\mathcal{D}(T)$ of a tournament $T = (A, \succ)$ is defined as a rooted tree whose nodes are non-empty subsets of $A$. The root of $\mathcal{D}(T)$ is $A$ and for each node $B$ with $|B| \geqslant 2$, the children of $B$ are defined as follows:

- If $T|_B$ is reducible, the children of $B$ are the components of the finest scaling decomposition of $T|_B$.
- If $T|_B$ is irreducible, the children of $B$ are the components of the minimal decomposition of $T|_B$.

It follows from Propositions 3.3 and 3.4 that every tournament has a *unique* decomposition tree. By definition, each node in $\mathcal{D}(T)$ is a component of $T$ and each leaf is a singleton. However, not all components of $T$ need to appear as nodes in $\mathcal{D}(T)$. An example of a decomposition tree is provided in Figure 3.1.

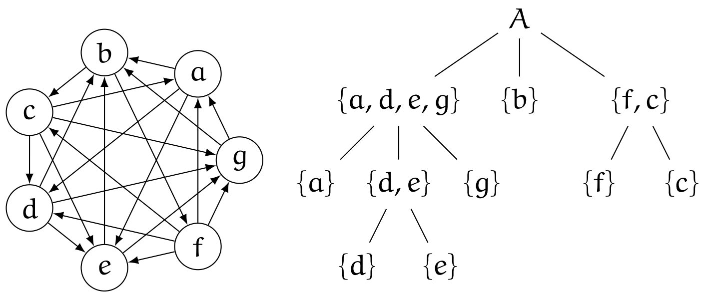



**Figure 3.1:** Example tournament with corresponding decomposition tree. Nodes $\{f, c\}$ and $\{d, e\}$ are reducible, all other nodes are irreducible.

An internal (i.e., non-leaf) node $B$ of $\mathcal{D}(T)$ with children $B_1, \dots, B_k$ corresponds to the tournament $T_B = (\{1, \dots, k\}, \tilde{\succ})$ where $i \tilde{\succ} j$ if and only if $B_i \succ B_j$, i.e., $T_B$ is the summary of $T|_B$ with respect to the minimal decomposition $\{B_1, \dots, B_k\}$. The order of $T_B$ is thus equal to the number of children of node $B$. Moreover, we call an internal node $B$ *reducible* (respectively, *irreducible*) if the tournament $T_B$ is reducible (respectively, irreducible).[16] If $B$ is reducible, we assume without loss of generality that the children $B_1, \dots, B_k$ are labeled according to their transitive summary, i.e., $B_i \succ B_j$ if and only if $i < j$. In particular, the maximum of $T_B$ is 1.

With the help decomposition tree $\mathcal{D}(T)$, we can argue that the complexity of the original tournament has been split up into its irreducible components. With this in mind, we define the decomposition degree of $T$ as the size of the largest irreducible component.

Definition 3.6

Let $Irr(\mathcal{D}(T))$ be the set of irreducible internal nodes of $\mathcal{D}(T)$. The *decomposition degree* $\delta(T)$ of a tournament $T$ is defined as

*decomposition degree*

$$\delta(T) = \begin{cases} \max\{|T_B| : B \in Irr(\mathcal{D}(T))\}, & \text{if } Irr(\mathcal{D}(T)) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition degree of the example tournament in Figure 3.1 is 3, attained by the nodes $A$ and $\{a, d, e, g\}$.

16 $T|_B$ is reducible (respectively, irreducible) if and only if its summary $T_B$ is.

## 3.3 COMPUTING THE DECOMPOSITION TREE OF A TOURNAMENT

Our aim, which we will pursue further in Chapter 7 it to beneficially use the decomposition tree to speed up the computation of functions defined on tournaments. A necessary prerequisite for this approach is that the decomposition tree can be computed efficiently in a preprocessing step. Fortunately, it was recently shown that that the full decomposition tree of a tournament can be computed in linear time.[17]

Proposition 3.7 *(McConnell and de Montgolfier, 2005)*
The decomposition tree of a tournament T can be computed in time $\mathcal{O}(|T|^2)$.

*factorizing permutation*

In fact, the results by McConnell and de Montgolfier and the related paper by Capelle et al. (2002) even hold for general digraphs. The idea is to first find a *factorizing permutation* $\sigma$, a permutation of the alternatives in which each component of the digraph forms a consecutive block. Then, with the help of $\sigma$, construct an approximation to the decomposition tree called the *fracture tree*. This tree might have some nodes in excess and may not properly represent all transitive components. After taking care of these, this gives the decomposition tree.

As a corollary to Proposition 3.7, $\delta(T)$ can be computed efficiently.

## 3.4 EXPERIMENTS: DECOMPOSABILITY

Similarly to our measuring of transitivity in Section 2.3, we experimentally studied the degree of decomposability of tournaments from the different stochastic models. We measure this in terms of the frequency of reducible tournaments and by the average (normalized) decomposition degree of the generated tournaments.

The results of our experiments are again shown as heat maps in Figures 3.2 and 3.4. For the reducibility criterion, we find that for the urn models (including IC and IAC), the results are essentially the same as for the frequency of Condorcet winners, cf. Figure 2.2.[18] For the distance-based models, we see that the sampled tournaments are often times reducible even when they do not exhibit a Condorcet winner.[19] Spatial models, which had the highest frequency of Condorcet winners already, almost always induce reducible tournaments.

17 The size of the representation of a tournament is already quadratic in the number of its alternatives.

18 Note that every tournament that admits a Condorcet winner is also reducible.

19 A small caveat: We used a different value for the parameter in the Condorcet noise model ($p = 0.65$ here compared to $p = 0.55$ in Figure 2.2) to capture the different phase transitions.

For the decomposability, the picture changes a bit as Condorcet winners not necessarily imply small decomposition degrees which is most notable for the spatial model. The results for distance-based models strongly resemble those for the degree of transitivity in Figure 2.1. With their strong tendency towards a particular reference ranking (or, in case of Mallows mixtures, a combination of few reference rankings), the decomposability of the obtained tournaments is tightly linked to their degree of transitivity.

## 3.5 SUMMARY

We discussed the decomposition tree of a tournament and defined the decomposition degree that informally captures the complexity reduction obtained through the decomposition. We examined the decomposability of tournaments generated through stochastic preference models. For the distance-based models, decompositions seem to be a valid technique to reduce complexity. Unsurprisingly, we found that models that do not impose any or little structure on the preferences will not give highly decomposable tournaments. In fact, it is not difficult to show that under the uniform random tournament model, the probability that just a single non-trivial component exists goes to zero (McKay, personal communication).

Overall, we can argue that all but the urn models (including IC and IAC) on average give tournaments that are properly decomposable.

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.93 | 0.77 | 0.90 | 0.73 | 0.83 | 0.90 | 0.90 | 0.87 | 0.93 | 0.97 | 0.93 |
| 10 | 0.87 | 0.77 | 0.83 | 0.90 | 0.60 | 0.90 | 0.70 | 0.90 | 0.83 | 0.77 | 0.87 |
| | 0.83 | 0.80 | 0.87 | 0.70 | 0.70 | 0.77 | 0.93 | 0.70 | 0.63 | 0.67 | 0.70 |
| 20 | 0.70 | 0.50 | 0.60 | 0.70 | 0.50 | 0.57 | 0.43 | 0.57 | 0.60 | 0.47 | 0.57 |
| | 0.67 | 0.47 | 0.53 | 0.40 | 0.37 | 0.37 | 0.40 | 0.37 | 0.43 | 0.50 | 0.33 |
| 30 | 0.50 | 0.43 | 0.33 | 0.53 | 0.40 | 0.20 | 0.50 | 0.40 | 0.47 | 0.40 | 0.53 |

impartial culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 20 | 0.87 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | 0.93 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30 | 0.90 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

spatial model (dim = 2)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.97 | 0.93 | 0.90 | 0.90 | 0.90 | 0.93 | 0.93 | 0.83 | 0.83 | 0.93 | 0.93 |
| 10 | 0.80 | 0.87 | 0.77 | 0.77 | 0.87 | 0.80 | 0.87 | 0.73 | 0.70 | 0.80 | 0.80 |
| | 0.77 | 0.83 | 0.63 | 0.80 | 0.40 | 0.67 | 0.67 | 0.70 | 0.63 | 0.47 | 0.73 |
| 20 | 0.70 | 0.60 | 0.60 | 0.60 | 0.53 | 0.57 | 0.63 | 0.33 | 0.60 | 0.53 | 0.47 |
| | 0.67 | 0.53 | 0.33 | 0.47 | 0.47 | 0.33 | 0.30 | 0.40 | 0.40 | 0.50 | 0.43 |
| 30 | 0.47 | 0.33 | 0.53 | 0.50 | 0.37 | 0.57 | 0.50 | 0.47 | 0.37 | 0.37 | 0.40 |

impartial anonymous culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.93 | 0.90 | 1.00 | 0.87 | 0.83 | 0.93 | 0.87 | 0.93 | 0.90 | 0.93 | 0.93 |
| 10 | 0.87 | 0.87 | 0.77 | 0.73 | 0.80 | 0.87 | 0.80 | 0.87 | 0.83 | 0.67 | 0.60 |
| | 0.70 | 0.73 | 0.70 | 0.70 | 0.87 | 0.67 | 0.73 | 0.63 | 0.63 | 0.63 | 0.73 |
| 20 | 0.80 | 0.43 | 0.50 | 0.60 | 0.80 | 0.73 | 0.63 | 0.60 | 0.50 | 0.63 | 0.60 |
| | 0.73 | 0.63 | 0.43 | 0.53 | 0.60 | 0.43 | 0.63 | 0.37 | 0.50 | 0.50 | 0.40 |
| 30 | 0.53 | 0.33 | 0.43 | 0.43 | 0.47 | 0.33 | 0.27 | 0.47 | 0.43 | 0.47 | 0.47 |

urn ($\alpha = 10$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1.00 | 0.83 | 0.93 | 1.00 | 0.93 | 0.97 | 0.90 | 0.93 | 0.97 | 0.93 | 0.93 |
| 10 | 0.83 | 0.87 | 0.90 | 0.93 | 0.97 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | 0.63 | 0.67 | 0.93 | 0.93 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 20 | 0.73 | 0.77 | 0.90 | 0.93 | 1.00 | 0.97 | 1.00 | 0.97 | 1.00 | 1.00 | 1.00 |
| | 0.67 | 0.80 | 0.90 | 0.97 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30 | 0.67 | 0.87 | 0.90 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Mallows-$\phi$ ($\phi = 0.95$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.93 | 1.00 | 0.90 | 0.90 | 0.93 | 0.87 | 0.93 | 0.97 | 0.90 | 0.97 | 0.97 |
| 10 | 0.83 | 0.80 | 0.93 | 1.00 | 0.97 | 0.93 | 1.00 | 1.00 | 1.00 | 1.00 | 0.93 |
| | 0.77 | 0.80 | 0.87 | 0.90 | 1.00 | 1.00 | 1.00 | 1.00 | 0.97 | 1.00 | 1.00 |
| 20 | 0.73 | 0.83 | 0.97 | 0.97 | 1.00 | 1.00 | 1.00 | 0.97 | 1.00 | 1.00 | 0.97 |
| | 0.63 | 0.73 | 0.87 | 0.93 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30 | 0.60 | 0.87 | 0.97 | 0.97 | 1.00 | 1.00 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 |

Mallows 4-mixture (uniform, $\phi = 0.95$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.40 | 0.67 | 0.83 | 0.83 | 0.83 | 0.93 | 0.83 | 0.90 | 0.97 | 0.93 | 0.90 |
| 10 | 0.00 | 0.20 | 0.27 | 0.37 | 0.47 | 0.70 | 0.80 | 0.60 | 0.87 | 0.83 | 0.93 |
| | 0.00 | 0.07 | 0.03 | 0.10 | 0.30 | 0.47 | 0.63 | 0.60 | 0.63 | 0.77 | 0.87 |
| 20 | 0.00 | 0.00 | 0.00 | 0.03 | 0.07 | 0.23 | 0.27 | 0.53 | 0.57 | 0.60 | 0.83 |
| | 0.00 | 0.00 | 0.03 | 0.03 | 0.13 | 0.10 | 0.23 | 0.37 | 0.43 | 0.70 | 0.70 |
| 30 | 0.00 | 0.00 | 0.03 | 0.00 | 0.07 | 0.10 | 0.27 | 0.47 | 0.30 | 0.40 | 0.43 |

Condorcet noise ($p = 0.55$)

**Figure 3.2**: Experimental results regarding the frequency of reducible tournaments obtained from various stochastic models. Green entries correspond to a high turnout of reducible tournaments.

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.78 | 0.65 | 0.69 | 0.70 | 0.61 | 0.67 | 0.69 | 0.67 | 0.67 | 0.69 | 0.70 |
| 10 | 0.38 | 0.37 | 0.28 | 0.26 | 0.32 | 0.27 | 0.37 | 0.22 | 0.22 | 0.35 | 0.24 |
| | 0.16 | 0.14 | 0.10 | 0.12 | 0.12 | 0.15 | 0.12 | 0.12 | 0.14 | 0.14 | 0.12 |
| 20 | 0.08 | 0.05 | 0.04 | 0.06 | 0.05 | 0.06 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 |
| | 0.06 | 0.05 | 0.04 | 0.04 | 0.04 | 0.02 | 0.04 | 0.03 | 0.03 | 0.03 | 0.02 |
| 30 | 0.04 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.03 | 0.02 | 0.02 | 0.03 |

impartial culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.96 | 0.95 | 0.99 | 0.99 | 0.98 | 1.00 | 1.00 | 0.99 | 0.98 | 0.99 | 0.97 |
| 10 | 0.72 | 0.84 | 0.89 | 0.90 | 0.93 | 0.92 | 0.91 | 0.94 | 0.96 | 0.95 | 0.95 |
| | 0.68 | 0.68 | 0.79 | 0.81 | 0.85 | 0.87 | 0.90 | 0.89 | 0.89 | 0.92 | 0.94 |
| 20 | 0.57 | 0.65 | 0.73 | 0.78 | 0.79 | 0.84 | 0.84 | 0.83 | 0.86 | 0.87 | 0.88 |
| | 0.51 | 0.62 | 0.69 | 0.73 | 0.76 | 0.80 | 0.79 | 0.82 | 0.84 | 0.86 | 0.83 |
| 30 | 0.46 | 0.53 | 0.60 | 0.68 | 0.74 | 0.75 | 0.78 | 0.78 | 0.80 | 0.81 | 0.83 |

spatial model (dim = 2)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.74 | 0.69 | 0.65 | 0.69 | 0.77 | 0.69 | 0.67 | 0.74 | 0.64 | 0.63 | 0.75 |
| 10 | 0.36 | 0.29 | 0.23 | 0.28 | 0.33 | 0.22 | 0.34 | 0.29 | 0.28 | 0.26 | 0.21 |
| | 0.27 | 0.13 | 0.11 | 0.11 | 0.11 | 0.14 | 0.12 | 0.10 | 0.12 | 0.11 | 0.10 |
| 20 | 0.13 | 0.05 | 0.06 | 0.07 | 0.04 | 0.04 | 0.05 | 0.06 | 0.06 | 0.07 | 0.04 |
| | 0.06 | 0.03 | 0.03 | 0.04 | 0.04 | 0.03 | 0.02 | 0.03 | 0.04 | 0.03 | 0.04 |
| 30 | 0.03 | 0.02 | 0.03 | 0.01 | 0.03 | 0.02 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 |

impartial anonymous culture

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.77 | 0.75 | 0.61 | 0.67 | 0.69 | 0.69 | 0.72 | 0.63 | 0.69 | 0.64 | 0.76 |
| 10 | 0.41 | 0.27 | 0.24 | 0.28 | 0.25 | 0.30 | 0.27 | 0.33 | 0.20 | 0.30 | 0.26 |
| | 0.17 | 0.11 | 0.12 | 0.12 | 0.10 | 0.12 | 0.13 | 0.10 | 0.12 | 0.09 | 0.11 |
| 20 | 0.08 | 0.07 | 0.03 | 0.09 | 0.07 | 0.06 | 0.04 | 0.05 | 0.04 | 0.05 | 0.06 |
| | 0.05 | 0.04 | 0.04 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.02 | 0.03 |
| 30 | 0.04 | 0.03 | 0.03 | 0.04 | 0.02 | 0.02 | 0.02 | 0.02 | 0.03 | 0.02 | 0.02 |

urn ($\alpha = 10$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.75 | 0.71 | 0.73 | 0.73 | 0.78 | 0.79 | 0.67 | 0.69 | 0.79 | 0.77 | 0.73 |
| 10 | 0.40 | 0.32 | 0.31 | 0.46 | 0.49 | 0.51 | 0.59 | 0.57 | 0.54 | 0.69 | 0.67 |
| | 0.20 | 0.22 | 0.30 | 0.43 | 0.44 | 0.48 | 0.52 | 0.57 | 0.63 | 0.65 | 0.71 |
| 20 | 0.10 | 0.21 | 0.29 | 0.34 | 0.47 | 0.55 | 0.59 | 0.66 | 0.68 | 0.69 | 0.75 |
| | 0.11 | 0.12 | 0.27 | 0.42 | 0.45 | 0.52 | 0.66 | 0.65 | 0.66 | 0.74 | 0.74 |
| 30 | 0.04 | 0.10 | 0.28 | 0.39 | 0.47 | 0.62 | 0.63 | 0.70 | 0.73 | 0.74 | 0.78 |

Mallows-$\phi$ ($\phi = 0.95$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.70 | 0.62 | 0.80 | 0.66 | 0.78 | 0.77 | 0.71 | 0.73 | 0.75 | 0.83 | 0.71 |
| 10 | 0.34 | 0.47 | 0.36 | 0.47 | 0.52 | 0.49 | 0.48 | 0.58 | 0.58 | 0.57 | 0.70 |
| | 0.20 | 0.22 | 0.35 | 0.39 | 0.39 | 0.34 | 0.51 | 0.51 | 0.55 | 0.49 | 0.58 |
| 20 | 0.11 | 0.16 | 0.20 | 0.29 | 0.33 | 0.42 | 0.47 | 0.48 | 0.53 | 0.56 | 0.50 |
| | 0.07 | 0.13 | 0.21 | 0.26 | 0.32 | 0.43 | 0.46 | 0.49 | 0.53 | 0.58 | 0.58 |
| 30 | 0.09 | 0.09 | 0.18 | 0.23 | 0.36 | 0.45 | 0.44 | 0.51 | 0.49 | 0.51 | 0.57 |

Mallows 4-mixture ($\phi = 0.95$)

| m \ n | 5 | 35 | | | 125 | | | 215 | | | 305 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.39 | 0.46 | 0.43 | 0.54 | 0.60 | 0.69 | 0.74 | 0.74 | 0.75 | 0.87 | 0.86 |
| 10 | 0.03 | 0.04 | 0.09 | 0.15 | 0.20 | 0.22 | 0.37 | 0.41 | 0.48 | 0.43 | 0.57 |
| | 0.00 | 0.00 | 0.02 | 0.03 | 0.06 | 0.10 | 0.12 | 0.20 | 0.23 | 0.29 | 0.31 |
| 20 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.02 | 0.06 | 0.09 | 0.11 | 0.17 | 0.22 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.03 | 0.05 | 0.06 | 0.09 | 0.13 |
| 30 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.06 | 0.09 |

Condorcet noise ($p = 0.55$)

**Figure 3.4**: Experimental results on the decomposability of tournaments generated by various stochastic models. The numbers correspond to $1 - \delta(T)/|T|$. Green entries corresponds to a high degree of decomposability with the maximum being achieved by transitive tournaments; red entries indicate that either no or only large components exist.

# 4 ON MAJORITY RELATIONS OF FEW VOTERS

> [C]ontrary to what common sense might expect, even societies having millions of voters (and whose voters have millions of distinct preference orderings over the alternatives) can often be faithfully represented by a relatively small group of representatives.
>
> Scott L. Feld and Bernard Grofman, 1986

In many scenarios, such as voting in a committee, the number of voters is limited to a small constant. In this chapter, we study to which extent structured or arbitrary digraphs can be induced as the majority graph of a small electorate.

## 4.1 MAJORITY DIMENSION AND EXPRESSIVENESS

Given a (weighted) digraph we are interested in the minimal number of voters needed such that the digraph represents the (weighted) majority relation of the voters' preferences. This is captured in the majority dimension of the digraph.[20] Formally, the *majority dimension* of a digraph $G = (V, E)$ or a weighted digraph $G = (V, w)$ is the smallest number of voters in a profile that induces $G$, i.e.,

*majority dimension*

$$\dim(G) = \min\{k \in \mathbb{N} : G \text{ is a (weighted) } k\text{-majority digraph}\}.$$

Also, let $k_{maj}(m)$ denote the minimum electorate size required to induce all digraphs of size $m$, i.e.,

$$k_{maj}(m) = \min\{k : \dim(G) \leqslant k \text{ for all } G \in \mathcal{G}_m\}.$$

If we restrict our attention to tournaments, we will write $k^{\mathcal{T}}_{maj}(m)$ instead. Note that $k^{\mathcal{T}}_{maj}(m) \leqslant k_{maj}(m)$ since $\mathcal{T} \subset \mathcal{G}$.

Conversely, define the *majoritarian expressiveness* of (electorates of size) $k$ to be the maximum integer $m^{\mathcal{T}}(k)$ such that every complete majority relation on up to $m^{\mathcal{T}}(k)$ alternatives is $k$-inducible. Since the work by Erdős and Moser (1964) that we will discuss in more detail in the following, it is known that $m^{\mathcal{T}}(k)$ is finite for every $k$. Note that this implies that the smallest tournament that cannot be induced by $k$ voters is of size $m^{\mathcal{T}}(k) + 1$.

*majoritarian expressiveness*

McGarvey (1953) showed that every digraph can be induced by

*McGarvey's result*

20 This complexity measure of digraphs can also be interpreted as a complexity measure for preference profiles.

some preference profile and gave a construction that requires exactly two voters per edge in the digraph. In our notation, this implies that $k_{maj}(m) \leqslant m(m-1) < \infty$ for all $m$. Our first observation about $\dim(G)$ is that it has to be odd or even, depending on whether $G$ is a tournament or not.

Lemma 4.1
The majority dimension $\dim(G)$ is odd if $G$ is a tournament and even otherwise.

*Proof.* Due to McGarvey's result, $\dim(G)$ has to be finite for every digraph $G$. Let $T$ be a tournament and assume that $\dim(T) = k$ was even. Then there exists a preference profile $R$ with $k$ voters that induces $T$. Since $k$ is even, the majority margin must be even for every pair of alternatives and can furthermore never be zero as $T$ is a tournament. Therefore, removing any single voter from $R$ gives a profile $R'$ with just $k-1$ voters that still induces $T$, a contradiction.

For non-complete digraphs, the statement follows directly from the fact that for all preference profiles $R$ with an odd number of voters $k$, the majority relation $\succsim_R$ is complete and anti-symmetric (as no majority ties can occur). □

*bounds on $k_{maj}(m)$*

The work by McGarvey has been followed up by Stearns (1959) who showed that $k_{maj}(m) \leqslant m+2$ which was later improved by Fiol (1992) to $k_{maj}(m) \leqslant m - \lfloor \log m \rfloor + 1$. For larger $m$, Erdős and Moser (1964) gave the asymptotically better bound $k_{maj}(m) \leqslant c \cdot \frac{m}{\log m}$ for some constant $c$. Their work nicely complemented a second result by Stearns (1959) who proved that $k_{maj}(m) > 0.55 \cdot \frac{m}{\log m}$ for large $m$. Together, this gives that $k_{maj}(m)$ is in $\Theta(\frac{m}{\log m})$. Note that the lower bound on $\dim(G)$ gives an upper bound to $m^{\mathcal{T}}(k)$, i.e., it proves that for every electorate size $k$, there exist digraphs that are not $k$-inducible. Still, $k^{\mathcal{T}}_{maj}(m)$ could be bounded by a constant and all tournaments could be inducible by some constant electorate size. The following lemma shows that this is not the case by an argument similar to the one by Stearns.

Lemma 4.2
If $k^{T}_{maj}(m) = k \geqslant 3$, then

$$\binom{m}{2} \cdot \ln(2) \leqslant k \cdot \left(\ln(2) + \sum_{i=2}^{m} \ln(i)\right) - \ln(k!). \qquad (1)$$

*Proof.* If every tournament on $m$ vertices can be induced by $k$ voters, then for every $T \in \mathcal{T}$, there needs to be at least one anonymous $k$-voter profile that induces $T$. There are $m!$ possible preference orders over $m$ alternatives, and the number of anonymous $k$-voter profiles

| $k$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|
| $m^{\mathcal{T}}(k)$ | 19 | 42 | 67 | 94 | 123 | 153 | 184 | 217 | 250 | 283 |

**Table 4.1:** Upper bounds on the size $m^{\mathcal{T}}(k)$ of the smallest tournament that is not $k$-majority for small odd $k$.

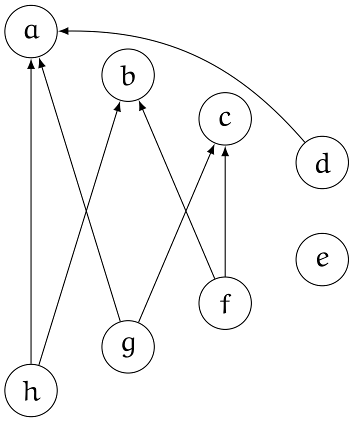



**Figure 4.1:** A tournament on 8 vertices with majority dimension 5. This is a smallest tournament that cannot be induced by three voters. Omitted edges point downwards.

is $\binom{m!+k-1}{k}$. Also, the number of labeled tournaments on $m$ vertices is $2^{\binom{m}{2}}$ implying that

$$2^{\binom{m}{2}} \leqslant \binom{m!+k-1}{k} \leqslant \frac{(2(m!))^k}{k!}$$

where the last inequality follows from Fiol's bound stated before. The result follows immediately. □

Using the lemma, we can search for an upper bound on $m^{\mathcal{T}}(k)$ for a given $k$ efficiently by finding the minimal $m$ such that (1) is violated. The results, for some small $k$ can be found in Table 4.1. It shows, for example, that there exists a tournament of size 42 that is not 5-inducible.[21]

It is clear, however, that these bounds are not tight. For example, the results in the table imply there has to exist a tournament of size 19 that is not 3-inducible. In fact, Shepardson and Tovey (2009) proved that every tournament that contains a certain 8-vertex digraph as a subgraph is not 3-inducible. In section 4.4.1, we will argue that there are no smaller tournaments with this property. An example of such a tournament is shown in Figure 4.1.

Also, our bounds for the size of the smallest tournament that is not $k$-inducible are obtained non-constructively. Alon (2006) pursued a systematic approach to construct tournaments with high majority dimension by analyzing dominating sets. His argument goes as fol-

*construct tournaments with high majority dimension*

21 A slightly tighter analysis even gives the existence of such a tournament of size 41.

lows.

A *dominating set* of a digraph $G = (V, E)$ is a set $U \subseteq V$ such that for all $v \in V \setminus U$, there exists a $u \in U$ such that $(u, v) \in E$. Alon (2006) showed that the size of the smallest dominating set of any $k$-majority graph for odd $k$ is bounded from above by a function $\mathcal{F}(k)$ with $\mathcal{F}(k) \in \mathcal{O}(k \log k)$ and $\mathcal{F}(k) \in \Omega(\frac{k}{\log k})$ with rather large constants hidden in the Landau notation (80 for $\mathcal{O}$). This means that if a given tournament $T$ does not have a dominating set of size $\mathcal{F}(k)$, then $T$ is not inducible by $k$ voters.

This can be leveraged to construct a tournament not inducible by $k$ voters due to the following constructive result by Graham and Spencer (1971). Let $f(x) = p > x^2 2^{2x-2}$ where $x$ is a positive integer and $p$ is the smallest prime congruent to $3 \mod 4$ satisfying the inequality (the construction works for any such $p$). Then, the *quadratic residue tournament* $Q_p$ of size $p$ does not exhibit a dominating set of size $x$.[22]

*quadratic residue tournament*

Together, this gives us a construction for a tournament on $(f \circ \mathcal{F})(k)$ vertices that is not a $k$-majority graph for any odd $k$. Unfortunately, $f(x)$ is exponential in $x$, and the value of $\mathcal{F}(k)$ is known precisely only for $k = 3$ where we have $\mathcal{F}(3) = 3$. To our knowledge, the best currently available bound for $k = 5$ is $\mathcal{F}(5) \leqslant 12$ (Fidler, 2011). Together, we get that for the smallest (or any other) prime $p$ congruent to $3 \pmod 4$ fulfilling the inequality

$$p > 12^2 \cdot 2^{2 \cdot 12 - 2} = 603{,}979{,}776$$

$Q_p$ is not inducible by 5 voters. Bounds on $\mathcal{F}(k)$ for larger odd $k$ give wildly worse values: for 7 voters, we already have $\mathcal{F}(7) \leqslant 44$ due to Fidler (2011).

## 4.2 MAJORITY RELATIONS OF FEW VOTERS

In this section, we analyze the structure of digraphs that are $k$-majority digraphs for a constant $k$. Building on earlier work by Dushnik and Miller (1941) that implied a characterization of 2-majority digraphs, we give a characterization for the case of three voters. In addition, we present sufficient conditions for larger majority dimensions.

### 4.2.1 Two and Three Voters

*Pareto relation*

Given a preference profile $R$, the *Pareto relation* holds between two alternatives $v$ and $w$ if all voters prefer $v$ over $w$. Dushnik and Miller (1941) specified sufficient and necessary conditions for relations to be

*characterization of 2-majority digraphs*

22 $Q_p = (A, \succ)$ with $A = (a_1, \ldots, a_p)$ and $a_i \succ a_j$ if and only if

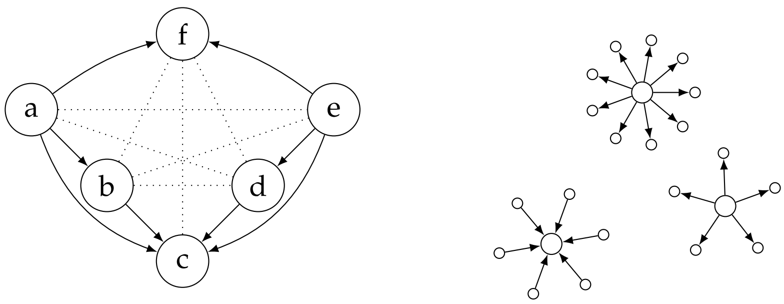
$$(i - j)^{\frac{p-1}{2}} \equiv 1 \mod p$$

induced as the Pareto relation of a 2-voter preference profile. As for two voters the majority relation and the Pareto relation coincide, we can rephrase their result for majority graphs as follows.

LEMMA 4.3 *(Dushnik and Miller, 1941)*

A majority graph $(V, E)$ is induced by a 2-voter preference profile if and only if it is transitive and its *incomparability graph* $(V, \widetilde{E})$ is transitively orientable. Moreover, the weight of every edge is 2.

See Figure 4.2a for an example of a digraph that is not 2-inducible even though it is transitive. If it was 2-inducible there would have to exist a transitive reorientation $E'$ of $\widetilde{E}$. We can assume w.l.o.g. that $(b, d) \in E'$. But then $(a, d)$ and $(b, e)$ also have to be in $E'$ leaving no option to orient $\{a, e\}$ without getting a contradiction to the assumed transitivity of $E'$.




**(a)** This digraph cannot be induced by a 2-voter profile. Dotted edges denote the incomparability graph.

**(b)** Every forest of directed stars is 2-inducible.

**Figure 4.2:** Examples of transitive digraphs.

If, on the other hand, a graph $(V, E)$ is in fact induced by a 2-voter profile $(R_1, R_2)$, then $R_1$ and $R_2$ coincide on $E$ and are opposed on $\widetilde{E}$, i.e., $R_1 \cap R_2 = E$. As $R_1$ and $R_2$ are both transitive, so is $E$. If $E'$ is the respective reorientation of $\widetilde{E}$, then $R_1 = E \cup E'$ and $R_2 = E \cup \overline{E'}$, or *vice versa*. As a useful notational convention we adopt $\|E\| = E \cup \overline{E}$, where $\overline{E}$ is the *converse* of $E$, i.e., $\overline{E} = \{(w, v) : (v, w) \in E\}$. *converse*

A graph $(V, E)$ is a *unidirected star* if there is some $v^* \in V$ such that either $E$ or $\overline{E}$ equals $\{v^*\} \times (V \setminus \{v^*\})$. Clearly, $(V, E)$ is transitive as there are no $v, w, u \in V$ such that both $(v, w), (w, u) \in E$. Moreover, every transitive relation over the leaves of $(V, E)$ serves as a transitive orientation of $\widetilde{E}$. With Lemma 4.3 this gives us the following which is a special case of a result by Erdős and Moser (1964, Lemma 1). *unidirected star*

LEMMA 4.4

Every unidirected star is 2-inducible.

Another insight that follows from Lemma 4.3, is that the union of pairwise disjoint graphs that are induced by 2-voter profiles is also induced by a 2-voter profile.

LEMMA 4.5
Let $V_1, \dots, V_k$ be pairwise disjoint and $(V_1, E_1), \dots, (V_k, E_k)$ majority graphs induced by 2-voter profiles. Then, $(V_1 \cup \dots \cup V_k, E_1 \cup \dots \cup E_k)$ is also induced by a 2-voter profile.

*Proof.* Let $V = V_1 \cup \dots \cup V_k$ and $E = E_1 \cup \dots \cup E_k$ and consider the graph $(V, E)$. As each of $(V_1, E_1), \dots, (V_k, E_k)$ is induced by a 2-voter profile, by Lemma 4.3, each of $E_1, \dots, E_k$ is transitive and each of $\widetilde{E}_1, \dots, \widetilde{E}_k$ is transitively orientable. Let $E'_1, \dots, E'_k$ be the respective transitive reorientations of $\widetilde{E}_1, \dots, \widetilde{E}_k$. Since $V_1, \dots, V_k$ are pairwise disjoint, $E_1 \cup \dots \cup E_2$ can readily be seen to be transitive as well. Let furthermore $E^* = \bigcup_{1 \leqslant i < j \leqslant k} (V_i \times V_j)$. Observe that $\widetilde{E} = \widetilde{E}_1 \cup \dots \cup \widetilde{E}_k \cup E^*$ and that $E'_1 \cup \dots \cup E'_k \cup E^*$ is a transitive reorientation of $\widetilde{E}$. The claim then follows by another application of Lemma 4.3. □

Consequently, every forest of (unidirected) stars such as the one shown in Figure 4.2b is 2-inducible.[23]

Apart from a family of tournaments of order eight that are *not* 3-majority (Shepardson and Tovey, 2009), little was known about the majority graphs that are induced by 3-voter profiles. In a much similar vein as Lemma 4.3, we now provide a characterization of these graphs.

*characterization of 3-majority digraphs*

LEMMA 4.6
A tournament $(V, E)$ is induced by a 3-voter profile if and only if there are disjoint sets $E_1, E_2$ with $E = E_1 \cup E_2$ such that $E_1$ is transitive and $E_2$ is both acyclic and transitively reorientable. Then, the weight of every edge in $E_1$ is either 1 or 3 and that of each edge in $E_2$ is 1.

*Proof.* For the if-direction, assume that there are disjoint sets $E_1, E_2$ with $E = E_1 \cup E_2$ such that $E_1$ is transitive and $E_2$ is both acyclic and transitively reorientable. Consider the graph $(V, E_1)$ and observe that for the corresponding incomparability graph $(V, \widetilde{E}_1)$, $\widetilde{E}_1 = \|E_2\|$. It follows that $\widetilde{E}_1$ is transitively orientable and, by Lemma 4.3, that $(V, E_1)$ is induced by a 2-voter profile $(R_1, R_2)$ and that all edges in $E_1$ have weight 2. As $E_2$ is acyclic, there is a (strict) preference relation $R_3$ with $E_2 \subseteq R_3$. Now consider the majority graph induced by the preference profile $(R_1, R_2, R_3)$, which apparently coincides with $(V, E)$. $E_1$ is determined by $R_1$ and $R_2$ and each of its edges obtains weight 1

23 Erdős and Moser (1964) gave a different class of graphs that are 2-inducible which they call *bilevel graphs*. A bilevel graph is the union of a finite number of vertex-disjoint digraphs $(V_1, E_1), (V_2, E_2), \cdots$ such that each $(V_i, E_i)$ is complete bipartite and unidirected, i.e., there is a partition into vertex sets $V_{i,1}, V_{i,2}$ such that $E_i = V_{i,1} \times V_{i,2}$.

or 3 depending on whether $R_3$ agrees with both $R_1$ and $R_2$ or not. Moreover, $E_2$ is determined by $R_3$, as $R_1$ and $R_2$ can be assumed to specify contrary preferences on this part.

For the only-if-direction, assume that $(V, E)$ is the majority graph induced by the 3-voter preference profile $(R_1, R_2, R_3)$. Let furthermore $(V, E_1)$ be the majority graph induced by $(R_1, R_2)$ and $E_2 = R_3 \cap ((V \times V) \setminus \|E_1\|)$. By Lemma 4.3, $(V, E_1)$ is transitive and $\widetilde{E_1}$ is transitively (re)orientable, where $(V, \widetilde{E_1})$ is the incomparability graph of $(V, E_1)$. As $R_3$ is transitive (and strict) $E_2$ is obviously acyclic. Observe furthermore that $\|R_3 \cap ((V \times V) \setminus \|E_1\|)\| = \|\widetilde{E}_1\|$. It follows that $E_2$ is transitively reorientable. □

### 4.2.2 More than Three Voters

Extensions of these results provide useful sufficient conditions for a graph to be induced by a constant larger number of voters. We say that two edge sets $E_1$ and $E_2$ are *orientation compatible* if

*orientation compatibility*

$$E_1 \cap (\|E_1\| \cap \|E_2\|) = E_2 \cap (\|E_1\| \cap \|E_2\|).$$

We show that if the edge set of a graph can be decomposed into pairwise orientation compatible sets such that each satisfies the conditions of Lemma 4.3, the graph is induced by a profile with two voters per set.

LEMMA 4.7

Let $(V, E_1), \ldots, (V, E_k)$ be majority graphs induced by 2-voter profiles such that $E_1, \ldots, E_k$ are pairwise orientation compatible. Then, $(V, E_1 \cup \cdots \cup E_k)$ is induced by a 2k-voter profile.

*Proof.* Let for each $i$ with $1 \leqslant i \leqslant k$, $(R_1^i, R_2^i)$ be a 2-voter profile that induces $(V, E_i)$. By Lemma 4.3, for every $(v, w) \in E_i$ we know that both $v\ R_1^i\ w$ and $v\ R_2^i\ w$ and for every $(v, w) \notin E_i$, $v\ R_1^i\ w$ if and only if $w\ R_2^i\ v$. Now consider the preference profile $(R_1^1, R_2^1, \ldots, R_1^k, R_2^k)$ and the majority graph $(V, E)$ it induces. We argue that $E = E_1 \cup \cdots \cup E_k$. First assume that $(v, w) \in E_i$ for some $i$ with $1 \leqslant i \leqslant k$. Then, both $v\ R_1^i\ w$ and $v\ R_2^i\ w$. Since, $E_1, \ldots, E_k$ are pairwise orientation compatible, $(w, v) \in E_j$ for no $j$ with $1 \leqslant j \leqslant k$, i.e., for all $j$ with $1 \leqslant j \leqslant k$ either $v\ R_1^j\ w$ and $v\ R_2^j\ w$ or $v\ R_1^j\ w$ if and only if $w\ R_2^j\ v$. It follows that a majority prefers $v$ over $w$ and thus $(v, w) \in E$. Now assume that $(v, w) \in E_i$ for no $i$ with $1 \leqslant i \leqslant k$. Then for all $i$ with $1 \leqslant i \leqslant k$ either both $w\ R_1^i\ v$ and $w\ R_2^i\ v$ or $w\ R_1^j\ v$ if and only if $v\ R_2^j\ w$. it is easy to see that $v$ is not majority preferred to $w$, i.e., $(v, w) \notin E$. □

Next, we show that a similar condition suffices for a graph to be inducible by a certain odd number of voters.[24]

24 The if-direction of Lemma 4.6 can also be obtained as a special case of this lemma.

Lemma 4.8
Let $(V, E)$ be a tournament and $(V, E_1), \ldots, (V, E_k)$ be majority graphs induced by 2-voter profiles such that $E, E_1, \ldots, E_k$ are orientation compatible. Let, moreover, $E_{k+1} \supseteq E \setminus (E_1 \cup \cdots \cup E_k)$ be acyclic. Then, $(V, E)$ is induced by a $2k+1$-voter profile.

*Proof.* In virtue of Lemma 4.7 we know that $(V, E_1 \cup \cdots \cup E_k)$ is induced by a 2k-voter profile $(R_1^1, R_2^1, \ldots, R_1^k, R_2^k)$. Inspection of the proof also reveals that every edge $(v, w) \in E_1 \cup \cdots \cup E_k$ has a positive even weight of at least two. As $E_{k+1}$ is acyclic and asymmetric, there is some (strict) preference relation $R^{k+1}$ with $E_{k+1} \subseteq R^{k+1}$. Moreover, since $E_{k+1}$ corresponds to only one voter and every edge in $E_1 \cup \cdots \cup E_k$ has a majority of at least two, $E_{k+1}$ does not have to be orientation compatible with any of $E_1, \ldots, E_k$. It can then easily be seen that the majority graph induced by $(R_1^1, R_2^1, \ldots, R_1^k, R_2^k, R^{k+1})$ equals $(V, E)$, $E_1 \cup \cdots \cup E_k$ being determined by majorities of at least one in $(R_1^1, R_2^1, \ldots, R_1^k, R_2^k, R^{k+1})$ and $E \setminus (E_1 \cup \cdots \cup E_k)$ by $R^{k+1}$, each edge in which has then weight one. □

## 4.3 DETERMINING THE MAJORITY DIMENSION OF A DIGRAPH

In this section, we address the computational problem of computing the majority dimension. To this end, we define the problem of checking whether for a given digraph G there exists a preference profile with k voters that induces G, i.e., whether G is a k-majority digraph.

Check-k-Majority
*Input:* A digraph G and a positive integer k.
*Question:* Is G a k-majority digraph?

Recall that for a digraph G, whether $\dim(G)$ is odd or even depends on whether G is complete (i.e., a tournament) or not, according to Lemma 4.1.

In the following, we provide an implementation for computing the minimal number of voters that is required to induce a given digraph. This implementation relies on an encoding of the problem as a Boolean satisfiability (SAT) problem which is then solved by a SAT solver. This technique turns out to be surprisingly efficient and easily outperforms an implementation for 3-majority digraphs based on the graph-theoretic characterization in Lemma 4.6.

### 4.3.1 Computing the Majority Dimension via SAT

The number of objects potentially involved in the Check-k-Majority problem are given in Table 4.2. It is immediately clear that a naïve

| | Preference profiles | | | Tournaments |
| --- | --- | --- | --- | --- |
| | $k = 1$ | $k = 3$ | $k = 5$ | (unlabeled) |
| $m = 5$ | 120 | $\sim 1.7 \cdot 10^{6}$ | $\sim 2.5 \cdot 10^{10}$ | 12 |
| $m = 10$ | $\sim 3.6 \cdot 10^{6}$ | $\sim 4.8 \cdot 10^{19}$ | $\sim 6.3 \cdot 10^{32}$ | $\sim 9.7 \cdot 10^{6}$ |
| $m = 25$ | $\sim 1.6 \cdot 10^{25}$ | $\sim 3.7 \cdot 10^{75}$ | $\sim 9.0 \cdot 10^{125}$ | $\sim 1.3 \cdot 10^{65}$ |
| $m = 50$ | $\sim 3.0 \cdot 10^{64}$ | $\sim 2.8 \cdot 10^{193}$ | $\sim 2.6 \cdot 10^{322}$ | $\sim 1.9 \cdot 10^{305}$ |
| $m = 100$ | $\sim 9.3 \cdot 10^{157}$ | $\sim 8.1 \cdot 10^{473}$ | $\sim 7.1 \cdot 10^{789}$ | $> 10^{1332}$ |

Table 4.2: Number of objects involved in the CHECK-k-MAJORITY problem for one, three, and five voters.

```
Input: digraph (A, ≻), positive integer k
Output: whether (A, ≻) is a k-majority digraph
/* Encoding of problem in CNF                                   */
File cnfFile;
foreach voter i do
    cnfFile += Encoder.reflexivePreferences(i);
    cnfFile += Encoder.completePreferences(i);
    cnfFile += Encoder.transitivePreferences(i);
    cnfFile += Encoder.antisymmetricPreferences(i);
cnfFile += Encoder.majorityImplications((A, ≻));
if ≻ is not complete then
    cnfFile += Encoder.indifferenceImplications((A, ≻));
/* SAT solving                                                  */
satisfiable = SATsolver.solve(cnfFile);
if instance is satisfiable then
    return true;
else
    return false
```

**Algorithm 4.1:** SAT-CHECK-k-MAJORITY

algorithm will not solve the problem in a satisfactory manner. We describe our algorithmic efforts to solve this problem for reasonably large instances.

