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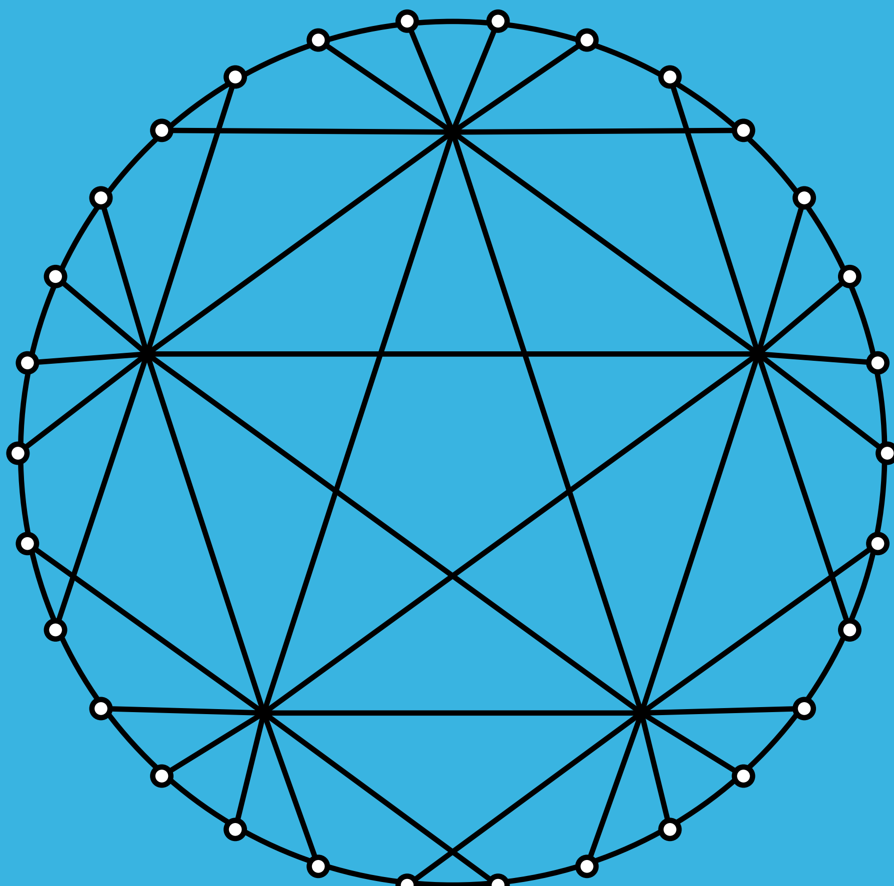

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Regular d-handicap graphs of even order

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Abstract: Let $G = (V, E)$ be a simple, undirected graph of order n. Let ℓ be a bijection from V to the natural numbers, and for all $i \in \{1, 2, ..., n\}$, define the weight of the vertex x_i as $w(x_i) = \sum_{x_i x_j \in E} \ell(x_j)$. If ℓ has the property that $\ell(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), ..., w(x_n)$ is d-arithmetic for some $d \geq 1$, then we say that G is a d-handicap graph. The motivation for studying these graphs stems from tournament design. A k-regular, d-handicap graph on n vertices corresponds to a tournament on n teams, each ranked according to strength with the natural numbers, in which each team plays exactly k opponents, and the strength of schedule (the sum of opponent rankings) increases d-arithmetically with strength of team. In this paper, we provide necessary and sufficient conditions for the existence of regular d-handicap graphs for all orders $n \equiv 0 \pmod{2^{d+2}}$. In addition, we construct a regular d-handicap graph of lowest possible regularity for all orders $n \equiv 0 \pmod{2^d}$ with a small number of exceptions.

1 Motivation

Consider a tournament of n teams, each ranked with one of the first n natural numbers (with no repeats) according to strength. Let $w(i)$ be the sum of the opponents of the team ranked i. We call w the strength of schedule. If all possible games are played, we say the tournament is a round-robin or complete tournament. In a round-robin tournament, the sequence $s = w(1), w(2), ..., w(n)$ is arithmetic with constant difference -1. This

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means the weakest team has the most difficult schedule, the second weakest team has the second most difficult schedule, and so on. In particular, the strongest team has the weakest schedule.