*translation to propositional logic*

In order to answer CHECK-k-MAJORITY, we follow a similar approach as Tang and Lin (2009), Geist and Endriss (2011), and Brandt and Geist (2014): we translate the problem to propositional logic (on a computer) and use state-of-the-art SAT solvers to find a solution. At a glance, the overall solving steps are shown in Algorithm 4.1.

Generally speaking, the problem at hand can be understood as the problem of finding a preference profile that satisfies certain conditions, in this case: inducing a given digraph. Thus, a satisfying instance of the propositional formula to be designed should represent a preference profile. To capture this, a surprisingly simple formalization involving just one type of variable suffices: in our encoding the boolean variable $r_{i,a,b}$ represents $a\ R_i\ b$, i.e., voter $i$ ranking alternative $a$ at least as high as alternative $b$. As it turns out, this one variable type also suffices for the additional condition of inducing the given digraph.

More specifically, the following conditions or axioms need to be formalized:

1. All $k$ voters have linear orders over the $m$ alternatives as their preferences (short: linear preferences).

2. For each majority edge $a \succ b$ in the digraph, a majority of voters needs to prefer $a$ over $b$ (short: majority implications).

3. For each missing edge ($a \not\succ b$ and $b \not\succ a$) in the digraph, *exactly* half the voters need to prefer $a$ over $b$ (short: indifference implications).[25]

The details of the encoding are given in our papers (Brandt et al., 2014c; Bachmeier et al., 2014). Ultimately, this leads to a formula in conjunctive normal form with a total of

$$m^2 \cdot \left(k + \binom{k}{m(k)}\right)$$

variables for the case of tournaments and

$$m^2 \cdot \left(k + \binom{k}{m(k)} + \binom{k}{k/2}\right)$$

variables for incomplete digraphs. The number of clauses is equal to

$$k \cdot (m^3 + m^2) + \frac{m^2 - m}{2} \cdot \left(1 + \binom{k}{m(k)} \cdot m(k)\right)$$

for tournaments and at most

$$k \cdot (m^3 + m^2) + (m^2 - m) \cdot \left(1 + \binom{k}{k/2} \cdot \frac{k}{2}\right)$$

for incomplete digraphs.

This formalization of all axioms in propositional logic puts us in a position where we can analyze arbitrary digraphs $G$ for their majority dimension $\dim(G)$. Before we do so, however, we describe an optimization technique for tournament graphs, which, for certain instances, speeds up the computation significantly.

*optimization through decomposition*

Recognizing all components in a tournament can be done efficiently as shown in Proposition 3.7. Here, we demonstrate how the knowledge about the full decomposition tree of a tournament $T$ can be used to optimize the computation of the majority dimension $\dim(T)$. In particular, we show that the majority dimension of a tournament is equal to the maximum of the majority dimensions of its components and the corresponding summary.

25 Note that this axiom is only required for incomplete digraphs.

LEMMA 4.9

Let $T$ be a tournament and $\widetilde{B} = \{B_1, \dots, B_p\}$ a decomposition of $T$. Also, let $T_B$ the summary of $T$ with respect to $\widetilde{B}$ and for each $j \in \{1, \dots, p\}$, let $T_{B_j}$ denote the summary of $T|_{B_j}$ with respect to its minimal decomposition. Then,

$$\dim(T) = \max_j \{\dim(T_{B_j}), \dim(T_B)\}.$$

*Proof.* Let $R$ be a minimal profile inducing $T$. Then, $R|_{B_j}$ induces $T_{B_j}$ for every $B_j$ establishing $\dim(T) \geqslant \dim(T_{B_j})$. That $\dim(T) \geqslant \dim(T_B)$ holds is also easy to see by considering a variant of $R$ in which from each component all but one alternative are arbitrarily chosen and removed. The remaining profile then induces $T_B$. For the other direction, let $\nu'(T) = \max_j\{\dim(T_{B_j}), \dim(T_B)\}$. We know, by Lemma 4.1, that $\dim(T')$ and every $\dim(T_{B_j})$ is odd, as these are all tournaments. Each $T_{B_j}$ (and $T_B$) has a minimal profile $R^j$ (and $R$, respectively). We can add pairs of voters with opposing preferences to each profile without changing its majority relation. This way, we get profiles $R'^j$ (and $R'$) that still induce $T_{B_j}$ (or $T_B$) but now all have the same number of voters $\nu'(T)$. Now, create a new profile $\hat{R}$ from $R'$ in which $R^j_i$ replaced alternative $j$ as a segment in $R'_i$ for each voter $i$ and every alternative $j$ as in (Laffond et al., 1996). It is easy to check that $\hat{R}$ has $\nu'(T)$ voters and still induces $T$, i.e., $\dim(T) \geqslant \nu'(T) = \max_j \dim(T_{B_j})$. □

We have implemented this optimization and found that many real-world majority digraphs exhibit proper decompositions, speeding up the computation of SAT-CHECK-k-MAJORITY.

### 4.3.2 Computational Efficiency

*algorithm for checking 3-inducability*

The characterization of 3-majority digraphs in Section 4.2 allows for a straightforward algorithm, which is expected to have a much better running time than any naïve implementation enumerating all preference profiles (also compare Table 4.2). The corresponding algorithm 2-PARTITION-CHECK-3-MAJORITY is given in Algorithm 4.2. Besides enumerating all 2-partitions of the majority relations, the only non-trivial part is to check whether a relation has a transitive reorientation. This can be done efficiently using an algorithm by Pnueli et al. (1971).

We compared the running times of 2-PARTITION-CHECK-3-MAJORITY with the ones of our implementation via SAT as described in Section 4.3.1 (see also Algorithm 4.1).[26] It turns out that—even though it is much more universal—SAT-CHECK-3-MAJORITY offers significantly better running times (see Table 4.3). Note that in addition to being

26 As a programming language, Java was used in both cases.

| $m$ | SAT | 2-PARTITION |
|---|---|---|
| 5 | $< 0.1$s | $< 0.1$s |
| 6 | $< 0.1$s | $< 0.1$s |
| 7 | $< 0.1$s | $0.1$s |
| 8 | $< 0.1$s | $530$s |
| 9 | $< 0.1$s | — |
| 10 | $< 0.1$s | — |
| 20 | $0.1$s | — |
| 50 | $1.5$s | — |
| 100 | $12.5$s | — |

**Table 4.3:** Runtime comparison of the SAT implementation for $k = 3$ and 2-PARTITION-CHECK-3-MAJORITY for complete digraphs (tournaments) of different sizes $m$ with a cutoff time of one hour.

more efficient, SAT-CHECK-k-MAJORITY is even able to return a preference profile with $k$ voters that induces the given digraph (without the need for additional computations).

Further runtimes, which exhibit the practical power of our SAT approach (and its limits), can be obtained from Table 4.4. All experiments were run on an Intel Core i5, 2.66GHz (quad-core) machine with 12 GB RAM using the SAT solver PLINGELING (Biere, 2013).

```
Input: digraph (A, ≻)
Output: whether (A, ≻) is a 3-majority digraph
if ≻ is incomplete then
    return false;
else
    foreach 2-partition {≻1, ≻2} of ≻ do
        if ≻1 is transitive and ≻2 is acyclic and ≻2 has a transitive
        reorientation then
            return true;
    return false;
```

**Algorithm 4.2:** 2-PARTITION-CHECK-3-MAJORITY

## 4.4 ANALYZING MAJORITY DIMENSIONS

With the method described in the previous section, we are in a position to analyze the majority dimension of digraphs. In this section, we report on our findings for different sources of digraphs.

| m\k | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | .04 | .04 | .03 | .04 | .04 | .04 | .04 | .05 | .08 | .10 |
| 4 | .03 | .04 | .03 | .04 | .04 | .04 | .05 | .07 | .10 | .18 |
| 5 | .03 | .04 | .03 | .04 | .06 | .05 | .06 | .09 | .16 | .35 |
| 6 | .03 | .04 | .04 | .04 | .05 | .06 | .08 | .12 | .27 | .63 |
| 7 | .04 | .04 | .04 | .05 | .05 | .07 | .10 | .17 | .45 | 1.10 |
| 8 | .04 | .05 | .05 | .05 | .07 | .08 | .13 | .23 | .69 | 1.80 |
| 9 | .04 | .05 | .05 | .64 | .07 | .10 | .17 | .33 | 1.06 | 2.83 |
| 10 | .05 | .05 | .06 | .67 | .09 | .12 | .23 | .46 | 1.56 | 4.25 |
| 11 | .06 | .06 | .06 | 1.92 | .10 | .14 | .30 | .63 | 2.22 | 6.37 |
| 12 | .06 | .07 | .07 | 3.35 | .12 | .19 | .40 | .85 | 3.18 | 8.48 |
| 13 | .07 | .07 | .09 | 3.93 | .15 | .27 | .52 | 1.16 | 4.44 | 12.30 |
| 14 | .07 | .09 | .10 | 4.15 | .18 | .36 | .64 | 1.51 | 5.99 | 16.84 |
| 15 | .08 | .10 | .13 | 3.89 | .21 | .88 | .79 | 2.22 | 7.67 | — |
| 16 | .09 | .11 | .14 | 4.12 | .25 | 4.55 | .99 | 2.90 | 9.80 | — |
| 17 | .10 | .12 | .19 | 4.41 | .29 | 7.15 | 1.23 | 4.69 | 12.48 | — |
| 18 | .11 | .14 | .23 | 4.76 | .35 | 17.51 | 1.53 | 8.25 | 15.97 | — |
| 19 | .12 | .15 | .35 | 4.97 | .43 | — | 1.80 | — | 19.99 | — |
| 20 | .13 | .17 | .54 | 5.04 | .47 | — | 2.21 | — | — | — |
| 21 | .14 | .18 | 5.87 | 6.15 | .63 | — | 2.71 | — | — | — |
| 22 | .16 | .20 | 11.07 | 5.43 | .96 | — | 3.24 | — | — | — |
| 23 | .17 | .23 | 18.95 | 5.76 | 1.57 | — | 4.12 | — | — | — |
| 24 | .20 | .26 | — | 5.87 | 2.56 | — | 4.60 | — | — | — |
| 25 | .22 | .29 | — | 6.12 | 4.21 | — | 5.85 | — | — | — |

**Table 4.4:** Runtime in seconds of SAT-CHECK-k-MAJORITY for different number of alternatives and different number of voters $k$ when average runtimes did not exceed 20 seconds. For this table, averages were taken over 5 samples from the uniform random tournament model.

### 4.4.1 Exhaustive Analysis

Using the tournament generator from the nauty toolkit (McKay and Piperno, 2013a), we generated all tournaments with up to 10 alternatives and found that all of these are 5-inducible. In fact, all tournaments of size up to seven are even 3-inducible, confirming a conjecture by Shepardson and Tovey (2009). They also showed that there exist tournaments of size 8 that are not 3-inducible. We confirmed that the exact number of such tournaments is 96 (out of 6880) as mentioned by Eggermont et al. (2013). One of these is depicted in Figure 4.1.

We have not encountered a single tournament for which we could show that it is not 5-inducible. Since quadratic residue tournaments of enormous size are the only concrete tournament of which we know that they have higher majority dimension (see Section 4.1), we examined small tournaments of this kind as well and found that

$$\dim(Q_{11}) = 3 \quad \text{and} \quad \dim(Q_{19}) = 5.$$

Unfortunately, we were not able to check whether the majority dimension of $Q_{23}$ is equal to 5 or larger as the SAT solver did not terminate in reasonable time.[27]

Another specific tournament that we considered is a tournament on 24 alternatives that will be presented in Section 10.2 and serves as the current minimal counterexample to a now disproved conjecture by Schwartz (1990) in social choice theory. We found it to be a 5-majority tournament, implying that the negative theoretical consequences of the counterexample already hold for scenarios with only 5 voters (and at least 24 alternatives).

### 4.4.2 Empirical Analysis

In the preference library PrefLib (Mattei and Walsh, 2013), scholars have contributed data sets from real world scenarios ranging from preferences over movies or sushi via Formula 1 championship results to real election data. Accordingly, the number of voters whose preferences originally induced these data sets vary heavily between 4 and 44000. At the time of writing, PrefLib contained 354 tournaments induced from pairwise majority comparisons as well as 185 incomplete majority digraphs.

Among the tournaments in PrefLib, 58 are 3-inducible. Out of the two largest tournaments in the data set with 240 and 242 alternatives, respectively, the first is a 5-majority tournament while on the second the Sat solver did not terminate within one day. The remaining tournaments are transitive and thus 1-inducible. Therefore, all checkable tournaments in PrefLib are inducible by only 5 voters.

For the non-complete majority digraphs in PrefLib, we found that the indifference constraints which are imposed on all missing edges change the picture. Not only does it negatively affect the running time of Sat-Check-$k$-Majority in comparison to tournaments which made us restrict our attention to instances with at most 40 alternatives, but it also seems to result in higher voter complexities of up to 8 among the 85 feasible instances. However, given that the number of voters in the profiles that originally induced these majority digraphs are often in the hundreds or thousands, we still consider these low majority dimensions.

### 4.4.3 Stochastic Analysis

Additionally, we considered stochastic models to generate tournaments of a given size $m$ as described in Section 2.2.

27 We terminated the solving process after a total of 6 weeks.

For up to 21 alternatives, we sampled preference profiles (each consisting of 51 voters[28]) from five of the models described in Section 2.2 and examined the corresponding majority graphs for their majority dimension using Sat-Check-k-Majority. The average complexities over 30 instances of each size are shown in Table 4.5. We see that the unbiased models (IC, IAC, uniform) tend to induce digraphs with higher majority dimension.

Again, we encountered no tournament that was not a 5-majority tournament.[29]

| $m$ | uniform | IC | IAC | Mallows-$\phi$ ($\phi = 0.95$) | spatial (dim = 2) |
|---|---|---|---|---|---|
| 3 | 1.40 | 1.13 | 1.13 | 1.13 | 1.00 |
| 5 | 3.00 | 1.67 | 2.13 | 1.33 | 1.13 |
| 7 | 3.00 | 2.67 | 2.67 | 2.47 | 1.33 |
| 9 | 3.13 | 3.00 | 3.00 | 2.67 | 1.60 |
| 11 | 3.93 | 3.07 | 3.00 | 2.87 | 2.33 |
| 13 | 4.80 | 3.07 | 3.20 | 2.93 | 2.53 |
| 15 | 5.00 | 3.27 | 3.40 | 3.00 | 2.67 |
| 17 | 5.00 | 3.40 | 3.80 | 2.93 | 2.80 |
| 19 | 5.00 | 4.27 | 4.20 | 3.00 | 2.80 |
| 21 | 5.00 | 4.47 | 4.33 | 3.00 | 2.87 |

**Table 4.5:** Average majority dimension in tournaments generated by stochastic (preference) models. The given values are averaged over 30 samples each.

28 We found that this size turned out to be sufficiently large to discriminate the different underlying stochastic models, cf. Section 9.3.

29 Our efforts also included checking more than 8 million uniform random tournaments with 12 alternatives.

# Part II

# TOURNAMENT SOLUTIONS

# 5 DEFINITIONS

> The advantage of making economics more mathematical is that it introduces order, precision, and a sense of objectivity into what would otherwise be considered a vague social science.
>
> — Ariel Rubinstein, 2007

Many problems in multiagent decision making can be addressed using tournament solutions, i.e., functions that associate with each complete and asymmetric relation on a set of alternatives a non-empty subset of the alternatives. Tournament solutions are most prevalent in social choice theory, where—as in this thesis—the binary relation is typically assumed to be given by pairwise majority comparisons (e.g., Moulin, 1986; Laslier, 1997). Other application areas include zero-sum games (Fisher and Ryan, 1995; Laffond et al., 1993b; Duggan and Le Breton, 1996), argumentation theory (Dunne, 2007; Dung, 1995), multi-criteria decision analysis (Arrow and Raynaud, 1986; Bouyssou et al., 2006), and coalitional games (Brandt and Harrenstein, 2010).

Recent years have witnessed an increasing interest in tournament solutions both in terms of their axiomatic as well as algorithmic properties by the multiagent systems community (Brandt and Fischer, 2008; Faliszewski et al., 2009; Brandt et al., 2010b; Brandt et al., 2014a; Brandt et al., 2013b) and the theoretical computer science community (Woeginger, 2003; Alon, 2006; Baumeister et al., 2013).

In this chapter, we define, illustrate and discuss a good number of tournament solutions that will be studied with different perspectives in the later chapters of this part. Axiomatic properties which rightfully play a very important role in the evaluation of tournament solutions are only mentioned in passing in this chapter. Those needed will be introduced in Chapter 10 for the concepts discussed there.

For an excellent overview, more details on most concepts and a more thorough treatment of axiomatic properties, we refer to Laslier (1997). Computational issues are discussed by Brandt (2009) and Hudry (2009). The presentation in this chapter is based on the corresponding section in our paper (Brandt et al., 2015b) with inspirations taken from the book chapters by Brandt et al. (2015a) and Fischer et al. (2015).

## 5.1 UNWEIGHTED SOLUTION CONCEPTS

An (unweighted) tournament solution is a function that maps a tournament to a nonempty subset of its alternatives and disregards the names of alternatives. In addition, we require that a Condorcet winner must be chosen whenever one exists.[30] Here, we deviate from Laslier's definition who required that Condorcet winners have to be chosen *uniquely* whenever they exist.[31]

DEFINITION 5.1

A function $S$ is a tournament solution if it associates each tournament $T = (A, \succ)$ with a subset $S(T) \subseteq A$ and it holds for all $T \in \mathcal{T}$ that

- $S(T) \neq \emptyset$,
- $\pi(S(T)) = S(\pi(T))$ for all isomorphisms $\pi$ on $\mathcal{T}$, and
- $CW(T) \subseteq S(T)$.

*neutrality*

Note that by the second property, every tournament solution automatically satisfies neutrality. Part of this is that every tournament solution has to select either every alternative of an orbit or none.

*refinement*

Let $S_1, S_2$ be two tournament solutions. We say that $S_1$ is *finer* than $S_2$ or is a *refinement* of $S_2$ and write $S_1 \subseteq S_2$ if $S_1$ always returns subsets of the choice sets of $S_2$, i.e.,

$$S_1 \subseteq S_2 \quad \Leftrightarrow \quad S_1(T) \subseteq S_2(T) \text{ for all } T \in \mathcal{T}.$$

*coarsening*

Conversely, $S_2$ is *coarser* than $S_1$ or a *coarsening* of $S_1$ in such a case.

We start with three non-complex tournament solutions that would not be seriously considered as choice functions in practical scenarios.

*trivial tournament solution*

The *trivial tournament solution* (*TRIV*) always selects all alternatives of a tournament. By definition, it is the coarsest tournament solution and does not even rule out alternatives in transitive tournaments. A first albeit very conservative approach to excluding alternatives is reflected in *Condorcet Non-Losers* (*CNL*) that excludes Condorcet losers whenever they exist and chooses the whole set of alternatives otherwise. An interesting aspect about *CNL* is that it is a non-trivial representative of the family of *simple* tournament solutions, defined by their property to never exclude more than one alternative. A third tournament solution that will serve as a more serious baseline for comparisons in terms of discriminativity is what we call the *Condorcet solution* (*COND*). It chooses the Condorcet winner whenever one exists and the whole set of alternatives otherwise.[32]

*simple tournament solution*

*Condorcet solution*

30 The last requirement is the property of *Condorcet-consistency* which is therefore automatically satisfied by every tournament solution.
31 This property is then called *strong Condorcet-consistency*.
32 Note that *COND* differs from *CW* (cf. Section 2.1.2) on all tournaments $T = (A, \succ)$ that exhibit no Condorcet winner. Then, $COND(T) = A$ whereas $CW(T) = \emptyset$ which is why *CW* does not constitute a tournament solution.

For the upcoming definitions, let $T = (A, \succ)$ be a tournament and let $M(T)$ denote the *adjacency matrix* of $T$ where entries $(m_{ab})_{a,b \in A}$ are 1 whenever $a \succ b$ and 0 otherwise.

*adjacency matrix*

### 5.1.1 Solutions Based on Scores

Several solution concepts are defined via scores being attached to alternatives with the choice set consisting of those alternatives that either tie for the highest score or have a non-negative score.

The *Copeland set* $CO(T)$ of $T$ consists of all alternatives whose dominion is of maximal size, i.e.,

*Copeland set*

$$CO(T) = \arg\max_{a \in A} |D(a)|.$$

This set can be easily computed in linear time by determining all out-degrees and choosing the alternatives with maximum out-degree.

The *Slater set* $SL(T)$ of $T$ consists of the maximal elements of those linear orders that have as many edges as possible in common with $T$ (Slater, 1961), i.e.,

*Slater set*

$$SL(T) = \{\max(L) : L \in \arg\max_{L' \in \mathcal{L}(A)} |L' \cap \succ|\}$$

Finding these linear orders is equivalent to solving an instance of the NP-complete problem *feedback arc set* (Alon, 2006; Charbit et al., 2007; Conitzer, 2006), which implies that checking membership in the Slater set is NP-hard (Charon and Hudry, 2010). Yet, there are implementations that are sufficiently fast on small instances (e.g., Charon and Hudry, 2011).

The *Markov set* of a tournament is defined as those alternatives that have maximum probability in the unique stationary distribution of a Markov chain associated with $T$ in the following way. Laslier (1997) used the tournament matrix to define the transition matrix of a Markov chain as

*Markov set*

$$N = \frac{1}{|T| - 1} \cdot (M + I_{CO})$$

where $I_{CO}$ is the diagonal matrix of the (Copeland) scores. Then, the Markov set $MA(T)$ of $T$ is defined as

$$MA(T) = \arg\max_{a \in A} \{p(a) : p \in \Delta(A) \text{ and } N \cdot p = p\}$$

where $\Delta(A)$ denotes the set of all probability distributions over $A$. Computing $p$ as the eigenvector of $N$ associated with the eigenvalue 1 is governed by matrix multiplication (Hudry, 2009) and therefore is in $\mathcal{O}(|T|^{2.3729})$ (Vassilevska Williams, 2012).

The *bipartisan set* of $T$ is defined as the support of the unique mixed Nash equilibrium of a zero-sum game, associated with $T$. To this end, let $G(T)$ denote the *skew-adjacency matrix* of a tournament $T$. This

*bipartisan set*

*skew-adjacency matrix*

skew-symmetric matrix is defined as the difference of the adjacency matrix and its transpose, i.e.,

$$G(T) = M(T) - M(T)^t.$$

This matrix is interpreted as the payoff matrix of a symmetric (two-player) zero-sum game.[33] Laffond et al. (1993b) and Fisher and Ryan (1995) have shown independently that every such game has a unique mixed Nash equilibrium $(p_T, p_T)$ which is equivalent to every tournament $T$ admitting a unique probability distribution $p_T \in \Delta(A)$ such that

$$\sum_{a,b \in A} p_T(a)q(b)G(T)_{a,b} \geqslant 0 \text{ for all } q \in \Delta(A).$$

The bipartisan set $BP(T)$ of $T$ is defined as the support of this equilibrium, i.e.,

$$BP(T) = \{a \in A : p_T(a) > 0\}.$$

Brandt and Fischer (2008) have shown that $BP(T)$ can be computed in polynomial time using a linear feasibility program.

### 5.1.2 Uncovered Set and Banks Set

If all majority relations were transitive, there would be little dispute about always picking the Condorcet winner as a singleton choice set. However, as we have seen throughout the first part of this thesis, the majority relation does not need to be transitive and may in fact be isomorphic to any binary relation (cf. Chapter 4). Two approaches to gain back transitivity are captured in the Banks set which considers inclusion-maximal transitive subsets and in the covering relation which is transitive subset of the dominance relation.

*Banks set*

The *Banks set* $BA(T)$ of $T$, introduced by Banks (1985) is defined as the maximal elements of maximal transitive subsets in $T$[34], i.e.,

$$BA(T) = \{a \in A : \exists B \in \mathcal{B}_T(a) \text{ such that } \nexists b \colon b \succ B\}.$$

Computing $BA$ is known to be NP-hard (Woeginger, 2003). Brandt et al. (2010b) gave an arguably simpler proof of this hardness result which we will use for new hardness arguments in Section 6.2 and Section 10.3.1. Our implementation is based on a recent algorithm by Gaspers and Mnich (2010) that enumerates all *feedback vertex sets*, each of which is the complement of a maximal transitive subset.

*covering relation*

The *covering relation* is always defined with respect to a subset $B \subsetneq A$. For all distinct $a, b \in B$, $a$ *covers* $b$ in $B$, if $D_B(b) \subset D_B(a)$.

*uncovered set*

Then, the *uncovered set* $UC(T)$ of $T$ is defined as the set of vertices who

33 With the payoffs being only within $\{-1, 0, 1\}$, this has also been called a *tournament game*.

34 Banks's original motivation was slightly different as his aim was to characterize the set of outcomes under sophisticated voting in the amendment procedure.

are maximal elements according to the covering relation in $A$ (Fishburn, 1977; Miller, 1980). Equivalently, a vertex is not covered and thereby in the uncovered set if and only if it can reach every other vertex in the tournament via a path of length at most two.[35] It is easily seen from the second definition that $a \in UC(T)$ if and only if $\left(M^2 + M\right)_{ab} \neq 0$ for all $b \in A \setminus \{a\}$. Consequently, the running time for computing *UC* is governed by matrix multiplication, i.e., it is in $\mathcal{O}(|T|^{2.3729})$ (Vassilevska Williams, 2012).

The uncovered set as a tournament solution is not *idempotent*, i.e., it does not necessarily hold that $UC(UC(T)) = UC(T)$, and one can therefore define a sequence of tournament solutions by letting

$$UC^k = UC(UC^{k-1}(T)) \quad \text{and} \quad UC^1(T) = UC(T).$$

The *iterated uncovered set* $UC^\infty(T)$ of a tournament $T$ is then defined as

*iterated uncovered set*

$$UC^\infty(T) = \bigcap_{k \in \mathbb{N}} UC^k(T).$$

Due to the finiteness of $T$, computing $UC^\infty$ requires at most $|T|$ successive *UC*-computations. Therefore, $UC^\infty$ can be computed in time $\mathcal{O}(|T|^{1+2.3729})$.

### 5.1.3 Solutions based on Stability

A subset of alternatives $B \subseteq A$ is called *S-stable* for a tournament solution $S$ if $a \notin S(B \cup \{a\})$ for all $a \in A \setminus B$. Stable sets can be used to define a new tournament solution $\widehat{S}$ that returns the union of all minimal $S$-stable sets, i.e.,

*S-stable*

$$\widehat{S}(T) = \bigcup \{B \text{ is } S\text{-stable} : \forall C \subsetneq B : C \text{ is not } S\text{-stable}\}.$$

Defining new tournament solutions via the $\widehat{\ }$-operator is most appealing for tournament solutions $S$ that always admit a *unique* minimal set.

Recall that a set $B \subseteq A$ is dominant in $T$ if $B \succ A \setminus B$. Using the notion of stable sets, we see that this is equivalent to $B$ being *CNL*-stable in $T$. The *top cycle* $TC(T)$ of a tournament $T$ is defined as the unique minimal *CNL*-stable (or dominant) set, i.e.,

*top cycle*

$$TC = \widehat{CNL}.$$

Uniqueness of the minimal dominant set is straightforward and was first shown by Good (1971). Referring to our treatment of components in majority graphs in Chapter 3, the top cycle of $T$ coincides with the top-most component of the scaling decomposition of $T$. If $T$ is irreducible, $TC(T) = A$. The top cycle can be computed in linear time by identifying the strongly connected components of $T$(Tarjan, 1972).

Dutta (1980) has shown that every tournament admits a unique *UC*-stable set and defined the *minimal covering set* $MC(T)$ of $T$ as

*minimal covering set*

35 In graph theory, such vertices are often called *kings*.

$$MC(T) = \widehat{UC}(T).$$

A polynomial-time algorithm for computing *MC* using the *BP* algorithm as a subroutine was proposed by Brandt and Fischer (2008).

*extending sets*

Brandt (2011b) was the first to consider *BA*-stable sets which he called *extending sets*. He conjectured that minimal extending sets were always unique and defined

$$ME(T) = \widehat{BA}(T).$$

The conjecture has been disproved by Brandt et al. (2013a) and we address the implications of this on *ME* at length in Section 10.3.[36]

Computing the minimal extending set is a tedious task. We show in Section 10.3.1 that the problem is NP-hard but it may very well be harder than that. The best known upper bound is $\Sigma_3^p$ as verifying whether a set is *BA*-stable already seems to require solving the coNP-complete problem of deciding whether an alternative is *not* contained in *BA* (Brandt, 2009). We compute minimal extending sets using a naïve implementation, which already takes about 3 minutes on instances of 25 alternatives.

### 5.1.4 Solutions Based on Retentiveness

*S-retentive set*

A nonempty subset of alternatives $B \subseteq A$ is called *S-retentive* for tournament solution $S$ if $S(\overline{D}(b)) \subseteq B$ for all $b \in B$ such that $\overline{D}(b) \neq \emptyset$. Just like stable sets, retentive sets can be used to define a new tournament solution $\mathring{S}$ that returns the union of all minimal $S$-retentive sets, i.e.,

$$\mathring{S}(T) = \bigcup\{B \text{ is } S\text{-retentive} : \forall C \subsetneq B : C \text{ is not } S\text{-retentive}\}.$$

Just like $\widehat{\ }$ before, $\mathring{\ }$ is an operator that maps from the space of tournament solutions to itself. It was introduced by Schwartz (1990) in the course of his definition and analysis of the *tournament equilibrium set* (*TEQ*) which is defined as the unique fixed point of $\mathring{\ }$, i.e.,

*tournament equilibrium set*

$$TEQ = \mathring{TEQ}.$$

This recursive definition is well-defined since the dominator sets become strictly smaller in each level of the recursion.

While $\mathring{S}$ could be considered for any tournament solution $S$, the concept is most appealing when there is a unique inclusion-minimal $S$-retentive set. This was shown by Brandt et al. (2014a) for $S = TC$, resulting in $\mathring{TC}$ or *TC-ring*.

*TC-ring*

A general method for computing $\mathring{S}$, given an implementation for $S$, is to compute the corresponding $S$-relation $\xrightarrow{S}$ where $a \xrightarrow{S} b$ if and only if $a \in S(\overline{D}(b))$ and then return the maximal elements of that

36 Originally, the name of the concept was *the minimal extending set* of a tournament. In light of the possible non-uniqueness, we will only call it *ME*.

relation's transitive closure, as suggested by Brandt et al. (2010b). In case of $\mathring{TC}$, this takes polynomial time. Due to its recursive nature, computing *TEQ* is much harder than computing $\mathring{TC}$. The problem is known to be NP-hard while the best known upper bound is PSPACE (Brandt et al., 2010b). For general tournaments with more than 100 alternatives, computing *TEQ* is currently out of reach. For structured tournaments this changes drastically as we will see in Chapter 7.

## 5.2 WEIGHTED SOLUTION CONCEPTS

*Borda*

The *Borda* solution (*BO*) is typically used in a voting context, where it is construed as based on voters' rankings of the alternatives: each alternative receives $|A|-1$ points for each time it is ranked first, $|A|-2$ points for each time it is ranked second, and so forth. The solution concept then chooses the alternatives with the highest total number of points (Borda, 1784). In the more general setting of weighted tournaments, the *Borda score* of alternative $x \in A$ in $G = (A, w)$ generalizes the Copeland score by taking the weights into account:

$$s_{BO}(a, G) = \sum_{b \in A \setminus \{a\}} w(a, b)$$

Then, the *Borda winners* are the alternatives with the highest Borda score.[37] If $w(a, b)$ represents the number of voters that rank $a$ higher than $b$, the two definitions of Borda are equivalent. Computing the set of Borda winners runs along the same lines as computing the Copeland set and can be done in linear time.

*maximin solution*

The *maximin score* $s_{MM}(a, T)$ of an alternative $a$ in a weighted tournament $T = (A, w)$, is given by its worst pairwise comparison, i.e., $s_{MM}(x, T) = \min_{y \in A \setminus \{x\}} w(x, y)$. The *maximin solution*, also known as the *Simpson-Kramer method* and denoted by *MM*, returns the set of all alternatives with the highest maximin score (Simpson, 1969; Kramer, 1977; Young, 1977).

*ranked pairs*

There are two solution concepts in the literature under the name *ranked pairs* that differ by their inherent tie-breaking rule.[38] Given a weighted tournament $T = (A, w)$, both variants construct transitive tournaments $T'$ on $A$ in the following manner. First order the (directed) edges of $T$ in decreasing order of weight. Then consider the edges one by one according to this ordering. If the current edge can be added to $T'$ without creating a cycle, then do so; otherwise discard the edge. Then, the unique undominated alternative of $T'$ is chosen from $T$.

The difference lies in how ties in the ranking of edge weights are handled. In the original definition by Tideman (1987), no tie-breaking

37 Borda winners and the Copeland set coincide for unweighted tournaments.

38 See Brill and Fischer (2012) and Brill (2012, Chapter 8.1) for a more thorough discussion of these two variants.

rule is fixed and the procedure returns the set of *all* tournaments $T'$ that are obtained through *some* tie-breaking rule. We refer to the weighted tournament solution that returns all alternatives that are undominated in any such $T'$ as *RP*. Brill and Fischer (2012) have shown that the winner determination problem for *RP* is NP-hard.

In the second variant, ties are broken according to some pre-defined tie-breaking rule $\tau$. The corresponding solution concepts $RP_\tau$ is the only resolute concept we consider in this thesis: for any weighted tournament $T$, it chooses a singleton, i.e., $|RP_\tau(T)| = 1$. It is obvious that determining the sole winner according to $RP_\tau$ is tractable.

*Kemeny's rule*

A very prominent weighted tournament solution that we include for completeness in this list is *Kemeny's rule* (Kemeny, 1959). While usually being considered for preference profiles, it actually only operates on the weighted majority relation and can be seen as an extension of Slater's rule to weighted tournaments. In this perspective, an alternative $a$ is a Kemeny winner in $T = (A, w)$ if it is undominated in a *Kemeny ranking* $\succ^K$ that maximizes the total weight on edges shared with $T$, i.e.,

*Kemeny ranking*

$$\succ^K \in \arg\max_{\succ} \sum_{\substack{a,b \in A \\ a \succ b}} w(a, b).$$

Virtually all problems related to Kemeny's rule are at least NP-hard (Bartholdi, III et al., 1989; Hemaspaandra et al., 2005). See Fischer et al. (2015) for a good overview about what is known for Kemeny's rule.

## 5.3 SUMMARY

Tournament solutions as functions that chose a set of alternatives from complete weighted or unweighted graphs can be applied in many scenarios, one of which is social choice. There is a plethora of unweighted and weighted tournament solutions that differ, for example, in terms of their axiomatic properties, their ability to discriminate between alternatives, and their computational complexity. We briefly introduced all solution concepts that we are concerned with in this thesis. For later reference, the abbreviated names are listed in Table 5.1.

| | |
|---|---|
| *TRIV* | trivial tournament solution |
| *CNL* | Condorcet non-losers |
| *COND* | Condorcet solution |
| *CO* | Copeland set |
| *SL* | Slater set |
| *MA* | Markov set |
| *BP* | bipartisan set |
| *BA* | Banks set |
| *UC* | uncovered set |
| $UC^{\infty}$ | iterated uncovered set |
| *TC* | top cycle |
| *MC* | minimal covering set |
| *ME* | *ME* |
| *TEQ* | tournament equilibrium set |
| $\mathring{TC}$ | *TC*-ring |
| *BO* | Borda |
| *MM* | maximin, Simpson-Kramer |
| *RP* | ranked pairs (Tideman's variant) |
| $RP_\tau$ | ranked pairs (resolute variant) |

**Table 5.1:** List of unweighted and weighted tournament solutions.

# 6 HARDNESS OF VOTING WITH A CONSTANT NUMBER OF VOTERS

> [T]o my knowledge, when not trivial, the complexity for lower values of $m$ [the number of voters] remains unknown. In particular, it would be interesting to know whether some of the problems [...] remain NP-hard if $m$ is a given constant.
>
> Olivier Hudry, 2008

In this chapter, we show that the winner determination problem of four well-studied voting rules remains NP-hard even if the number of voters is a small constant. Our general method is to analyze the existing hardness constructions of four common social choice rules with respect to their susceptibility to the sufficient conditions in Lemma 4.7 or Lemma 4.8. In all cases, we slightly modify the original construction to get better bounds on the number of voters that are required to induce it.

In Section 6.1, we introduce for two new decision problems that they are NP-complete which will be used in the results in the next sections. In Section 6.2, Section 6.3, Section 6.4, and Section 6.5 that the winner determination problem remains NP-hard even with electorates of a small constant size for the Banks set, the tournament equilibrium set, the Slater set and the ranked pairs method, respectively. Section 6.6 summarizes our findings.

## 6.1 TWO NP-COMPLETE PROBLEMS

Before we proceed further, we introduce two new constrained classes of propositional formulae (Ordered3-CNF to be used for the results in Sections 6.2 and 6.3, and ReducedFew-CNF to be used for the result in Section 6.4) and show for both that the problem of deciding whether a given formula is satisfiable is NP-complete.

A formula of propositional logic in conjunctive normal form (CNF) is in 3-CNF if each clause has at most three literals. We say that a formula $\varphi$ from 3-CNF is in Ordered3-CNF if its clauses all contain exactly three distinct literals and are ordered within $\varphi$ in such a way that for each propositional variable $p$, all clauses containing the literal $p$ precede all clauses containing $\neg p$. It is known that 3Sat, the *Ordered3-CNF* problem of deciding whether a given formula in 3-CNF is satisfiable, is NP-complete (Karp, 1972). For formulae in ORDERED3-CNF, we call the corresponding decision problem ORDERED3SAT .

ORDERED3SAT

LEMMA 6.1
ORDERED3SAT is NP-complete.

*Proof.* Membership in NP is obvious. For the hardness, we reduce from 3SAT. Let $\varphi$ be some formula in 3-CNF. Let $P$ denote the set of variables of the propositional language in which $\varphi$ is formulated and let $C = (c_1, \dots, c_{|C|})$ denote the clause set of $\varphi$. We may assume w.l.o.g. that no clause contains the same variable twice, that all literals in a clause are ordered according to a fixed ordering $(p_1, p_2, \dots)$, and that every clause is of size three. The latter is due to the fact that clauses of size one can be easily used to simplify $\varphi$ and the remaining clauses $(p \vee q)$ of size two can be padded with a new variable $x$ to $(p \vee q \vee x) \wedge (p \vee q \vee \neg x)$. We call all variables that occur at least once in $\varphi$ *original* variables.

For the reduction, we construct an ordered formula $\varphi'$ in 3-CNF with $6 \cdot |C|$ clauses and $4 \cdot |C|$ additional variables that is satisfiable if and only if $\varphi$ is. For every clause $c_i = (\ell_1 \vee \ell_2 \vee \ell_3)$, define a set of new clauses $\varphi_i = \bigwedge_{j=1}^{6} c_i^j$ with

$$c_i^1 = (\ell_1 \vee x_i \vee x_i'), \qquad c_i^2 = (\ell_2 \vee \neg x_i \vee y_i),$$
$$c_i^3 = (\ell_2 \vee \neg x_i' \vee y_i), \qquad c_i^4 = (\ell_3 \vee \neg y_i \vee z_i),$$
$$c_i^5 = (\neg x_i \vee \neg y_i \vee \neg z_i), \text{ and} \qquad c_i^6 = (\neg x_i' \vee \neg y_i \vee \neg z_i)$$

where $x_i, x_i', y_i$, and $z_i$ are new propositional variables. It is easy to check that $c_i$ is satisfiable if and only if $\varphi_i$ is. Since the literals associated with original variables are spread over different $\varphi_i$ just as they were over the different clauses $c_i$ in $\varphi$, this implies that $\bigwedge_i \varphi_i$ is satisfiable if and only if $\varphi$ is.

What remains to be shown is that all the clauses $c_i^j$ can be arranged in such a way that the resulting formula is ordered. To this end, we define for each original variable $p$ and $j \in \{1, \dots, 4\}$ the clause sets

$$C^{p,j} = \bigcup_i \{c_i^j : p \in c_i^j\} \quad \text{and} \quad C^{\neg p,j} = \bigcup_i \{c_i^j : \neg p \in c_i^j\}$$

as well as

$$C^5 = \bigcup_i c_i^5 \quad \text{and} \quad C^6 = \bigcup_i c_i^6.$$

We are now in a position to define $\varphi'$ to be

$$\varphi' = \bigwedge_{i=1}^{|P|} \left( \left( \bigwedge_{j=1}^{4} \bigwedge_{c \in C^{p_i,j}} c \right) \wedge \left( \bigwedge_{j=1}^{4} \bigwedge_{c \in C^{\neg p_i,j}} c \right) \right) \wedge \bigwedge_{c \in C^5 \cup C^6} c.$$

We claim that $\varphi'$ is ordered. We show this for original and new variables separately. For each original variable $p$, all positive occurrences are in the $C^{p,j}$, preceding the negative occurrences in the $C^{\neg p,j}$.

For all new variables, the clauses in $C^5 \cup C^6$ only contain negative occurrences and are at the back of $\varphi$. Therefore, we only have to check that orderedness holds in the first part of $\varphi'$. For each $z_i$, this is trivially the case as it only occurs once (as a positive literal) outside of $C^5 \cup C^6$. For the others that we denoted by $x_i, x_i'$, and $y_i$, the positive occurrences in $C^{p_\ell,j} \cup C^{\neg p_\ell,j}$ for some $\ell$ and $j \in \{1,2,3\}$ always precede the single negative occurrence in $C^{p_L,J} \cup C^{\neg p_L,J}$ for some $J \in \{2,3,4\}$ and $L \neq \ell$: due to the fixed ordering of the literals within a clause we have that $L > \ell$. □

We say that a formula from 3-CNF is in Few-CNF if each literal appears at most twice, and each variable appears at most thrice. We call the problem of checking whether a formula given in Few-CNF is satisfiable FewSat. Tovey (1984) has shown that FewSat is NP-complete. We follow his proof to show that this still holds for formulae in ReducedFew-CNF where we additionally require that every variable occurs in at most one three-literal clause and every literal in at most one two-literal clause. Denote the corresponding decision problem by ReducedFewSat .

ReducedFew-*CNF*

ReducedFewSat

Lemma 6.2 *(cf. Tovey, 1984, Thm. 2.1)*
ReducedFewSat is NP-complete.

*Proof.* Membership in NP is obvious. For hardness, we reduce from 3Sat. Let $\varphi := \bigwedge_{i=1}^{n} (x_i \vee y_i \vee z_i)$ be some formula in 3-CNF where no clause contains the same variable twice. For every variable $v$ occurring in $\varphi$, replace each of its $L$ occurrences with a new variable $v_j$ where $1 \leqslant j \leqslant L$. Now add the clauses

$$\varphi_v = (\neg v_L \vee v_1) \wedge \bigwedge_{j=1}^{L-1} (\neg v_j \vee v_{j+1})$$

which are equivalent to $(v_L \Rightarrow v_1) \wedge \bigwedge_{j=1}^{L-1} (v_j \Rightarrow v_{j+1})$. Call the resulting formula $\mathtt{red}(\varphi)$. Note that $\mathtt{red}(\varphi)$ only contains clauses with three literals (original clauses with replaced variables) or two literals (the new clauses) and denote these clause sets by $C_3$ and $C_2$, respectively. Also observe that every variable occurs exactly once in $C_3$ and every literal exactly once in $C_2$, i.e., $\mathtt{red}(\varphi)$ is in ReducedFew-CNF.

For every old variable $v$, we can only satisfy $\varphi_v$ by setting all $v_j$ to the same value. Since setting all $v_j$ to the same value $t$ satisfies $\varphi_v$ and has the same effect on the original part of $\mathtt{red}(\varphi)$ that setting $v$ to $t$ has on $\varphi$, it follows that $\varphi$ is satisfiable if and only if $\mathtt{red}(\varphi)$ is satisfiable. □

## 6.2 THE BANKS SET

Although finding a random alternative in the Banks set can be done in polynomial time (Hudry, 2004), deciding whether an alternative belongs to the Banks set is NP-complete as shown by Woeginger (2003). Brandt et al. (2010b) gave an arguably simpler proof of this result by a reduction from 3SAT: every formula $\varphi$ in 3-CNF can be transformed in polynomial time into a tournament $T_\varphi^{BA}$ with a decision node $c_0$ such that $c_0$ is in the Banks set of $T_\varphi^{BA}$ if and only if $\varphi$ is satisfiable. Due to Lemma 6.1, this reduction works just as well if $\varphi$ is taken to be ordered as well. Again, we have $P$ denote the set of variables of the propositional language in which $\varphi$ is formulated.

*the class $\mathcal{G}^{BA}$*

A tournament $(V, E)$ is in *the class* $\mathcal{G}^{BA}$ if it satisfies the following properties. There is an odd integer $m$ such that,

$$V = C \cup U_1 \cup \cdots \cup U_m,$$

where $C, U_1, \ldots, U_m$ are pairwise disjoint and $C = \{c_0, \ldots, c_m\}$. We have $C_i$ denote the singleton $\{c_i\}$ and $U = \bigcup_{i=1}^m U_i$. If $i$ is odd, $U_i = \{u_i^1, u_i^2, u_i^3\}$ whereas if $i$ is even $U_i$ is a singleton $\{u_i\}$. Let $X = \bigcup\{U_i : i \text{ is odd}\}$ and $Y = \bigcup\{U_i : i \text{ is even}\}$. Intuitively, $(V, E)$ is $T_\varphi^{BA}$ for some $\varphi$ in ordered 3-CNF with $\frac{1}{2}(m+1)$ clauses. If $i$ is odd, $U_i$ corresponds to a clause of $\varphi$ and the nodes it contains represent (tokens of) literals. We assume each of these nodes $u_i^j$ to be labeled by the literal $\lambda(u_i^j)$ it represents. For *odd* $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$ we define,

$$\begin{aligned} U_i^j &= \{u_i^j\} \\ U_i^p &= \{u \in U_i : \lambda(u) = p\} \\ U_i^{\neg p} &= \{u \in U_i : \lambda(u) = \neg p\} \end{aligned}$$

Moreover, for *even* $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$, we stipulate,

$$U_i^j = U_i^p = U_i^{\neg p} = \emptyset.$$

Observe that $\bigcup_{\substack{p \in P \\ 1 \leqslant i \leqslant m}} (U_i^p \cup U_i^{\neg p}) = X$.

We are now in a position to define the edge set $E$, almost as in Brandt et al. (2010b).[39] Let

$$E = \bigcup_{i<j} (C_j \times C_i) \cup \bigcup_{i<j} ((U_i \times U_j) \setminus \overline{E^\varphi}) \cup \bigcup_{1 \leqslant i \leqslant m} ((U_i^1 \times U_i^2) \cup (U_i^2 \times U_i^3) \cup (U_i^1 \times U_i^3)) \cup \bigcup_{i \neq j} (C_i \times U_j) \cup \bigcup_i (U_i \times C_i) \cup E^\varphi,$$

where

$$E^\varphi = \bigcup_{\substack{p \in P \\ i<j}} (U_j^p \times U_i^{\neg p}) \cup \bigcup_{\substack{p \in P \\ i<j}} (U_j^{\neg p} \times U_i^p).$$

Figure 6.1 illustrates this type of tournament. We also refer to $E^\varphi$ as the *formula dependent* of the tournament $T_\varphi^{BA}$. The edge set

$$(E \setminus E^\varphi) \cup \overline{E^\varphi}$$

we refer to as its *skeleton*.

We will show that the skeleton of each tournament $T_\varphi^{BA}$ is induced by a 3-voter profile such that the edges in $\overline{E^\varphi}$ all get a weight of one. At the same time, $E^\varphi$ is inducible by a 2-voter profile such that the weight on all edges is two. A little reasoning and an application of Lemma 4.8 then gives us the desired result.

*inducing every tournament in $\mathcal{G}^{BA}$ with 5 voters*

THEOREM 6.3

Computing the Banks set is NP-hard, if the number of voters is at least five.

*Proof.* Let $(V, E)$ be a tournament in $\mathcal{G}^{BA}$. It suffices to show that $(V, E)$ is induced by a 5-voter profile. To this end define:

$$E_1 = \bigcup_i (U_i \times C_i),$$

$$E_2 = E^\varphi,$$

$$E_3 = E \setminus (E_1 \cup E_2).$$

Observe that $E = E_1 \cup E_2 \cup E_3$ and that $E_1$, $E_2$, and $E_3$ are pairwise disjoint. In virtue of Lemma 4.8, it therefore suffices to show that $(V, E_1)$ and $(V, E_2)$ are induced by 2-voter profiles and that $(V, E_3)$ is acyclic.

39 There is only a slight change compared to the original construction by Brandt et al. (2010b). Specifically, we now have edges $U_i^1 \times U_i^3$ instead of the other way around. It is not difficult to check that the argument of the reduction is not affected—it is irrelevant whether the crucial transitive subtournament with $c_0$ as its maximal element may contain one, two, or three vertices from a $U_i$.

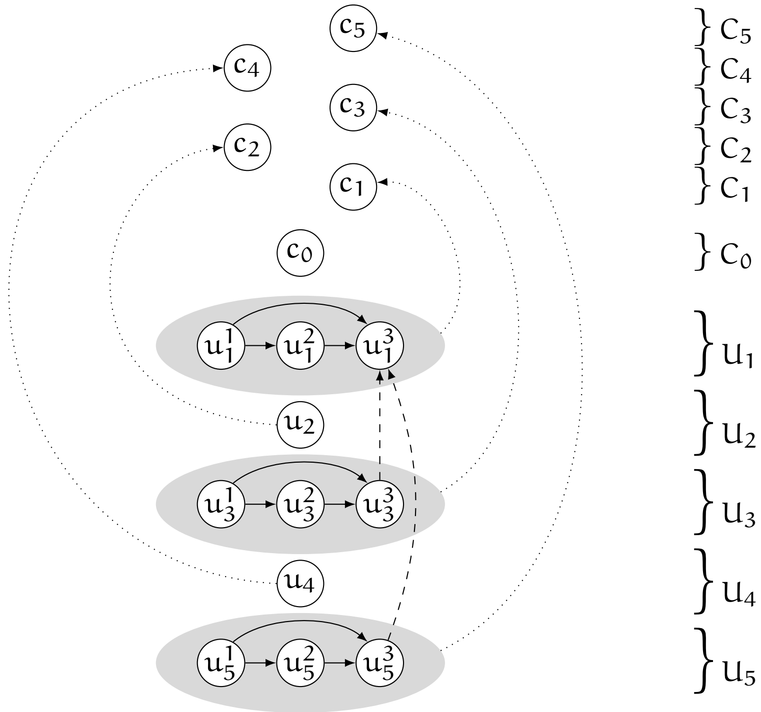



**Figure 6.1:** A tournament $T_\varphi^{BA} = (V, E)$ in the class $\mathcal{G}^{BA}$, where $E$ is given by the displayed edges of any kind and it is understood that missing edges point downwards. Moreover, $\lambda(u_5^3) = \lambda(u_3^3) = \bar{\lambda}(u_1^3)$. The dotted and dashed upward edges correspond to the edge sets $E_1$ and $E_2$ in Theorem 6.3, respectively. The remaining edges, i.e., all downward edges and the edges within the $U_i$ form an acyclic edge set and correspond to $E_3$.

For $(V, E_1)$ it is easy to see that it is a union of unidirected stars and therefore 2-inducible. For $(V, E_2)$, let

$$E_2^p = \bigcup_{i<j} \left(U_j^p \times U_i^{\neg p}\right) \cup \left(U_j^{\neg p} \times U_i^p\right)$$

be the edges in $E_2$ associated with a variable $p$. Note that $E_2 = \bigcup_{p \in P} E_2^p$ and that all $E_2^p$ are vertex-disjoint from each other. Recall that $(V, E)$ was in induced through a construction that was based on an *ordered* formula. This implies that whenever $U_j^p \neq \emptyset \neq U_i^{\neg p}$ we have that $i$ is greater than $j$. Therefore, $E_2^p$ can also be written as $\bigcup_{i,j}(U_i^{\neg p} \times U_j^p)$. In this representation, it is clear that $E_2^p$ is a complete, unidirected bipartite graph. But then, $E_2$ as a vertex-disjoint union of such graphs is a *bilevel graph* and 2-inducible according to Erdős and Moser (1964, Lemma 2, cf. footnote 23 on page 36).

To see that $E_3$ is acyclic, note that it forms a subset of

$$C \times U \cup \bigcup_{i<j}(U_i \times U_j) \cup \bigcup_{i>j}(C_i \times C_j) \cup \bigcup_{1 \leqslant i \leqslant m} \left((U_i^1 \times U_i^2) \cup (U_i^2 \times U_i^3) \cup (U_i^1 \times U_i^3)\right)$$

and corresponds to all (shown) horizontal and (missing) downward edges in Figure 6.1. □

## 6.3 THE TOURNAMENT EQUILIBRIUM SET

Brandt et al. (2010b) have shown that computing *TEQ* is NP-hard by a reduction from 3SAT. By Lemma 6.1, the very same construction is also a valid reduction from ORDERED3SAT. For every formula $\varphi$ in ordered 3-CNF, a tournament $T_\varphi^{TEQ}$ can be constructed such that *TEQ* selects a decision node $c_0$ from $T_\varphi^{TEQ}$ if and only if $\varphi$ is satisfiable. The class of these tournaments $T_\varphi^{TEQ}$ is denoted by $\mathcal{G}^{TEQ}$ and the tournaments in this class bear a strong structural similarity to those in $\mathcal{G}^{BA}$, which can be exploited to show that every tournament in $\mathcal{G}^{TEQ}$ is induced by a 7-voter profile.

A tournament $(V, E)$ is in *the class* $\mathcal{G}^{TEQ}$ if it satisfies the following properties. There is an odd integer $m$ with $m \equiv 1 \pmod 4$ such that, *the class* $\mathcal{G}^{TEQ}$

$$V = C \cup U_1 \cup \cdots \cup U_m,$$

where $C, U_1, \ldots, U_m$ are defined the same as in $\mathcal{G}^{BA}$. We have $C_i$ denote the singleton $\{c_i\}$. Moreover, let $X = \bigcup\{U_i : i \equiv 1 \pmod 4\}$, $Y = \bigcup\{U_i : i \text{ is even}\}$, and $Z = \bigcup\{U_i : i \equiv 3 \pmod 4\}$.

Intuitively, $(V, E)$ is $T_\varphi^{TEQ}$ for some $\varphi$ with $\frac{1}{4}(m+3)$ clauses in ordered 3-CNF. Every $U_i \in X$ corresponds to a clause of $\varphi$ and the nodes it contains represent (tokens of) literals. Again, we assume each of these nodes $u_i^j$ to be labeled by the literal $\lambda(u_i^j)$ it represents. For $i \in \{1, 5, \ldots, m\}$ and $j \in \{1, 2, 3\}$ we define,

$$\begin{aligned} U_i^j &= \{u_i^j\}, \\ U_i^p &= \{u \in U_i : \lambda(u) = \phantom{\neg}p\}, \text{ and} \\ U_i^{\neg p} &= \{u \in U_i : \lambda(u) = \neg p\}. \end{aligned}$$

Moreover, for the other values of $i$, and $j \in \{1, 2, 3\}$, we stipulate,

$$U_i^j = U_i^p = U_i^{\neg p} = \emptyset.$$

Observe that $\bigcup_{\substack{p \in P \\ 1 \leqslant i \leqslant m}} (U_i^p \cup U_i^{\neg p}) = X$.

We are now in a position to define the edge set $E$.

$$\begin{aligned} E = &\bigcup_{i<j} (C_j \times C_i) \cup \bigcup_{i \neq j} (C_i \times U_j) \cup \bigcup_{i=j} (U_j \times C_i) \cup \\ &\bigcup_{1 \leqslant i \leqslant m} \big((U_i^1 \times U_i^2) \cup (U_i^2 \times U_i^3) \cup (U_i^3 \times U_i^1)\big) \cup \\ &\bigcup_{i<j} \big((U_i \times U_j) \setminus (\overline{E^\varphi} \cup \overline{E^z})\big) \cup E^\varphi \cup E^z, \end{aligned}$$

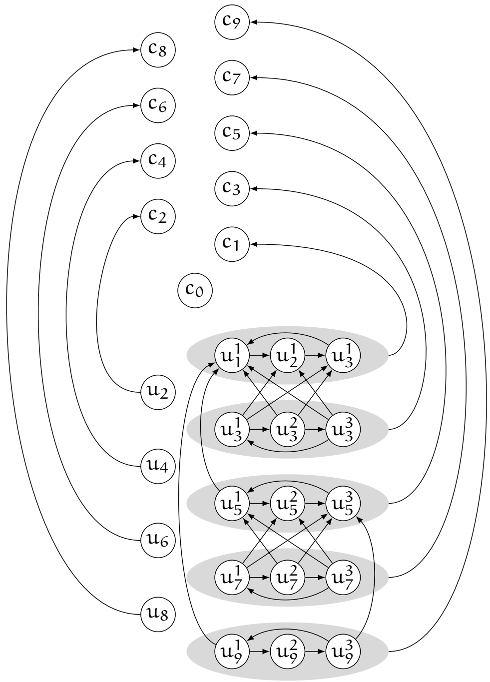



**Figure 6.2:** A tournament $T_\varphi^{TEQ} = (V, E)$ in the class $\mathcal{G}^{TEQ}$, where $E$ is given by solid edges and it is understood that missing edges point downwards.

where

$$E^\varphi = \bigcup_{\substack{p \in P \\ i > j}} (U_i^p \times U_j^{\neg p}) \cup \bigcup_{\substack{p \in P \\ i > j}} (U_i^{\neg p} \times U_j^p),$$

$$E^z = \bigcup_{\substack{l \neq l' \\ i = j+2}} (U_i^l \times U_j^{l'}).$$

*inducing every tournament in $\mathcal{G}^{TEQ}$ with 7voters*

An example of such a tournament is depicted in Figure 6.2. The notable structural differences to $\mathcal{G}^{BA}$ are the cycles in $U_i$ for odd $i$ and the edges $E^z$ between $Z$ and $X$. Next, we show that every tournament $T_\varphi^{TEQ}$ is induced by a 7-voter profile, using the same approach as in Theorem 6.3.

Theorem 6.4

Computing *TEQ* is NP-hard, if the number of voters is at least seven.

*Proof.* Similar to the proof for Theorem 6.3, it suffices to show that every tournament $(V, E)$ in $\mathcal{G}^{TEQ}$ is induced by a 7-voter profile. To achieve this, we partition $E$ into four disjoint edge sets $E_1, E_2, E_3, E_4 \subseteq E$ and show that the graphs $(V, E_1)$, $(V, E_2)$, and $(V, E_3)$ are each induced by 2-voter profiles as well as that $(V, E_4)$ is acyclic. Then the result follows from Lemma 4.8.

While the tournaments in $\mathcal{G}^{TEQ}$ are very similar to the ones in $\mathcal{G}^{BA}$, the introduction of new nodes and edges makes finding an appealing partition a bit trickier. We define

$$\begin{aligned}
E_1 = &\bigcup_{i>j} (C_i \times (C_j \cup U_j)) \cup \bigcup_i (U_i^3 \times U_i^1) \cup \\
&\bigcup_{\substack{i \equiv 3 \\ \text{mod } 4}} ((U_i^1 \cup U_i^3) \times U_{i-2}^2), \\
E_2 = &\quad E^{\varphi}, \\
E_3 = &\quad E^z \setminus E_1, \text{ and} \\
E_4 = &\quad E \setminus (E_1 \cup E_2 \cup E_3).
\end{aligned}$$

It can readily be appreciated that $E_1$, $E_2$, and $E_3$ are contained in $E$ (see Figure 6.3). Also, they are pairwise disjoint and therefore $\{E_1, E_2, E_3, E_4\}$ is proper partition of $E$.

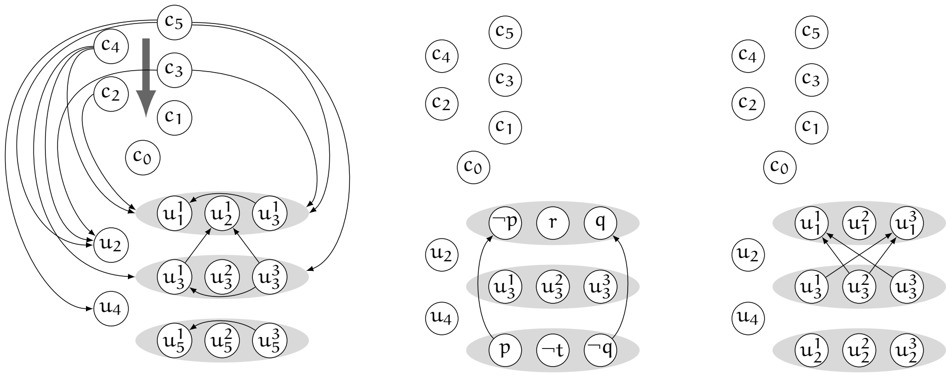



**Figure 6.3:** Illustration of the edge sets $E_1, E_2, E_3 \subset E$ in $T_{\varphi}^{TEQ} = (V, E)$. The thick arrow on the left represents all edges $\bigcup_{i<j}\{(c_j, c_i)\}$ being part of $E_1$.

To show that $(V, E_1)$ is 2-inducible, we define

$$\begin{aligned}
E_1' = &\bigcup_{\substack{i \leqslant j \\ U_j \subset X \cup Z}} (C_i \times U_j) \cup \bigcup_{\substack{i \leqslant j \\ U_j \subset Y}} (U_j \times C_i) \cup \\
&\bigcup_{\substack{i<j \\ U_i, U_j \subset X \cup Z}} ((U_i \times U_j) \setminus E_1) \cup \bigcup_{\substack{i<j \\ U_i, U_j \subset Y}} (U_j \times U_i) \cup \\
&\bigcup_{i \text{ odd}} ((U_i^1 \cup U_i^3) \times U_i^2) \cup (Y \times (X \cup Z)).
\end{aligned}$$

It is straightforward to check that $E_1'$ is a reorientation of $\widetilde{E}_1$. Also, it is easy but tedious, by making the obvious case distinctions, to show for $E_1$ and $E_1'$ that the dominion of each vertex is contained in the dominion of each of its dominators, implying that $E_1$ and $E_1'$ are both transitive. For example, consider a vertex $u_i^1 \in X$ in $E_1'$ for which

$$D(u_i^1) = U_i^2 \cup \bigcup_{\substack{j>i \\ j \text{ odd}}} U_j \qquad \text{and} \qquad \overline{D}(u_i^1) = Y \cup \bigcup_{\substack{j<i \\ j \text{ odd}}} U_j \cup \bigcup_{j \leqslant i} C_j$$

again denote the set of all out-neighbors and all in-neighbors of $u_i^1$ in $(V, E_1')$, respectively. It is straightforward to check that every vertex in $\overline{D}(u_i^1)$ also has an edge in $E_1'$ to every vertex in $D(u_i^1)$. Thus, in virtue of Lemma 4.3, $(V, E_1)$ is induced by a 2-voter profile.

The proof for $(V, E_2)$ being 2-inducible is analogous to the proof of the same statement in the Banks construction (see Theorem 6.3). This is also where the orderedness of $\varphi$ is exploited.

The graph $(V, E_3)$ is obviously transitive. We also observe that it consists of isomorphic and vertex-disjoint subgraphs $(U_i \cup U_{i-2}, E_{3,i})$ for $i \equiv 3 \pmod 4$ with $E_i = (U_i^l \times U_{i-2}^{l'})$ for $l \neq l'$. It is sufficient to find a general transitive reorientation $E_{3,i}'$ on such a subgraph because then every completion of $\bigcup_{i \equiv 3 (\text{mod } 4)} E_{3,i}'$ is a transitive reorientation of $\widetilde{E}_3$. We define

$$\begin{aligned} E_{3,i}' = {} & (U_i \times U_{i-2}^2) \cup \big((U_{i-2}^1 \cup U_{i-2}^3) \times U_{i-2}^2\big) \cup \\ & \big((U_{i-2}^3 \cup U_i^1) \times (U_{i-2}^1 \cup U_i^3)\big) \cup \\ & (U_i^1 \times U_i^2) \cup (U_i^2 \times U_i^3). \end{aligned}$$

This subgraph set is also shown in Figure 6.4 and it is easy to verify that it is indeed transitive.

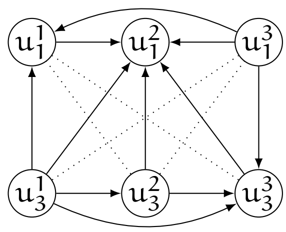



**Figure 6.4:** The edge set $E_{3,3}'$ which is part of the reorientation $E_3'$ of $\widetilde{E}_3$ in the proof of Theorem 6.4. Dotted edges denote the incomparability subgraph of $E_{3,3}'$.

Finally, to see the acyclicity of $(V, E_4)$, observe that

$$\begin{aligned} E_4 = {} & \bigcup_{i<j} (C_i \times U_j) \cup \bigcup_{i<j} \big((U_i \times U_j) \setminus (\overline{E^\varphi} \cup \overline{E^z})\big) \cup \\ & \bigcup_i \big((U_i^1 \times U_i^2) \cup (U_i^2 \times U_i^3)\big) \end{aligned}$$

and is thereby contained in the transitve closure of the ordering

$$(c_0, u_1^1, u_1^2, u_1^3, c_1, u_2, c_2, u_3^1, u_3^2, u_3^3, c_3, u_4, c_4, u_5^1, \ldots, c_m). \qquad \square$$

## 6.4 THE SLATER SET

The close relationship between Slater rankings and feedback arc sets can be used to easily show that computing Slater rankings is NP-hard in general digraphs. It was proved by Alon (2006), Conitzer (2006), and Charbit et al. (2007) that computing feedback arc sets is NP-hard even in tournaments. We will analyze the proof by Conitzer (2006), a reduction from MaxSat. The latter problem asks for an assignment to the propositional variables in a Boolean formula $\varphi$ such that at least a given number $s_1$ of clauses is satisfied. Due to Lemma 6.2, we can constrain $\varphi$ to be in ReducedFew-CNF without affecting the correctness of Conitzer's reduction. In there, a tournament $T_\varphi^{SL}$ is constructed for which a Slater ranking with at most $s_2$ inconsistent edges exists if and only if such an assignment for $\varphi$ exists, where $s_2$ depends on $\varphi$ and $s_1$.

Let $\mathcal{G}^{SL}$ denote the class of all tournaments $T_\varphi^{SL}$ obtained from a Boolean formula $\varphi$ in ReducedFew-CNF according to this construction. A tournament $(V, E)$ is in the class $\mathcal{G}^{SL}$ if it satisfies the following properties. There exist integers $m, l \geqslant 1$, such that

*the class $\mathcal{G}^{SL}$*

$$V = C \cup \bigcup_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant 6}} T_i^j,$$

where $C$ and all $T_i^j$ are pairwise disjoint and for $1 \leqslant i \leqslant m$

$$C = \{c_1, \ldots, c_{|C|}\},$$
$$T_i^j = \{t_i^{j,1}, \ldots, t_i^{j,l}\}.$$

Each subtournament $\left(T_i^j, E \cap \left(T_i^j \times T_i^j\right)\right)$ has to be a transitive component, i.e., it is a linear order and for a vertex $v \in V \setminus T_i^j$ and vertices $v_1, v_2 \in T_i^j$, either $\{(v_1, v), (v_2, v)\}$ or $\{(v, v_1), (v, v_2)\}$ have to be in $E$. For our purposes, we can treat $T_i^j$ as a single vertex denoted by $t_i^j$. Every $c_i$ is associated with a clause in $\varphi$. Abusing notation, we denote this clause with $c_i$ as well. Every $T_i$ corresponds to a variable $\lambda(T_i)$ in $\varphi$. For notational convenience, let

$$T^j = \bigcup_{1 \leqslant i \leqslant m} t_i^j \quad \text{and} \quad T_i = \bigcup_{1 \leqslant j \leqslant 6} t_i^j.$$

For $(V, E)$ to be in $\mathcal{G}^{SL}$, the edge set has to be of the form

$$\begin{aligned} E = E_A \cup \bigcup_i \{ (t_i^1, t_i^2), (t_i^2, t_i^3), (t_i^3, t_i^1) \} \cup \\ \left((T^5 \cup T^6) \times C\right) \cup \\ \left(C \times (T^1 \cup T^2 \cup T^3 \cup T^4)\right) \setminus \overline{E^\varphi} \cup E^\varphi \end{aligned}$$

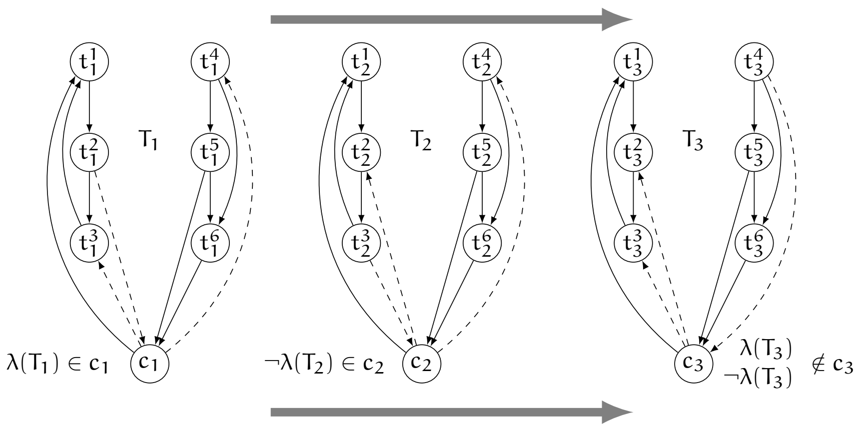



**Figure 6.5:** A schematic of a tournament $T_\varphi^{SL}$ in $\mathcal{G}^{SL}$ to illustrate the three different cases for the edges between $T^2 \cup T^3 \cup T^4$ and $C$. These edges are shown as dashed and are the only that depend on $\varphi$. The thick arrows below and above indicate the fixed order between and within the $T_i^j$, and in between the $c_i$. Many other edges are omitted in favor of comprehensibility.

where

$$E_A = \bigcup_{i<j}\{(c_i, c_j)\} \cup \bigcup_{i<j}\{(T_i, T_j)\} \cup \bigcup_{i} \bigcup_{\substack{J \in \{4,5,6\} \\ 1 \leqslant j < J}} \{(t_i^j, t_i^J)\}, \text{ and}$$

$$\begin{aligned} E^\varphi = &\{(t_i^2, c_j) : \lambda(T_i) \in c_j\} \cup \\ &\{(t_i^3, c_j) : \neg\lambda(T_i) \in c_j\} \cup \\ &\{(t_i^4, c_j) : \lambda(T_i), \neg\lambda(T_i) \notin c_j\}. \end{aligned}$$

We again refer to $E^\varphi$ as the *formula dependent* of the tournament $T_\varphi^{SL}$ and to $E^\sigma = E \setminus E^\varphi$ as its *skeleton*. An illustration of a tournament in $\mathcal{G}^{SL}$ is depicted in Figure 6.5.

*inducing every tournament in $\mathcal{G}^{SL}$ with 13 voters*

We show that every tournament $T_\varphi^{SL}$ is induced by a 13-voter profile.

THEOREM 6.5

> Computing the Slater set is NP-hard if the number of voters is at least 13.

*Proof.* For even numbers of voters greater than two, the result was shown by Dwork et al. (2001) and Biedl et al. (2009). For an odd number of voters, the majority digraph has to be a tournament.

Let $(V, E)$ be a tournament in $\mathcal{G}^{SL}$ that is constructed from a formula $\varphi$ in REDUCEDFEW-CNF. We decompose $E$ into disjoint sets $E_1, \dots, E_7$ and claim that each of $E_1, \dots, E_6$ is 2-inducible while $E_7$ is acyclic. Invoking Lemma 4.8 then gives the desired result.

Recall that since it is taken to be from REDUCEDFEW-CNF, $\varphi$ consists of clauses containing two or three literals and we denote these clause sets and (abusing notation) their corresponding $c_i \in V$ by $C_2$

and $C_3$, respectively. Also, each variable occurs exactly once (positively or negatively) in $C_3$, once as a positive literal in $C_2$, and once as a negative literal in $C_2$. Therefore, we can define two edge sets that conveniently partition $E \cap (C_2 \times T^4)$ as

$$(C_2 \times T^4)^+ = \bigcup_{p \in P} \{(c_j, t_i^4) : \ \lambda(T_i) \in c_j\} \qquad \text{and}$$
$$(C_2 \times T^4)^- = \bigcup_{p \in P} \{(c_j, t_i^4) : \neg\lambda(T_i) \in c_j\}.$$

Now, we are in a position to define

$$\begin{aligned}
E_1 &= E_A \cup \bigcup_i \{(t_i^1, t_i^2)\}, \\
E_2 &= E \cap (C_3 \times T^4), \\
E_3 &= E \cap (C_2 \times T^4)^+, \\
E_4 &= E \cap (C_2 \times T^4)^-, \\
E_5 &= E \cap \left((T^2 \cup T^3) \times C_3\right), \\
E_6 &= E \cap \left((T^2 \cup T^3) \times C_2\right), \text{ and} \\
E_7 &= E \setminus (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6).
\end{aligned}$$

First, we consider $(V, E_1)$ and find that it is easy but a bit tedious to check that it is transitive. For the reorientation of $(V, \widetilde{E}_1)$, define

$$E_1' = \bigcup_i \left(\{(t_i^1, t_i^3), (t_i^2, t_i^3)\} \cup (T_i \times C)\right).$$

Again, it is easy to see that $(V, E_1')$ is a transitive reorientation of $(V, \widetilde{E}_1)$. By Lemma 4.3, this makes $E_1$ 2-inducible.

For the edge sets $E_2, E_3, E_4, E_4$, and $E_6$ we find that they are all vertex-disjoint union of unidirected stars and therefore 2-inducible by Lemmas 4.4 and 4.5. These results all exploit that $\varphi$ is in REDUCED-FEW-CNF: for $E_2$ and $E_5$, the statement follows from the fact that every variable occurs at most once in $C_3$ whereas for $E_3, E_4$, and $E_6$, the statement holds due to each literal occuring at most once in $C_2$.

Finally, we find that

$$\begin{aligned}
E_7 \subseteq\ &((T^4 \cup T^5 \cup T^6) \times C) \cup \\
&(C \times (T^1 \cup T^2 \cup T^3)) \cup \\
&(T^2 \times T^3) \cup (T^3 \cup T^1).
\end{aligned}$$

Notice that the righthand side is acyclic, and so $E_7$ is also acyclic.

Since the defined edge sets $E_1, \ldots, E_7$ are pairwise disjoint, $E_7$ is acyclic, and all sets but $E_7$ are 2-inducible, we can apply Lemma 4.8 and find that $(V, E)$ is 13-inducible. This concludes the proof. □

## 6.5 RANKED PAIRS

The NP-hardness proof by Brill and Fischer (2012) for *RP*, the neutral variant of ranked pairs, is by a reduction from SAT. For each Boolean formula $\varphi$ in CNF they constructed a weighted graph $G_\varphi^{RP}$ such that a decision node $d$ is selected by *RP* from $G_\varphi^{RP}$ if and only if $\varphi$ is satisfiable. The construction, of course, works just as well for a reduction from 3SAT. We may also assume that in every formula $\varphi$ in 3-CNF no variable occurs more than once in each clause.

Since the original construction in Brill and Fischer (2012) does not yield a tournament, investigating it would give only results involving an even number of voters. However, a minor modification of the argument results in a tournament, providing the means to discuss an odd number of voters. We first define the class $\mathcal{G}^{RP}$ in which the weighted graphs $G_\varphi^{RP}$ for formulas $\varphi$ in 3-CNF are contained. Then we prove that every graph in this class is induced by an 8-voter profile, showing that deciding whether a given alternative is a ranked pairs winner is already NP-complete for eight voters. Later, we define the tournament class $\mathcal{T}^{RP}$ and show the same result for an odd number of voters. Finally we combine these two results into a corollary.

*the class $\mathcal{G}^{RP}$*

A weighted graph $(V, E)$ (with weight function $w$) belongs to $\mathcal{G}^{RP}$ if and only if it fulfills the following conditions. There are some integers $m, l \geqslant 1$ such that

$$V = D \cup U_1 \cup \cdots \cup U_m \cup X_1 \cup \cdots \cup X_l,$$

where, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant l$,

$$D = \{d\},$$
$$U_i = \{u_i^1, u_i^2, u_i^3, u_i^4\}, \text{ and}$$
$$X_j = \{x_j\}.$$

If $(V, E)$ is obtained as the graph $G_\varphi^{RP}$ for some $\varphi$ in 3-CNF, $l$ is the number of clauses, $m$ the number of variables occurring in $\varphi$, the $U_i$s are the variable gadgets, the $X_j$s the clause gadgets, and, finally, $D$ the decision node. Let $U_i^j = \{u_i^j\}$, $U^j = \bigcup_{i=1}^m \{u_i^j\}$, $U = \bigcup_{i=1}^m U_i$ and $X = \bigcup_{i=1}^l X_i$. Moreover, $E = E^\sigma \cup E^\varphi$, where $E^\sigma$ (the *skeleton*) and $E^\varphi$ (the *formula dependent part*) are disjoint such that

$$E^\sigma = (D \times (U^1 \cup U^3)) \cup (X \times D) \cup \bigcup_{i=1}^m \{(u_i^1, u_i^2), (u_i^2, u_i^3), (u_i^3, u_i^4), (u_i^4, u_i^1)\}$$

and $E^\varphi$ is such that for all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant l$:

$$E^\varphi \subset (U^2 \cup U^4) \times X,$$
$$|E^\varphi \cap (U^2 \cup U^4) \times X_j)| \leqslant 3, \text{ and}$$
$$|E^\varphi \cap (U_i^2 \cup U_i^4) \times X_j)| \leqslant 1,$$

i.e., every vertex in X has at most three incoming edges (intuitively corresponding to the literals $x$ contains) and at most one from every $U_i$ (intuitively corresponding to that no propositional variable occurs more than once in each clause). Finally, we check that the weight function $w$ is defined such that all edges in $E \cap ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 4 and all edges in $E \setminus ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 2. The reader is deferred to Figure 6.6 for an example illustrating this definition of the class $\mathcal{G}^{RP}$.

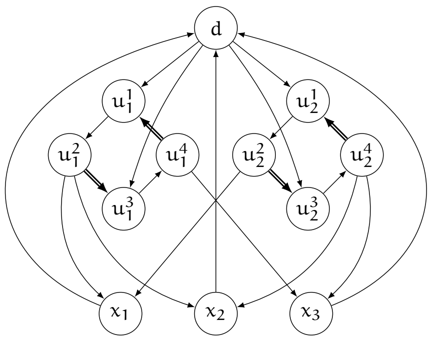



**Figure 6.6:** A graph $(V, E)$ in the class $\mathcal{G}^{RP}$. The thick edges have weight 4 whereas the thin edges have weight 2.

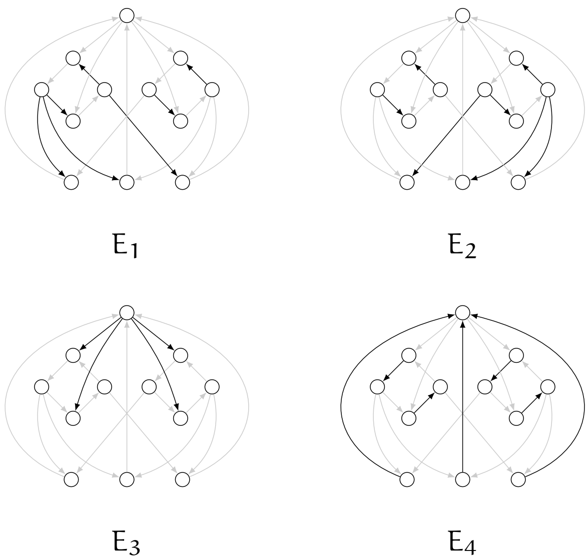



**Figure 6.7:** The sets $E_1$, $E_2$, $E_3$, and $E_4$ for the graph of Figure 6.6 as defined in the proof of Theorem 6.6.

*inducing every graph in $\mathcal{G}^{RP}$ with 8 voters*

Not being a complete graph, $G_{\varphi}^{RP}$ can only be induced by a profile involving an even number of voters. In fact, we will prove that only eight voters suffice to induce any graph in $\mathcal{G}^{RP}$.

THEOREM 6.6

Deciding whether a given alternative is a ranked pairs winner is NP-complete if the number of voters is even and at least 8.

*Proof.* Membership in NP follows from the fact that it is easy to verify whether a given ranking can be the outcome of the *RP* procedure, independent on the number of voters.

For hardness, let $(V, E)$ be a graph (with weight function $w$) in $\mathcal{G}^{RP}$. Intuitively, $(V, E) = G_{\varphi}^{RP}$ for some formula $\varphi$ in 3-CNF. It suffices to show that $(V, E)$ is induced by an 8-voter profile. As an auxiliary notion, let for each $1 \leqslant j \leqslant l$,

$$E^{\varphi} \cap ((U^2 \cup U^4) \times X_j) = E_{j,1}^{\varphi} \cup E_{j,2}^{\varphi} \cup E_{j,3}^{\varphi},$$

where $|E_{j,i}^{\varphi}| \leqslant 1$ for all $1 \leqslant i \leqslant 3$. Intuitively, $E_{j,1}^{\varphi}$, $E_{j,2}^{\varphi}$, and $E_{j,3}^{\varphi}$ impose an ordering on the incoming edges of vertex $x_j$. Also set

$$E_i^{\varphi} = \bigcup_{j=1}^{l} E_{j,i}^{\varphi}$$

for each $1 \leqslant i \leqslant 3$, i.e., $E_i^{\varphi}$ collects the $i$-th incoming edges of the vertices in $X$. Now define the following edge sets.

$$E_1 = E_1^{\varphi} \cup \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1)),$$

$$E_2 = E_2^{\varphi} \cup \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1)),$$

$$E_3 = E_3^{\varphi} \cup (D \times (U^1 \cup U^3)), \text{ and}$$

$$E_4 = (X \times D) \cup \bigcup_{i=1}^{m} ((U_i^1 \times U_i^2) \cup (U_i^3 \times U_i^4)).$$

Observe that $E = E_1 \cup E_2 \cup E_3 \cup E_4$ (see Figure 6.7). Moreover, each of $(V, E_1)$, $(V, E_2)$, $(V, E_3)$, and $(V, E_4)$ is a vertex-disjoint union of unidirected stars. Hence, by Lemma 4.5 we may assume they are induced by the 2-voter profiles $(R_1^1, R_2^1)$, $(R_1^2, R_2^2)$, $(R_1^3, R_2^3)$, and $(R_1^4, R_2^4)$, respectively. Moreover, $E_1$, $E_2$, $E_3$, and $E_4$ all contained in $E$ and therefore also pairwise orientation compatible. By Lemma 4.7 it thus follows that $(V, E)$ is induced by the 8-voter profile

$$R = (R_1^1, R_2^1, R_1^2, R_2^2, R_1^3, R_2^3, R_1^4, R_2^4).$$

Moreover, $E_1$, $E_3$, and $E_4$ as well as $E_2$, $E_3$, and $E_4$ are pairwise disjoint whereas $E_1 \cap E_2 = \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1))$. Thus, all edges in $E \setminus \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1))$ have weight 2, whereas those in $\bigcup_{i=1}^{m}\big((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1)\big)$ have weight 4. We may conclude that also the graph $(V, E)$ with its weights is induced by the 8-voter profile $R$. □

The original hardness construction contained edges with weights 2 or 4 and unspecified edges, defining a priority over the edges. It is easy to see that increasing all weights in such a graph by 1 to 3 and 5 does not change this priority. Similarly, adding edges with weight 1 is not harmful as the corresponding pairs are added to the bottom of the priority, making them irrelevent to determining whether $d$ is an *RP* winner or not. Therefore, by incorporating these observations into $G_\varphi^{RP}$, for each Boolean formula $\varphi$ in 3-CNF, we can create a weighted tournament (call it $T_\varphi^{RP}$) from which $d$ is selected by *RP* if and only if $\varphi$ is satisfiable. We denote the class of weighted tournaments that consist of these $T_\varphi^{RP}$ by $\mathcal{T}^{RP}$.

We adapt the same notation as for $\mathcal{G}^{RP}$. A weighted tournament $(V, E')$ (with weight function $w'$) belongs to $\mathcal{T}^{RP}$ if and only if it satisfies the following conditions. The set of alternatives can be written as

*the class $\mathcal{T}^{RP}$*

$$V = D \cup U_1 \cup \cdots \cup U_m \cup X_1 \cup \cdots \cup X_l$$

whereas the edge set $E'$ is the union of two disjoint sets $E'^{\sigma}$ (the *skeleton*) and $E'^{\varphi}$ (the *formula dependent part*). Assuming that $E$ is the edge set of $\mathcal{G}_\varphi^{RP}$, then $E'^{\varphi} = E^{\varphi}$ and $E'^{\sigma} = E^{\sigma} \cup E_c'^{\sigma}$ where

$$\begin{aligned} E_c'^{\sigma} = &\big(((D \times U) \cup (U \times X)) \setminus E\big) \cup \\ &\bigcup_{i=1}^{m} ((U_i^1 \times U_i^3) \cup (U_i^2 \times U_i^4)) \cup \\ &\bigcup_{i<j} (U_i \times U_j) \cup \bigcup_{i<j} (X_i \times X_j). \end{aligned}$$

$E_c'^{\sigma}$ can be equivalently described as a reorientation of $\widetilde{E}$. Moreover, we check that $w'$ is defined such that all edges in $E'^{\sigma} \cap ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 5, all edges in $(E'^{\varphi} \cup E'^{\sigma}) \setminus ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 3, and all edges in $E_c'^{\sigma}$ have weight 1.

We show 11 voters are sufficient to induce every tournament in the class $\mathcal{T}^{RP}$.

*inducing every tournament in $\mathcal{T}^{RP}$ with 11 voters*

THEOREM 6.7

Deciding whether a given alternative is a ranked pairs winner is NP-complete, if the number of voters is odd and at least 11.