If one wishes to design a more competitive tournament, obviously only incomplete tournaments (tournaments in which not all possible games are played) must be considered. If the sequence s is constant, we call the tournament an equalized incomplete tournament. If the sequence s is d-arithmetic for some $d \geq 1$, then we say the tournament is a d-handicap tournament. In this kind of tournament, the weakest team has the weakest schedule, the second team has the second weakest schedule, and so on. In particular, the strongest team has the most difficult schedule.

We model the problem with a graph in the natural way; team x plays team y in the tournament if and only if x is adjacent to y in the graph. Let $G = (V, E)$ be a simple, undirected graph of order n. Let ℓ be a bijection from V to $\{1, 2, ..., n\}$ and define the weight of the vertex x_i as $w(x_i) = \sum_{x_j \in N(x_i)} \ell(x_j)$, where $N(x) = \{v | xv \in E\}$ is the open neighborhood of the vertex x. If ℓ has the property that $\ell(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), ..., w(x_n)$ is d-arithmetic for some $d \geq 1$, then we say that ℓ is a d-handicap labeling. A graph G which admits such a labeling is called a d-handicap graph. d-Handicap graphs are members of a family of graphs called distance antimagic graphs.

Let $H(n, k, d)$ denote a k-regular, d-handicap graph on n vertices. The following necessary conditions for the existence of an $H(n, k, d)$ were established in the author's Ph.D. thesis [1].

Theorem 1. [1] *If an $H(n, k, d)$ exists, then all of the following are true.*

1. $w(x_i) = di + \frac{(k-d)(n+1)}{2}$, *for all* $i \in \{1, 2, ..., n\}$.
2. *If n is even, then* $k \equiv d \pmod 2$.
3. *If n is odd, then* $k \equiv 0 \pmod 2$.
4. $n \geq 4d + 4$.
5. $d + 2 \leq k \leq n - d - 4$.

2 Known results

Much attention has been paid to 1-handicap graphs. The first of the following two theorems completely settles the spectrum for regular 1-handicap graphs on an even number of vertices [5]. Theorem 3 provides the existence of an $H(n,k,1)$ for every feasible odd number n and some k [4].

Theorem 2. [5] *An $H(n,k,1)$ exists when $n \geq 8$ and*

i. $n \equiv 0 \pmod 4$ *if and only if* $3 \leq k \leq n-5$ *and k is odd*

ii. $n \equiv 2 \pmod 4$ *if and only if* $3 \leq k \leq n-7$ *and* $k \equiv 3 \pmod 4$,

except when $k = 3$ and $n \in \{10, 12, 14, 18, 22, 26\}$.

Theorem 3. [4] *Let n be an odd positive integer. Then an $H(n,k,1)$ exists for at least one value of k if and only if $n = 9$ or $n \geq 13$.*

Froncek was the first to consider d-handicap graphs for $d \neq 1$. Theorem 4 completely settles the existence of regular 2-handicap graphs for $n \equiv 0 \pmod{16}$ and was proved independently in [1] and [2]. Theorem 5 partially settles the existence for $n \equiv 8 \pmod{16}$ [3].

Theorem 4. [1], [2] *If $n \equiv 0 \,(\mathrm{mod}\, 16)$, then an $H(n,k,2)$ exists if and only if k is even and $4 \leq k \leq n-6$.*

Theorem 5. [3] *If $n \equiv 8 \,(\mathrm{mod}\, 16)$ and $n \geq 56$, then an $H(n,k,2)$ exists if k is even and $6 \leq k \leq n-50$.*

The author gave constructions for d-handicap graphs for all d in [1]. The following two theorems give a partial summary of the results therein. Observe that the necessary conditions for k, the regularity of the d-handicap graph, are only met when $d = 1$ (see Theorem 6) or $d = 2$ (see Theorem 7).

Theorem 6. [1] *For every odd d, there exists an $H(n,k,d)$ for every odd k such that $2d+1 \leq k \leq n-(2d+3)$ provided*

- $n \equiv 0 \,(\mathrm{mod}\, 4d+4)$, $n \geq (d+1)(d+3)$, *and* $d \equiv 1 \,(\mathrm{mod}\, 4)$ *or*
- $n \equiv 0 \,(\mathrm{mod}\, 4d+4)$, $n \geq (d+1)(d+5)$, *and* $d \equiv 3 \,(\mathrm{mod}\, 4)$ *or*
- $n \equiv 2d+2 \,(\mathrm{mod}\, 4d+4)$, $n \geq (d+1)(d+3)$, *and* $d \equiv 3 \,(\mathrm{mod}\, 4)$.

Theorem 7. [1] *Let $d \geq 2$ and $t, v \geq d+2$ all be even integers and let $n = vt$. If $d \equiv 0 \pmod 4$ or $v \equiv t \equiv 0 \pmod 4$, then there exists an $H(n, k, d)$ for all even k such that $2d \leq k \leq n - 2d - 2$.*

3 New results

In this section, we provide constructions for building d-handicap graphs from $(d-1)$-handicap graphs. We may refer to a vertex by its label to simplify notation. For any subgraph H of the graph G, let $w_H(x)$ denote the weight of vertex x induced by the subgraph H. For any graph G, let $\overline{G}$ denote its complement.

Theorem 8. *If an $H(n, d+2, d)$ exists, then an $H(2n, d+3, d+1)$ exists.*

Proof. Let H be an $H(n, d+2, d)$ with a d-handicap labeling f. Let $G = 2H$ be the union of two vertex-disjoint copies of H. Apply f to one of the copies of H and $f + n$ to the other copy of H. Define a 1-factor $F = \{i(i+n) | i = 1, 2, ..., n\}$. We claim that the graph $G \cup F$ is an $H(2n, d+3, d+1)$. Clearly $G \cup F$ has the correct order and regularity, so it suffices to show that the sequence of weights, $s = w(1), w(2), ..., w(2n)$ is $(d+1)$-arithmetic. For $i \in \{1, 2, ..., n\}$, $w_G(i) = di + n + 1$ and $w_G(i+n) = di + n + 1 + (d+2)n$ by Theorem 1. Also, $w_F(i) = i + n$ and $w_F(i+n) = i$. Therefore,

$$\begin{aligned} w(i) &= w_G(i) + w_F(i) \\ &= (d+1)i + 2n + 1, \end{aligned}$$

for all $i \in \{1, 2, ..., 2n\}$. Hence, the sequence s is $(d+1)$-arithmetic, which proves the result. □

Starting with the 1-handicap graphs from Theorem 2, repeated application of Theorem 8 yields d-handicap graphs of even orders and lowest possible regularity.

Theorem 9. *Let $m \geq 4$ and $m \notin \{5, 6, 7, 9, 11, 13\}$. Then an $H(2^d m, d+2, d)$ exists for every $d \geq 1$.*

Proof. The proof follows from Theorems 2 and 8. □

Theorem 9 proves the existence of new classes of 2-handicap graphs. In particular, a 4-regular, 2-handicap graph on n vertices exists for all $n \equiv 0 \pmod 4$ and $n \geq 68$.

Corollary 10. *An $H(n, 4, 2)$ exists if*

i. $n \equiv 0 \pmod{16}$ *and* $n \geq 16$ *or*

ii. $n \equiv 4 \pmod{16}$ *and* $n \geq 68$ *or*

iii. $n \equiv 8 \pmod{16}$ *and* $n \geq 40$ *or*

iv. $n \equiv 12 \pmod{16}$ *and* $n \geq 60$.

Suppose we wish to add teams to a d-handicap tournament without changing d and without changing the number of games each team plays. The next theorem shows this can be accomplished by adding 2^{d+2} teams.