*Proof.* The proof here is very similar to the previous theorem. Let $(V, E')$ be a tournament with weight function $w'$ in $\mathcal{T}^{RP}$. Intuitively, $(V, E') = T_\varphi^{RP}$ for some formula $\varphi$ in 3-CNF. It suffices to show that

$(V, E')$ is induced by an 11-voter profile. Using the notation provided in the proof of Theorem 6.6, we define the following edge sets.

$$\begin{aligned}
E'_1 =& E'^{\varphi}_1 \cup \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1)),\\
E'_2 =& E'^{\varphi}_2 \cup \bigcup_{i=1}^{m} ((U_i^2 \times U_i^3) \cup (U_i^4 \times U_i^1)),\\
E'_3 =& E'^{\varphi}_3 \cup (D \times (U^1 \cup U^3)),\\
E'_4 =& (X \times D) \cup \bigcup_{i=1}^{m} ((U_i^1 \times U_i^2) \cup (U_i^3 \times U_i^4)),\\
E'_5 =& (X \times D) \cup \bigcup_{i=1}^{m} \{(u_i^4, u_i^1)\}, \text{ and}\\
E'_6 =& (D \times U) \cup (D \times X) \cup (U \times X) \cup \bigcup_{\substack{1 \leqslant i \leqslant m\\ j<l}} (U_i^j \times U_i^l) \cup\\
& \bigcup_{i<j} (U_i \times U_j) \cup \bigcup_{i<j} (X_i \times X_j).
\end{aligned}$$

Observe that $E'_1$, $E'_2$, $E'_3$, $E'_4$, and $E'_5$ are contained in $E'$, making them pairwise orientation compatible, and that each of $(V, E'_1)$, $(V, E'_2)$, $(V, E'_3)$, $(V, E'_4)$, and $(V, E'_5)$ is a forest of stars. Therefore, in virtue of Lemma 4.5 we may assume that they are induced by the 2-voter profiles $(R^1_1, R^1_2)$, $(R^2_1, R^2_2)$, $(R^3_1, R^3_2)$, $(R^4_1, R^4_2)$,and $(R^5_1, R^5_2)$. Moreover, it can readily be appreciated that $E'_6 \supseteq E' \setminus (E'_1 \cup \ldots \cup E'_5)$. As $E'_6$ defines a transitive closure for an order over all of the alternatives in $V$ (see Figure 6.8), $(V, E'_6)$ is acyclic, and we may assume that it is induced by a voter with the preference relation $R^6 = E'_6$. Thus by Lemma 4.8, $(V, E')$ is induced by the 11-voter profile

$$R = (R^1_1, R^1_2, R^2_1, R^2_2, R^3_1, R^3_2, R^4_1, R^4_2, R^5_1, R^5_2, R^6).$$

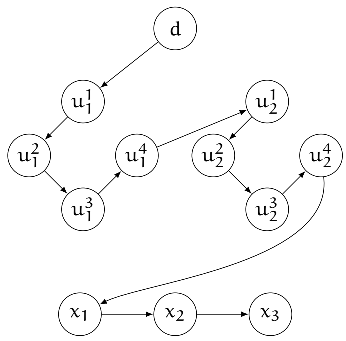



**Figure 6.8:** The order implied by the edge set $E'_6$ over the alternatives of a tournament $(V, E')$ in the class $\mathcal{T}^{RP}$.

Furthermore, note that there are some edges in common among the edge sets and that $E'_6$ is not orientation compatible with $E'$. Edges in

$E'^{\sigma} \cap (U^2 \times U^3)$ occur in $E'_1$, $E'_2$, and $E'_6$; edges in $E'^{\sigma} \cap (U^4 \times U^1)$ occur in $E'_1$, $E'_2$, and $E'_5$ while $E'_6$ includes edges in the opposing direction or, equivalently, includes $\bigcup_{i=1}^{m}(U_i^1 \times U_i^4)$; each edge in $(E'^{\varphi} \cup E'^{\sigma}) \setminus ((U^2 \times U^3) \cup (U^4 \times U^1))$ occurs in $E'_6$ and exactly one of the other edge sets; and, finally, edges in $E_c'^{\sigma}$ occur only in $E'_6$. Thus, a simple counting reveals that edges in $E'^{\sigma} \cap ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 5, edges in $(E'^{\varphi} \cup E'^{\sigma}) \setminus ((U^2 \times U^3) \cup (U^4 \times U^1))$ have weight 3, and edges in $E_c'^{\sigma}$ have weight 1. Therefore, we may conclude that $(V, E')$ together with its weights is induced by the 11-voter profile $R$. □

COROLLARY 6.8

Deciding whether a given alternative is a ranked pairs winner is NP-complete if the number of voters is either eight or at least ten.

*Proof.* This follows from Theorems 6.6 and 6.7. □

## 6.6 SUMMARY

We have shown that the winner determination problem for the Banks set, the tournament equilibrium set, the Slater set, and ranked pairs winners remains NP-hard even when the number of voters is limited to a small constant. Our finding are summarized in Table 6.1. It remains open whether any of these tournament solutions is already NP-hard for three voters or whether Kemeny's rule is NP-hard for *any* constant odd number of voters.

| Solution Concept | NP-hard for $n \geqslant$ |
|---|---|
| Banks set | 5 voters |
| Tournament equilibrium set | 7 voters |
| Slater set | 13 voters |
| Ranked pairs | 8 voters ($n \neq 9$) |

**Table 6.1:** Numbers of voters for which winner determination is NP-hard. The Banks set and the tournament equilibrium set are defined for an odd number of voters only.

# 7 COMPOSITION-CONSISTENCY OF TOURNAMENT SOLUTIONS

> But there are always several adequate formal sets of possible choices: just add tiny distinctions about which nobody cares. Then the "good choice for the society" should not depend upon the modeler's chosen mathematical formalization, if this formalization is adequate.
>
> — Jean-François Laslier 2000

Many tournament solutions are non-trivial to compute. We have seen in Chapter 6 that the winner determination problem for, e.g., the Banks set and the tournament equilibrium set is NP-hard even if the number of voters is limited to a small constant. This can be seen as evidence that the class of all tournaments—even those that correspond to majority relations of a rather small electorate—is excessively rich even though it is well-known that only a fraction of these tournaments occur in realistic settings (see, e.g., Feld and Grofman, 1992). Therefore, an important question is whether there are natural distributions of tournaments that admit more efficient algorithms for computing specific tournament solutions.

In this chapter, we seek to exploit the frequent existence of proper decompositions of tournaments as discussed in Section 3.4, combined with a consistency property (composition-consistency) that is satisfied by a good number of tournament solutions. As before, we use simulations to assess the effect in question on tournaments obtained from stochastic models.

In related work, Betzler et al. (2014) reviewed data reduction rules that facilitate the computation of *Kemeny rankings*. One of the techniques, the "Extended Condorcet criterion" corresponds to a special case of (weak) composition-consistency for Kemeny rankings where only reducible components are considered. Furthermore, a preprocessing technique that resembles the one proposed here has been used by Conitzer (2006) to speed up the computation of *Slater rankings*. Interestingly, even though Slater's solution is *not* composition-consistent, decompositions of the tournament can be exploited to identify a *subset* of the optimal rankings (see Laslier, 1997, Proposition 3.4.4).

Our results, on the other hand, allow us to compute *complete choice sets* and are applicable to *all* composition-consistent tournament solutions. For this chapter, we implicitly assume that a tournament solution also satisfies strong Condorcet-consistency, i.e., that Condorcet winners are chosen uniquely whenever they exist (see Section 5.1).[40]

## 7.1 COMPOSITION-CONSISTENCY

A tournament solution is composition-consistent if it chooses the "best" alternatives from the "best" components (Laffond et al., 1996).[41]

DEFINITION 7.1

*composition-consistent*

A tournament solution S is *composition-consistent* if for all tournaments T and $\widetilde{T}$ such that $\widetilde{T}$ is the summary of T with respect to some decomposition $\{B_1, \dots, B_k\}$,

$$S(T) = \bigcup_{i \in S(\widetilde{T})} S(T|_{B_i}).$$

Composition-consistent tournament solutions include the uncovered set, the minimal covering set, the bipartisan set, the Banks set, the tournament equilibrium set, and the minimal extending set (Laslier, 1997; Brandt, 2011b).[42] Recall that the former three admit polynomial-time algorithms whereas the latter three are computationally intractable. None of the concepts is known to admit a linear-time algorithm which, according to Proposition 3.7, is the time needed to compute the decomposition tree.

Before we proceed, we briefly introduce the most basic concepts of parameterized complexity theory (see, e.g., Niedermeier, 2006). In contrast to classical complexity theory, where only the size of problem instances is taken into account, parameterized complexity allows for a more fine-grained analysis by considering arbitrary parameters of the instances. A problem with parameter $k$ is said to be *fixed-parameter tractable* (or to belong to the class FPT) if there exists an algorithm that solves the problem in time $f(k) \cdot \text{poly}(|I|)$, where $|I|$ is the size of the input and $f$ is some computable function independent of $|I|$.

*fixed-parameter tractable*

For example, each (computable) problem is trivially fixed-parameter tractable with respect to the parameter $|I|$. Given an NP-hard problen, the crucial point is to identify a parameter that is reasonably small in realistic instances and to devise an algorithm that is at most superpolynomial in this parameter.

40 The only concepts considered in this thesis that violate this property are *TRIV* and *CNL*.

41 Composition-consistency is related to *cloning-consistency*, which was introduced by Tideman (1987) in the context of voting.

42 The top cycle is not composition-consistent but a decomposition could still be used to determine the choice set (Laslier, 1997, Prop. 2.4.9). Since the top cycle can always be computed in linear time, this approach is of limited use.

## 7.2 EXPLOITING COMPOSITION-CONSISTENCY

Let $S$ be a composition-consistent tournament solution and consider an arbitrary tournament $T = (A, \succ)$ together with its decomposition tree $\mathcal{D}(T)$. For an internal node $B$ of $\mathcal{D}(T)$, let $B_i(\mathcal{D}(T), B)$ denote the $i$-th children of $B$ in $\mathcal{D}(T)$. Composition-consistency implies that

$$S(T|_B) = \bigcup_{i \in S(T_B)} S(T|_{B_i(\mathcal{D}(T),B)}).$$

*recursive algorithm*

The choice set $S(T)$ can thus be computed by starting at the root of $\mathcal{D}(T)$ and iteratively applying the equation above. If $B$ is reducible, we immediately know that $S(T|_B) = S(T|_{B_1(\mathcal{D}(T),B)})$, since $1$ is the maximum of the transitive tournament $T_B$. A straightforward implementation of this approach is given in Algorithm 7.1.

**Input**: composition-consistent tournament solution $S$,
tournament $T$
**Output**: $S(T)$

```
compute 𝒟(T)
S ← ∅
Q ← (A)
while Q ≠ () do
    B ← Dequeue(Q)
    if |B| = 1 then
        S ← S ∪ B
    else
        if B is reducible then
            Enqueue(Q, B_1(𝒟(T), B))
        else B is irreducible
            compute S(T_B)
            foreach i ∈ S(T_B) do
                Enqueue(Q, B_i(𝒟(T), B))
return S
```

**Algorithm 7.1:** Compute $S(T)$ via decomposition tree

Algorithm 7.1 visits each node of $\mathcal{D}(T)$ at most once. The algorithm for computing $S$ is only invoked for tournaments $T_B$ for which $B$ is irreducible and $|B| \geqslant 2$. The order of such a tournament $T_B$ is equal to the number of children of node $B$ in $\mathcal{D}(T)$.

Let $f(m)$ be an upper bound on the running time of an algorithm that computes $S(T)$ for tournaments of order $|T| \leqslant m$. Then, the running time of Algorithm 7.1 can be upper-bounded by $f(\delta(T))$ times the number of irreducible nodes of $\mathcal{D}(T)$, i.e., with the decomposition degree $\delta(T)$ as a parameter. We thus obtain the following Theorem.

THEOREM 7.2

Let $S$ be a composition-consistent tournament solution and let furthermore $f(k)$ be an upper bound on the running time of an algorithm that computes $S$ for tournaments of order at most $k$. Then, $S(T)$ can be computed in $\mathcal{O}(m^2) + f(\delta(T)) \cdot (m-1)$ time, where $m$ is the order or $T$.

*Proof.* Let $T$ be a tournament and $m = |T|$. We show that Algorithm 7.1 computes $S(T)$ in $\mathcal{O}(m^2) + f(\delta(T)) \cdot |Irr(D(T))|$ time. Correctness follows from composition-consistency of $S$. The running time can be bounded as follows. Computing $\mathcal{D}(T)$ requires time $\mathcal{O}(m^2)$ (Proposition 3.7). During the execution of the while-loop, each node $B$ of $\mathcal{D}(T)$ is visited at most once. If $B$ is reducible or a singleton, there is no further computation. If $B \in Irr(D(T))$, $S(T_B)$ is computed. As $|T_B|$ is upper-bounded by $\delta(T)$, this can be done in $f(\delta(T))$ time. Finally, as the number of internal nodes in a tree with $m$ leafs is bounded by $m-1$, we have that $|Irr(D(T))| \leqslant m-1$. Summing up, this yields a running time of at most $\mathcal{O}(m^2) + f(\delta(T)) \cdot (m-1)$. □

In particular, Theorem 7.2 shows that the computation of $S(T)$ is fixed-parameter tractable with respect to the parameter $\delta(T)$.

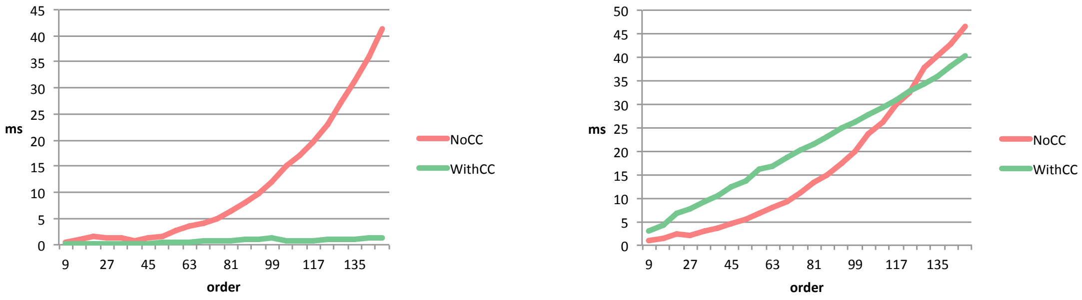
For a better understanding of this theorem, consider a composition-consistent tournament solution $S$ such that $f(m)$ is in $\mathsf{DTIME}(2^{\mathcal{O}(m)})$. This holds, for example, for the Banks set. For any given tournament $T$ of order $m$, Theorem 7.2 then implies that $S(T)$ can be computed efficiently (i.e., in time polynomial in $m$) whenever $\delta(T)$ is in $\mathcal{O}(\log m)$. Theorem 7.2 is of course also applicable to tractable tournaments solutions such as the minimal covering set and the bipartisan set. Although computing these solutions is known to be in P, existing algorithms rely on linear programming and may be too time-consuming for very large tournaments. For both concepts, a significant speed-up can be expected for distributions of tournaments that admit a small decomposition degree.

Generally, decomposing a tournament asymptotically never harms the running time, as the time required for computing the decomposition tree is only linear in the input size.[43]

## 7.3 EXPERIMENTAL RESULTS

It has been shown in the previous section that the problem of computing a composition-consistent tournament solution is fixed-parameter tractable with respect to the decomposition degree of a tournament. Actual values of the decomposition degree for tournaments from various stochastic preference models were already presented in Section 3.4. In this section, we use the same kind of simulations to ex-

43 Checking whether there exists a Condorcet winner already requires $\theta(m^2)$ time.




Banks set (*BA*) bipartisan set (*BP*)

**Figure 7.1:** Running time comparisons of straightforward implementations ("NoCC") and their enhancements by Algorithm 7.1 ("WithCC") on artificial, highly decomposable tournaments.

amine the effect on the actual running time of using Algorithm 7.1 ("WithCC") in comparison to the straightforward method ("NoCC") to compute composition-consistent tournament solution. The times given for the WithCC algorithm always include the time it took to compute the decomposition tree. As for tournament solutions, we considered *BA* and *TEQ* both of which are notoriously hard to compute but satisfy composition-consistency. The results are given in Figure 7.2 (for *BA*) and Figure 7.3 (for *TEQ*) at the end of this chapter.

It can be seen that the WithCC algorithm is never significantly worse and in many scenarios much better than the NoCC implementation. It is in line with our earlier findings on urn cultures (including IC and IAC) that the number of voters does not have a noteworthy effect on their induced tournaments. For *TEQ*, it is fair to say that it should never be computed with the NoCC algorithm. The gap between the two implementations for our sensible distance-based models can be huge.

*always respect components when computing TEQ*

For illustrative purposes, we also examined the speedup on artificial tournaments that were designed to have components. Specifically, we generated cyclones[44] of size up to 51 and replaced each vertex by a 3-cycle component. Results of this comparison for the Banks set and the bipartisan set are shown in Figure 7.1.

As can be seen, the effect for the computation of *BA* is dramatic! For *BP*, we can observe an interesting effect where the running time of the algorithm that exploits composition-consistency is larger for small instances. This is due to an overhead caused by the now multiple calls to the external LP solver that is used to compute *BP*. For sufficiently large instances, the increase in computational speed outweighs this (seemingly linear) effect.

*overhead due to number of problem instances*

44 For odd $m$, a cyclone (or cyclical tournament) $C_m = (\{a_1, \dots, a_m\}, \succ)$ is defined by $a_i \succ a_j$ if and only if $(j - i \mod m) \in \{1, \dots, \frac{m-1}{2}\}$

## 7.4 SUMMARY

We studied the algorithmic benefits of composition-consistent tournament solutions and show that computing any composition-consistent tournament solution is fixed-parameter tractable with respect to the decomposition degree. This is of particular relevance for tournament solutions that are known to be computationally intractable such as the Banks set and the tournament equilibrium set. For example, one corollary of this result is that the Banks set of a tournament can be computed efficiently whenever the decomposition degree is polylogarithmic in the number of alternatives. As a consequence, the speedup obtained by exploiting composition-consistency when computing tournament solutions for these instances is quite substantial as we showed experimentally for the tournament equilibrium set and the Banks set. Since computing a decomposition tree requires only linear time, decomposing a tournament never hurts, and often helps.

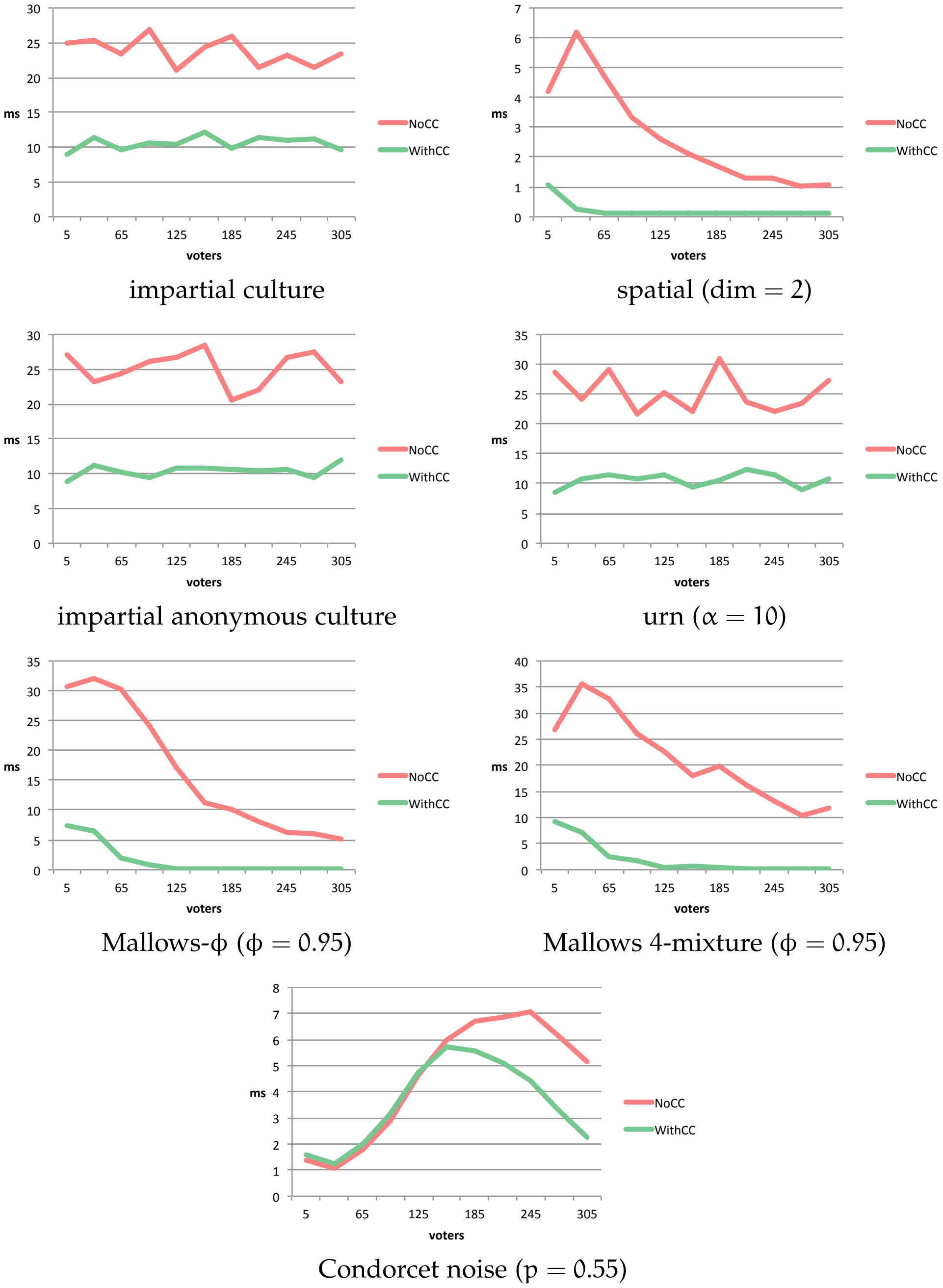



**Figure 7.2:** Running time comparisons of a straightforward implementation of the Banks set ("NoCC") and its enhancements due to Algorithm 7.1 ("WithCC") on tournaments of size 30 from various stochastic models for electorate sizes from 3 to 305.

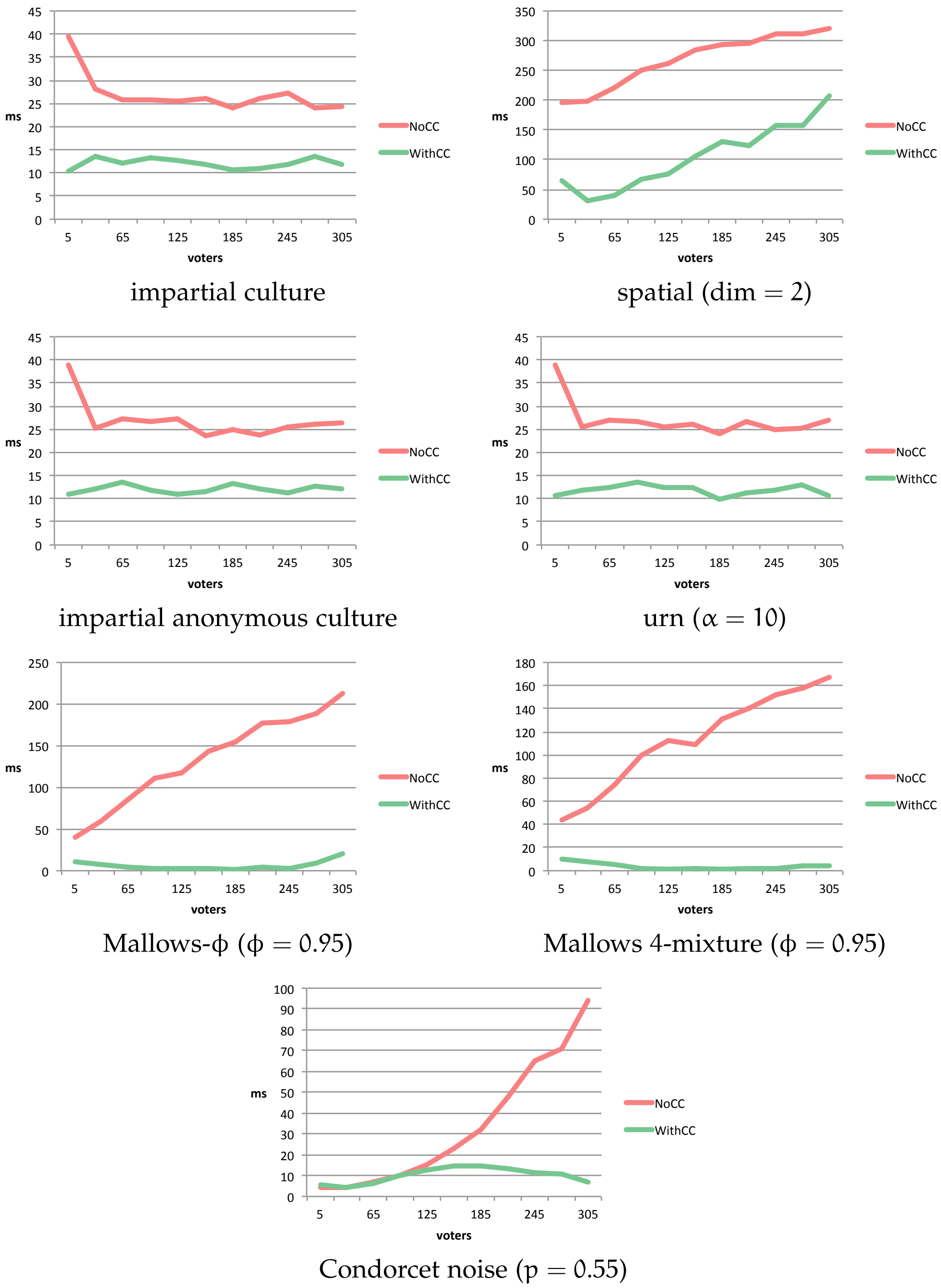



**Figure 7.3:** Running time comparisons of a straightforward implementation of the tournament equilibrium set ("NoCC") and its enhancements due to Algorithm 7.1 ("WithCC") on tournaments of size 20 from various stochastic models for electorate sizes from 3 to 305.

# 8 POSSIBLE AND NECESSARY WINNERS OF PARTIAL TOURNAMENTS

> People, for some reason, like to do combinatorial search in their spare time.
>
> Michael Trick, 2010

When choosing from a tournament, relevant information may only be partly available. This could be because some preferences are yet to be elicited, some matches yet to be played, or certain comparisons yet to be made. In such cases, it is natural to speculate which are the potential and inevitable outcomes on the basis of the information already at hand.

Given any solution concept on tournaments $S$, *possible winners* of a partial tournament $G$ are defined as alternatives that are selected by $S$ in *some* tournament completion of $G$, and *necessary winners* are alternatives that are selected in *all* such completions.

In this chapter, we address the computational complexity of identifying the possible and necessary winners for a number of solution concepts whose winner determination problem for complete tournaments is tractable. We consider five of the most common solution concepts for tournaments—namely, Condorcet winners (*CW*), Condorcet non-loses (*CNL*), the Copeland solution (*CO*), the top cycle (*TC*), and the uncovered set (*UC*)—and three common solutions for weighted tournaments—Borda (*BO*), maximin (*MM*) and resolute ranked pairs ($RP_\tau$). For each of these solution concepts, we characterize the complexity of the following problems: deciding whether a given alternative is a possible winner (*PW*), deciding whether a given alternative is a necessary winner (*NW*), as well as deciding whether a given subset of alternatives equals the set of winners in some completion (*PWS*). These problems can be challenging, as even unweighted partial tournaments may allow for an exponential number of completions.

## 8.1 RELATED WORK

Similar problems have been considered before. For Condorcet winners, voting trees and the top cycle, it was already shown that possible and necessary winners are computable in polynomial time (Lang et al., 2012; Pini et al., 2008; Pini et al., 2011). The same holds for

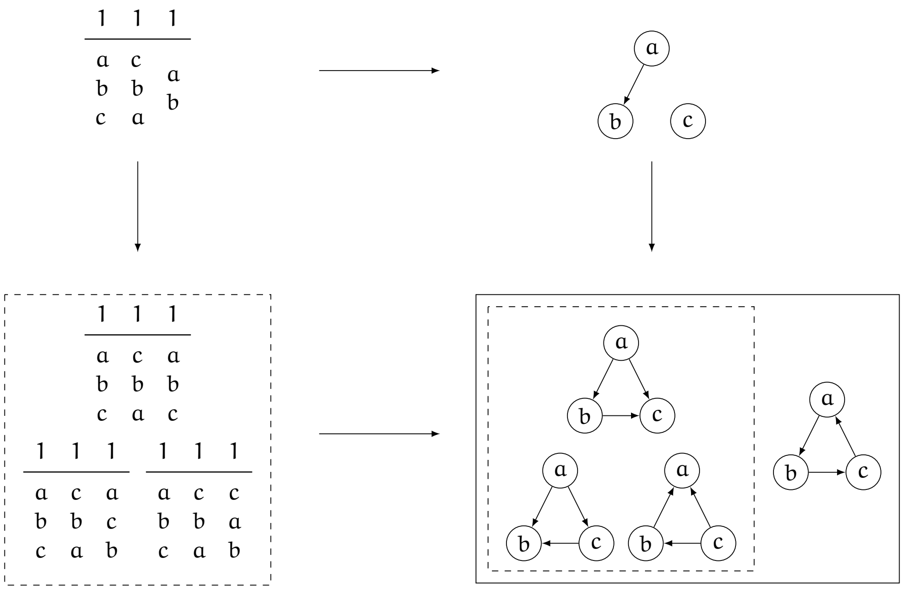


**Figure 8.1:** This non-commutative diagram illustrates the two approaches to possible and necessary winners of partial preference profiles for majoritarian social choice functions. First, the completion of the partial profile to full preference profiles is shown in the bottom left. The corresponding majority tournaments are in the dashed box on the bottom right. In this work, we start from the partial majority tournament on the top right which is induced by the partial preference profile. Then, we consider all possible completions to tournaments which are depicted in the solid box on the bottom right.

computing possible Copeland winners that were considered in the context of sports tournaments (Cook et al., 1998).

*possible and neccesary winners of partial preference profiles*

A more specific setting that is frequently considered within the area of computational social choice differs from our setting in a subtle but important way that is worth being pointed out. There, tournaments are assumed to arise from pairwise majority comparisons on the basis of a profile of individual voters' preferences.[45] Since a *partial* preference profile $R$ need not conclusively settle every majority comparison, it may give rise to a *partial* tournament only. There are two natural ways to define possible and necessary winners for a partial preference profile $R$ and solution concept $S$.[46] The first is to consider the completions of $R$ and the winners under $S$ in the corresponding tournaments. The second—covered by our more general setting—is to consider the completions of the incomplete tournament $G(R)$ corre-

45 See, e.g., Baumeister and Rothe (2010), Betzler and Dorn (2010), Konczak and Lang (2005), Walsh (2007), and Xia and Conitzer (2011) for the basic setting, Betzler et al. (2009) for parameterized complexity results, Hazon et al. (2012) and Kalech et al. (2011) for probabilistic settings, and Chevaleyre et al. (2010) and Xia et al. (2011) for settings with a variable set of alternatives.

46 These two ways of defining possible and necessary winners are compared (both theoretically and experimentally) in Lang et al. (2012) and Pini et al. (2011) for three solution concepts: Condorcet winners, voting trees and the top cycle.

sponding to R and the winners under S in these. Since every tournament corresponding to a completion of R is also a completion of $G(R)$ but not necessarily the other way round, the first definition gives rise to a *stronger* notion of a possible winner and a *weaker* notion of a necessary winner. Interestingly, and in sharp contrast to our results, determining these stronger possible and weaker necessary winners is computationally hard for many voting rules (Lang et al., 2012; Xia and Conitzer, 2011).

Here, we do not assume that tournaments arise from majority comparisons in voting or from any other specific procedure. This approach has a number of advantages. Firstly, it matches the diversity of settings to which solutions concepts on tournaments are applicable, which goes well beyond social choice and voting. For instance, our results also apply to a question commonly encountered in sports competitions, namely, which teams can still win the cup and which future results this depends on.[47] Secondly, (partial) tournaments provide an informationally sustainable way of representing the relevant aspects of many situations while maintaining a workable level of abstraction and conciseness. For instance, in the social choice setting described above, the partial tournament induced by a partial preference profile is a much more succinct piece of information than the preference profile itself. Finally, specific settings may impose restrictions on the feasible extensions of partial tournaments. The positive algorithmic results can be used to efficiently approximate the sets of possible and necessary winners in such settings, where the corresponding problems may be intractable. The voting setting discussed above serves to illustrate this point.

## 8.2 PRELIMINARIES

A *partial tournament* is a pair $G = (A, E)$ where $A$ is a finite set of alternatives and $E \subseteq A \times A$ an asymmetric relation on $A$.[48]

*partial tournament*

Let $G = (A, E)$ be a partial tournament. Another partial tournament $G' = (A', E')$ is called an *extension* of $G$, denoted $G \leqslant G'$, if $A = A'$ and $E \subseteq E'$. If $E'$ is complete, $G'$ is called a *completion of* $G$. We write $[G]$ for the set of completions of $G$, i.e., $[G] = \{T \in \mathcal{T} : G \leqslant T\}$. See Figure 8.2 for an example.

*extension*

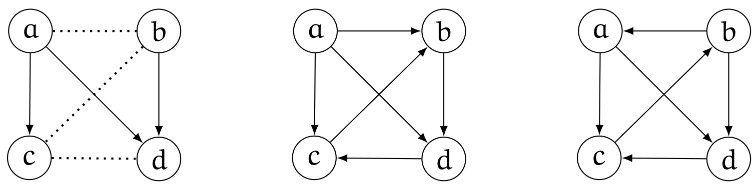
*completion*

We extend our definition of dominions and dominators to sets by defining $D_G(X) = \bigcup_{x \in X} D_G(x)$ and $\overline{D}_G(X) = \bigcup_{x \in X} \overline{D}_G(x)$.[49] For

47 See, e.g., Cook et al. (1998), Kern and Paulusma (2004), and Schwartz (1966).

48 The difference between partial tournaments and the majority graphs we defined earlier lies in the interpretation of non-existing edges. They are seen as (majority) ties in the case of majority graphs whereas we treat them as not yet elicited pairwise comparisons in this chapter.

49 In this chapter, we slightly depart from notational conventions used elsewhere in this thesis: outside of examples, alternatives are usually named $x, y$ instead of $a, b$ to avoid confusion with the capacities of b-matchings used in Section 8.5.1.




$G$ $\quad$ $T_1 \in [G]$ $\quad$ $T_2 \in [G]$

**Figure 8.2:** Example of a partial unweighted tournament $G$ and two of its possible completions. In $G$, the (dotted) edges between the pairs $\{a, b\}, \{b, c\}$, and $\{c, d\}$ are not yet specified.

given $G = (A, E)$ and $X \subseteq A$, we further write $E^{X\rightarrow}$ for the set of edges obtained from $E$ by adding all missing edges from alternatives in $X$ to alternatives not in $X$, i.e.,

$$E^{X\rightarrow} = E \cup \{(x, y) \in X \times A : y \notin X \text{ and } (y, x) \notin E\}.$$

We use $E^{X\leftarrow}$ as an abbreviation for $E^{A\setminus X\rightarrow}$, and respectively write $E^{x\rightarrow}$, $E^{x\leftarrow}$, $G^{X\rightarrow}$, and $G^{X\leftarrow}$ for $E^{\{x\}\rightarrow}$, $E^{\{x\}\leftarrow}$, $(A, E^{X\rightarrow})$, and $(A, E^{X\leftarrow})$.

*partial $n$-weighted tournament*

Let $n$ be a positive integer. A *partial $n$-weighted tournament* is a pair $G = (A, w)$ consisting of a finite set of alternatives $A$ and a weight function $w : A \times A \rightarrow \{0, \dots, n\}$ such that for each pair $(x, y) \in A \times A$ with $x \neq y$, $w(x, y) + w(y, x) \leqslant n$. We say that $T = (A, w)$ is a *(complete) $n$-weighted tournament* if for all $x, y \in A$ with $x \neq y$, $w(x, y) + w(y, x) = n$. A (partial or complete) *weighted* tournamentweighted tournament is a (partial or complete) $n$-weighted tournament for some $n \in \mathbb{N}$. The class of $n$-weighted tournaments is denoted by $\mathcal{T}[n]$. Observe that with each partial 1-weighted tournament $(A, w)$ we can associate a partial tournament $(A, E)$ by setting $E = \{(x, y) \in A : w(x, y) = 1\}$. Thus, (partial) $n$-weighted tournaments can be seen to generalize (partial) tournaments, and we may identify $\mathcal{T}[1]$ with $\mathcal{T}$.

*weighted tournament*

The notations $G \leqslant G'$ and $[G]$ can be extended naturally to partial $n$-weighted tournaments $G = (A, w)$ and $G' = (A', w')$ by letting $(A, w) \leqslant (A', w')$ if $A = A'$ and $w(x, y) \leqslant w'(x, y)$ for all $x, y \in A$, and $[G] = \{T \in \mathcal{T}[n] : G \leqslant T\}$.

For given $G = (A, w)$ and $X \subseteq A$, we further define $w^{X\rightarrow}$ such that for all $x, y \in A$,

$$w^{X\rightarrow}(x, y) = \begin{cases} n - w(y, x) & \text{if } x \in X \text{ and } y \notin X, \\ w(x, y) & \text{otherwise,} \end{cases}$$

and set $w^{X\leftarrow} = w^{A\setminus X\rightarrow}$. Moreover, $w^{x\rightarrow}$, $w^{x\leftarrow}$, $G^{X\rightarrow}$, and $G^{X\leftarrow}$ are defined in the obvious way.

We use the term *solution concept* for functions $S$ that associate with each (complete) tournament $T = (A, E)$, or alternatively with each (complete) weighted tournament $T = (A, w)$, a choice set $S(T) \subseteq A$.[50] A solution concept $S$ is called *resolute* if $|S(T)| = 1$ for each tournament $T$. In this chapter we will consider the following solution concepts: *Condorcet winners* (*CW*), *Condorcet non-losers* (*CNL*), *Copeland* (*CO*), *top cycle* (*TC*), and *uncovered set* (*UC*) for tournaments, and *maximin* (*MM*), *Borda* (*BO*), and resolute *ranked pairs* ($RP_\tau$) for weighted tournaments. Of these, no concepts besides $RP_\tau$ is resolute.

*resolute*

## 8.3 POSSIBLE & NECESSARY WINNER PROBLEMS

A solution concept selects alternatives from complete tournaments or complete weighted tournaments. A partial (weighted) tournament, on the other hand, can be extended to a number of complete (weighted) tournaments, and a solution concept selects a (potentially different) set of alternatives for each of them.

For a given solution concept $S$, we can thus define the set of *possible winners* for a partial (weighted) tournament $G$ as the set of alternatives selected by $S$ from *some* completion of $G$, i.e., as $PW_S(G) = \bigcup_{T \in [G]} S(T)$. Analogously, the set of *necessary winners* of $G$ is the set of alternatives selected by $S$ from *every* completion of $G$, i.e., $NW_S(G) = \bigcap_{T \in [G]} S(T)$. We write $PWS_S(G) = \{S(T) : T \in [G]\}$ (*possible winning sets*) for the set of *sets* of alternatives that $S$ selects for the different completions of $G$.

*possible winners*

*necessary winners*

*possible winning sets*

Note that $NW_S(G)$ may be empty even if $S$ selects a non-empty set of alternatives for each tournament $T \in [G]$, and that $|PWS_S(G)|$ may be exponential in the number of alternatives of $G$. It is also easily verified that $G \leqslant G'$ implies

$$PW_S(G') \subseteq PW_S(G) \text{ and } NW_S(G) \subseteq NW_S(G')$$

and that it holds that

$$PW_S(G) = \bigcup_{G \leqslant G'} NW_S(G') \text{ and } NW_S(G) = \bigcap_{G \leqslant G'} PW_S(G').$$

If a solution concept $S$ refines a solution concept $S'$, that is, $S(G) \subseteq S'(G)$ for all $G$ (which we denote by $S \subseteq S'$), then $PW_S(G) \subseteq PW_{S'}(G)$ and $NW_S(G) \subseteq NW_{S'}(G)$.[51]

Deciding membership in the sets $PW_S(G)$, $NW_S(G)$, and $PWS_S(G)$ for a given solution concept $S$ and a partial (weighted) tournament $G$ are natural computational problems. We will respectively refer to these problems as $PW_S$, $PWS_S$, and $NW_S$.

50 We avoid the term *tournament solution* in this chapter as in our common definition (Def. 5.1 on 50), it requires the concept to never consist of the empty set. This would exclude one of the concepts we study, namely *CW*.

51 $S \subseteq S'$ does however *not* imply $PWS_S(G) \subseteq PWS_{S'}(G)$. The following holds, though: if $S \subseteq S'$ then for all $X \in PWS_S(G)$ there is a $X' \in PWS_{S'}(G)$ such that $X \subseteq X'$.

$PW_S$ (Possible Winners)
*Input:* A partial tournament $G = (A, E)$ or an $n$-weighted partial tournament $G = (A, w)$ with the number $n$; an alternative $x \in A$.
*Question:* Does there exist a completion $T \in [G]$ such that $x \in S(T)$?

$PWS_S$ (Possible Winning Set)
*Input:* A partial tournament $G = (A, E)$ or an $n$-weighted partial tournament $G = (A, w)$ with the number $n$; a subset of alternatives $X \subseteq A$.
*Question:* Does there exist a completion $T \in [G]$ such that $X = S(T)$?

$NW_S$ (NecessaryWinners)
*Input:* A partial tournament $G = (A, E)$ or an $n$-weighted partial tournament $G = (A, w)$ with the number $n$; an alternatives $x \in A$.
*Question:* Is $x$ contained in $S(T)$ for all completions $T \in [G]$?

The problem *NWS* (Necessary Winning Set) is not considered because it can be reduced to *PW* and *NW*:

$$X \in NWS_S(G) \Leftrightarrow X = PW_S(G) = NW_S(G).$$

For irresolute solution concepts, $PWS_S$ may appear a more complex problem than $PW_S$. We are, however, not aware of a polynomial-time reduction from $PW_S$ to $PWS_S$. The relationship between all of these problems may also be of interest for the "classic" possible winner setting with partial preference profiles.

For complete tournaments $T$ we have $[T] = \{T\}$ and thus $PW_S(T) = NW_S(T) = S(T)$ and $PWS_S(T) = \{S(T)\}$. As a consequence, for solution concepts $S$ with an NP-hard winner determination problem—like *Banks*, *Slater*, and *TEQ*—the problems $PW_S$, $NW_S$, and $PWS_S$ are NP-hard as well.[52] We therefore restrict our attention to solution concepts for which winners can be computed in polynomial time.

## 8.4 UNWEIGHTED TOURNAMENTS

In this section, we consider the following well-known solution concepts for unweighted tournaments: Condorcet winners, Condorcet Non-Losers, the Copeland set, the top cycle, and the uncovered set.

### 8.4.1 Condorcet Winners & Condorcet Non-Losers

Condorcet winners and Condorcet non-losers are very simple solution concepts and will provide a nice warm-up. Recall that an alterna-

52 This does not exclude the possibility that computing *some (arbitrary)* possible winner or possible winning set for some of these solution concepts could be done in polynomial time.

tive $x \in A$ is a *Condorcet winner* of a complete tournament $T = (A, E)$ if it dominates all other alternatives, i.e., if $(x, y) \in E$ for all $y \in A \setminus \{x\}$. The set of Condorcet winners of tournament T will be denoted by $CW(T)$; obviously this set is always either a singleton or empty.

By contrast, alternative x is a *Condorcet non-loser* in T if v dominates some other alternatives in A, i.e., if $(x, y) \in E$ for some $y \in A \setminus \{x\}$. The set of Condorcet non-losers of a tournament T will be denoted by $CNL(T)$.

Let $G = (A, E)$ be a partial tournament. If some alternative x is dominant in G, then x will obviously be the Condorcet winner in all completions of G. On the other hand, if for some $y \in A \setminus \{x\}$ it is not the case that $(x, y) \in E$, there is some completion of G in which x is no Condorcet winner. Hence,

$$x \in NW_{CW}(G) \text{ iff } (x, y) \in E \text{ for all } y \in A \setminus \{x\}$$

and

$$x \in PW_{CW}(G) \text{ iff } (y, x) \in E \text{ for no } y \in A \setminus \{x\}.$$

Obviously, the criteria on the right-hand side of the equivalences can be checked in polynomial time.

Each of the sets in $PWS_{CW}(G)$ is either a singleton or the empty set, and determining membership for a singleton is obviously tractable. Checking whether $\emptyset \in PWS_{CW}(G)$ is not quite that simple. The following result gives an exact characterization of $PWS_{CW}(G)$, which is interesting *per se*.

LEMMA 8.1

Let U be the set of undominated alternatives of a partial tournament $G = (A, E)$. Then,

- for every alternative $x \in A$, $\{x\} \in PWS_{CW}(G)$ if and only if $x \in U$;
- $\emptyset \notin PWS_{CW}(G)$ if and only if $1 \leqslant |U| \leqslant 2$ and U is dominant.

*Proof.* Because a complete tournament has either one Condorcet winner or none, any set in $PWS_{CW}(G)$ has cardinality 0 or 1. Clearly, $\{x\} \in PWS_{CW}(G)$ if and only if $x \in U$. It remains to be shown that $PWS_{CW}(G)$ contains $\emptyset$ if and only if $U = \emptyset$, or $|U| \geqslant 3$, or $1 \leqslant |U| \leqslant 2$ and U is not dominant.

If $U = \emptyset$, $CW(T) = \emptyset$ for every $T \in [G]$. It follows that $\emptyset \in PWS_{CW}(G)$. If $|U| \geqslant 3$, consider any cycle $C \subseteq U \times U$. Then, the set of undominated alternatives in $G' = (A, E \cup C)$ is empty. It again follows that $\emptyset \in PWS_{CW}(G)$.

If $U = \{x\}$ and x is dominant, then x is a Condorcet winner in every $T \in [G]$, therefore $\emptyset \notin PWS_{CW}(G)$.

If $U = \{x\}$ and $\{x\}$ is not dominant, then $(x, y) \notin E$ for some $y \neq x$. Consider a completion of $G$ containing $(y, x)$. In this completion, the set of undominated alternatives is empty. It follows that $\emptyset \in PWS_{CW}(G)$.

If $U = \{x, y\}$ and $\{x, y\}$ is dominant, then for every $T \in [G]$, either $(x, y) \in T$ and $x$ is a Condorcet winner in $T$, or $(y, x) \in T$ and $y$ is a Condorcet winner in $T$. It follows that $\emptyset \notin PWS_{CW}(G)$.

Lastly, if $U = \{x, y\}$ and $\{x, y\}$ is not dominant, then for some $z \neq x, y$ we have $(x, z) \notin E$ or $(y, z) \notin E$. Without loss of generality, assume $(x, z) \notin E$. Consider a completion of $G$ containing $(z, x)$ and $(x, y)$. Such a completion exists, because $(x, z) \notin E$, and $(y, x) \notin E$ (since $x \in U$). In this completion, the set of undominated alternatives is empty. It follows that $\emptyset \in PWS_{CW}(G)$. □

Theorem 8.2

$PW_{CW}$, $NW_{CW}$, and $PWS_{CW}$ can be solved in polynomial time.

*Condorcet winners redcuces to maximin*

The results for $PW_{CW}$ and $NW_{CW}$ also follow from Proposition 2 of Lang et al. (2012) and Corollary 2 of Konczak and Lang (2005). We further note that Theorem 8.2 is a corollary of corresponding results for maximin in Section 8.5.2. The reason is that a Condorcet winner is the maximin winner of a 1-weighted tournament, and a tournament does not admit a Condorcet winner if and only if all alternatives are maximin winners.

*Condorcet Non-Losers reduces to Condorcet winners*

We conclude this section by observing that the problems $PW_{CNL}$, $NW_{CNL}$, and $PWS_{CNL}$ each are reducible to $NW_{CW}$, $PW_{CW}$, and $PWS_{CW}$, respectively. It can straightforwardly be checked that for all partial tournaments $G = (A, E)$ and all $X \subseteq A$,

$$X \in PWS_{CNL}(G) \text{ if and only if } A \setminus X \in PWS_{CW}(G^{-1}),$$

where $G^{-1} = (A, E^{-1})$ is $G$ with all of its set edges inverted, i.e., $E^{-1} = \{(x, y) : (y, x) \in E\}$. It also follows that,

$$PW_{CNL}(G) = A \setminus NW_{CW}(G^{-1}), \text{ and}$$
$$NW_{CNL}(G) = A \setminus PW_{CW}(G^{-1}).$$

Since set complementation and edge reversal can be achieved in polynomial time and by Theorem 8.2, we obtain the following result as a corollary of Theorem 8.2.

Theorem 8.3

$PW_{CNL}$, $NW_{CNL}$, and $PWS_{CNL}$ are all polynomial-time solvable.

*example for CW and CNL*

As an example for *CW* and *CNL*, consider the partial tournament $G$ depicted in Figure 8.2 in which there is no dominating alternative while the set of undominated alternatives in $G$ is $U = \{a, b\}$. Therefore,

$$PW_{CW}(G) = \{a, b\} \text{ and}$$
$$NW_{CW}(G) = \emptyset.$$

For $PWS_{TC}(G)$, note that the set $U$ is not dominant because $(b, c) \notin E$. By Lemma 8.1, this gives

$$PWS_{CW}(G) = \{\{a\}, \{b\}, \emptyset\}.$$

For Condorcet non-losers, we observe that $G^{-1} = (A, E^{-1})$ with $E^{-1} = \{(c, a), (d, a), (d, b)\}$. Now, we have that

$$PW_{CW}(G^{-1}) = \{c, d\},$$
$$NW_{CW}(G^{-1}) = \emptyset, \text{ and}$$
$$PWS_{CW}(G^{-1}) = \{\{c\}, \{d\}, \emptyset\} \text{ (by Lemma 8.1)}$$

which implies that

$$PW_{CNL}(G) = \{a, b, c, d\},$$
$$NW_{CNL}(G) = \{a, b\}, \text{ and}$$
$$PWS_{CNL}(G) = \{\{a, b, d\}, \{a, b, c\}, \{a, b, c, d\}\}.$$

### 8.4.2 Copeland

To illustrate the determination of possible and necessary Copeland winners, consider again the partial tournament $G$ shown in Figure 8.2. *example for CO* In completions of $G$ where $a$ (resp. $b$) is a Condorcet winner, $a$ (resp. $b$) is the sole Copeland winner as in the completion shown in Figure 8.2b. The only two completions in which neither $a$ nor $b$ is a Condorcet winner are

$$\{(a, c), (a, d), (b, a), (b, d), (c, b), (c, d)\},$$

where the set of Copeland winners is $\{a, b, c\}$, and

$$\{(a, c), (a, d), (b, a), (b, d), (c, b), (d, c)\},$$

also depicted in Figure 8.2c, where the set of Copeland winners is $\{a, b\}$. Therefore,

$$PW_{CO}(G) = \{a, b, c\},$$
$$NW_{CO}(G) = \emptyset, \text{ and}$$
$$PWS_{CO}(G) = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}.$$

Since Copeland scores coincide with Borda scores in the case of 1-weighted tournaments, the following is a direct corollary of the results in Section 8.5.1.[53]

THEOREM 8.4
*$PW_{CO}$, $NW_{CO}$, and $PWS_{CO}$ can be solved in polynomial time.*

From $PWS_{CO}$ being solvable in polynomial time, we get the following corollary that may be of independent interest to graph theorists.

COROLLARY 8.5
There exists a polynomial-time algorithm to check whether a partial tournament admits a regular completion.

### 8.4.3 Top Cycle

*alternative definition for TC*

Lang et al. (2012, Corollaries 1 and 2) have shown that possible and necessary winners for *TC* can be computed efficiently by greedy algorithms. Still, we give the following characterization that will be useful for our $PWS_{TC}$ considerations. An alternative is a possible winner if and only if it can reach every other alternative via existing or unspecified edges. Formally, given a partial tournament $G = (A, E)$, an alternative $x_0 \in A$ is in $PW_{TC}(G)$ if and only if for every other alternative $y \in A \setminus \{x_0\}$, there exists a path $(x_0, x_1, x_2, \dots, x_k)$ in $A$ with $x_k = y$ such that $(x_{i+1}, x_i) \notin E$ for all $i \in \{0, \dots, k-1\}$. We call such a path a *possible path*.

*possible path*

For $PWS_{TC}$, we not only have to check whether there exists a completion such that the set in question is dominating, but also that there is no smaller dominating set. It turns out that this can still be done in polynomial time.

THEOREM 8.6
*$PWS_{TC}$ can be solved in polynomial time.*

*Proof.* Let $G = (A, E)$ be a partial tournament and $X \subseteq A$. If $X = \emptyset$, then $X \notin PWS_{TC}(G)$ as the set of all alternatives is always dominant. If $|X| = 1$, then the $X \in PWS_{TC}(G)$ if and only if $X \in PW_{CW}(G)$. If $|X| = 2$, then $X \notin PWS_{TC}(G)$ because there exists no top cycle set of size two. Therefore we can assume that $|X| \geqslant 3$.

Consider the graph $G^{X\rightarrow}$. If $X$ does not dominate $A \setminus X$ in $G^{X\rightarrow}$, then $X \notin PWS_{TC}(G)$ because an alternative in $A \setminus X$ beats an alternative in $X$. Therefore, we now need to check whether $X \in PWS_{TC}(G_{|X})$, i.e., whether $X$ is a possible top cycle set in the partial tournament $G$ restricted to $X$. In essence, the problem $PWS_{TC}$ is reduced to the restricted problem $PWS_{TC}$ for the *set of all alternatives*.

*Good set*

For a partial tournament $G = (A, E)$, the *Good set* $GO(G)$ is defined

53 $PW_{CO}$ can alternatively be solved via a polynomial-time reduction to maximum network flow (see, e.g., Cook et al., 1998, p. 51).

as the set of all alternatives that can reach every other alternative via a possible path (Good, 1971). Note that for a (complete) tournament $T$, $GO(T) = TC(T)$. We prove that $A \in PWS_{TC}(G)$ if and only if $GO(G) = A$. Obviously, if $A \neq GO(G)$ then $A \notin PWS_{TC}(G)$. For the other direction, we start with a partial tournament $G = (A, E)$ with $GO(G) = A$ and successively add new edges to $G$ while the Good set is still everything until $G$ is a tournament. Pick an arbitrary unspecified edge between alternatives $x$ and $y$. We claim that either $G' = (A, E')$ with $E' = E \cup \{(x, y)\}$ or $G'' = (A, E'')$ with $E'' = E \cup \{(y, x)\}$ maintains the Good set invariance. Assume the latter is not the case. Then there have to exist alternatives $a$ and $b$ such that the only possible path from $a$ to $b$ in $G$ contained $(x, y)$. In particular, $(b, a) \in E$. Now, since $G'$ also fails to have $GO(G') = A$, there have to exist alternatives $c$ and $d$ such that the only possible path from $c$ to $d$ in $G$ contained $(y, x)$. The situation is depicted in Figure 8.3. But then we have a contradiction because there is in fact a possible path from from $c$ to $d$ via $y, b, a$, and $x$ in $G'$. □

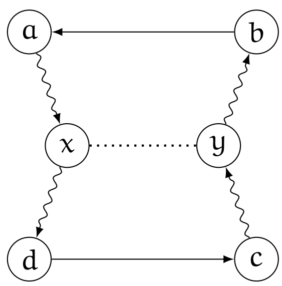



**Figure 8.3:** Illustration of the final step in the proof of Theorem 8.6. Possible paths between alternatives are shown as snaked edges. The shown edges stem from the assumption that neither $(x, y)$ nor $(y, x)$ can be fixed without making any alternative lose the property that they can reach every other alternative on a possible path. But then, there even has to exist a circle of possible paths that circumvents the connection between $x$ and $y$ altogether.

*example for TC*

As an example, we again consider the partial tournament $G$ depicted in Figure 8.2a, for which we show that

$$PW_{TC}(G) = \{a, b, c, d\},$$
$$NW_{TC}(G) = \emptyset, \text{ and}$$
$$PWS_{TC}(G) = \{\{a\}, \{b\}, \{a, b, c\}, \{a, b, c, d\}\}.$$

The result for $PW_{TC}(G)$ is witnessed by the completion shown in Figure 8.2c where every alternative is in the top cycle. For $NW_{TC}(G)$, the statement follows from the observation that for every alternative, there exists a completion in which another alternative is a Condorcet winner. Regarding $PWS_{TC}(G)$, we consider each subset separately. Since $PWS_{CW} \subseteq PWS_{TC}$, we get that $\{a\}$ and $\{b\}$ are in $PWS_{TC}(G)$. For

$\{a, b, c\}$, we apply Theorem 8.6: $a, b, c$ are undominated by $d$, and the Good set of $G|_{\{a,b,c\}}$ is $\{a, b, c\}$. Likewise, the Good set of $G$ is $\{a, b, c, d\}$. It remains to be shown that the other subsets of size three are not in $PWS_{TC}(G)$. To this end, note that the Good set of $G|_{\{a,b,d\}}$ is only $\{a, b\}$ and that $\{a, c, d\}$ and $\{b, c, d\}$ are both not undominated in $G$.

### 8.4.4 Uncovered Set

Recall that a second definition of the uncovered set states that an alternative is not covered if and only if it can reach every other alternative in at most two steps. Formally, $x \in UC[T]$ if and only if for all $y \in A \setminus \{x\}$, either $(x, y) \in E$ or there is some $z \in A$ with $(x, z), (z, y) \in E$. We denote the two-step dominion $D_E(D_E(x))$ of an alternative $x$ by $D_E^2(x)$.

We first consider $PW_{UC}$, for which we check for each alternative whether it can be reinforced to reach every other alternative in at most two steps.

Theorem 8.7
*$PW_{UC}$ can be solved in polynomial time.*

*Proof.* For a given partial tournament $G = (A, E)$ and an alternative $x \in A$, we check whether $x$ is in $UC(T)$ for some completion $T \in [G]$.

Consider the graph $G' = (A, E'')$ where $E''$ is derived from $E$ as follows. First, we let $D(x)$ grow as much as possible by letting $E' = E^{x\rightarrow}$. Then, we do the same for its two-step dominion by defining $E''$ as $E'^{D_{E'}(x)\rightarrow}$. We claim that $x \in PW_{UC}(G)$ if and only if $A = \{x\} \cup D_{E''}(x) \cup D^2_{E''}(x)$.

($\Rightarrow$) First, let $x \in PW_{UC}(G)$. By definition, there is a completion $(A, E^*)$ such that for all $y \in A \setminus \{x\}$ we have $y \in D_{E^*}(x) \cup D_{E^*}(x)$. But from the definition of $E''$, it holds that $D_{E^*}(x) \subseteq D_{E''}(c)$ and $D^2_{E^*}(x) \subseteq D^2_{E''}(x)$. Consequently, $y$ is also in $D_{E''}(x) \cup D^2_{E''}(x)$.

($\Leftarrow$) For the other direction, let $y \in A \setminus \{x\}$, $y \in D_{E''}(x) \cup D^2_{E''}(x)$. In any completion $T$ of $G'$, $x$ is trivially in $UC(T)$, implying that $x \in PW_{UC}(G)$. □

A similar argument yields the following.

Theorem 8.8
*$NW_{UC}$ can be solved in polynomial time.*

*Proof.* For a given partial tournament $G = (A, E)$ and an alternative $x \in A$, we check whether $x$ is in $UC(T)$ for all completions $T \in [G]$.

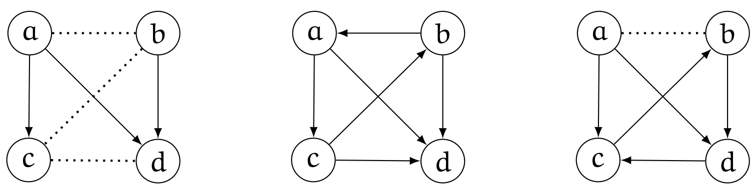
Consider the graph $G' = (A, E'')$ with $E''$ defined as follows. First, let $E' = E^{x\leftarrow}$. Then, expand it to $E'' = E'^{\overline{D}_{E'}(x)\rightarrow}$. Intuitively, this makes it as hard as possible for $x$ to beat alternatives outside of its dominion in two steps.

We claim that $x \in NW_{UC}(G)$ if and only if $A = \{x\} \cup D_{E''}(x) \cup D^2_{E''}(x)$.

($\Rightarrow$) First, let $x \in NW_{UC}(G)$. Assume for contradiction that there exists a $y \in A \setminus \{x\}$ such that $y \notin D_{E''}(x) \cup D^2_{E''}(x)$. Then, in any completion $(A, E^*)$ of $G'$, $x$ cannot reach $y$ in two steps and consequently $x \notin UC(A, E^*)$, a contradiction.

($\Leftarrow$) Now, let $A \setminus \{x\} = D_{E''}(x) \cup D^2_{E''}(x)$. In any completion $(A, E^*)$ of $G$, we have $D_{E''}(x) \subseteq D_{E^*}(x)$ and $D^2_{E''}(x) \subseteq D^2_{E^*}(x)$. Consequently, $x \in UC(A, E^*)$ and $x \in NW_{UC}(G)$.

As it can be checked in polynomial time whether $A = \{x\} \cup D_{E''}(x) \cup D^2_{E''}(x)$, this completes the proof. □




**(a)** A partial tournament G

**(b)** An extension of G strengthening c and its dominion

**(c)** An extension of G strengthening d and its dominion

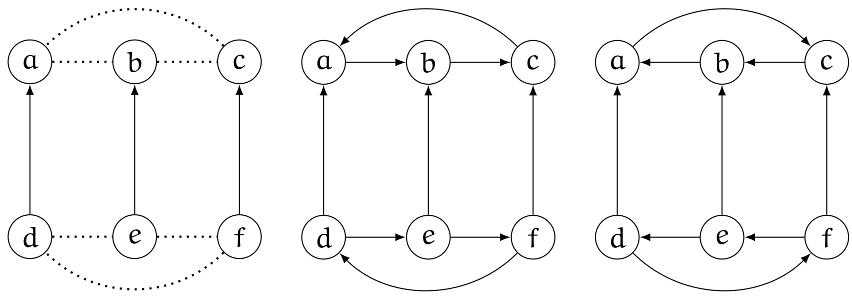
**Figure 8.4:** A partial unweighted tournament G and possible extensions. In the center, the alternative c and its dominion was maximally reinforced resulting in c reaching every other alternative in at most two steps. Therefore, $c \in PW_{UC}(G)$. On the right, the same was done for alternative d that cannot reach a in two steps and is therefore not contained in $PW_{UC}(G)$.

*example for UC*

Again, consider the partial tournament G from Figure 8.4a as an example. It is straightforward to check that G contains no necessary winners for *UC* which is also a consequence of $NW_{TC}(G) = \emptyset$ and $UC \subseteq TC$. For $PWS_{UC}$, we consider each alternative separately. For $a$, we have $E' = E^{a\rightarrow} = \{(a, b), (a, c), (a, d), (b, d)\}$, and $E'' = E'$, therefore $D_{E''}(a) = \{b, c, d\}$ and $a \in PW_{UC}(G)$. Likewise, $b \in PW_{UC}(G)$. Now, for $c$, we have $E' = \{(a, c), (a, d), (b, d), (c, b), (c, d)\}$ and $E'' = \{(a, c), (a, d), (b, d), (c, b), (c, d), (b, a)\}$, see also Figure 8.4b. This gives us $D_{E''}(c) = \{b, d\}$ and $D^2_{E''}(c) = \{a\}$, and therefore, $c \in PW_{UC}(G)$. Lastly, for $d$, we have $E' = \{(a, c), (a, d), (b, d), (d, c)\}$ and $E'' = \{(a, c), (a, d), (b, d), (d, c), (c, b)\}$ as depicted in Figure 8.4c. This gives us $D_{E''}(d) = \{c\}$ and $D^2_{E''}(d) = \{b\}$, implying that $d \notin PW_{UC}(G)$. Together, we get that

$$PW_{UC}[G] = \{a, b, c\},$$
$$NW_{UC}(G) = \emptyset, \text{ and}$$
$$PWS_{UC}(G) = \{\{a\}, \{b\}, \{a, b, c\}\}$$

where an ad-hoc reasoning was used to obtain $PWS_{UC}(G)$.




(a) (b) (c)

**Figure 8.5:** A partial tournament G and the only two completions of G for which the uncovered set is given by $\{a, b, c\}$. Dotted edges are missing and omitted edges point downwards. This partial tournament is used as a gadget in the proof of Theorem 8.9.

For all solution concepts considered so far—Condorcet winners, Condorcet non-losers, Copeland, and the top cycle—*PW* and *PWS* have the same complexity. One might wonder whether a result like this holds more generally, and whether there could be a polynomial-time reduction from *PWS* to *PW*. In the following, we show that this is not the case, unless P = NP.

*P-time reduction from PWS to PW?*

*gadget for $PWS_{UC}$*

First, consider the partial tournament $G = (\{a, \dots, f\}, E)$ depicted in Figure 8.5a. It is not hard to see that there are (exactly) two completions T of G such that $\{a, b, c\} = UC(T)$. The first is pictured in Figure 8.5b and the other in Figure 8.5c.

To see that there are no other such complectios, consider an arbitrary completion $(\{a, \dots, f\}, E')$. Then, either $(d, e) \in E'$ or $(e, d) \in E'$. If the former, observe that d must be covered by c. Hence, $(c, a) \in E'$ and $(f, d) \in E'$. It now follows that f is covered by b. Therefore, also $(b, c) \in E'$ and $(e, f) \in E'$. This entails that a covers e and, with $(e, b)$ we finally obtain $(a, b) \in E'$. It can now readily be appreciated that $(\{a, \dots, f\}, E')$ is the complete tournament depicted in Figure 8.5b. By an analogous argument it follows that $(\{a, \dots, f\}, E')$ must be the complete tournament depicted in Figure 8.5c if we assume that $(d, e) \in E'$.

This observation forms the basis of the construction used to prove NP-hardness of $PWS_{UC}$, the problem of deciding whether a subset of alternatives of a partial tournament G is the uncovered set of some completion of G.

**Theorem 8.9**
$PWS_{UC}$ is NP-complete.

*Proof.* Let $G = (A, E)$ be a partial tournament. Given a set $X \subseteq A$ and a completion $T \in [G]$, it can be checked in polynomial time whether $X = UC(T)$. Hence, $PWS_{UC}$ is obviously in NP.

NP-hardness can be shown by a reduction from SAT. Let $\varphi$ be a formula in conjunctive normal form. We construct a partial tournament $G_\varphi = (A_\varphi, E_\varphi)$ as follows. For each propositional variable p we introduce five alternatives denoted by $p$, $p^+$, $p^-$, $\underline{p}^+$, and $\underline{p}^-$. For each clause $c$, we also introduce two alternatives denoted by $c$ and $\underline{c}$. In the construction, $p^+$ is associated with the positive literal $p$ and $p^-$ with the negative literal $\bar{p}$. We will argue that a literal node covering a $\underline{c}$ corresponds to this literal satisfying the associated clause $c$. We also have two auxiliary alternatives denoted by $1$ and $\underline{0}$.

For each propositional variable $p$, we have edges from $\underline{p}^+$ to $p^+$, from $\underline{p}^-$ to $p^-$, from $p^+$ to $\underline{p}^-$, and from $p^-$ to $\underline{p}^+$. Moreover, there is an edge from each of the alternatives $p^+, p^-, \underline{p}^+$, and $\underline{p}^-$ to $p$. The edges between $p^+$, $p^-$, and $1$ and those between $\underline{p}^+$ and $\underline{p}^-$ are missing. For distinct propositional variables $p$ and $q$, for each of $p$, $p^+$, and $p^-$ there is an edge to $\underline{q}^+$ and $\underline{q}^-$.

For every clause $c$, there are edges from alternative $\underline{c}$ to $c$, and from $\underline{c}$ to $1$. For every propositional variable $p$, the edges between $\underline{c}$ and $\underline{p}^+$ and those between $\underline{c}$ and $\underline{p}^-$ are missing. Observe that, for each propositional variable $p$, the subgraph induced by $p^-$, $p^+$, $1$, $\underline{p}^-$, $\underline{p}^+$, and $\underline{c}$ is isomorphic to the partial tournament depicted in Figure 8.5a, above. Otherwise $\underline{c}$ is dominated by every other alternative (but the relation to other $\underline{c}'$ is left unspecified). Moreover, if $p$ occurs as a (positive) literal in $c$, there is an edge from $p^+$ to $c$. This is a prerequisite for $\underline{c}$ to be covered by $p^+$. Similarly, there is an edge from $p^-$ to $c$ whenever $\bar{p}$ occurs as a negative literal in $c$. Otherwise there are edges from $p$ to $c$ as well as from $c$ to $\underline{p}^+$, to $\underline{p}^-$, and to every other $p^+$ and $p^-$.

Moreover, there are edges from $\underline{0}$ to $p$ for every propositional variable $p$. Otherwise, there is an edge from every other alternative to $\underline{0}$.

Lastly, for every clause $c$ and every propositional variable $p$ that does not occur in $c$, there are edges from $\underline{c}$ to $\underline{p}^+$ and to $\underline{p}^-$. Any edges not specified in the above description can be set arbitrarily. For an example of this construction the reader is referred to Figure 8.6.

Finally, define

$$X = \{p, p^+, p^- : p \text{ a propositional variable}\} \cup \{c : c \text{ a clause}\} \cup \{1\}.$$

Table 8.1 summarizes which alternatives can reach which other alternatives in at most two steps. We thus find that, for every completion $T$ of $G_\varphi$, the set $X$ is contained in $UC(T)$ and that $\underline{0}$ is covered by $1$. Whether, for propositional variables $p$, the alternatives $\underline{p}^+$ and $\underline{p}^-$ and whether, for clauses $c$, alternative $\underline{c}$ are in $UC(T)$ depends on the way in which $T$ completes $G_\varphi$.

We conclude the proof by showing that

$$X = UC(T) \text{ for some } T \in [G_\varphi] \quad \text{if and only if} \quad \varphi \text{ is satisfiable.}$$

First assume that $\varphi$ is satisfiable and let $v$ be the satisfying assignment for $\varphi$. For each propositional variable $p$ that $v$ sets to true add edges $(p^-, p^+)$, $(p^+, 1)$, and $(1, p^-)$ as well as $(\underline{p}^-, \underline{p}^+)$, $(\underline{p}^+, c)$, and $(c, \underline{p}^-)$, for each clause $c$ in which $p$ occurs as a literal. For each propositional variable $q$ that $v$ sets to false, add edges $(1, q^+)$, $(q^+, q^-)$, and

| | $p$ | $q$ | $p^-$ | $p^+$ | $1$ | $q^-$ | $q^+$ | $c$ | $c'$ | $\underline{p}^-$ | $\underline{p}^+$ | $\underline{c}$ | $\underline{c}'$ | $\underline{q}^-$ | $\underline{q}^+$ | $\underline{0}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\cdot$ | $q^-$ | $c[p]$ | $c[\bar{p}]$ | $\underline{c}$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $c$ | $c$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $c$ |
| $p^-$ | $\cdot$ | $\underline{0}$ | $\cdot$ | $\underline{p}^+$ | $\underline{c}$ | $p$ | $p$ | $p$ | $p$ | $c[\bar{p}]$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $p^+$ | $\cdot$ | $\underline{0}$ | $\underline{p}^-$ | $\cdot$ | $\underline{c}$ | $p$ | $p$ | $\underline{c}$ | $\underline{c}'$ | $\cdot$ | $c[p]$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $1$ | $\cdot$ | $\cdot$ | $\underline{p}^-$ | $\underline{p}^+$ | $\cdot$ | $\underline{q}^-$ | $\underline{q}^+$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $p$ | $p$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $c[p]$ | $\underline{0}$ | $\underline{0}$ | $\cdot$ | $\underline{p}^+$ | $\underline{c}'$ | $\underline{q}^-$ | $\underline{q}^+$ | $\cdot$ | $\underline{c}'$ | $\cdot$ | $\cdot$ | $p^-$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $c[\bar{p}]$ | $\underline{0}$ | $\underline{0}$ | $\underline{p}^-$ | $\cdot$ | $\underline{c}'$ | $\underline{q}^-$ | $\underline{q}^+$ | $\cdot$ | $\underline{c}'$ | $\cdot$ | $\cdot$ | $p^+$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\underline{p}^-$ | $\cdot$ | $\underline{0}$ | $\cdot$ | $\square$ | $\square$ | $p$ | $p$ | $p$ | $p$ | $\cdot$ | $p^-$ | $p$ | $p$ | $p$ | $p$ | $\cdot$ |
| $\underline{p}^+$ | $\cdot$ | $\underline{0}$ | $\square$ | $\cdot$ | $\square$ | $p$ | $p$ | $p$ | $p$ | $p^+$ | $\cdot$ | $p$ | $p$ | $p$ | $p$ | $\cdot$ |
| $\underline{c}[p]$ | $\underline{0}$ | $\underline{0}$ | $c$ | $\square$ | $\cdot$ | $c$ | $c$ | $1$ | $c$ | $1$ | $1$ | $\cdot$ | $c$ | $1$ | $1$ | $\cdot$ |
| $\underline{c}[\bar{p}]$ | $\underline{0}$ | $\underline{0}$ | $\square$ | $c$ | $\cdot$ | $c$ | $c$ | $1$ | $c$ | $1$ | $1$ | $\cdot$ | $c$ | $1$ | $1$ | $\cdot$ |
| $\underline{0}$ | $\cdot$ | $\cdot$ | $q$ | $q$ | $-$ | $p$ | $p$ | $p$ | $p$ | $q$ | $q$ | $p$ | $p$ | $p$ | $p$ | $\cdot$ |

**Table 8.1:** Table summarizing which types of alternatives reach which other types of alternatives in one or two steps in completions of the partial tournament $G_\varphi$. An alternative $x$ in the entry for row $r$ and column $c$ means that $r$ can reach $c$ via $x$. If the entry is a dot ("$\cdot$"), $r$ can reach $c$ directly, i.e., in one or zero steps. A box ("$\square$") signifies that it depends on how $G_\varphi$ is being completed, if and via which alternative $r$ can reach $c$. The minus in the entry for $\underline{0}$ and $1$ means that $\underline{0}$ cannot reach $1$ in at most two steps, no matter how $G_\varphi$ is completed. Thus, $\underline{0}$ is covered by $1$ in every completion of $G_\varphi$. We assume $p$ and $q$ to be distinct variables. Furthermore, $c[p]$ simply denotes clause $c$ on the understanding that $p$ occurs as a literal in $c$. Similarly for $c[\bar{p}]$. An alternative $\underline{c}$ reaches $q^+$ ($q^-$) via $c$ or $\underline{q}^+$ ($\underline{q}^-$) depending on whether $q$ ($\bar{q}$) occurs in $c$ or not. We may assume that no clause contains both a literal and its negation as well as that every literal occurs in at least one clause.

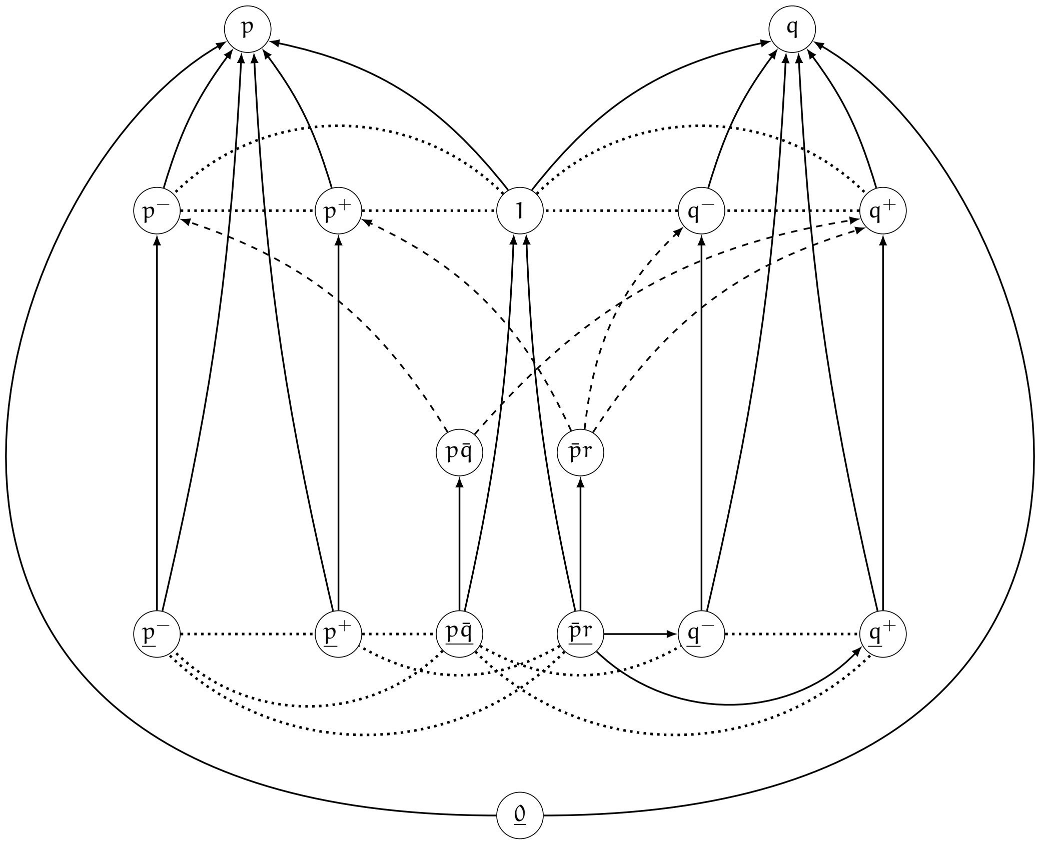



**Figure 8.6:** Part of the dominance relation of the partial tournament $G_\varphi$ associated with $\varphi = (p \vee \neg q) \wedge (\neg p \vee r)$. The part involving variable $r$, i.e., the nodes $r, r^+, r^-, \underline{r}^+$, and $\underline{r}^-$, has been omitted. The dashed edges are dependent on the clauses of $\varphi$. Omitted edges point downwards or, when on the same level, in an arbitrary direction. Dotted edges are missing.

$(q^-, 1)$ as well as $(c, \underline{q}^+)$, $(\underline{q}^+, \underline{q}^-)$, and $(\underline{q}^-, c)$, for each clause $c$ in which $\bar{q}$ occurs as a literal. Then, $\underline{p}^-$ is covered by $1$, $\underline{p}^+$ by $p^-$ and every $\underline{c}$ with $p$ occurring in it as a literal by $p^+$. Similarly, $\underline{q}^-$ is covered by $q^+$, $\underline{q}^+$ by $1$, and $\underline{c}$ by $q^-$ provided that $\bar{q}$ occurs in $c$.

For the opposite direction, assume that, there is some completion $T$ of $G_\varphi$ such that for every propositional variable $p$ and for every clause $c$, alternatives $\underline{p}^-$, $\underline{p}^+$, and $\underline{c}$ are covered. Define assignment $\nu_T$ such that it sets propositional variable $p$ to true if there is some clause $c$ such that $p^+$ covers $\underline{c}$ in $T$ and sets $p$ to false, otherwise. Observe that $\nu_T$ is well-defined as an assignment.

Finally, we have to show that $\nu_T$ satisfies $\varphi$. To this end consider an arbitrary clause $c$. It suffices to prove that $\nu_T$ satisfies $c$. By assumption, $\underline{c}$ is covered either by some $p^+$ where $p$ is some propositional variable that occurs as a (positive) literal in $c$ or $c$ is covered by some

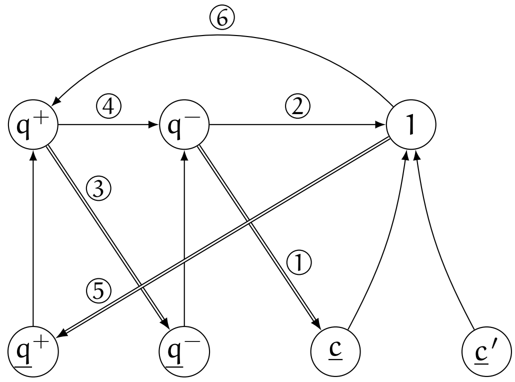



**Figure 8.7:** Illustration of the concluding argument of the proof of Theorem 8.9. A double edge from alternative $x$ to alternative $y$ indicates that $x$ covers $y$. The numbers some of the edges are labelled with correspond to the order in which their existence is demonstrated in the proof of Theorem 8.9.

other alternative $x$. If the former, $\nu_T$ sets $p$ to true and accordingly satisfies $c$. In the latter case, recall that $c$ reaches all alternatives in at most two steps apart from alternatives $p^+$ such that $p$ occurs in $c$ and from alternatives $q^-$ such that $\bar{q}$ occurs as a literal in $c$ (also see Table 8.1). We may therefore assume that $x = q^-$ for some $q$ such that $\bar{q}$ occurs as a literal in $c$. It suffices to show that $\nu_T$ sets $q$ to false. To this end, consider an arbitrary $c'$. We prove that $q^+$ does not cover $\underline{c}'$, see Figure 8.7 for an illustration of the reasoning. As $q^-$ covers $\underline{c}$ in $T$ and because there is an edge from $\underline{c}$ to $1$ by construction, there is also an edge from $q^-$ to $1$. This edge, together with the one from $\underline{q}^-$ to $q^-$, preclude that $1$ covers $\underline{q}^-$. Reaching every other alternative in at most two steps, $\underline{q}^-$ must therefore be covered by $q^+$. As there is an edge from $\underline{q}^-$ to $q^-$, it follows that $T$ also has an edge from $q^+$ to $q^-$. This being established and there being an edge from $\underline{q}^+$ to $q^+$, we may conclude that $q^-$ does not cover $\underline{q}^+$ in $T$. Rather, $\underline{q}^+$ reaches every alternative except $1$ in at most two steps in $T$. Therefore, $\underline{q}^+$ is covered by $1$. Because there is an edge from $\underline{q}^+$ to $q^+$, there is also an edge from $1$ to $q^+$ in $T$. At this point observe that, by construction, there is an edge from $\underline{c}'$ to $1$. Therefore, $q^+$ does not cover $\underline{c}'$ in $T$. This concludes the proof. □

## 8.5 WEIGHTED TOURNAMENTS

We now turn to weighted tournaments, and in particular consider the solution concepts Borda, maximin, and ranked pairs.

### 8.5.1 Borda

Recall that $BO(T)$ is defined as the set of alternatives with the highest total weight on outgoing edges.

Before we proceed further, we define the notion of a b-matching, which will be used in the proofs of several of our results in this section. Let $H = (A_H, E_H)$ be an undirected graph with vertex capacities $b : A_H \to \mathbb{N}_0$. Then, a b-*matching* of H is a function $\mu : E_H \to \mathbb{N}_0$ such that for all $v \in A_H$, $\sum_{e\in\{e'\in E_H : v\in e'\}} \mu(e) \leqslant b(v)$. The *size* of b-matching $\mu$ is defined as $\sum_{e\in E_H} \mu(e)$. It is easy to see that if $b(v) = 1$ for all $v \in A_H$, then a maximum-size b-matching is equivalent to a maximum-cardinality matching. In a b-matching problem with upper *and* lower bounds, there further is a function $a : A_H \to \mathbb{N}_0$. A feasible b-matching then is a function $\mu : E_H \to \mathbb{N}_0$ such that $a(v) \leqslant \sum_{e\in\{e'\in E_H : v\in e'\}} \mu(e) \leqslant b(v)$. b-*matching*

If H is bipartite, then the problem of computing a maximum size feasible b-matching with lower and upper bounds can be solved in strongly polynomial time (Schrijver, 2003, Chapter 21). We will use this result to show that $PW_{BO}$ and $PWS_{BO}$ can both be solved in polynomial time. While the following result for $PW_{BO}$ can also be shown using Theorem 6.1 of Kern and Paulusma (2004), we still give a direct proof that will then be extended to $PWS_{BO}$.

THEOREM 8.10

*$PW_{BO}$ can be solved in polynomial time.*

*Proof.* Generally, we observe that making a *BO*-winner $x$ stronger by increasing weight on an edge to another alternative, cannot make $x$ a losing alternative.[54] Now, let $G = (A, w)$ be a partial $n$-weighted tournament and $x \in A$. By the observation before, $x \in PW_{BO}(G)$ if and only if $x \in PW_{BO}(G^{x\to})$. We give a polynomial-time algorithm for checking whether the latter holds via a reduction to the problem of computing a maximum-size b-matching of a bipartite graph. Let $s^* = s_{BO}(x, G^{x\to})$ be the Borda score of $x$ in $G^{x\to}$. We construct a bipartite graph $H = (A_H, E_H)$ with vertices $A_H = A \setminus \{x\} \cup E^x$, where $E^x = \{\{i,j\} \subseteq A \setminus \{x\}\}$[55] and edges $E_H = \{\{v, e\} : v \in A \setminus \{x\} \text{ and } v \in e \in E^x\}$. We further define vertex capacities $b : A_H \to \mathbb{N}_0$ such that $b(\{i,j\}) = n - w(i,j) - w(j,i)$ for $\{i,j\} \in E^x$ and $b(v) = s^* - s_{BO}(v, G^{x\to})$ for $v \in A \setminus \{x\}$.

Now observe that in any completion $T = (A, w') \in [G^{x\to}]$, it holds that $w'(i,j) + w'(j,i) = n$ for all $i, j \in A$ with $i \neq j$. The sum of the Borda scores in T is therefore $n|A|(|A|-1)/2$. Some of the weight has already been used up in $G^{x\to}$; the weight which has not yet been used up is equal to

$$\alpha = n|A|(|A|-1)/2 - \sum_{v\in A} s_{BO}(v, G^{x\to}).$$

54 This means that *BO* satisfies *monotonicity*.
55 Note that $w(i,j) = w^{x\to}(i,j)$ for alternatives $i, j \in A \setminus \{x\}$.

We claim that $x \in PW_{BO}(G^{x\rightarrow})$ if and only if H has a b-matching of size at least $\alpha$.

($\Rightarrow$) Let $T = (A, w') \in [G^{x\rightarrow}]$ be a completion with $x \in BO(T)$. Consider the b-matching $\mu$ with $\mu(i, \{i,j\}) = w'(i,j) - w(i,j)$. We verify that $\mu$ is a feasible b-matching. Let $v \in A_H$. If $v \in A \setminus \{x\}$, we have that

$$\begin{aligned}\sum_{e \in \{e' \in E_H : v \in e'\}} \mu(e) &= s_{BO}(v, T) - s_{BO}(v, G^{x\rightarrow}) \\ &\leqslant s^* - s_{BO}(v, G^{x\rightarrow}) \\ &= b(v).\end{aligned}$$

Otherwise, $v = \{i,j\} \in E^x$ and

$$\begin{aligned}\sum_{e \in \{e' \in E_H : \{i,j\} \in e'\}} \mu(e) &= \mu(\{i, \{i,j\}\}) + \mu(\{j, \{i,j\}\}) \\ &= n - w(i,j) - w(j,i) \\ &= b(\{i,j\}).\end{aligned}$$

As the the size of $\mu$ is

$$\begin{aligned}\sum_{e \in E_H} \mu(e) &= \sum_{i \neq j} \big(w'(i,j) + w'(j,i) - w(i,j) - w(j,i)\big) \\ &= \sum_{i \neq j} n - \sum_{i \in A} \sum_{j \in A \setminus \{i\}} w(i,j) \\ &= \alpha,\end{aligned}$$

the statement is shown.