Theorem 11. *If there exists an $H(n, d+2, d)$ and $d \geq 2$, then there exists an $H(n + 2^{d+2}, d + 2, d)$.*

Proof. Let $d \geq 2$ be given. The proof is by construction. Let G be any $H(n, d+2, d)$ with 2^{d+1} added to the label of each vertex. Let $H = 2H'$ be two vertex-disjoint copies of an $H(2^{d+1}, d+1, d-1)$ (such a graph exists by Theorem 9). Leave the $(d-1)$-handicap labeling of one of the copies of H' unchanged and add $n + 2^{d+1}$ to the label of each vertex of the other copy of H'. Define a 1-factor $F = \{i(i + n + 2^{d+1}) | i = 1, 2, ..., 2^{d+1}\}$ spanning the two copies of H'. We claim that $G \cup (H \cup F)$ is an $H(n + 2^{d+2}, d + 2, d)$. We use Theorem 1 and the regularity of G and H to determine the weight of each vertex in $G \cup (H \cup F)$. For $i \in V(G)$, we have

$$\begin{aligned} w(i) &= d(i - 2^{d+1}) + n + 1 + (d+2)2^{d+1} \\ &= di + 2^{d+2} + n + 1. \end{aligned}$$

If $i \in V(H \cup F)$, then

$$\begin{aligned} w(i) &= (d-1)i + 2^{d+1} + 1 + n + 2^{d+1} + i \\ &= di + 2^{d+2} + n + 1, \end{aligned}$$

for $i = 1, 2, ..., 2^{d+1}$, and

$$\begin{aligned} w(i) &= (d-1)(i - n - 2^{d+1}) + 2^{d+1} + 1 \\ &\quad +(d+1)(n + 2^{d+1}) + i - (n + 2^{d+1}) \\ &= di + 2^{d+2} + n + 1, \end{aligned}$$

for $i = n + 1 + 2^{d+1}, n + 2 + 2^{d+1}, ..., n + 2^{d+2}$. Hence, $G \cup (H \cup F)$ is an $H(n + 2^{d+2}, d + 2, d)$. $\square$

Theorem 11 does not produce any new results over Theorem 9. However, applying Theorem 11 to a d-handicap graph (with lowest possible regularity) of an order not covered by Theorem 9 will produce new results. For example, if an $H(m, 4, 2)$ can be found for some $m \equiv 2 \pmod{16}$, then Theorem 11 gives the existence of an $H(n, 4, 2)$ for every $n \equiv 2 \pmod{16}$ and $n \geq m$, which would be a new class of 2-handicap graphs.

Now that we have established classes of d-handicap graphs of lowest feasible regularity for every $d \geq 1$, our next direction will be to increase the regularity of these graphs. We will accomplish this by adding constant weight 2-factors to the graph. Before doing so, we recall a graph product and make some observations regarding the 1-factorability of our graphs. A 1-factor of a graph G is a vertex-disjoint union of edges which partition the edge set of G. If the edge set of G can be partitioned into 1-factors, we say the graph G is 1-factorable (or allows a 1-factorization).

For any two graphs G and H with vertex sets $V(G)$ and $V(H)$, respectively, the lexicographic product $G \circ H$ has vertex set $V(G) \times V(H)$, and two vertices (g, h) and (g', h') are adjacent in $G \circ H$ if and only if either g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H.

Informally speaking, the graph $G \circ H$ may be constructed by replacing each vertex of G with an isomorphic copy of H, and then placing a complete bipartite graph between two copies of H for each corresponding edge in G.

Theorem 12. [6] *Let G and H be regular graphs at least one of which is 1-factorable. Then the lexicographic product $G \circ H$ is 1-factorable.*

Observation 13. *There exists a graph $G \cong H(2^{d+2}, d+2, d)$ for all $d \geq 1$ such that $\overline{G}$ allows a 1-factorization.*

Proof. If $d = 1$, let G be the graph in Figure 1. One can easily check that $G \cong H(8, 3, 1)$.

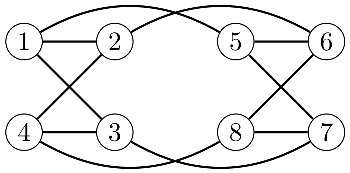


Figure 1: The graph $G \cong H(8, 3, 1)$.