($\Leftarrow$) For the other direction, assume that a feasible b-matching of size at least $\alpha$ exists. We construct a completion $T = (A, w') \in [G^{x\rightarrow}]$ with $x \in BO(T)$. Let $w'(i,j) = \mu(i, \{i,j\}) + w(i,j)$ for all $\{i,j\} \subseteq A \setminus \{x\}$ as well as $w'(x,i) = w(x,i)$ and $w'(i,x) = w(x,i)$ for all $i \in A \setminus \{x\}$. As $w(i,j) \leqslant w'(i,j)$ and $w'(i,j) + w'(j,i) \leqslant w(i,j) + w(j,i) + b(\{i,j\}) = n$ for all $\{i,j\} \subseteq A$, T is an extension of $G^{x\rightarrow}$. From

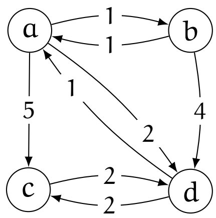
$$\alpha = \sum_{\{i,j\} E^x} b(\{i,j\}) \geqslant \sum_{e \in E_H} \mu(e) = \sum_{i \neq j} \geqslant \alpha,$$

we know that the upper capacities $b(\{i,j\})$ of all $\{i,j\} \in E^x$ are exactly met by $\mu$ (and that there cannot be a matching with size more than $\alpha$). This implies that

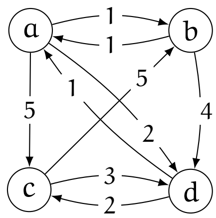
$$w'(i,j) + w'(j,i) = w(i,j) + w(j,i) + b(\{i,j\}) = n,$$

showing that T is indeed a completion of $G^{x\rightarrow}$.

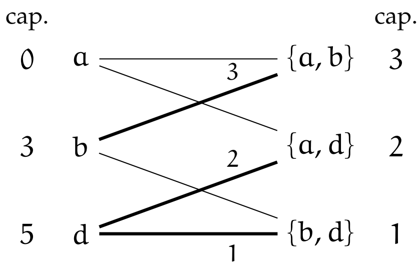
Since H can be constructed efficiently, and since a maximum size b-matching can be computed in strongly polynomial time, our algorithm runs in polynomial time. □



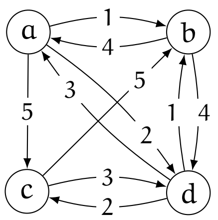

**(a)** A partial 5-weighted tournament G.




**(b)** The partial 5-weighted tournament $G^{c\rightarrow}$.




**(c)** The constructed bipartite graph H for target Borda score $s^* = s_{BO}(c, G^{c\rightarrow}) = 8$. Capacities are given next to the vertices. Thick edges with weights indicate the unique maximum b-matching.




**(d)** The completion T of G that corresponds to the maximum b-matching. In this case, $BO(T) = \{a, b, c\}$.

**Figure 8.8:** Illustration of the algorithm for checking whether an alternative c is contained in $PW_{BO}(G)$ for a partial 5-weighted tournament G.

Figure 8.8 illustrates the described steps for determining whether an alternative is contained in $PW_{BO}(G)$.

This idea can be extended to a polynomial-time algorithm for $PWS_{BO}$ where we use a similar construction for a given $G = (A, w)$, a candidate set $X \subset X$ and a target Borda score $s^*$. Binary search can be used to efficiently search the interval of possible target scores.

THEOREM 8.11

$PWS_{BO}$ can be solved in polynomial time.

*Proof.* Let $G = (A, w)$ be a partial $n$-weighted tournament, and $X \subseteq A$. We give a polynomial time algorithm for checking whether $X \in PWS_{BO}(G)$, via a bisection method and a reduction to the problem of computing a maximum $b$-matching of a graph with lower and upper bounds.

Assume that there is a target Borda score $s^*$ and a completion $T \in [G]$ with $X \in PWS_{BO}(T)$ and $s_{BO}(x, T) = s^*$ for all $x \in X$. Then, the maximum Borda score of an alternative not in $X$ is $s^* - 1$.

For a given target Borda score $s^*$, we construct a bipartite graph $H = (A_H, E_H)$ with vertices $A_H = A \cup E^x$, where $E^x = \{\{i, j\} \subseteq A : i \neq j, w(i, j) + w(j, i) < n\}$, and edges $E_H = \{\{v, e\} : v \in A \text{ and } v \in e \in E^x\}$. Only the lower bounds $a_{s^*} : A_H \to \mathbb{N}_0$ and upper bounds $b_{s^*} : A_H \to \mathbb{N}_0$ depend on $s^*$ and are defined as follows: For vertices $x \in X$, lower and upper bounds coincide and are given by $a_{s^*}(x) = b_{s^*}(x) = s^* - s_{BO}(x, G)$. All other vertices $v \in A_H \setminus X$ have a lower bound of $a_{s^*}(v) = 0$. Upper bounds for these vertices are defined such that $b_{s^*}(v) = s^* - s_{BO}(v, G) - 1$ for $v \in A \setminus X$, and $b_{s^*}(\{i, j\}) = n - w(i, j) - w(j, i)$ for $\{i, j\} \in E^x$. As in the proof of Theorem 8.10, it holds that a feasible $b$-matching in $H$ corresponds to an extension of $G$. Such an extension is a completion $T \in [G]$ if and only if the $b$-matching has size $\alpha = n|A|(|A| - 1)/2 - \sum_{v \in A} s_{BO}(v, G)$, which equals the weight not yet used up in $G$. Then, $T$ satisfies $X \in PWS_{BO}(T)$ and $s_{BO}(x, T) = s^*$ for all $x \in X$. If, on the other hand, no $s^*$ gives rise to a graph that has a $b$-matching of size $\alpha$, then $X \notin PWS_{BO}(G)$.

In order to obtain a polynomial-time algorithm, we need to check whether there exists a target score $s^*$ for which the corresponding graph $H$ with upper and lower bounds admits a $b$-matching of size $\alpha$. It is easily verified that any such $s^*$ is contained in the integer interval

$$I = [\max_{x \in X} s_{BO}(x, G), n(|A| - 1)].$$

Observe that $|I|$ depends on $n$ and thus is *not* polynomially bounded in the size of $G$. Checking every integer $s \in I$ is therefore not feasible in polynomial time. However, we now show that we can perform binary search in order to find $s^*$ efficiently. We need the following two observations about the interval $I$. For $s \in I$, we say that $s$ *admits a feasible $b$-matching* if the corresponding graph $H$ has a feasible $b$-matching.

*structure of score interval $I$*

First, if an $s' \in I$ admits a feasible b-matching, then every $s'' \in I$ with $s'' \leqslant s'$ also admits a feasible b-matching. This is because removing all weight from edges that exceeds the (reduced) upper bounds gives a feasible b-matching for $s''$.

Second, with $s'$ as before and $\alpha'$ the size of the corresponding *maximum* feasible b-matching $\mu'$, there cannot be an $s'' \in I$ with $s'' \geqslant s'$ such that the size $\alpha''$ of a maximum feasible b-matching $\mu''$ for $s''$ is smaller than $\alpha'$. This is because either *(i)* no such $\mu''$ exists since not all lower bounds can be met, or *(ii)* such an $\mu''$ exists and its size is at least $\alpha'$. To see the latter, note that a decrease in the size of a maximum feasible matching cannot be caused by upper bounds as $b_{s''}(v) \geqslant b_{s'}(v)$ for all $v \in A_H$. It remains to be shown that the increase in $a_{s''}(v)$ for $v \in X$ does not result in a smaller maximum b-matching. Since the weight of all edges incident to a vertex in X in the b-matching is completely determined by the bounds and increases from $\mu'$ to $\mu''$, a total decrease in size can only be due to edges $\{j, \{i, j\}\}$ with $i \in A \setminus X, j \in A$ whose weight is bounded by $b_{s''}(\{i, j\}) - \mu''(i, \{i, j\})$. But then,

$$\begin{aligned} \mu''(i, \{i,j\}) + \mu''(j, \{i,j\}) &= b_{s''}(\{i,j\}) \\ &\geqslant b_{s'}(\{i,j\}) \geqslant \mu'(i, \{i,j\}) + \mu'(j, \{i,j\}) \end{aligned}$$

and therefore $\alpha'' \geqslant \alpha'$.

These two observations show that the interval I consists of two subintervals where the lower part admits feasible b-matchings of increasing size, whereas the upper part does not admit feasible b-matchings. Therefore, $s^*$ is either at the upper end of the lower part or it does not exist.

Algorithmically, we can check the existence of $s^*$ with the following *binary search* algorithm. Let $[I_{min}, I_{max}]$ be an interval that is initialized to $I = [\max_{x \in X} s_{BO}(x, G), n(|A| - 1)]$. Consider the median value s of this interval. If the corresponding graph H has no feasible b-matching, continue with the interval $[I_{min}, s - 1]$. Otherwise, if the maximum feasible b-matching has size at least $\alpha$, return "yes". If its size is less than $\alpha$, continue with $[s + 1, I_{max}]$. If $[I_{min}, I_{max}]$ is empty, return "no."

*binary search*

The number of queries of this algorithm is bounded by $\lceil \log_2 |I| \rceil \leqslant \lceil \log_2 n|A| \rceil$ and, therefore, polynomial in the size of G. □

To conclude this section, we give a proof for $NW_{BO}$ being solvable in polynomial time as well. It is worth noting that this result does not follow directly from the polynomial-time result for $NW_{BO}$ for the case of preference profiles (Xia and Conitzer, 2011).

THEOREM 8.12

$NW_{BO}$ can be solved in polynomial time.

*Proof.* Let $G = (A, w)$ be a partial weighted tournament, $x \in A$. We give a polynomial-time algorithm for checking whether $x \in NW_{BO}(G)$.

Let $G' = G^{x\rightarrow}$. We want to check whether some other alternative $y \in A \setminus \{x\}$ can achieve a Borda score of more than $s^* = s_{BO}(x, G')$. This can be done separately for each $y \in A \setminus \{x\}$ by reinforcing it as much as possible in $G'$. If for some $y$, $s_{BO}(y, G'^{y\rightarrow}) > s^*$, then $x \notin NW_{BO}(G)$. If, on the other hand, $s_{BO}(y, G'^{y\rightarrow}) \leqslant s^*$ for all $y \in A \setminus \{x\}$, then $x \in NW_{BO}(G)$. □

*example for BO*

As an example, consider the partial 5-weighted tournament $G$ in Figure 8.8a. The fact that $\{a, b, c\} \subseteq PW_{BO}(G)$ follows already from the completion shown in Figure 8.8d. Also note that this was the only completion in which $c$ was chosen. Alternative $d$ cannot be a possible Borda winner since $s_{BO}(d, G^{d\rightarrow}) = 7 < 8 = s_{BO}(a, G)$. To determine $PWS_{BO}(G)$, we still have to check which subsets of $\{a, b, c\}$ are possible winning sets. For singletons, it is easy to see that only $\{a\}$ and $\{b\}$ are in $PWS_{BO}(G)$. For $\{a, b\}$, we could employ the binary search method described in Theorem 8.11. Here, we just argue that moving one unit of weight from $(c, d)$ to $(d, c)$ in the completion shown in Figure 8.8d, gives another completion in which $\{a, b\}$ is the winning set. For $NW_{BO}(G)$, it is straightforward to check that no alternative is a necessary Borda winner. Together, we have that

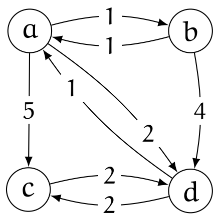
$$\begin{aligned} PW_{BO}(G) &= \{a, b, c\} \\ NW_{BO}(G) &= \emptyset \\ PWS_{BO}(G) &= \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}. \end{aligned}$$

### 8.5.2 Maximin

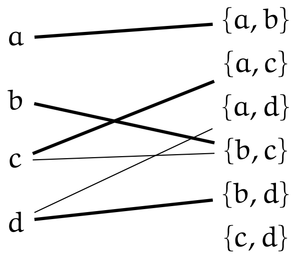
*example for MM*

For maximin, consider again the partial 5-weighted tournament depicted in Figure 8.9a as an example. It is easy to see that $a$ (or $b$) are the unique maximin winners in all completions of $G^{a\rightarrow}$ (or $G^{b\rightarrow}$). Also, $c$ can not be a possible maximin winner as it will always have a maximin score of 0 whereas $a$ always has at least 1. Similarly, alternative $d$ can never have a higher maximin score than $a$. Figure 8.9c shows a completion in which $\{a, d\}$ is the set of maximin winners. If one unit of weight is shifted from $(c, b)$ to $(b, c)$, the resulting completion has $\{a, b, d\}$ as the maximin winners. It is also straightforward to find a completion of $G^{\{a,b\}\rightarrow}$ with $\{a, b\}$ as the set of maximin winners. It is easy to verify that no alternative is a necessary maximin winner. Together, this gives

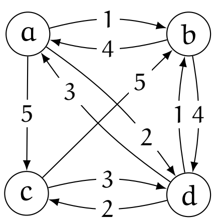
$$\begin{aligned} PW_{MM}(G) &= \{a, b, d\} \\ NW_{MM}(G) &= \emptyset \\ PWS_{MM}(G) &= \{\{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}. \end{aligned}$$



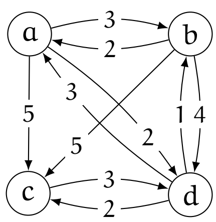

**(a)** A partial 5-weighted tournament $G$.




**(b)** The constructed bipartite graph $H^{s^*}$ for $s^* = 1$ and $X = \{a, d\}$ as in the proof of Theorem 8.14. A maximum cardinality matching is given by the thick edges.




**(c)** A completion $T$ of $G$ that could be obtained from the matching. Indeed, $MM(T) = \{a, d\}$ with $s_{MM}(T) = 1$.




**(d)** This completion $T$ of $G$ is a witness for $\{a, b\} \in PWS_{MM}(G)$.

**Figure 8.9:** Example of a 5-weighted partial tournament and completions relevant for possible maximin winners.

We first show that $PW_{MM}$ is polynomial-time solvable by reducing it to the problem of finding a maximum-cardinality matching of a graph.

THEOREM 8.13

$PW_{MM}$ can be solved in polynomial time.

*Proof.* We show how to check whether $x \in PW_{MM}(G)$ for a partial n-weighted tournament $G = (A, w)$. Consider the graph $G^{x\rightarrow} = (A, w^{x\rightarrow})$. Then, $s_{MM}(x, G^{x\rightarrow})$ is the best possible maximin score $x$ can get among all completions of G. If $s_{MM}(x, G^{x\rightarrow}) \geqslant \frac{n}{2}$, then we have $s_{MM}(y, T) \leqslant w^{x\rightarrow}(y, x) \leqslant \frac{n}{2}$ for every $y \in A \setminus \{x\}$ and every completion $T \in [G^{x\rightarrow}]$ and therefore $x \in PW_{MM}(G)$.

*reduction to maximum cardinality matching*

Now consider $s_{MM}(x, G^{x\rightarrow}) < \frac{n}{2}$. We will reduce the problem of checking whether $x \in PW_{MM}(G)$ to that of finding a maximum cardinality matching, which is known to be solvable in polynomial time (Edmonds, 1965). We want to find a completion $T \in [G^{x\rightarrow}]$ such that $s_{MM}(x, T) \geqslant s_{MM}(y, T)$ for all $y \in A \setminus \{x\}$. In other words, we want to complete the weights of edges between vertices in $A \setminus \{x\}$ in such a balanced way so that $x$ is still a winner. If there exists a $y \in A \setminus \{x\}$ such that $s_{MM}(y, G^{x\rightarrow}) > s_{MM}(x, G^{x\rightarrow})$, then we already know that $x \notin PW_{MM}(G)$. Otherwise, each $y \in A \setminus \{x\}$ derives its maximin score from at least one particular edge $(y, z)$ where $z \in A \setminus \{x, y\}$ and $w(y, z) \leqslant s_{MM}(x, G^{x\rightarrow})$. Moreover, it is clear that in any completion, $y$ and $z$ cannot both achieve a maximin score of less than $s_{MM}(x, G^{x\rightarrow})$ from edges $(y, z)$ and $(z, y)$ at the same time. Construct the following undirected and unweighted graph $H = (A_H, E_H)$ where $A_H = A \setminus \{x\} \cup \{\{i, j\} \subseteq A : i \neq j\}$. Build up $E_H$ such that: $\{i, \{i, j\}\} \in E_H$ if and only if $i \neq j$ and $w^{x\rightarrow}(i, j) \leqslant s_{MM}(x, G^{x\rightarrow})$. In this way, if $i$ is matched to $\{i, j\}$ in H, then $i$ derives a maximin score of less than or equal to $s_{MM}(x, G^{x\rightarrow})$ from his comparison with j. Clearly, the size of H is polynomial in the size of G. We show that $x \in PW_{MM}(G)$ if and only if there exists a matching of cardinality $|A| - 1$ in H.

(⇒) First, assume that $x \in PW_{MM}(G)$. Then there exists a completion $T = (A, w')$ of $G^{x\rightarrow}$ in which the maximin score of each $y \in A \setminus \{x\}$ is at most $s_{MM}(x, G^{x\rightarrow}) < \frac{n}{2}$. If alternative $i$ derives its maximin score from a comparison with $j \neq i \in A \setminus \{x\}$, i.e., $s_{MM}(i, T) = w'(i, j)$, then j cannot derive its maximin score from a comparison with $i$ because $w'(j, i) \geqslant n - s_{MM}(x, G^{x\rightarrow})$ implies $w'(j, i) > \frac{n}{2}$. Therefore, in H, each $i \in A_H \cap A$ can be matched to a $\{i, j\} \in A_H$ such that $\{i, j\}$ is not matched to any other vertex in $A_H$. The resulting matching in H has cardinality $|A| - 1$.

(⇐) Now, assume that there exists a matching M of cardinality $|A| - 1$ in H. Then, each $i \in A \setminus \{x\}$ has to be matched to an $\{i, j\}$ where $w(i, j) \leqslant s_{MM}(x, G^{x\rightarrow})$. Consider a completion $T = (A, w') \in [G^{x\rightarrow}]$ in which for all $(i, j) \in A \times A$ such that $\{i, \{i, j\}\} \in M$, we set $w'(i, j) = w(i, j)$ and $w'(j, i) = n - w(i, j)$. Moreover, the weights of all other edges in T are set by any arbitrary completion of edges in

$G^{x\rightarrow}$. Clearly, T is a proper completion of $G^{x\rightarrow}$ and therefore of G. In T, the maximin score of each $y \in A \setminus \{x\}$ is less than or equal to the maximin score of x. Therefore $x \in MM(G)$ which implies that $x \in PW_{MM}(G)$. □

Next, we show that $PWS_{MM}$ can be solved in polynomial time. The proof proceeds by identifying the maximin values that could potentially be achieved simultaneously by all elements of the set in question, and solving the problem for each of these values using similar techniques as in the proof of Theorem 8.13. Only a polynomially bounded number of problems need to be considered.

THEOREM 8.14
$PWS_{MM}$ can be solved in polynomial time.

*Proof.* Let $G = (A, w)$ be a partial n-weighted tournament, and $X \subseteq A$. We give a polynomial time algorithm for checking whether $X \in PWS_{MM}(G)$.

If $X \in PWS_{MM}(G)$ there must be a completion $T \in [G]$ and $s^* \in \{0, \dots, n-1\}$ such that $s_{MM}(x, T) = s^*$ for all $x \in X$ and $s_{MM}(i, T) < s^*$ for all $y \in A \setminus X$. First, we note that if $s^* > n - w(j, i)$ for some $i \in X, j \in A$ or $s^* \leqslant w(i, j)$ for some $i \notin X, j \in A$, then $X \notin PWS_{MM}(G)$. Therefore, assume that

$$\begin{aligned} n - w(j,i) \geqslant s^* &\text{ for all } i \in X, j \in A \text{ and} \\ w(i,j) < s^* &\text{ for all } i \notin X, j \in A. \end{aligned}$$

We treat the cases $s^* > \frac{n}{2}$, $s^* = \frac{n}{2}$, and $s^* < \frac{n}{2}$ separately.

First, assume that $s^* > \frac{n}{2}$. Then, $X \in PWS_{MM}$ if and only if X is a singleton $\{x\}$ and $w^{x\rightarrow}(x, j) > \frac{n}{2}$ for all $j \in A \setminus \{x\}$.

Now assume that $s^* = \frac{n}{2}$. With the assumptions above, we can define $G' = (A, w')$ as an extension of $G^{X\rightarrow}$ with $w'(i, j) = w'(j, i) = \frac{n}{2} = s^*$. Note that in every completion T of $G'$, $s_{MM}(i, T) = s^*$ for all $i \in X$ and that $X \in PWS_{MM}(G)$ with maximum maximin score $\frac{n}{2}$ in the corresponding completion if and only if $X \in PWS_{MM}(G')$ with the same maximum maximin score in the respective completion.

In addition, we need to check whether alternatives not in X can be forced to have a strictly smaller maximin score than $\frac{n}{2}$. To this end, construct an unweighted undirected bipartite graph $H = (A_H, E_H)$ with $A_H = A \cup \{\{i, j\} \subseteq A : i \neq j\}$. For $i \in A \setminus X$ and $j \neq i$, $E_H$ contains an edge $\{i, \{i, j\}\}$ if $w(i, j) < s^*$. Otherwise, $E_H$ contains no edges. We claim that $X \in PWS_{MM}(G')$ with a maximin score of $s^* = \frac{n}{2}$ in the corresponding completion if and only if there is a maximum cardinality matching of size $|A \setminus X|$ in H.

(⇒) Let $T = (A, w'')$ a completion of $G'$ (and thereby of G) in which X is the set of maximin winners with $s_{MM}(i, T) = s^* = \frac{n}{2}$ for all $i \in X$. For each $i \notin X$, there needs to be a $j \neq i$ with $w''(i, j) < s^*$.

Collecting $\{i, \{i, j\}\}$ for each such pair gives a matching of size $|A \setminus X|$ in $H$ which is maximum since each vertex on one side of the bipartite graph is contained in it.

($\Leftarrow$) For the other direction, assume that there is a maximum matching of size $|A \setminus X|$. We construct a completion $T = (A, w'')$ of $G'$ such that $X$ is the set of maximin winners. Note that every $i \in (A_H \cap A) \setminus X$ has to be contained in an edge $\{i, \{i, j\}\}$ in the matching. For each such edge, let $w''(i, j) = w'(i, j)$ and $w''(j, i) = n - w''(i, j)$. These weights witness that $s_{MM}(i, T) < s^*$. Otherwise, $T$ is an arbitrary completion of $G$. Together, we have that $s_{MM}(i, T) = s^*$ for all $i \in X$ and $s_{MM}(i, T) < s^*$ for all $i \notin X$. Figure 8.10 illustrates the procedure for a 2-weighted tournament and the set $X = \{a\}$.

Lastly, assume that $s^* < \frac{n}{2}$. For a given $s^*$, we construct an undirected unweighted bipartite graph $H^{s^*} = (A_H, E_H^{s^*})$. Let $A_H$ as before. An edge $\{i, \{i, j\}\}$ with $i \in A, j \in A \setminus \{i\}$ is contained in $E_H^{s^*}$ if

$$i \in X \text{ and } w(i, j) \leqslant s^* \leqslant n - w(j, i) \text{ or}$$
$$i \notin X \text{ and } w(i, j) \leqslant s^* - 1.$$

Otherwise, $E_H^{s^*}$ contains no edges. Note that $H^{s^*}$ is bipartite and contains at most $|A|^2$ edges.

We claim that $X \in PWS_{MM}(G)$ with a maximin score of $s^* < \frac{n}{2}$ in the corresponding completion if and only if there is a maximum cardinality matching of size $|A|$ in $H^{s^*}$.

($\Rightarrow$) Let $T = (A, w')$ a completion of $G$ in which $X$ is the set of maximin winners with the maximum maximin score $s^*$. For every vertex $i \in A$, there has to be an $j \neq i$ such that $w'(i, j)$ accounts for the maximin score of $i$. Also, since $s^* < \frac{n}{2}$, it can not be the case that $j$ also derives its maximin score from $w'(j, i)$. Therefore, the set of all such pairs $\{i, \{i, j\}\}$ is a valid matching of size $|A|$. It is obviously maximal.

($\Leftarrow$) For the other direction, assume that there is a maximum matching of size $|A|$. Note that every $i \in (A_H \cap A)$ has to be contained in an edge $\{i, \{i, j(i)\}\}$ in the matching. We construct a completion $T = (A, w')$ in which $X$ is the set of maximin winners. For $i \in X$, define $w'(i, j(i)) = s^*$ and $w'(j(i), i) = n - s^*$. Similarly, for $i \in A \setminus X$, define $w'(i, j(i)) = s^* - 1$ and $w'(j(i), i) = n - (s^* - 1)$. As long as there are unspecified edges $(i, j)$ in the completion, define

$$w'(i, j) = \max\{w(i, j), s^*\}, \qquad \text{if } i \in X, j \in A, \text{ and}$$
$$w'(i, j) = \max\{w(i, j), s^* - 1\}, \qquad \text{otherwise}$$

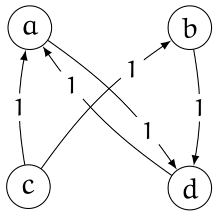
together with $w'(j, i) = n - w(i, j)$.

Note that $T$ is a proper completion of $G$. Now, we have $s_{MM}(i, T) = s^*$ for all $i \in X$ and $s_{MM}(i, T) < s^*$ for all $i \notin X$.

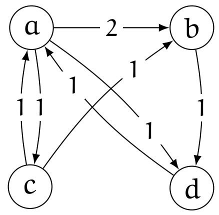
It remains to be shown that in the last case only a limited number of possible $s^*$ have to be considered. To see this, note that when $s^*$

is gradually incremented from 0 to $\frac{n}{2}-1$, whether an edge $\{i,\{i,j\}\}$ is contained in $E_H^{s^*}$ or not changes at most twice. Therefore, it is sufficient to only consider target maximin scores $s^* \in \{w(i,j), n - w(j,i) : i \in X, j \in A \setminus \{x\}\} \cup \{w(i,j)+1 : i \in A, j \in A \setminus \{i\}\}$, i.e., at most $2|A|^2$ different values to examine all distinct $H^{s^*}$.

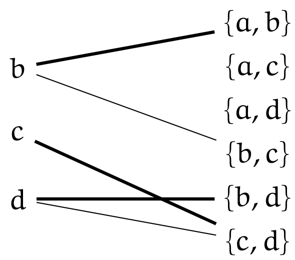
All cases can be completed in polynomial time. □



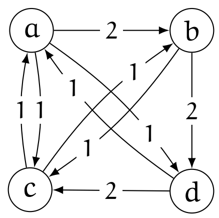

**(a)** A partial 2-weighted tournament G.




**(b)** An extension G′ reinforcing {a}.




**(c)** The constructed undirected bipartite graph H. Thick edges indicate a maximum cardinality matching.




**(d)** A completion of G′ and G with a as the sole alternative with the maximum maximin score of 1.

**Figure 8.10:** Illustration of the algorithm for checking whether a singleton $\{a\}$ is contained in $PWS_{MM}(G)$ for a partial 2-weighted tournament G. It is obvious that a cannot have a maximin score of 2 in any completion or be the sole maximin winner with a maximin score of 0. Therefore, we check the for the case $s^* = \frac{n}{2} = 1$.

Lastly, we consider $NW_{MM}$ for which we apply a similar technique as for $NW_{BO}$: to see whether $x \in NW_{MM}(G)$, we start from the graph $G^{x\leftarrow}$ and check whether some other alternative can achieve a higher maximin score than $x$ in a completion of $G^{x\leftarrow}$.

THEOREM 8.15

$NW_{MM}$ can be solved in polynomial time.

*Proof.* We show how to check whether $x \in NW_{MM}(G)$ for a partial $n$-weighted tournament $G = (A, w)$. The maximin score of $x$ in $G^{x\leftarrow}$ is the worst case maximin score of $x$ among all proper completions of G.

For each $y \in A \setminus \{x\}$, the maximin score of $y$ in $G^{y\rightarrow}$ is the best possible maximin score of $y$ among the completions of G. If the maximin score of each $y$ in the corresponding $G^{y\rightarrow}$ is not more than the maximin score of $x$ in $G^{x\leftarrow}$, then $x \in NW_{MM}(G)$, otherwise $x \notin NW_{MM}(G)$. □

### 8.5.3 Ranked Pairs

$S \in P \not\Rightarrow PW_S \in P$

It is readily appreciated that the ranked pairs procedure for $RP_\tau$, and thus its winner determination problem, is computationally tractable. The possible winner problem, on the other hand, turns out to be NP-hard. This also shows that tractability of the winner determination problem, while necessary for tractability of $PW$, is not generally sufficient.

For the proofs of the results in this section, we refer to our paper (Aziz et al., 2012) and the thesis by Brill (2012).

THEOREM 8.16
$PW_{RP_\tau}$ is NP-complete.

Since the ranked pairs method is resolute, hardness of $PWS_{RP_\emptyset}$ follows immediately.

COROLLARY 8.17
$PWS_{RP_\tau}$ is NP-complete.

Computing necessary ranked pairs winners turns out to be coNP-complete. This is again somewhat surprising, as computing necessary winners is often considerably easier than computing possible winners, both for partial tournaments and partial preference profiles (Xia and Conitzer, 2011).

THEOREM 8.18
$NW_{RP_\tau}$ is coNP-complete.

## 8.6 POSSIBLE WINNING SUBSETS

*PWSS*

*open relation to PWS*

We considered the problem *PWS*—whether a subset of alternatives is a possible winning set. In addition, it may be of interest whether a subset of alternatives is *among* the winners in some completion, i.e., they are all in the choice set but there may be more winning alternatives. We will refer to the latter problem as *PWSS* (possible winning subset). We note that an oracle to solve *PWSS* can be used to solve *PW*. If we want to check whether $i \in PW(G)$, we simply check whether $\{i\} \in PWSS(G)$. We are not aware of any algorithmic relation between the problems *PWS* and *PWSS*.

We examined the computational complexity of $PWSS_S$ for most of the solution concepts considered in this chapter. Since the arguments are often very similar to proofs already given, we briefly summarize our findings here.

*CW* As there is never more than one Condorcet winner, every $X \in PWSS_{CW}(G)$ is a singleton and the problem reduces to computing $PW_{CW}(G)$.

*CNL* For $PWSS_{CNL}$, note that $X \notin PWSS_{CNL}(G)$ if and only if every completion $T^{-1} \in [G^{-1}]$ has a Condorcet winner which is furthermore located in X. Therefore,

$$X \in PWSS_{CNL}(G) \Leftrightarrow \begin{cases} \emptyset \in PWS_{CW}(G^{-1}) & \text{or} \\ PW_{CW}(G^{-1}) \setminus X \neq \emptyset. \end{cases}$$

*CO* Just as for the other problems, polynomial computability of $PWSS_{CO}$ follows from the corresponding result for $PWSS_{BO}$.

*TC* For the top cycle, the problem $PWSS_{TC}$ can be solved in polynomial time. In fact, it can be shown that for a partial tournament G and a set of alternatives X, it is sufficient to check whether $X \subseteq PW_{TC}(G)$ (with an additional argument if $|X| = 2$) in order to determine whether $X \in PWSS_{TC}(G)$.

*BO* The argument and algorithm for checking whether X is contained in $PWSS_{BO}(G)$ is almost the same as the argument for $PWS_{BO}$ in Theorem 8.11. The only difference is that the $s_{BO}(v, T)$ may now be up to $s^*$ instead of $s^* - 1$ for $v \in A \setminus X$ in $T \in [G]$. Consequently, we only need to redefine $b_{s^*}(v)$ to $s^* - s_{BO}(v, G)$ for all $v \in A \setminus X$. The rest is analogous.

*MM* The proof for efficient computability of checking whether $X \in PWS_{MM}(G)$ can be modified to accommodate $PWSS_{MM}$. More precisely, the second basic assumption is now $w(i, j) \leqslant s^*$ for $i \notin X, j \in A$. For $s^* = \frac{n}{2}$ it is sufficient to check whether $G'$ is a proper extension of G. For $s^* < \frac{n}{2}$, edges $\{i, \{i, j\}\}$ with $i \in X$ are now contained in $E_H^{s^*}$ of $w(i, j) \leqslant s^*$. The rest of the argument can be adjusted appropriately. For $s^* > \frac{n}{2}$, nothing changes.

*$RP_\tau$* Since $PW_{RP_\tau}$ is NP-complete by Theorem 8.16, we get NP-hardness of $PWSS_{RP_\tau}$ by the oracle argument above. The problem is also NP-complete as membership in NP is obvious.

*complexity of $PWSS_{UC}$ is open*

The complexity of $PWSS_{UC}$ is currently open. Minor modification to our hardness proof of $PW_{UC}$ will not do the trick. In that argument, the crucial question was whether there is a completion that *excludes* certain alternatives from the choice set. This does not help for $PWSS_{UC}$.

## 8.7 SUMMARY AND DISCUSSION

The problem of computing possible and necessary winners for partial preference profiles has recently received a lot of attention. We have investigated this problem in a setting where partially specified (weighted or unweighted) *tournaments* instead of profiles are given as input. Our findings are summarized in Table 8.2.

| S | $PW_S$ | | $NW_S$ | | $PWS_S$ | |
|---|---|---|---|---|---|---|
| *CW* | in P | Lang et al., 2012 | in P | Lang et al., 2012 | in P | (Th. 8.2) |
| *CNL* | in P | (Th. 8.3) | in P | (Th. 8.3) | in P | (Th. 8.3) |
| *CO* | in P | (Th. 8.4)[a] | in P | (Th. 8.4)[a] | in P | (Th. 8.4) |
| *TC* | in P | Lang et al., 2012[a] | in P | Lang et al., 2012 | in P | (Th. 8.6) |
| *UC* | in P | (Th. 8.7) | in P | (Th. 8.8) | NP-C | (Th. 8.9) |
| *BO* | in P | (Th. 8.10)[a] | in P | (Th. 8.12) | in P | (Th. 8.11) |
| *MM* | in P | (Th. 8.13)[a] | in P | (Th. 8.15) | in P | (Th. 8.14) |
| *RP* | NP-C | (Th. 8.16) | coNP-C | (Th. 8.18) | NP-C | (Cor. 8.17) |

[a] This P-time result contrasts with the intractability of the same problem for partial preference profiles (Lang et al., 2012; Xia and Conitzer, 2011).

**Table 8.2:** Complexity of computing possible winners (PW) and necessary winners (NW) and of checking whether a given subset of alternatives is a possible winning set (PWS) under different solution concepts given partial tournaments.

A key conclusion is that computational problems for partial tournaments can be significantly easier than their counterparts for partial profiles. For example, possible Borda or maximin winners can be found efficiently for partial tournaments, whereas the corresponding problems for partial profiles are NP-complete (Xia and Conitzer, 2011).

While tractability of the winner determination problem is necessary for tractability of the possible or necessary winners problems, the results for ranked pairs in Section 8.5.3 show that it is not sufficient. We further considered the problem of deciding whether a given subset of alternatives equals the winner set for some completion of the partial tournament. The results for the uncovered set in Section 8.4.4 imply that this problem cannot be reduced to the computation of possible or necessary winners, but whether a reduction exists in the opposite direction remains an open problem.

Partial tournaments have also been studied in their own right, independent of their possible completions. For instance, Peris and Subiza (1999) and Dutta and Laslier (1999) have generalized several solution concepts on tournaments to incomplete tournaments by directly adapting their definitions. The common point with the approach we follow here is the nature of the input, namely, incomplete tournaments. However, solution concepts for incomplete tournaments in Peris and Subiza (1999) are defined by a direct generalization of the usual definition on complete tournaments in contrast to our definitions, which are based on the completions of the input partial tournament. In this context, the notion of possible winners suggests a canonical way to generalize a solution concepts defined on tournament to incomplete tournaments. The way of extending tournament solutions to partial tournaments is referred to as the "conservative extension" and inherits various axiomatic properties which the original tournament solutions satisfies for tournaments (Brandt et al., 2014b). The positive computational results presented in this chapter are an indication that this may be a promising approach.

Furthermore, we have not examined the complexity of computing possible and necessary winners for some attractive tournament solutions such as the minimal covering set, the bipartisan set (Laslier, 1997) and weighted versions of the top cycle and the uncovered set (De Donder et al., 2000).

# 9

# COMPARING CHOICE SETS OF TOURNAMENT SOLUTIONS

> Perhaps more important is that methods used here to furnish tools for analysing structural properties of dominance which could be applied, without really undue difficulty, to give a complete treatment of the cases, say $m \leqslant 12$. The cases previously so treated—$m \leqslant 4$—were exceptional, while some of the more variegated aspects of dominance set in for $m = 5$ and $m = 6$.
>
> Robert L. Davis, 1954

As we have seen in Section 5, a wide variety of tournament solutions have been proposed in the literature. Even though many of them are based on vastly different ideas, they share some similarities. For instance, most tournament solutions uniquely select the Condorcet winner whenever it exists. Moreover, some tournament solutions return completely identical or at least overlapping choice sets if the number of alternatives is sufficiently small whereas some have a reputation for excluding only few alternatives.

In this chapter, we study the differences of tournament solutions in matters of their choice sets and choice set sizes. First, we review the known inclusion relations among tournament solutions in Section 9.1. Then, in Section 9.2, we aim at formalizing and systematically investigating the similarity of any given pair of tournament solutions by studying the minimal number of alternatives that are required for the disparity and the separation of the corresponding choice sets. In Section 9.3, we turn to stochastic simulations to assess the typical size of the different choice sets. Generally, we consider all tournament solutions from Section 5.1 that satisfy strong Condorcet-consistency (besides *COND* which will only serve as a baseline in Section 9.3), which are *TC*, *UC*, $UC^\infty$, *MC*, *BP*, $\mathring{TC}$, *BA*, *ME*, *TEQ*, *CO*, *SL*, and *MA*.

## 9.1 SET-THEORETIC RELATIONS

All of the aforementioned tournament solutions return subsets of *TC* and all except *TC* and $\mathring{TC}$ return subsets of *UC*. On top of that, the following inclusion relationships are known (see Laslier, 1997):

$$BP \subseteq MC \subseteq UC^\infty \quad \text{and} \quad TEQ \subseteq BA.$$

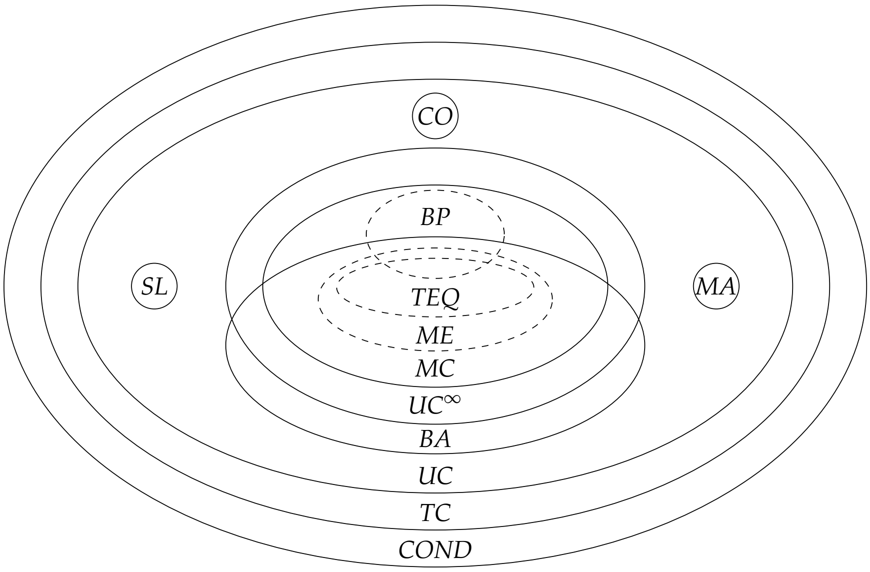



**Figure 9.1:** Set-theoretic relationships between tournament solutions. If the ellipses of two tournament solutions $S$ and $S'$ intersect, then $S(T) \cap S'(T) \neq \emptyset$ for all tournaments $T$. If the ellipses for $S$ an $S'$ are disjoint, however, this signifies that $S(T) \cap S'(T) = \emptyset$ for some tournament $T$. The exact location of *BP*, *TEQ*, and *ME* in this diagram are unknown but *BP* is contained in *MC* and intersects with *TEQ* in all known instances whereas *TEQ* and *ME* are contained in *BA*, but their inclusion in *MC* (and in each other) is uncertain.

Furthermore, it has been shown that

$$BA(MC) \subseteq BA \quad \text{and} \quad TEQ(UC^\infty) = TEQ,$$

which implies that

$$BA \cap MC \neq \emptyset \text{ , and } TEQ \subseteq BA \cap UC^\infty \neq \emptyset.$$

Also, by the upcoming Theorem 10.22, we know that

$$ME \subseteq BA.$$

The set-theoretic relationships of these concepts are depicted in Figure 9.1. *BA*, *ME*, and *TEQ* are the only tournament solutions that are capable of discriminating in regular tournaments, i.e., tournaments in which all alternatives have the same degree. All other tournament solution, besides possibly $\mathring{TC}$ for which regularity is open, always select all alternatives in regular tournaments.

## 9.2 DISPARITY AND SEPARATION OF TOURNAMENT SOLUTIONS

*disparity index* For two tournament solutions $S_1$ and $S_2$, we define the *disparity index*

$d(S_1, S_2)$ as the size of the smallest tournament T for which $S_1$ and $S_2$ differ, i.e.,

$$d(S_1, S_2) = \min\{m \in \mathbb{N} : \exists T \in \mathcal{T}_m \text{ such that } S_1(T) \neq S_2(T)\}.$$

Similarly, we define the *separation index* $s(S_1, S_2)$ as the size of the smallest tournament T for which the two choice sets are disjoint. Formally,

*separation index*

$$s(S_1, S_2) = \min\{m \in \mathbb{N} : \exists T \in \mathcal{T}_m \text{ such that } S_1(T) \cap S_2(T) = \emptyset\}.$$

Isolated bounds on both values for selected tournament solutions have been provided in previous work. In particular, the construction of tournaments for which certain tournament solutions return disjoint choice sets has occupied researchers. For example, the first described tournament for which the Banks set and the Slater set are disjoint consists of 75 alternatives (Laffond and Laslier, 1991). Later, this bound on the separation index was improved to 16 alternatives by Charon et al. (1997) and, quite recently, to 14 alternatives by Östergård and Vaskelainen (2010). Östergård and Vaskelainen have also provided a lower bound of 11 by means of an exhaustive computer analysis. In other work, Hudry (1999) has shown that the separation index for the Banks set and the Copeland set is 13. Dutta (1990) provided a tournament of order 8 in which the Banks set and the tournament equilibrium set are both strictly contained in the minimal covering set. Among other facts, our study has shown that Dutta's example is minimal.

$s(BA, SL) \in [11, 14]$

$s(BA, CO) = 13$

$d(BA, TEQ) \leqslant 8$

Perhaps the most interesting open problem regarding the relationships between tournament solutions concerns the bipartisan set and the Banks set. In all examples studied so far, either the Banks set is contained in the bipartisan set or the Banks set is contained in the bipartisan set (see, e.g., Laslier, 1997). In particular, it is unknown whether these tournament solutions always intersect. Here, we provide the first tournament in which the bipartisan set and the Banks set are *not* contained in each other.

*open relation between BA and BP*

### 9.2.1 Methodology

For some pairs of tournament solutions, we can easily show that they always intersect. As a consequence, their separation index is $\infty$.

Proposition 9.1
The following statements hold:

1. $s(MC, ME) = \infty$
2. $s(UC^{\infty}, ME) = \infty$
3. $s(\mathring{TC}, TEQ) = \infty$

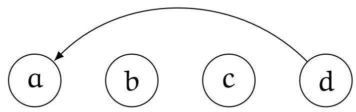



**Figure 9.2:** In this tournament, $MA(T) = SL(T) = \{a\} \subsetneq CO(T) = \{a, b\} \subsetneq UC(T) = \{a, b, d\} \subsetneq TC(T) = \{a, b, c, d\}$. All other tournament solutions considered here coincide with *UC*. Omitted edges point rightwards.

*Proof.* We prove each statement separately.

1. Since $BA \subseteq UC$, every *UC*-stable set is also *BA*-stable.
2. Since $MC \subseteq UC^\infty$, this follows from Statement 1.
3. Since $TEQ \subseteq TC$, every *TC*-retentive set is also *TEQ*-retentive. □

*generation of non-isomorphic tournaments*

Apart from these theoretical results, we exhaustively searched for minimal examples with disparate or disjoint choice sets. Obviously, the number of non-isomorphic tournaments of order $m$ grows exponentially (Moon, 1968, p. 87). We generated all non-isomorphic tournaments of of order ten or less using McKay's *nauty* toolkit (McKay, 2009). In total, we analyzed about $10^7$ tournaments. For each pair of tournament solutions and all tournaments in increasing order, we examined the choice sets for disparity and disjointness. Some of the most interesting tournaments we encountered were rearranged using a graphical tournament tool until the respective statements seemed most intuitive. Figures of these tournaments are included in Sections 9.2.2 and 9.2.3.

### 9.2.2 Experimental Results

Our results are summarized in Table 9.1 on page 129. When the exact value of an index is unknown, we provide lower and upper bounds.

#### *TC, CO, SL, MA vs. the rest*

*most discriminating tournament solutions*

*MA*, *SL*, and *CO* tend to select significantly smaller choice sets than the other tournament solutions whereas *TC* is not very discriminative. This is witnessed by the tournament of order 4 depicted in Figure 9.2 where *CO*, *SL*, and *MA* are smaller and *TC* is larger than all the remaining tournament solutions. This tournament accounts for all '4' entries in Table 9.1.

#### *UC, BA vs. $UC^\infty$, MC, BP, TEQ, ME, $\mathring{T}C$*

A smallest tournament for which *BA* (and *UC*) differs from *MC* (as well as $UC^\infty$, *BP*, *TEQ*, *ME*, $\mathring{T}C$) is shown in Figure 9.3. It is easy to verify that $\{a, b, d\}$ is *UC*-stable. Alternative $c$, however, is in $BA(T)$ because $B = \{c, d, e\} \in \mathcal{B}_T(c)$ and neither $a$ nor $b$ dominates $B$.

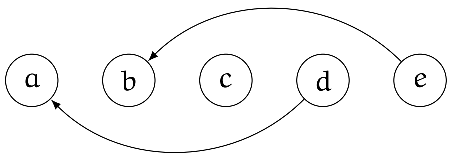



**Figure 9.3:** In this tournament, $UC^\infty(T) = MC(T) = BP(T) = TEQ(T) = ME(T) = \mathring{TC}(T) = \{a, b, d\}$ whereas $UC(T) = BA(T) = \{a, b, c, d\}$. Omitted edges point rightwards.

### *UC vs. BA*

There is an interesting family of tournaments that serve as minimal examples for a number of set-theoretic relationships between different tournament solutions. The first is the disparity of *UC* and *BA*—two solutions that return identical choice sets for all tournaments of order up to six.

The basic variant of this tournament family is shown in Figure 9.4 and constitutes a minimal tournament for which $BA \subsetneq UC$ (Miller et al., 1990). The difference is that $d \notin BA(T)$ as for all $B \in \mathcal{B}_T(d)$ there is some $x \in \overline{D}(d)$ with $x \succ B$. Note that in this tournament $|D(x)| = 4$ for all $x \in BA$ and $|D(x)| \leqslant 3$ for all $x \notin BA$, i.e., $CO(T) = BA(T)$.

When each grey alternative is replaced by the unique tournament of order 2, the resulting tournament of order 10 is a minimal example for $BA \subsetneq CO$, as $CO(T) = \{a, b, c, d\}$. (This result is not part of Table 9.1.)

If we go one step further and replace each grey alternative with any tournament of order 3, the resulting tournament has order 13 and is a known minimal example for the separation of *BA* and *CO* proposed by Moulin (1986) and Hudry (1999).

Finally, if we put any tournament of order 4 in place of the grey alternatives, we get a tournament of order 16 where still $BA(T) = \{a, b, c\}$ but $MA(T) = \{d\}$. Since any one of the alternatives from the new components can be removed without changing the result, this gives an upper bound of 15 for the separation of *MA* and *BA*.

### *BP vs. MC, TEQ, ME*

Consider the tournament in Figure 9.5. The unique equilibrium strategy of the tournament game $G(T)$ is $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0)$ and therefore $BP(T) = \{a, b, c, d, e\}$. However, this set is not *UC*-stable as $f$ can reach every other alternative in $BP(T) \cup \{f\} = A$ in at most two steps. This is a minimal tournament for which *MC* differs from *BP*. The same holds for *TEQ* and *ME* as they coincide with *MC* for tournaments up to size 7.

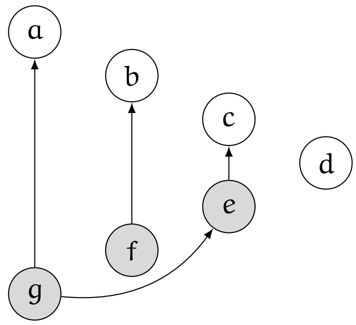



**Figure 9.4:** Minimal example for $BA(T) = \{a, b, c\} \subsetneq UC(T) = \{a, b, c, d\}$ (Miller et al., 1990). If $e, f$, and $g$ each get replaced by any tournament of order 3, the resulting tournament of order 13 is the minimal example for $BA \cap CO = \emptyset$ by Hudry (1999). If two of $e, f$, and $g$ are instead replaced with tournaments of order 4, we get a tournament of order 15 in which $MA(T) = \{d\}$ is disjoint from $BA(T) = \{a, b, c\}$. Omitted edges point downwards.

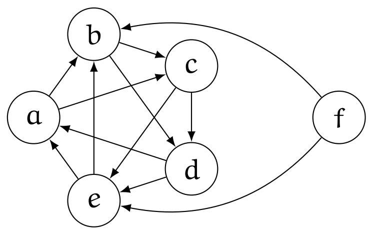



**Figure 9.5:** In this tournament, $BP(T) = \{a, b, c, d, e\} \subsetneq A = TEQ(T) = ME(T) = MC(T)$. Omitted edges point rightwards. Note that the subtournament on $BP(T)$ constitutes the only regular tournament of order 5.

### *MC* vs. *TEQ*, *ME*

A minimal tournament for which *TEQ* and *ME* differ from *MC* is of order 8 and depicted in Figure 9.6. This tournament is again a variant of the tournament from Figure 9.4, this time expanded with an additional vertex $h$. In this tournament $B = A \setminus \{d\}$ is the only *BA*-stable set. It is easy to check that $B$ is not *UC*-stable as $d$ does reach every other vertex in $A$ in at most two steps. In fact, only $A$ is *UC*-stable and therefore $MC(T) = A$. This implies that $d(ME, MC) = 8$. The reader can also verify that $d$ does not dominate any vertex according to the *TEQ*-relation and therefore $d \notin TEQ(T)$, implying $d(TEQ, MC) = 8$. While *TEQ* and *ME* actually coincide for this tournament, a small modification gives a minimal tournament $T'$ for which this is not the case, similar to the one reported by Brandt (2009). The only necessary change in the dominance relation is $e \succ g$, then $TEQ(T') = A \setminus \{d\} \subsetneq ME(T') = A$, accounting for $d(TEQ, ME) = 8$.

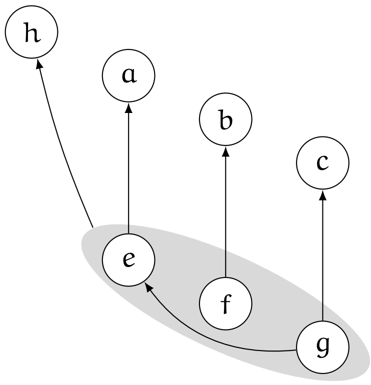



**Figure 9.6:** Minimal tournament for which $TEQ(T) = ME(T) \neq MC(T)$. Here, $TEQ(T) = ME(T) = A \setminus \{d\}$ whereas $MC(T) = A$. The ellipse indicates $\{e, f, g\} \succ h$ and omitted edges point downwards. If we change the dominance relation slightly to $e \succ g$, we get a minimal tournament for which *TEQ* and *ME* do not coincide.

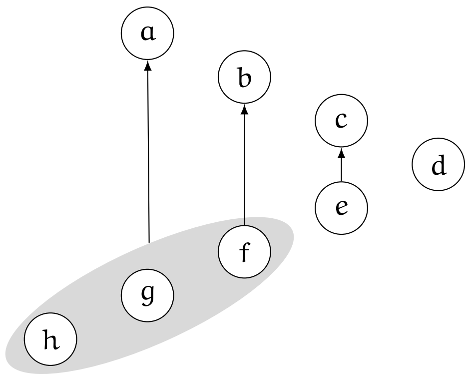



**Figure 9.7:** In this tournament, $CO(T) = MA(T) = \{b\}$ whereas $UC^{\infty}(T) = MC(T) = BP(T) = \mathring{TC}(T) = TEQ(T) = ME(T) = \{a, c, d, e, f\}$. This is the smallest tournament for which these choice sets are separate. The ellipse indicates $\{f, g, h\} \succ a$ and omitted edges point downwards.

*CO, MA vs. UC$^{\infty}$, MC, BP, $\mathring{TC}$, TEQ, ME*

For the separation of these tournament solutions, we found the tournament T depicted in Figure 9.7. It is easy to verify that alternative b has the largest dominion but is not contained in the *UC*-stable set $\{a, c, d, e, f\}$. Therefore, $CO(T) \cap MC(T) = \emptyset$ which gives $s(CO, MC) = 8$. As for this tournament $CO(T) = MA(T)$ and $MC(T) = UC^{\infty}(T) = BP(T) = \mathring{TC}(T) = ME(T) = TEQ(T)$, this also induces a few other separation indices in our table.

### 9.2.3 Further Findings

Apart from values and bounds for the disparity and separation index, our exhaustive search also revealed a number of other tournaments with interesting properties.

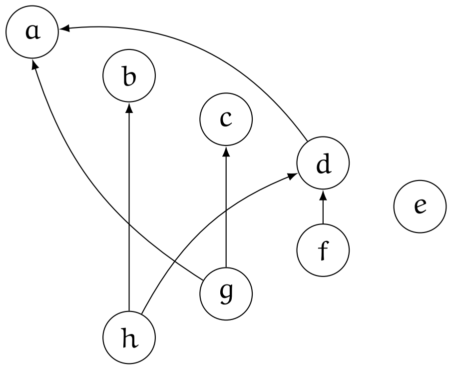



**Figure 9.8:** The first reported tournament where *BA* and *BP* are not contained in each other. In this tournament, $BP(T) = A \setminus \{e\}$ whereas $BA(T) = A \setminus \{f\}$. Omitted edges point downwards.

### *BP and BA*

For instance, we have found the first tournament where *BP* and *BA* have a proper intersection, i.e., are not contained in each other. The tournament is depicted in Figure 9.8, has 8 alternatives, and is minimal. The equilibrium strategy is $(\frac{7}{23}, \frac{3}{23}, \frac{1}{23}, \frac{7}{23}, 0, \frac{1}{23}, \frac{1}{23}, \frac{3}{23})$, i.e., $BP(T) = A \setminus \{e\}$. It is, however, easy to verify that $e \in BA$ as no other alternative dominates $\{e, f, g, h\} \in \mathcal{B}_T(e)$. At the same time, every set in $\mathcal{B}_T(f)$ is dominated by some alternative in $\{b, c, e\} \subseteq \overline{D}(f)$ and therefore $f \notin BA$. In fact, $BA = A \setminus \{f\}$.

### *BA and MC*

It was known already that *BA* and *MC* always intersect but none of them always chooses a subset of the other (Laslier, 1997). Our experiments showed that a proper intersection can only be observed for tournaments of order at least 10. A tournament of this kind is depicted in Figure 9.9. The reader can easily check that $A \setminus \{c, i\}$ is *UC*-stable. On the other hand, $i$ obviously is in $BA(T)$, witnessed by the maximal transitive subset $\{i, j, c\}$. Alternative $f$, however, is not in $BA(T)$ as for each $B \in \mathcal{B}_T(f)$, there is an alternative from $\{b, d, e\} \subseteq \overline{D}(f)$ that dominates $B$. In fact, $MC(T) = A \setminus \{c, i\}$ and $BA(T) = A \setminus \{c, f\}$. The choice sets overlap.

## 9.3 DISCRIMINATIVE POWER OF TOURNAMENT SOLUTIONS

Neutrality, as a very basic property we require from a social choice function, implies that it may be possible that several alternatives qual-

| s \ d | TC | UC | $UC^\infty$ | MC | BP | $\mathring{TC}$ | BA | ME | TEQ | CO | SL | MA |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TC | — | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| UC | ∞ | — | 5 (9.3) | 5 (9.3) | 5 (9.3) | 5 (9.3) | 7[a] (9.4) | 5 (9.3) | 5 (9.3) | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| $UC^\infty$ | ∞ | ∞ | — | 6 | 6 | 6 | 5 (9.3) | 6 | 6 | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| MC | ∞ | ∞ | ∞ | — | 6[b] | 6 | 5 (9.3) | 8 (9.6) | 8[c] (9.6) | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| BP | ∞ | ∞ | ∞ | ∞ | — | 6 | 5 (9.3) | 6 | 6[b] | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| $\mathring{TC}$ | ∞ | [11, ∞] | [11, ∞] | [11, ∞] | [11, ∞] | — | 5 (9.3) | 6 | 6 | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| BA | ∞ | ∞ | ∞ | ∞ | [11, ∞] | [11, ∞] | — | 5 (9.3) | 5 (9.3) | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| ME | ∞ | ∞ | ∞ | ∞ | [11, ∞] | [11, ∞] | ∞ | — | 8[d] (9.6) | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| TEQ | ∞ | ∞ | ∞ | [11, ∞] | [11, ∞] | ∞ | ∞ | [11, ∞] | — | 4 (9.2) | 4 (9.2) | 4 (9.2) |
| CO | ∞ | ∞ | 8 (9.7) | 8 (9.7) | 8 (9.7) | 8 (9.7) | 13[e] (9.4) | 8 (9.7) | 8 (9.7) | — | 4 (9.2) | 4 (9.2) |
| SL | ∞ | ∞ | 8[b] | 8[b] | 8[b] | 8 | [11, 14][f] | 8 | 8[b] | 6[g] | — | 5 |
| MA | ∞ | ∞ | 8 (9.7) | 8 (9.7) | 8 (9.7) | 8 (9.7) | [11, 15] (9.4) | 8 (9.7) | 8 (9.7) | 8 | 6 | — |

[a] Shown by Miller et al. (1990)
[b] Shown (without minimality) by Laslier (1997)
[c] Shown (without minimality) by Dutta (1990)
[d] Shown by Brandt (2009)
[e] Shown by Hudry (1999)
[f] Shown by Östergård and Vaskelainen (2010)
[g] Shown by Charon et al. (1996)

**Table 9.1:** Overview of all disparity indices and separation indices currently known for the tournament solutions considered. The names of coorresponding figures are given in round brackets.

ify equally well to be chosen.[56] Depending on the rationalization of the social choice function, this might be a rare exception or a common phenomenon. Since alternatives are generally assumed to be mutually exclusive, it is typically understood that ties will eventually be broken by some procedure that is independent of the agents' preferences. While it seems desirable to narrow down the choice as much as possible based on the preferences of the voters alone, the uncertainty the agents face when it comes to the final selection process can also be used as a powerful tool to satisfy certain formal criteria (such as impartiality, consistency, or strategyproofness) that would otherwise be impossible to attain. The trade-off between discriminative power and axiomatic foundations is especially evident for tournament solutions as many of them can be axiomatically characterized as the *most discriminating* functions that satisfy certain desirable properties.[57].

56 For example, every tournament solution has to chose all alternatives in a 3-cycle or, more generally in every *vertex-homogeneous* tournament, i.e., when all alternatives are in the same orbit.

57 For example, *TC* is the most discriminating tournament solution satisfying expansion-consistency. Similar characterizations are known for *UC*, *BA*, *MC*, and *BP* (see, e.g., Brandt et al., 2013b, Chapter 6, Section 2.2.2)

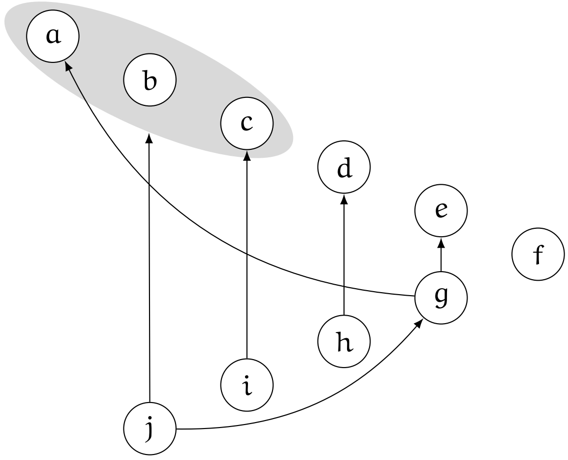



**Figure 9.9:** A minimal tournament for which *BA* and *MC* properly intersect. $BA(T) = A \setminus \{c, f\}$ whereas $MC(T) = A \setminus \{c, i\}$. Omitted edges point downwards.

In this section, we study the discriminative power of various social choice functions—i.e., how many tied alternatives are returned—when preferences are drawn from common distributions that have been proposed in the literature.

### 9.3.1 Motivation

*analytical results: large choice sets*

Analytical results about the discriminative power of tournament solutions for realistic distributions of preferences are very difficult to obtain. To the best of our knowledge, all existing papers explicitly or implicitly consider a uniform distribution over all tournaments of a fixed size. Under this assumption, it was shown by Fey (2008) that *BA* almost always selects all alternatives as the number of alternatives goes to infinity. By the above-mentioned inclusion relationship this implies the same statement for *UC* and *TC*. Later, an analogous result was shown for *MC* by Scott and Fey (2012). More precise results for *BP* have been given by Fisher and Reeves (1995) who identified the whole distribution of $|BP|$ for any fixed number of alternatives $m$. They found that the probability that *BP* returns exactly $k$ alternatives is $2^{-(m-1)}\binom{m}{k}$ if $k$ is odd and zero otherwise. This directly implies that on average, *BP* returns half of the alternatives for tournaments with an odd number of alternatives. In fact, for large tournaments, *BP* almost always chooses close to half of the alternatives (Scott and Fey, 2012).

*empirical results: Condorcet winners everywhere*

These analytical results stand in sharp contrast to empirical observations that Condorcet winners exist in many real-world situations (see, e.g., Feld and Grofman, 1992; Regenwetter et al., 2006), implying that tournament solutions very frequently return singletons. At the time of writing, the preference library PrefLib (Mattei and Walsh, 2013), contained 354 tournaments induced from pairwise majority comparisons. Out of these, all except 9 exhibit a Condorcet winner. The remaining tournaments are still very structured as the uncovered set never contains more than 4 alternatives (even in the largest of the remaining tournaments with 242 alternatives).

Our aim is to fill the gap between analytical and empirical results by means of stochastic simulations.

### 9.3.2 Experimental Results and Discussion

Our informal measure for discriminative power of a tournament solution on a specific model is the ratio of its average choice set size to the average size of *COND* which serves as a baseline being the least discriminative tournament solution that satisfies—as the other concepts—strong Condorcet consistency. We examined the average choice set sizes of the aforementioned tournament solutions for a fixed number of voters $m = 51$. The results are shown in Figure 9.10.

Due to the large number of Condorcet winners in these samples, the standard deviations of the measured choice set sizes are rather large. In cases of very high or very low average choice set sizes as in the spatial or in the uniform random tournament model, the standard deviation is, of course, low. A notable exception from this behavior is *BP* in case of the uniform random tournament model and the similar Condorcet noise model with $p = 0.55$. There, *BP* on average chooses less than half of the alternatives with low standard deviation.

The following conclusions can be drawn from our results.

- *TC* is almost as undiscriminating as *COND*.
- All other tournament solutions are much more discriminating than the analytical results for uniform random tournaments suggest. In fact, for all reasonable parameterizations of the considered models with transitive individual preferences and at least ten alternatives (including impartial culture) all tournament solutions except *TC* discarded at least 75% of the alternatives on average.
- All tournament solutions except *TC* behave similarly in terms of discriminative power. One may conclude that the decision which one to use in practical applications should not be based on discriminative power, but rather on axiomatic properties such as monotonicity or (group-) strategyproofness. The uncovered set, for example, has a particularly appealing axiomatization (Moulin, 1986).
- Using a more fine-grained analysis, tournament solutions can be divided into five clusters based on their discriminative power.

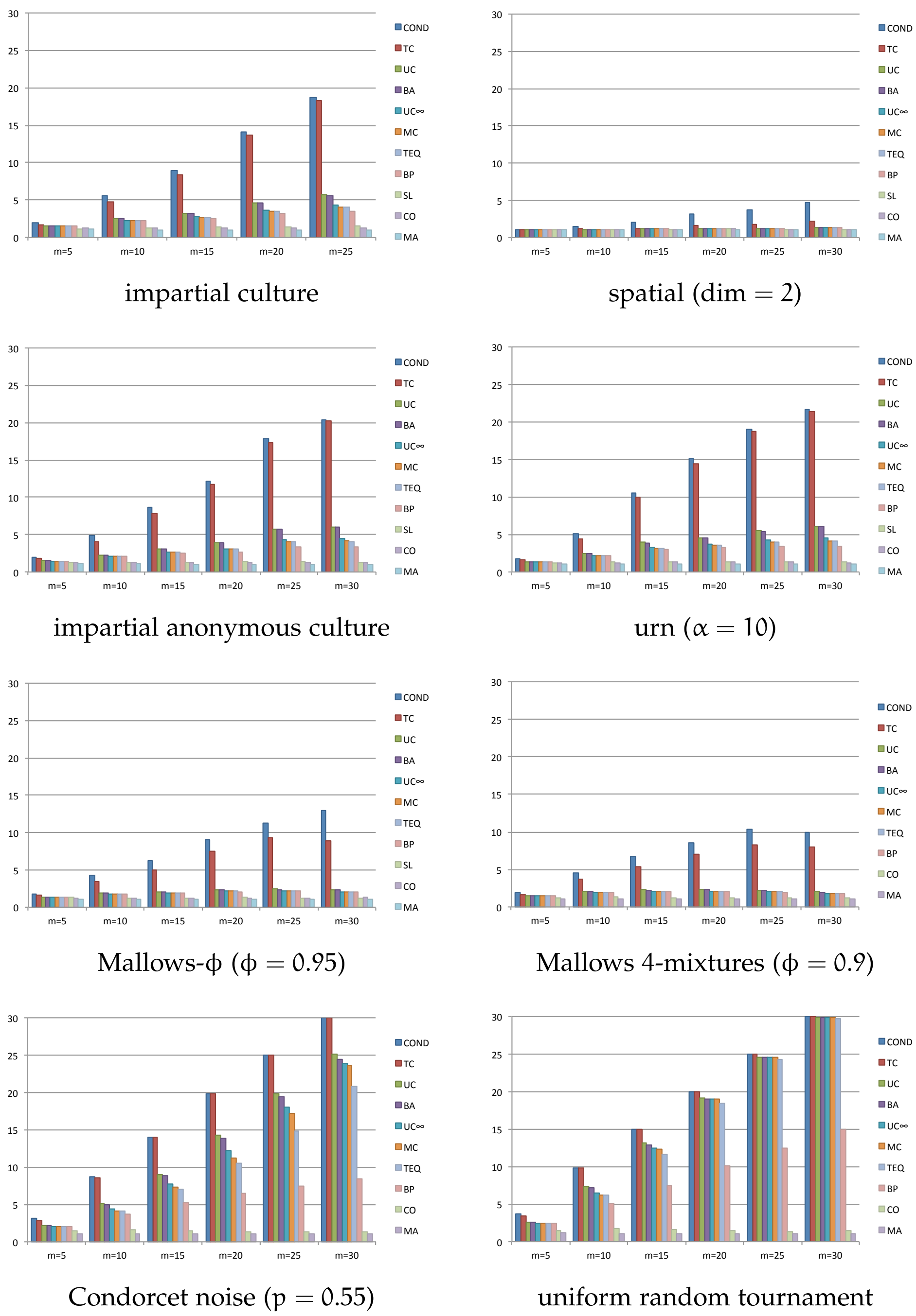



**Figure 9.10:** Comparison of average absolute choice set sizes for various stochastic preference models. The number of alternatives is on the horizontal axis, the number of voters is $m = 51$. Averages are taken over 100 runs. The Slater set (*SL*) is omitted whenever its computation was infeasible.

The first cluster merely consists of *TC*. The second cluster contains *UC* and *BA*. $UC^{\infty}$, *MC*, and *TEQ* are contained in the third cluster. *BP* forms a cluster of its own. Finally, tournament solutions based on scoring (*SL*, *CO*, and *MA*) are much more discriminating than all other tournament solutions and form the fifth cluster. Out of these, *MA* stands out as the most selective one. It is almost always unique.

- $UC^{\infty}$ (and thereby also *MC*) discriminates more than *BA*. This observation could not be deduced from the set-theoretic relationships between tournament solutions.
- *BP* is not only remarkably discriminating in uniform random tournaments (which already follows from the analytical results), but even more discriminating in the Condorcet noise model with $p = 0.55$. Within the group of tournament solutions with appealing characterizations, it discriminates the most (and is efficiently computable).

Aside from our insights into the discriminative powers of tournament solutions, we observed differences worth mentioning across the various stochastic models.

- The choice set sizes on the uniform random tournament model that was used in earlier analytical results differ significantly from the choice set sizes under the other models considered in this work.
- Impartial culture, urn model (including the impartial anonymous culture), and Mallows mixtures all result in similar choice set size distributions and lead to more discrimination than the uniform random tournament model.

## 9.4 SUMMARY

In the first part of this chapter, we exhaustively searched for smallest examples where the tournament solutions we consider do not coincide or are even disjoint. We believe that these might also be of didactic value when teaching the basics of tournament solutions.

In a second step we addressed a glaring discrepancy between analytical results suggesting that many tournament solutions almost always select the whole set of alternatives and empirical statements regarding the ubiquity of Condorcet winners or small dominant sets. Resorting to sensible stochastic models and simulations, we were able to show that the implicit assumptions for the negative analytic results describe an extreme case and that tournament solutions in most cases do select rather small choice sets.

# 10 STABILITY OF TOURNAMENT SOLUTIONS

> We say that a tournament has property $S(k)$ if for every set $A$ of $k$ players some other player beats all $x \in A$. Note the difficulty of determining the top $k$ players in such a tournament.
>
> — Paul Erdős and Joel Spencer, 1974

Besides very basic symmetry properties such as anonymity and neutrality, one can think of many different properties that seem to be desirable for a social choice function. Other innocuous properties include *monotonicity* which requires that chosen alternatives still have to be chosen when a pairwise comparison is flipped in their favor and *independence of unchosen alternatives* which requires the same for changes in the binary relation among non-chosen alternatives. Composition-consistency which featured prominently in Chapter 7 not only has an intuitive appeal but also relates choices of different tournaments (of different size), namely those that can be transformed into each other by replacing components with other components. Each of these three properties demands for a *consistency of choice* across different tournaments.

*monotonicity*

*independence of unchosen alternatives*

*consistency of choice*

In contrast, the property of *stability* which we are focusing on in this chapter relates choices made from different subtournaments of the same tournament. Only few tournament solutions are known to be stable, including the top cycle, the minimal covering set, and the bipartisan set.

In this chapter, we first review the relations of different properties around stability, including a new property that we call *local reversal symmetry* in Section 10.1. Then, we discuss the consequences of a recently found counterexample for a long-standing conjecture for *TEQ* in Section 10.2. In particular, we provide a much smaller counterexample that actually may have consequences for our assessment of *TEQ* as a tournament solution. In Section 10.3, we address many open questions regarding *ME*, including whether it satisfies stability—it does not. Lastly, in Section 10.4 we present ideas of our ongoing efforts to characterize *BP* using some of the properties mentioned in this chapter.

## 10.1 STABILITY AND RELATED PROPERTIES

Stability of a tournament solution is a property defined by Brandt and Harrenstein (2011). It requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets.

DEFINITION 10.1

*stable* A tournament solution $S$ is *stable* or satisfies stability (*STA*) if for all $(A, \succ)$ and nonempty $B, C, X \subseteq A$ with $X \subseteq B \cap C$,

$$X = S(B) = S(C) \quad \text{if and only if} \quad X = S(B \cup C).$$

As mentioned before, stability is satisfied by only a handful of the tournament solutions usually considered in the literature. These are (besides *TRIV*) the top cycle, the minimal covering set, the bipartisan set and $\mathring{TC}$.

We consider various properties related to stability.

### 10.1.1 Weakenings of Stability

*$\widehat{\alpha}$ and $\widehat{\gamma}$* Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately.

The former is also known as Chernoff's *postulate* $5^*$ (Chernoff, 1954), the *strong superset property* (Bordes, 1979), or *outcast* (Aizerman and Aleskerov, 1995) (see Monjardet, 2008, for a more thorough discussion of the origins of this condition).

*strong superset property*

A tournament solution $S$ satisfies $\widehat{\alpha}$, if for all sets of alternatives $A, B$, and $X$ with $X \subseteq A \cap B$, $X = S(A \cup B)$ implies $X = S(A) = S(B)$. Equivalently, $S$ satisfies $\widehat{\alpha}$ if for all sets of alternatives $A, B$, $S(A) \subseteq B \subseteq A$ implies $S(A) = S(B)$.

A tournament solution $S$ satisfies $\widehat{\gamma}$, if for all sets of alternatives $A, B$, $S(A) = S(B)$ implies $S(A \cup B) = S(A) = S(B)$.

Conveniently, these properties are automatically satisfied by all tournament solutions $\widehat{S}$ if the underlying solution concept $S$ always admits a unique minimal $S$-stable set (Brandt and Harrenstein, 2011).

For a finer analysis, we split $\widehat{\alpha}$ and $\widehat{\gamma}$ into two conditions (Brandt and Harrenstein, 2011, Remark 1).

DEFINITION 10.2

A tournament solution $S$ satisfies

- $\widehat{\alpha}_{\subseteq}$ if for all $A, B$, it holds that $S(A) \subseteq B \subseteq A$ implies $S(B) \subseteq S(A)$,[58]
- $\widehat{\alpha}_{\supseteq}$ if for all $A, B$, it holds that $S(A) \subseteq B \subseteq A$ implies $S(B) \supseteq S(A)$,

- $\widehat{\gamma}_{\subseteq}$ if for all $A, B$, and $X$, it holds that $X = S(A) = S(B)$ implies $X \subseteq S(A \cup B)$, and $\widehat{\gamma}_{\subseteq}$
- $\widehat{\gamma}_{\supseteq}$ if for all $A, B$, and $X$, it holds that $X = S(A) = S(B)$ implies $X \supseteq S(A \cup B)$. $\widehat{\gamma}_{\supseteq}$

Obviously, we have

$$\widehat{\alpha} \Leftrightarrow \widehat{\alpha}_{\subseteq} \wedge \widehat{\alpha}_{\supseteq} \text{ and}$$
$$\widehat{\gamma} \Leftrightarrow \widehat{\gamma}_{\subseteq} \wedge \widehat{\gamma}_{\supseteq}.$$

A tournament solution is *idempotent* if the choice set is invariant under repeated application of the solution concept, i.e., $S(T|_{S(T)}) = S(T)$ for all tournaments $T$. We show that $\widehat{\alpha}_{\supseteq}$ is stronger than idempotency. *idempotent*

LEMMA 10.3
If a tournament solution $S$ satisfies $\widehat{\alpha}_{\supseteq}$, then $S$ is idempotent.

*Proof.* In any tournament $T$, we have $S(T|_{S(T)}) \supseteq S(T)$ because of $\widehat{\alpha}_{\supseteq}$. The result follows. □

In Figure 10.1, the implications of the different properties are shown.

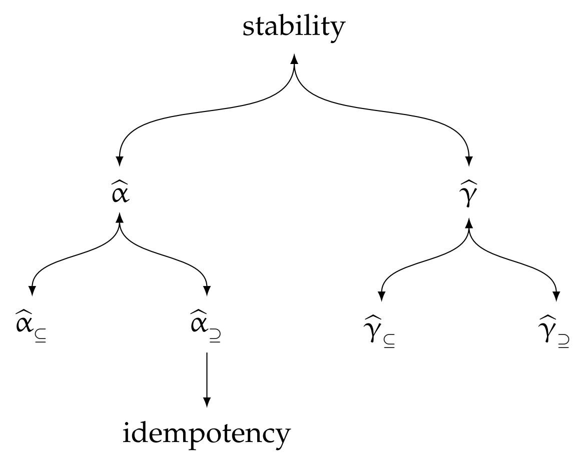



**Figure 10.1:** Implications of stability properties.

### 10.1.2 Local Reversal Symmetry

We propose a new property that is intimately connected to a flip operation on tournaments. Given a tournament $T = (A, \succ)$ and an alternative $a \in A$, we define $T^a$ as the tournament obtained from $T$ by a *local reversal* at alternative $a$, i.e., $T^a = (A, \succ^a)$ with *local reversal*

$$i \succ^a j \quad \text{if and only if} \quad (i \succ j \text{ and } a \notin \{i, j\}) \text{ or } (j \succ i \text{ and } a \in \{i, j\}).$$

58 The property $\widehat{\alpha}_{\subseteq}$ has been called the *weak superset property* or the *Aïzerman property* before.

The effect of local reversals is illustrated in Figure 10.2. Note that $T = (T^a)^a$ and $(T^a)^b = (T^b)^a$ for all alternatives $a$ and $b$.

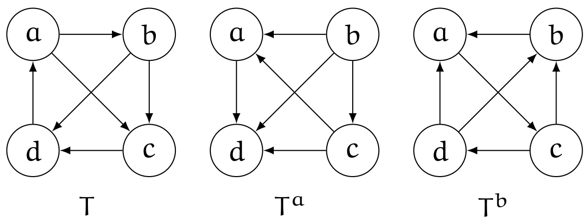



**Figure 10.2:** Local reversals at alternatives $a$ and $b$ in a tournament $T$ result in $T^a$ and $T^b$, respectively.

We say that a tournament solution satisfies *local reversal symmetry* (*LRS*) whenever reversing all relations involving an alternative $a$ changes the membership of $a$ in the choice set. We consider entering and leaving the choice set separately.

DEFINITION 10.4

*$LRS_{IN}$* A tournament solution $S$ satisfies *$LRS_{IN}$* if for all $T$ and for all alternatives $a$

$$a \notin S(T) \Rightarrow a \in S(T^a).$$

*$LRS_{OUT}$* Conversely, $S$ satisfies *$LRS_{OUT}$* if for all $T$ and for all $a$

$$a \in S(T) \Rightarrow a \notin S(T^a).$$

*LRS* $S$ satisfies *LRS* if it satisfies *$LRS_{IN}$* and *$LRS_{OUT}$*. In short, for all $T$ and for all $a$

$$a \in S(T) \Leftrightarrow a \notin S(T^a).$$

It is easy to see that *$LRS_{IN}$* (or *$LRS_{OUT}$*) for a tournament solution $S$ carries over for coarsenings (or refinements) of $S$.

LEMMA 10.5

Let $S, S'$ be two tournament solutions with $S \subseteq S'$. If $S$ satisfies *$LRS_{IN}$*, then so does $S'$. Conversely, if $S'$ satisfies *$LRS_{OUT}$* then $S$ does as well.

The notions of *LRS* and stability are not independent of each other. In fact, *$LRS_{IN}$* is implied by stability.

THEOREM 10.6

If a tournament solution $S$ is stable, then $S$ satisfies *$LRS_{IN}$*.

*Proof.* Assume that $S$ is stable but violates *$LRS_{IN}$*. Then there exists a tournament $T = (A, \succ)$ and an alternative $a \in A$ such that $a \notin S(T)$ and $a \notin S(T^a)$. From this, we construct a tournament with two disjoint $S$-stable sets.

| S | $\widehat{\alpha}$ | $\widehat{\gamma}$ | *STA* | $LRS_{IN}$ | $LRS_{OUT}$ | *LRS* |
|---|---|---|---|---|---|---|
| *TRIV* | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ |
| *TC* | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ |
| *UC* | ✗ | ✗ | ✗ | ✓ | ✗ | ✗ |
| *MC* | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ |
| *BP* | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| *BA* | ✗ | ✓ | ✗ | ✗ | ✗ | ✗ |
| *CO* | ✗ | ✗ | ✗ | ✗ | ✓ | ✗ |
| *SL* | ✗ | ✗ | ✗ | ✗ | ✓ | ✗ |
| $\mathring{TC}$ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ |
| *TEQ* | ✗ | ✗ | ✗ | ✗ | ✗ | ✗ |

**Table 10.1:** Properties of tournament solutions.

Let $T' = (A', \succ')$ with $A' = X \uplus Y$ and each of $T'|_X$ and $T'|_Y$ is isomorphic to $T|_{A\setminus\{a\}}$. Also, partition $X = X_0 \uplus X_1$ and $Y = Y_0 \uplus Y_1$ where $X_0$ and $Y_0$ consist of the alternatives that are mapped to $\overline{D}_T(a)$ by the isomorphism. Adding $X_0 \succ' Y_1, Y_1 \succ' X_1, X_1 \succ' Y_0$, and $Y_0 \succ' X_0$ completes the definition of $T'$. The structure of the construction which, for obvious reasons, we call a *shoelace construction* is depicted in Figure 10.3. We claim that both X and Y are S-stable. For this, we note that for every alternative $x \in X$ (or $y \in Y$) every subtournament $T|_{X\cup\{y\}}$ (or $T|_{Y\cup\{x\}}$) is isomorphic either to T or to $T^a$ with x (or y) being mapped to a. By assumption, a is neither chosen in T nor in $T^a$ and therefore X and Y are both S-stable in $T'$. Thus, S is not stable, a contradiction. □

*shoelace construction*

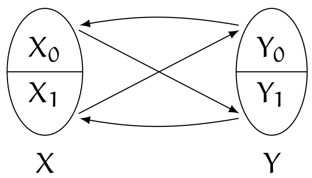



**Figure 10.3:** Shoelace construction of $T'$ with two S-stable sets X and Y used in the proof of Theorem 10.6 to show that stability implies $LRS_{IN}$.

A natural question to ask at this point is whether the studied tournament solutions satisfy $LRS_{IN}$ or $LRS_{OUT}$ (and $\widehat{\alpha}$ or $\widehat{\gamma}$ for that matter). The answers we found for this question are collected in Table 10.1.

Many of these results follow from existing statements on stability properties in Brandt and Harrenstein (2011) or are not difficult to prove[59]. Others are a bit more involved (e.g., that *SL* satisfies $LRS_{OUT}$ but not $\widehat{\gamma}$ or the violation of $LRS_{IN}$ by *BA*[60]) but are omitted here.

59 For example, the tournament that witnesses the violation of $\widehat{\gamma}$ by *UC* and *BA* has only five alternatives, in case of *SL* and $\widehat{\alpha}$, even four alternatives suffice.

60 The proof of the latter statement is similar to an argument in the proof of the upcoming Theorem 10.19.

Not surprisingly, $LRS_{IN}$ which is *inclusive* in nature is satisfied by all solution concepts that, according to our results in Chapter 9 do not discriminate that much, whereas the *exclusive* property $LRS_{OUT}$ is satisfied by those who tend to select smaller choice sets.[61]

From the table, it is apparent that only one of these tournament solutions satisfies both $LRS_{IN}$ and $LRS_{OUT}$ (and thereby *LRS*), namely the bipartisan set which, again referring to Chapter 9, is the smallest tournament solution that does not in most cases select singletons.

Proposition 10.7
*BP* satisfies *LRS*.

*Proof.* Since *BP* is stable, Theorem 10.6 implies that *BP* satisfies $LRS_{IN}$. Now, assume that *BP* violates $LRS_{OUT}$, i.e., there is a tournament T and an alternative $a$ s.t. $a \in BP(T)$ and $a \in BP(T^a)$. Let $p_a(T)$ be the total equilibrium scores of $T \setminus \{a\}$. It is known that $a \in BP(T)$ if and only if $p_a(D_T(a)) > p_a(\overline{D}_T(a))$ (Laslier, 1997, Prop. 6.4.8). But then $p_a(D_T(a)) > p_a(\overline{D}_T(a))$ and $p_a(D_{T^a}(a)) > p_a(\overline{D}_{T^a}(a))$. This is a contradiction as $D_T(a) = \overline{D}_{T^a}(a)$ and $\overline{D}_T(a) = D_{T^a}(a)$. □

## 10.2 THE CASE OF TEQ

Whether *TEQ* satisfies stability or not was shown (Brandt, 2009, Theorem 7) to be subject to Schwartz's long-standing conjecture that was recently disproved. This conjecture by Schwartz and its weakenings formally reads as follows.[62]

Conjecture 10.8 *(Schwartz, 1990)*
Every tournament admits a unique minimal *TEQ*-retentive set.

Conjecture 10.9 *(Brandt et al., 2010a, see Theorem 3)*
There is no tournament with two disjoint *BA*-retentive sets.

Conjecture 10.10 *(Brandt et al., 2013a)*
No tournament $T = (A, \succ)$ admits two disjoint sets $X_0, X_1$ such that for every $i \in \{0, 1\}$ and every transitive subset B of $B_i$, there exists an $a \in X_{1-i}$ such that $a \succ B$.

We call such sets $X_i$ from Conjecture 10.10 that contain only transitive subsets that can be extended by an outside alternative *chain-free sets*.

*chain-free sets*

Brandt et al. (2013a) disproved Conjecture 10.10 by non-constructively proving the existence of a tournament $T^{CF}$ with two disjoint chain-free sets. Not even the exact size of the counterexample can be derived from the proof, only that the two sets $X_1$ and $X_2$ have about $2^{15}$ and $10^{103}$ vertices, respectively.

61 We have not considered *MA* in this study but would suspect that it satisfies $LRS_{OUT}$.
62 To clarify again, all of these "conjectures" are disproved already. We will argue about the connections between their respective counterexamples.

It is not difficult to verify that $T^{CF}$ can be used to obtain a counterexample to Conjecture 10.9. In fact, we use the shoelace construction from Figure 10.3 to get a tournament $T^{TEQ}$ twice the size of $T^{CF}$ in which both sets X and Y are *BA*-retentive. Since $TEQ \subseteq BA$, it immediately follows that the two sets are also *TEQ*-retentive and $T^{TEQ}$ is a counterexample to Conjecture 10.8. Still, the size of $T^{TEQ}$ is unknown which is unfortunate as the size $m_{TEQ}$ of a *smallest* counterexample to Conjecture 10.8 bears some importance for the following reason: *TEQ* satisfies many desirable properties, including stability, for all tournaments that are smaller than the smallest counterexample to Schwartz's conjecture (Laffond et al., 1993a; Houy, 2009a; Houy, 2009b).

*a much smaller counterexample*

We provide a small tournament with two disjoint *TEQ*-retentive sets and thereby lower the upper bound on $m_{TEQ}$ from $\sim 10^{103}$ to 24. The counterexample was found by exhaustive searching for $LRS_{IN}$-violations of *TEQ*. We found a tournament T on 13 alternatives where $TEQ(T) = A \setminus a = TEQ(T^a)$ for a distinct alternative $a$. Using again the shoelace construction as described in the proof of Theorem 10.6, we obtained a tournament on 24 alternatives with two isomorphic *TEQ*-stable sets X and Y. It turned out that these are also *TEQ*-retentive, giving us a much smaller counterexample to Conjecture 10.8.

The definition of the tournament is as follows. Let $T = (A, \succ)$ with $A = X \uplus Y$ where $X = \{x_1, \dots, x_{12}\}$ and $Y = \{y_1, \dots, y_{12}\}$. Also, let $X_0 = \{x_1, \dots, x_6\}$, $1_2 = \{x_7, \dots, x_{12}\}$, $Y_0 = \{y_1, \dots, y_6\}$, and $Y_1 = \{y_7, \dots, y_{12}\}$. For $\in X, y \in Y$, the dominance relation is defined as in the shoelace construction illustrated in Figure 10.3.