Let $L = \{1,2,3,4\} \subseteq V$ and $U = \{5,6,7,8\} \subseteq V$. The edges in $\overline{G}$ of the form xy where $x, y \in L$ or $x, y \in U$ clearly form a 1-factor. The remaining edges in $\overline{G}$ form a regular bipartite graph with partite sets L and U. Since it is well known that every regular bipartite graph allows a 1-factorization, $\overline{G}$ allows a 1-factorization. Repeated application of the construction from Theorem 8 now gives the result. □

Observation 14. *Let $d \geq 2$ and $G \cong 2H \cup F \cong H(2^{d+2}, d+2, d)$ be a $(d+2)$-regular, d-handicap graph as constructed in Observation 13. Let I be any 1-factor which spans the two disjoint copies of H. Then $\overline{G \cup I}$ allows a 1-factorization.*

Proof. By Observation 13, each copy of $\overline{H}$ allows a 1-factorization. The remaining edges in $\overline{G \cup I}$ have one vertex in each copy of H, so these edges form a bipartite graph. Hence, $\overline{G \cup I}$ allows a 1-factorization. □

We are ready to prove the main theorem.

Theorem 15. *Let $d \geq 1$ be given and let $n \equiv 0 \pmod{2^{d+2}}$. Then an $H(n, k, d)$ exists if and only if $d + 2 \leq k \leq n - d - 4$ and $k \equiv d \pmod 2$.*

Proof. The proof is by construction and induction on d. Let $n = 2^{d+2}c$. If $d = 1$ the claim is true by Theorem 2, so we can assume $d \geq 2$. Let $H_1 \cong H(2^{d+1}, d+1, d-1)$ as constructed in Observation 13. Write $H_1 = H' \cup H'' \cup F'$, where $H' \cong H''$ and F' is the 1-factor which spans H' and H''. Define $H = H_1 \cup H_2 \cup ... \cup H_{2c}$, a mutually disjoint union of isomorphic copies of H_1 with $(j-1)2^{d+1}$ added to the label of each vertex of every H_j. Define a 1-factor $F = \{i(i + (2c - 2j + 1)2^{d+1}) | i \in H_j\}$ spanning each pair (H_j, H_{2c+1-j}) for $j = 1, 2, ..., c$.

We claim the graph $H \cup F$ is an $H(n, d+2, d)$. The order and regularity of $H \cup F$ match the parameters, so we need only show the sequence of weights is d-arithmetic. If $i \in H_j$, then

$$\begin{aligned} w(i) &= (d-1)(i - (j-1)2^{d+1}) + 2^{d+1} + 1 + (j-1)(d+1)2^{d+1} \\ &\quad + \begin{cases} i + (2c - 2j + 1)2^{d+1}, & j \in \{1, 2, ..., c\} \\ i - (2j - 2c + 1)2^{d+1}, & j \in \{c+1, c+2, ..., 2c\} \end{cases} \\ &= di + n + 1, \end{aligned}$$

which shows $H \cup F$ is an $H(n, d+2, d)$. Now that we have constructed the graph of lowest regularity, we will increase the regularity of the graph by adding constant weight 2-factors until the highest regularity is achieved.

Let J_v be the empty graph on v vertices. Notice that for each $j \in \{1, 2, ..., c\}$, the graph $H_j \cup H_{2c+1-j}$ is a subgraph of the lexicographic product $H_1 \circ J_2$, with the additional property that each pair of isolated vertices in each J_2 sum to $n+1$. That is, H is a subgraph of $c(H_1 \circ J_2) \cup F$, (where $c(H_1 \circ J_2)$ is the union of c vertex-disjoint copies of $H_1 \circ J_2$) and each 1-factor of $\overline{c(H_1 \cup I)}$, (where I is the 1-factor spanning H' and H'' corresponding to F) gives rise to a constant weight 2-factor F_1 which can be added to $H \cup F$, forming the graph $H \cup F \cup F_1 \cong H(n, d+4, d)$. Therefore, we have reduced the problem to finding 1-factors of $\overline{c(H_1 \cup I)}$.

Each copy of $\overline{(H_1 \cup I)}$ allows a 1-factorization by Observation 14. If $c = 1$, we are done. If $c \geq 2$, then the remaining edges in $\overline{c(H_1 \cup I)}$ form the lexicographic product $K_c \circ J_{2^{d+1}}$, so these edges allow a 1-factorization by Theorem 12. Hence, $\overline{c(H_1 \cup I)}$ allows a 1-factorization, and we can write $\overline{c(H_1 \cup I)} = F_1 \cup F_2 \cup ... \cup F_{c2^{d+1}-d-3}$, where each F_i is a 1-factor. We have proven the existence of an $H(n, k, d)$ for every $k = d + 2 + 2t$ for $t = 1, 2, ..., c2^{d+1} - d - 3$, or equivalently, for every $d + 2 \leq k \leq n - d - 4$ and $k \equiv d \pmod 2$. The fact that these are the only feasible values of k follows from Theorem 1. $\square$

Example 16. *An* $H(32, 6, 2)$.

The graph $H \cup F$ shown in Figure 2 is an $H(32, 4, 2)$. The arched vertical edges comprise the 1-factor, F. The connected component on the left contains H_1 and H_4, while the connected component on the right contains H_2 and H_3. Figure 3 shows how a 1-factor of $\overline{2(H_1 \cup I)}$ may be used to add 2 to the regularity of the tournament to obtain an $H(32, 6, 2)$. Each dashed edge in Figure 3b represents a constant weight $K_{2,2}$ shown in Figure 3c to be added to the tournament.

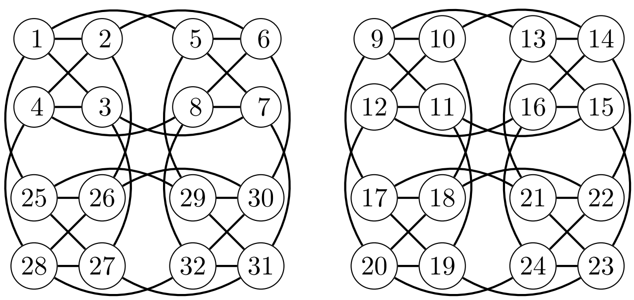
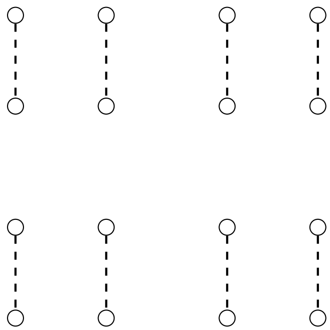
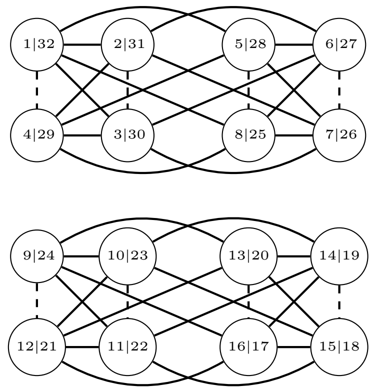


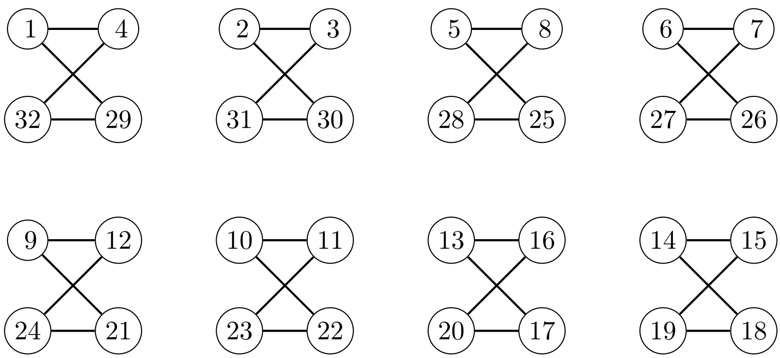
Figure 2: The graph $H \cup F \cong H(32, 4, 2)$.



(a) A 1-factor of $\overline{2(H_1 \cup I)}$.



(b) The 1-factor added to $2(H_1 \cup I)$.



(c) Games added to the tournament.

Figure 3: Adding two games to each team's schedule.

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