The two subtournaments $T|_X$ and $T|_Y$ are isomorphic. For $T|_X$, the dominator sets are defined as

$$\begin{aligned}
\overline{D}_X(x_1) &= \{x_4, x_5, x_6, x_8, x_9, x_{12}\}, & \overline{D}_X(x_2) &= \{x_1, x_6, x_7, x_{10}, x_{12}\},\\
\overline{D}_X(x_3) &= \{x_1, x_2, x_6, x_7, x_9, x_{10}\}, & \overline{D}_X(x_4) &= \{x_2, x_3, x_7, x_8, x_{11}\},\\
\overline{D}_X(x_5) &= \{x_2, x_3, x_4, x_8, x_{10}, x_{11}\}, & \overline{D}_X(x_6) &= \{x_4, x_5, x_9, x_{11}, x_{12}\},\\
\overline{D}_X(x_7) &= \{x_1, x_5, x_6, x_{11}, x_{12}\}, & \overline{D}_X(x_8) &= \{x_2, x_3, x_6, x_7, x_{12}\},\\
\overline{D}_X(x_9) &= \{x_2, x_4, x_5, x_7, x_8\}, & \overline{D}_X(x_{10}) &= \{x_1, x_4, x_6, x_7, x_8, x_9\},\\
\overline{D}_X(x_{11}) &= \{x_1, x_2, x_3, x_8, x_9, x_{10}\}, \text{ and} & \overline{D}_X(x_{12}) &= \{x_3, x_4, x_5, x_9, x_{10}, x_{11}\}.
\end{aligned}$$

A rather tedious check reveals that

$$\begin{aligned}
TEQ(\overline{D}_A(x_1)) &= \{x_4, x_8, x_{12}\}, & TEQ(\overline{D}_A(x_2)) &= \{x_6, x_{10}, x_{12}\},\\
TEQ(\overline{D}_A(x_3)) &= \{x_6, x_7, x_9\}, & TEQ(\overline{D}_A(x_4)) &= \{x_2, x_7, x_{11}\},\\
TEQ(\overline{D}_A(x_5)) &= \{x_2, x_8, x_{10}\}, & TEQ(\overline{D}_A(x_6)) &= \{x_4, x_9, x_{11}\},\\
TEQ(\overline{D}_A(x_7)) &= \{x_1, x_5, x_{11}\}, & TEQ(\overline{D}_A(x_8)) &= \{x_3, x_6, x_{12}\},\\
TEQ(\overline{D}_A(x_9)) &= \{x_2, x_5, x_7\}, & TEQ(\overline{D}_A(x_{10})) &= \{x_4, x_6, x_7\},\\
TEQ(\overline{D}_A(x_{11})) &= \{x_1, x_2, x_8\}, \text{ and} & TEQ(\overline{D}_A(x_{12})) &= \{x_3, x_4, x_9\}.
\end{aligned}$$

Obviously, $TEQ(\overline{D}_A(x)) \subseteq X$ for all $x \in X$. Hence, X is *TEQ*-retentive in T. Moreover, it can be checked that $TEQ(\overline{D}_A(y_i)) \subset Y$ for all $i \in \{1, \dots, 12\}$, which implies that Y is *TEQ*-retentive as well. In fact, we even have that, for all $i, j \in \{1, \ldots, 12\}$, $y_j \in \mathit{TEQ}(\overline{D}_A(y_i))$ if and only if $x_j \in \mathit{TEQ}(\overline{D}_A(x_i))$.

Since it has been shown in earlier computer experiments by Brandt et al. (2010b) that Conjecture 10.8 holds for all tournaments of size 12 or less[63], this gives us

$$13 \leqslant m_{\mathit{TEQ}} \leqslant 24.$$

## 10.3 THE CASE OF ME

Similarly as for *TEQ* and Conjecture 10.8, it was shown that *ME* satisfied stability and other desirable properties a weakening of Conjecture 10.8 was correct. However, these statements were only shown in one direction and after this "new" conjecture was disproved (as well by the conterexample in Brandt et al. (2013a), it remained open whether *ME* satisfies these properties or not.

Using the counter-example by Brandt et al. (2013a) we show that *ME* fails to satisfy most properties, in particular stability while it does satisfy $\widehat{\alpha}_{\subseteq}$, irregularity, and membership in the Banks set.[64]

Recall that if a set is *BA*-stable, then it is called an *extending set* and that the union of all inclusion minimal extending sets defines the tournament solution *ME* (Brandt, 2011b), i.e.,

$$\mathit{ME}(T) = \bigcup \{B \text{ is } \mathit{BA}\text{-stable} : \forall C \subsetneq B : C \text{ is not } \mathit{BA}\text{-stable}\}.$$

For an illustrative example where *ME* and *BA* do not coincide, see Figure 9.3 on 125.

### 10.3.1 Minimal extending sets

Minimal extending sets, while commonly not considered as a solution concept in their own right, satisfy a number of interesting properties.

First, it is obvious that an extending set remains an extending set when outside alternatives are removed. Moreover, when the set was a *minimal* extending set before, it is still minimal in the reduced tournament.

LEMMA 10.11

> In a tournament $T = (A, \succ)$ with (minimal) extending set $B \subset A$ in $T$ and $C \subseteq A$ with $B \subseteq C$, $B$ is also a (minimal) extending set in $T|_C$.

63 There is little hope in improving the lower bound significantly just by means of exhaustive search: according to the author of the nauty toolkit, B. McKay (personal communcation, Aug 26, 2008), generating all tournaments of size 13 "could be done if the future of life on earth depended on it".

64 Previously, the two statements on computational (in)tractability and membership in the Banks set were only known to hold if the (now disproved) conjecture Conjecture 10.13 had been true (Brandt, 2009).

*Proof.* Let $B$ be an extending set in $T$. Then, $a \notin BA(B \cup \{a\})$ for all $a \in A \setminus B$. As $C \subseteq A$, the result follows. Now let $B$ be a *minimal* extending set in $T$. Assume that $B$ is not minimal in $T|_C$. Then there exists $B' \subsetneq B$ such that $B'$ is extending in $T|_C$. Let $Q$ be maximal in $\mathcal{B}_{T|_{B' \cup \{a\}}}(a)$ for an $a \in A \setminus C$. Such $Q$ and $a$ have to exist as $B'$ is not extending in $T$. However, since $B$ is an extending set, there is a $b \in B \setminus B'$ with $b \succ Q \cup \{a\}$. For the transitive subset $Q \cup \{b\}$, there is a $b' \in B'$ with $b' \succ Q \cup \{b\}$ because $B'$ is extending in $T|_C$. If $b' \succ a$, then $b'$ extends $Q \cup \{a\}$ which cannot be the case. If $a \succ b'$, then $Q$ was not maximal. Therefore, such a $b'$ cannot exist, a contradiction.□

Secondly, from the definition of extending sets, it is immediate that a minimal extending set is unaffected by modifying the dominance relation among outside alternatives.

Lemma 10.12
A minimal extending set $B$ in a tournament $T = (A, \succ)$ is also a minimal extending set in every tournament $T' = (A, \succ')$ if $a \succ' b \Leftrightarrow a \succ b$ for all $a \in A$ and $b \in B$.

Brandt (2011b) conjectured that every tournament contains a unique minimal extending set and showed that, if the conjecture holds, *ME* satisfies a large number of desirable properties.

Conjecture 10.13 *(Brandt, 2011b)*
Every tournament admits a unique *BA*-stable set.

However, the tournament $T^{TEQ}$ derived from the counterexample to Conjecture 10.10 by the method described in 10.2, also serves as a counterexample to Conjecture 10.13: its two *BA*-retentive sets are also *BA*-stable and thereby disjoint extending sets (Brandt et al., 2013a).

For some of our proofs we need to know that there is a tournament with exactly two minimal extending sets that are disjoint. It is unknown whether the tournaments described by Brandt et al. (2013a) satisfy this property. To this end, we first show that from a tournament with more than two or non-disjoint minimal extending sets we can construct a strictly smaller tournament that still has multiple minimal extending sets.[65]

Lemma 10.14
Let $T$ be a tournament with multiple minimal extending sets $B_1, B_2, \ldots, B_k$. If $k > 2$ or $B_1 \cap B_2 \neq \emptyset$ then there is a tournament $T'$ with multiple minimal extending sets and $|T'| < |T|$.

*Proof.* Let $T$ be a tournament with multiple minimal extending sets that violates

(i) $A$ can be partitioned into only two minimal extending sets or

65 Part of the proof is inspired by a similar proof due to Brandt (2011b, Lemma 2).

(ii) the minimal extending sets of $T$ are disjoint.

If (i) is violated by $T$, let $T' = T|_{B_1 \cup B_2}$. Obviously, $|T'| \leqslant |T|$ and by Lemma 10.11, $B_1$ and $B_2$ are still minimal extending sets in $T'$.

If (ii) is violated, let $C = B_1 \cap B_2 \neq \emptyset$. Due to minimality of $B_1$ (and $B_2$), $C$ is not an extending set. Hence, there have to be $Q \subseteq C$ and $a \in A \setminus C$ such that $a \succ Q$ and $Q \cup \{a\}$ is maximal in $\mathcal{B}_{C \cup \{a\}}$ and cannot be extended from an alternative in $C$. Define $B_1' = \{b \in B_1 : b \succ Q\}$ and $B_2' = \{b \in B_2 : b \succ Q\}$.

Assume without loss of generality that $a \notin B_1$. Then there has to be a $b_1 \in B_1$ that extends $a \cup Q$ because $B_1$ is an extending set, i.e., $B_1'$ is not empty. To show that $B_1'$ and $B_2'$ are disjoint, assume for contradiction that there is a $b \in B_1' \cap B_2'$. It is easy to check that no matter whether $a \succ b$ or $b \succ a$, $Q \cup \{a\}$ is not maximal in $\mathcal{B}_{C \cup \{a\}}$. Hence, $B_1' \cap B_2' = \emptyset$ and by stability of $B_2$, there has to be a $b_2 \in B_2$ that extends $Q \cup \{b_1\}$, i.e., $B_2'$ is not empty as well. The situation is depicted in Figure 10.4.

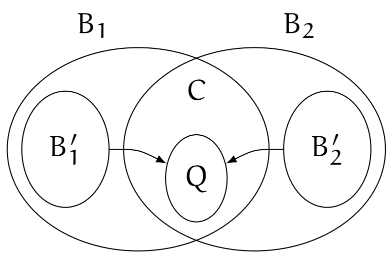



**Figure 10.4:** Relevant subsets in the argument to construct a tournament with disjoint minimal extending sets $B_1', B_2'$, given a tournament with overlapping minimal extending sets $B_1, B_2$.

Next, we show that $B_1'$ and $B_2'$ are extending sets in $T|_{B_1' \cup B_2' \cup Q}$. To this end, consider $a' \in B_2'$ and $R$ a maximal transitive subset of $B_1' \cup Q$ such that $a' \succ R$. It is easy to see that $Q \subseteq R$ due to $B_1' \succ Q$, $B_2' \succ Q$, and maximality of $R$. As $B_1$ is an extending set in $T$, there has to be a $c \in B_1$ that extends $R \cup \{a'\}$. By $c \succ Q \subseteq R$, $c$ is contained in $B_1'$, i.e., $B_1'$ (and analogously $B_2'$) is an extending set in $T|_{B_1' \cup B_2' \cup Q}$. Due to Lemma 10.11, $B_1'$ and $B_2'$ are also extending sets in $T' = T_{B_1' \cup B_2'}$ which is of strictly smaller order than $T$. □

This insight allows us to deduce properties of a smallest tournament that has multiple minimal extending sets which will later prove useful when reasoning about *ME*.

Corollary 10.15
Let $T^* = (A, \succ^*)$ be a smallest tournament with multiple minimal extending sets. Then $T^*$ has exactly two minimal extending sets and they partition $A$.

For the remainder of this section, let $T^*$ to be such a tournament of minimal order with multiple extending sets.[66]

### 10.3.2 Properties of ME

We analyze *ME* with respect to two different types of properties: dominance-based properties and choice-theoretic properties. Both serve as important benchmarks for the evaluation of decision-theoretic and choice-theoretic concepts. We furthermore investigate *ME*'s relationship to other tournament solutions. Finally, we also establish the computational complexity of deciding whether an alternative is in *ME* for a given tournament.

*Dominance-based properties*

In this section, we consider two properties that are based on the dominance relation. The first property is called *monotonicity* and corresponds to a well-established standard condition in social choice theory. It prescribes that a chosen alternative should still be chosen if it is reinforced. Formally, a tournament solution $S$ satisfies monotonicity if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $\succ_{A\setminus\{a\}} = \succ'_{A\setminus\{a\}}$ and for all $b \in A \setminus \{a\}$, $a \succ b$ implies $a \succ' b$.

*monotonicity*

THEOREM 10.16
*ME* does not satisfy monotonicity.

*Proof.* Consider $T^*$ from Corollary 10.15 with its two (disjoint) minimal extending sets $B_1$ and $B_2$ and alternatives $b_1 \in B_1, b_2 \in B_2$ with $b_2 \succ b_1$. Let $T^*_{b_1}$ be the modified tournament where $b_1 \succ (B_1 \setminus \{b_1\})$. By the remark after Lemma 10.11, $B_2$ is still a minimal extending set in $T^*_{b_1}$. By minimality of $T^*$ and Lemma 10.14, any other extending set would be disjoint from $B_2$, i.e., contained in $B_1$. The set $B_1$ itself is no longer an extending set as no alternative in $B_1$ extends $\{b_2, b_1\}$. If there is an extending set $B'_1 \subsetneq B_1$, then the tournament $T^*_{b_1}|_{B'_1 \cup B_2}$ contradicts the minimality of $T^*$. Therefore, no such $B'_1$ exists and $ME(T^*_{b_1}) = B_2$, i.e., the strengthened alternative $b_1$ is no longer contained in *ME*. □

The second property, *independence of unchosen alternatives*, states that the choice set should be unaffected by changes in the dominance relation between unchosen alternatives. Formally, a tournament solution $S$ is independent of unchosen alternatives if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $D_{\succ}(a) = D_{\succ'}(a)$ for all $a \in S(T)$.

*independence of unchosen alternatives*

66 Interestingly, the order of this tournament is unknown. By exhaustive analysis and the existence proof by Brandt et al. (2013a), we can say that it has at least 13 vertices and less than $10^1 04$.

THEOREM 10.17
*ME* does not satisfy independence of unchosen alternatives.

*Proof.* Consider again $T^*$. Let $T^*_{lin}$ be the modified tournament where the alternatives in $B_1$ are in a linear order L. Then, there is a $b_1 = \max(L)$ with $b_1 \succ_{B_1} B_1 \setminus \{b_1\}$. By the same argument as in the proof of Theorem 10.16, $B_2$ is the unique minimal extending set in $T^*_{lin}$ and therefore $ME(T^*_{lin}) = B_2$. If now edges in $T^*_{lin}$ are iteratively reversed towards the original $T^*$, we get a sequence of tournaments $T^*_{lin}, \dots, T^*$ in each of which $B_2$ is a minimal extending set by Lemma 10.12. Also, there can never be another minimal extending $B_1' \neq B_1$ because by Corollary 10.15, this would violate the minimality of $T^*$. This implies that at some point in this sequence, reversing an edge between two alternatives not in *ME* makes the whole set $B_1$ a (new) minimal extending set. □

An interesting aspect of minimal extending sets is that, according to Lemma 10.12, they satisfy a local variant of independence of unchosen alternatives, a property that their union (*ME*) fails to satisfy.

*set-monotonicity*

Brandt (2011a) has shown that *set-monotonicity*, defined as the invariance of a choice set under weakening unchosen alternatives, implies independence of unchosen alternatives. Consequently, *ME* is not set-monotonic.

COROLLARY 10.18
*ME* violates set-monotonicity.

### *Choice-theoretic properties*

An important class of properties concern the consistency of choice and relate choices from different subtournaments of the same tournament to each other.

THEOREM 10.19
*ME* satisfies (i) $\widehat{\alpha}_{\supseteq}$ but neither (ii) $\widehat{\alpha}_{\subseteq}$ nor (iii) $\widehat{\gamma}_{\supseteq}$.

*Proof.* We show each statement separately.

For (i), let $T = (A, \succ)$ be a tournament with minimal extending sets $B_1, \dots, B_k$ and let $T' = (C, \succ|_C)$ be a subtournament of T with $ME(T) = \bigcup_{i \in \{1,\dots,k\}} B_i \subseteq C \subseteq A$. By Lemma 10.11, every $B_i$ is still a minimal extending set in $T'$ and therefore $ME(T') \supseteq ME(T)$.

For (ii), consider again $T^* = (A, \succ^*)$ from Corollary 10.15 with its two minimal extending sets $B_1$ and $B_2$. We create a larger tournament $T^*_x$ by adding an alternative x such that $B_1 \succ x$ and $x \succ B_2$. The tournament is depicted in Figure 10.5. Obviously, $B_1$ still is a minimal extending set and we claim that there is no other. Assume for contradiction that there is another minimal extending set $B' \neq B_1$ in $T^*_x$. If $x \notin B'$ then $B'$ is also a minimal extending set in $T^*$ by Lemma 10.11. As $T^*$ has not minimal extending sets besides $B_1$ and $B_2$, it follows that $B' = B_2$. But $B_2$ cannot be an extending set in $T^*_x$ because for

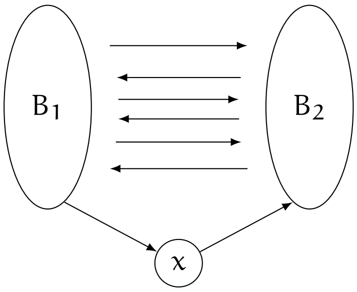



**Figure 10.5:** Structure of the tournament $T_x^*$ used to show that *ME* violates $\widehat{\alpha}_{\subseteq}$. Without alternative $x$, this is a minimal tournament $T^*$ among those with multiple minimal extending sets.

any $b_2 \in B_2$ there is no $b_2'$ that extends $\{x, b_2\}$. If, on the other hand, $x \in B'$, then $B' \cap B_1 \neq \emptyset$ since otherwise $\{b_1, x\}$ cannot be extended from within $B'$ for all $b_1 \in B_1$ due to $B_1 \succ x$. By Lemma 10.14, there exists a subtournament $T'$ of $T_x^*$ with two disjoint minimal extending sets and $|T'| \leqslant |T_x^*| - 2 < |T^*|$. The first inequality holds because $x$ and at least one alternative from $B_1 \cap B'$ are not contained in $T'$ by the construction. This contradicts the minimality of $T^*$.

For (iii), consider $T^* = (A^*, \succ^*)$ with its minimal extending sets $B_1$ and $B_2$. For all $b \in B_2$ let $T_b^* = (B_1 \cup \{b\}, \succ^*_{B_1 \cup \{b\}})$. By Lemma 10.11, $B_1$ is still a minimal extending set in all $T_b^*$. There cannot be another minimal extending set $B_2$ (containing $b$) because otherwise $T_b^*$ would have multiple extending sets, contradicting the minimality of $T^*$. Therefore $ME(T_b^*) = B_1$ for all $b \in B_2$. If $\widehat{\gamma}_{\supseteq}$ holds for *ME*, then $ME(T^*) = ME(T_b^*) = B_1$ because $T^*$ is the union of all $T_b^*$. However, $ME(T^*) = A$, a contradiction. This concludes the proof. □

It is open whether *ME* satisfies $\widehat{\gamma}_{\subseteq}$. Still, the fact that *ME* violates $\widehat{\alpha}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ immediately implies that it violates stability.

Corollary 10.20
*ME* does not satisfy $\widehat{\alpha}$ or $\widehat{\gamma}$ and is therefore not stable.

Interestingly, minimal extending sets themselves are stable sets. They satisfy a *local* version of stability, namely

- removing alternatives outside of a minimal extending set has no effect on it by Lemma 10.11 and
- a set that is minimal extending in several tournaments is also minimal extending in the union of these by the definition of minimal extending sets.

From Theorem 10.19 and Lemma 10.3, we also get that *ME* is idempotent.

Corollary 10.21
*ME* satisfies idempotency.

*Relationships to other tournament solutions*

Besides the axiomatic properties of *ME*, we are also interested in its set-theoretic relationships to other tournament solutions. Assuming that every tournament has only one minimal extending set, Brandt (2011b) showed that *ME* always selects subsets of *BA* and subsets of *MC*. Under an even stronger conjecture, he also proved that *ME* always selects supersets of *TEQ*. Since the conjectures turned out to be incorrect, these questions are open again. We can now answer one of these in the affirmative, namely that *ME* indeed chooses from *BA*.

Theorem 10.22
For all tournaments T,

$$\mathit{ME}(T) \subseteq \mathit{BA}(T).$$

*Proof.* Let $T = (A, \succ)$ be a tournament. It suffices to show that, if $B \subseteq A$ is a minimal extending set, all alternatives of B are also in the Banks set. Assume $a \in \mathit{ME}(T)$. Then there is some minimal extending set B such that $a \in B$. Also assume for contradiction that $a \notin \mathit{BA}(T)$. Then, for every $X \in \mathcal{B}_T(a)$ there is some $x \in A$ with $X \cup \{x\} \in \mathcal{B}_T(x)$. We show that $B \setminus \{a\}$ is also an extending set, which contradicts minimality of B.

To this end, consider an arbitrary $z \in A \setminus (B \setminus \{a\})$. First assume that $z = a$ and consider an arbitrary $Z \subseteq B \setminus \{a\}$ with $Z \cup \{a\} \in \mathcal{B}_T(a)$. Without loss of generality we may assume that Z is maximal, in the sense that there is no $Z' \subseteq B \setminus \{a\}$ with $Z \subsetneq Z'$ such that $Z' \cup \{a\} \in \mathcal{B}_T(a)$. As $a \notin \mathit{BA}(T)$, there is also an $x \in A$ with $Z \cup \{a, x\} \in \mathcal{B}_T(x)$. If, $x \in B \setminus \{a\}$ we are done immediately. If $x \notin B \setminus \{a\}$, by virtue of B being an extending set, there is some $b \in B$ with $Z \cup \{b, x\} \in \mathcal{B}_T(b)$. Also observe that $b \succ a$. Otherwise, $Z \cup \{a, b\} \in \mathcal{B}_T(a)$, which would contradict maximality of Z in B. It follows that $Z \cup \{a, b\} \in \mathcal{B}_T(b)$, and we are done for this case.

Now assume that $z \neq a$ and again consider an arbitrary maximal $Z \subseteq B \setminus \{a\}$ with $Z \cup \{z\} \in \mathcal{B}_T(z)$. If there is some $b \in B \setminus \{a\}$ with $Z \cup \{b, z\} \in \mathcal{B}_T(b)$, we are done. Thus, since B is an extending set, we may assume without loss of generality that $Z \cup \{a, z\} \in \mathcal{B}_T(a)$. Since $a \notin \mathit{BA}(T)$, there is some $x \in A$ such that $Z \cup \{a, x, z\} \in \mathcal{B}_T(x)$. If, $x \in B \setminus \{a\}$ we are done immediately. If $x \notin B \setminus \{a\}$, by virtue of B being an extending set, there is some $b \in B$ with $Z \cup \{a, b, x\} \in \mathcal{B}_T(b)$. Moreover, $b \succ z$. Otherwise, $Z \cup \{b, z\} \in \mathcal{B}_T(z)$, contradicting the assumed maximality of Z. Hence, $Z \cup \{b, x, a, z\} \in \mathcal{B}_T(b)$. Since obviously $b \neq a$, we may conclude that $Z \cup \{b, z\} \in \mathcal{B}_T(b)$ and also for this case we are done. □

Laslier (1997, Theorem 7.1.3) showed that *BA* is irregular by presenting a corresponding tournament on 45 alternatives. By Theorem 10.22, *ME* also has to be irregular. In Figure 10.6, we give a smallest regular tournament in which *ME* and *BA* exclude alternatives. The tournament is of order 13 and was found by exhaustively checking all tournaments of increasing order where the tournaments were generated with the help of McKay's *nauty* package (McKay and Piperno, 2013b).

Corollary 10.23
*ME* is irregular.

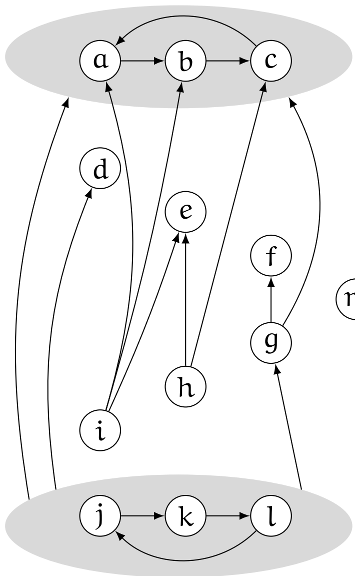



**Figure 10.6:** A regular tournament on 13 vertices. Omitted edges point downwards. Vertex m is not in *BA* and thereby not in *ME*. This is the smallest tournament where *ME* (as well as *BA* and *TEQ*!) is irregular.

An important property of every tournament solution is whether it can be computed efficiently. This is typically phrased as a decision problem which asks whether a given alternative is contained in the choice set of a given tournament. While it is known that this problem is NP-hard for *BA* (Woeginger, 2003), this has no immediate consequence on the complexity of the problem for *ME*.[67] We show that computing *ME* is indeed NP-hard.

*computational complexity of ME*

Theorem 10.24
Deciding whether an alternative in a tournament is contained in *ME* is NP-hard.

*Proof.* The proof is a reduction from 3Sat that uses the very same construction that was used to show the hardness of computing *TEQ*, even for seven voters (cf. Section 6.3).

Thus, let φ be a formula in 3-CNF

$$(x_1^1 \vee x_1^2 \vee x_1^3) \wedge \cdots \wedge (x_m^1 \vee x_m^2 \vee x_m^3)$$

67 For example, Hudry (2004) has pointed out that random members of *BA* can be found efficiently (in linear time).

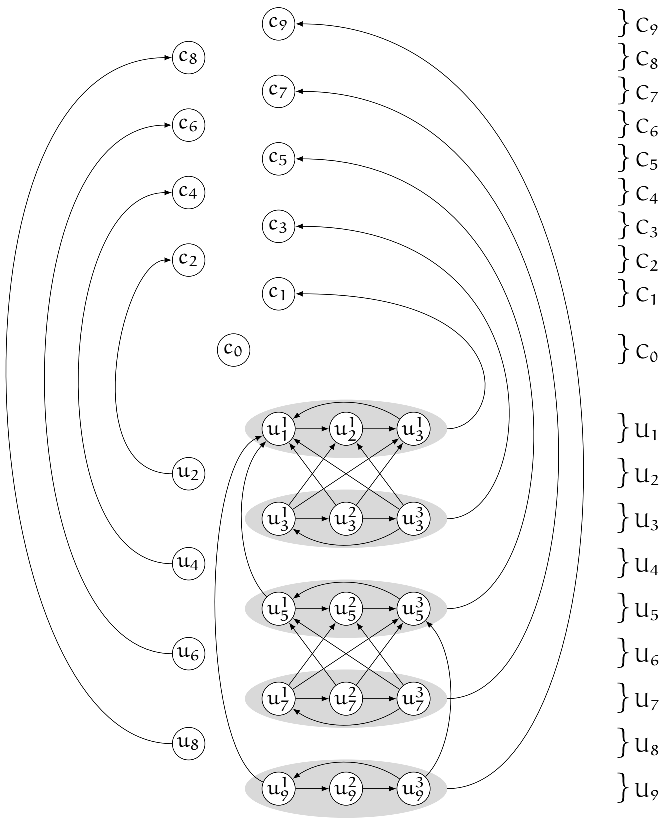



**Figure 10.7:** A tournament $T_\varphi$ as used in the proof of Theorem 10.24. Missing edges point downwards.

and consider the tournament $T_\varphi = (C \cup U, \succ)$. We prove that

$$\varphi \text{ is satisfiable} \quad \text{if and only if} \quad c_0 \in ME(T_\varphi).$$

Recall that $C = \{c_0, c_1, \ldots, c_{4m-3}\}$ and $U = \bigcup_{1 \leqslant i \leqslant 4m-3} U_i$, where

$$U_i = \begin{cases} \{u_i^1, u_i^2, u_i^3\} & \text{if } i \text{ is odd,} \\ \{u_i\} & \text{if } i \text{ is even.} \end{cases}$$

The dominance relation $\succ_\varphi$ has to satisfy

$$\begin{aligned} c_i &\succ c_j \text{ for all } i > j, \\ c_i &\succ U_j \text{ for all } i \neq j, \text{ and} \\ U_i &\succ c_i \text{ for all } i > 0. \end{aligned}$$

For our purposes in this chapter, we ignore further constraints on $\succ_U$ and refer the reader to (Brandt et al., 2010b) for details of the construction. A full example of such a tournament $T_\varphi$ is shown in Figure 10.7.

First assume that $\varphi$ is not satisfiable. Then, by a simple argument (Brandt et al., 2010b, proof of Theorem 2) we have $c_0 \notin BA(T_\varphi)$. From Theorem 10.22, it also follows that $c_0 \notin ME(T_\varphi)$.

For the opposite direction, assume that $\varphi$ is satisfiable and let $\alpha$ be the witnessing assignment.

Let

$$U^- = \bigcup_{0 \leqslant k \leqslant m} \{u^j_{4k+1} : \alpha \text{ sets } x^j_k \text{ to false}\}$$

and define

$$U^+ = U \setminus U^-,$$

and let $U^+_i = U^+ \cap U_i$. Let, furthermore, for all $i$ with $1 \leqslant i \leqslant 4m-3$,

$$U_{\geqslant i} = U_i \cup \cdots \cup U_{4m-3} \qquad \text{and} \qquad C_{\geqslant i} = C_i \cup \cdots \cup C_{4m-3}.$$

Let $U^+_i = U^+ \cap U_i$ and $U^+_{\geqslant i} = U^+ \cap U_{\geqslant i}$.

We say that a subset $V \subseteq U$ is $i$-*leveled* if $V \cap U_j \neq \emptyset$ for all $j$ with $4m-3 \geqslant j \geqslant i$. As $\alpha$ is a satisfying assignment, we then have that $U^+_i$ is 1-leveled. Moreover, for every $i$ with $4m-3 \geqslant i \geqslant 1$ and all $u \in U^+_i$, there is a transitive $i+1$-leveled set $X_u$ in $U^+_{\geqslant i+1}$ of which $u$ is a maximal element.

Let $B$ be a minimal extending set of $T_\varphi$. We show by induction on $i$ that for all $i$ with $4m-3 \geqslant i \geqslant 1$ we have

$$C_{\geqslant i} \cup U^+_{\geqslant i} \subseteq B.$$

For the basis, i.e., if $i = 4m-3$, we have to show that $c_{4m-3} \in B$ as well as that $U_{4m-3} \subseteq B$. To prove the former, assume for a contradiction that $B \cap C = \emptyset$. Then, $B \subseteq U$. Accordingly, there is some $u \in U$ with $u \in B$. Observe, however, that $c_0 \succ u$ and that there is no $u' \in U$ with $u' \succ c_0$. Therefore, there is no $u' \in B$ with both $u' \succ u$ and $u' \succ c_0$, a contradiction. We may conclude that there is some $c \in C$ with $c \in B$. If $c = c_{4m-3}$, we are done. Otherwise, $c_{4m-3} \succ c$. Observe that there is no alternative $a \in C \cup U$ with both $a \succ c_{4m-3}$ and $a \succ c$. It follows that $c_{4m-3} \in B$.

Second, we show that for each $u \in U_{4m-3}$ we have $u \in B$. Consider an arbitrary $u \in U_{4m-3}$. Without loss of generality we may assume that $u = u^1_{4m-3}$ and, for a contradiction, that $u^1_{4m-3} \notin B$. Also consider $u^2_{4m-3}$ and observe that $u^2_{4m-3} \succ c_{4m-3}$. If $u^2_{4m-3} \notin B$, then $u^1_{4m-3}$ is the *only* alternative $a$ in $C \cup U$ with both $a \succ u^2_{4m-3}$ and $a \succ c_{4m-3}$. It then follows that $u^1_{4m-3} \in B$. If, on the other hand, $u^2_{4m-3} \in B$, observe that both $u^1_{4m-3} \succ u^2_{4m-3}$ and $u^1_{4m-3} \succ c_{4m-3}$. Moreover, there is no alternative $a \in C \cup U$ with $a \succ \{u^1_{4m-3}, u^2_{4m-3}, c_{4m-3}\}$. Again, it follows that $u^1_{4m-3} \in B$.

For the induction step, we may assume that

$$C_{\geqslant i} \cup U^+_{\geqslant i} \subseteq B$$

and we show that

$$C_{\geqslant i-1} \cup U^+_{\geqslant i-1} \subseteq B.$$

First consider $c_{i-1}$ and assume for a contradiction that $B \cap (C \setminus C^+_{\geqslant i}) = \emptyset$. Recall that $U^+_{\geqslant i} \cap U_i \neq \emptyset$ and let $u \in U^+_{\geqslant i} \cap U_i$. By the induction hypothesis we have that $\{u\} \cup X_u \subseteq B$. Then, $c_0$ is the maximal element of the transitive set $\{c_0, u\} \cup X_u$. Observe that there is no $u' \in U$ that is the maximal element of $\{u', c_0, u\} \cup X_u$ because $c_0 \succ u'$. As $\{u\} \cup X_u$ is $i$-leveled, neither is there a $c \in C_{\geqslant i}$ such that $c$ is the maximal element of $\{c, c_0, u\} \cup X_u$. A contradiction follows. Accordingly, there is some $c \in B \cap (C \setminus C^+_{\geqslant i})$. If $c = c_{i-1}$ we are done. Otherwise, $c_{i-1} \succ c$. Observe that there is no alternative $a \in C \cup U$ with both $a \succ c_{i-1}$ and $a \succ c$. It follows that $c_{i-1} \in B$.

Now consider an arbitrary $u_{i-1} \in U^+_{\geqslant i-1}$. There is a $k$ with $0 \leqslant k \leqslant m$ such that either

(i) $U_{i-1} = U_{2k}$,

(ii) $U_{i-1} = U_{4k+1}$, or

(iii) $U_{i-1} = U_{4k+3}$.

If (i), observe that $u_{i-1}$ is the maximal element of the transitive set $\{u_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$. As $X_{u_{i-1}} \subseteq U^+_{\geqslant i}$, by the induction hypothesis and the previous argument, we may assume that $\{c_{i-1}\} \cup X_{u_{i-1}} \subseteq B$. Observe, however, that in this case, there is no alternative $a \in C \cup U$ distinct from $u_{i-1}$ that is the maximal element of $\{a, u_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$.

If (ii), $U_{i-1} = \{u^1_{i-1}, u^2_{i-1}, u^3_{i-1}\}$. Without loss of generality we may assume that $u_{i-1} = u^1_{i-1}$ and observe that $u_{i-1}$ is the maximal element of the transitive set $\{u_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$. By the induction hypothesis and the first part of the induction step, $\{c_{i-1}\} \cup X_{u_{i-1}} \subseteq B$. As $X_{u_{i-1}}$ is $i$-leveled there is some $u'' \in X_{u_{i-1}} \cap U_{i+1}$. By construction, moreover, $u'' = u^1_{i+1}$. Again, there is no alternative $a \in C \cup U$ distinct from $u_{i-1}$ that is the maximal element of $\{a, u_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$. In particular, it is not the case that $u^3_{i-1}$ is the maximal element of $\{a, u_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$. To see this, recall that $u^1_{i+1} \in X_{i+1}$. Then observe that, by construction of $T_\varphi$, also $u^1_{i+1} \succ u^3_{i-1}$.

Finally, if (iii), again $U_{i-1} = \{u^1_{i-1}, u^2_{i-1}, u^3_{i-1}\}$. Without loss of generality, we may assume that $u_{i-1} = u^1_{i-1}$. Observe that $u^1_{i-1}$ is the maximal element of the transitive set $\{u^1_{i-1}, c_{i-1}\} \cup X_{u_{i-1}}$. Moreover, by the induction hypothesis and the first part of the induction step, $\{c_{i-1}\} \cup X_{u_{i-1}} \subseteq B$. For contradiction assume that $u^1_{i-1} \notin B$. Also consider $u^2_{i-1}$. Then, also $u^2_{i-1} \succ c^1_{i-1}$. If $u^2_{i-1} \notin B$, then $u^1_{i-1}$ is the *only* alternative $a$ in $C \cup U$ with both $a \succ u^2_{i-1}$ and $a \succ c_{i-1}$. It then follows that $u^1_{i-1} \in B$. If, on the other hand, $u^2_{i-1} \in B$, then both $u^1_{i-1} \succ u^2_{i-1}$ and $u^1_{i-1} \succ c_{i-1}$. Moreover, there is no alternative $a \in C \cup U$ with $a \succ u^1_{i-1}$, $a \succ u^2_{i-1}$, and $a \succ c_{i-1}$. Again, it follows that $u^1_{i-1} \in B$.

To conclude the proof, let $u \in U^+_1$ and consider also $X_u$. Observe that $c_0$ is the maximal element of the transitive set $\{c_0, u\} \cup X_u$. As we have seen above, $\{c_0, u\} \cup X_u \subseteq B$. Also observe that there is no alternative $a \in C \cup U$ distinct from $c_0$ that is the maximal element of the set $\{a, c_0, u\} \cup X_u$. It follows that $c_0 \in B$, as desired. □

### 10.3.3 Discussion on ME

We have analyzed the axiomatic as well as computational properties of the tournament solution *ME*. Results were mixed. In conclusion, *ME*

1. is not monotonic,
2. is not independent of unchosen alternatives,
3. satisfies $\widehat{\alpha}_{\supseteq}$ and idempotency,
4. does not satisfy $\widehat{\alpha}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ and is not stable,
5. satisfies irregularity,
6. is contained in the Banks set,
7. is NP-hard to compute, and
8. satisfies composition-consistency.

The statement 8 was shown in Brandt (2011b), the others are new. Two relationships of *ME* with other tournament solutions are still open. It is unknown whether the tournament equilibrium set is always contained in *ME* and whether *ME* is always contained in the minimal covering set. These results, together with the recent findings mentioned in Section 10.2 regarding *TEQ*—which satisfies irregularity but also fails stability—, suggest that stability and irregularity may be incompatible in general. We intend to further pursue this question in future work.

We observed that many of the properties that are violated by *ME* are nevertheless satisfied by *individual* minimal extending sets (also see Table 10.2). It is an interesting issue whether extending sets could perhaps still be used as the basis for choice in tournaments. Selecting *one* of the extending sets in a way such that the axioms considered in here are still satisfied appears to be problematic.

## 10.4 THE CASE OF BP

In this last section, we give an outlook on ongoing efforts to characterize the bipartisan set as the finest tournament solution satisfying properties such as stability, *LRS*, regularity, and the like.

A helpful tool to reason about the *LRS* properties is the *LRS diagram*. It is based on the observation that every alternative $a_{T^1}$ in a

*LRS diagram*

| property | *ME* | minimal extending sets |
|---|---|---|
| set-monotonicity | ✗ | ✓ |
| independence of unchosen alternatives | ✗ | ✓ |
| stability ($\widehat{\alpha}$ and $\widehat{\gamma}$) | ✗ | ✓ |

**Table 10.2:** Comparison of properties of *ME* and minimal extending sets as tournament correspondences.

tournament $T^1$ is mapped to an alternative $b_{T^2}$ in a tournament $T^2$ of the same size as $T^1$ by a local reversal at $a_{T^1}$, and *vice versa*. More precisely, since this operation respects tournament isomorphisms, it is a bijection on orbits and partitions the set

$$\mathcal{A}_m = \{o \in \mathcal{O}_T : T \in \mathcal{T}_m\}$$

*local reversal pairs*

into pairs of orbits $(o^1_{T^1}, o^2_{T^2})$ that we call *local reversal pairs*. The pairs of $\mathcal{A}_4$ are depicted in Figure 10.8.

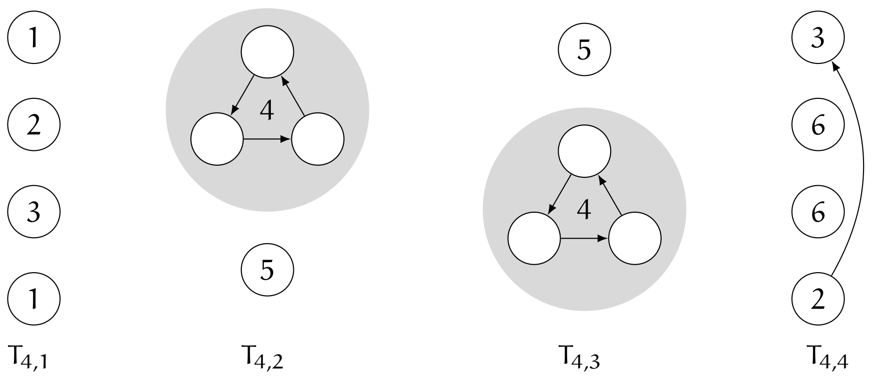



**Figure 10.8:** A visualization of the partitioning of the orbits in $\mathcal{T}_4$ into local reversal pairs. Omitted edges within each of the four tournaments point downwards.

The benefit of this is approach is that the *LRS* properties can be formulated in terms of choice from local reversal pairs.

Proposition 10.25
Every tournament solution $S$ satisfies $LRS_{IN}$ ($LRS_{OUT}$) if and only if it chooses at least (at most) one orbit from every local reversal pair.

Consequently, *LRS* is satisfied precisely by those tournament solutions which select *exactly* one orbit from every pair. Disregarding orbit sizes—which are rarely larger than one according to Gao et al. (2000)—this gives that a tournament solution satisfying *LRS* on average has to select half of the alternatives in the set of all tournaments, which is almost exactly the statement which Fisher and Reeves (1995) proved for *BP*.

Another consequence of Proposition 10.25 is that the local reversal operation cannot map an alternative $a$ in a tournament $T$ to itself and we do actually get reversal *pairs*. Assume for contradiction that such an $a$ exist. Then there could be no tournament solution satisfying *LRS* because no matter whether it selects $a$ or not, this entails a violation of *LRS*. But since *BP does* satify *LRS*, this can never happen.

The most interesting effect of Proposition 10.25 is that every tournament solution $S$ satisfying *LRS* is a finest *LRS*-satisfying tournament solution. This is because for another tournament solution $S'$ to be finer than $S$, it needs to select strictly less orbits from at least one local reversal pair than $S$. But since both satisfy *LRS*, they both always choose exactly one of the two orbits. This is remarkable since typical characterizations of tournament solutions identify a concept as the *finest* satisfying several properties.[68] In case of *LRS*, inclusion-minimality is inherent already.

## 10.5 SUMMARY AND DISCUSSION

We have reviewed several conjectures related to the tournament solutions *TEQ* and *ME*. As a consequence of the found counterexamples to these conjectures, *TEQ* and *ME* fail to satisfy most of the usually considered desirable properties. However, the practical relevance of these findings remains unclear. While we successfully lowered the size of the smallest counterexample to the *TEQ*-related conjecture to 24, this achievement required new insight and significant computational efforts. Everything indicates that these counterexamples are extremely rare and even more so for *ME*. Here, still no actual counterexample is known and the upper bound on the number of alternatives from when on *ME* fails to be an axiomatically very appealing tournament solution currently stands at roughly $10^{103}$. In effect, *ME does* satisfy these properties in all scenarios in which tournaments only admit a unique minimal extending set and it is fair to say that *ME* satisfies the considered properties for all practical purposes. This, in turn, may be interpreted as a questioning the axiomatic method in general: For what does it mean if a tournament solution (or any other mathematical object) in principle violates some desirable properties, but no concrete example of a violation is known and will perhaps ever be known?

*questioning the axiomatic method*

68 See the characterizations of *TC* (Bordes, 1976), *UC* (Moulin, 1986), *MC* (Dutta, 1988), and *BA* (Brandt, 2011b).

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