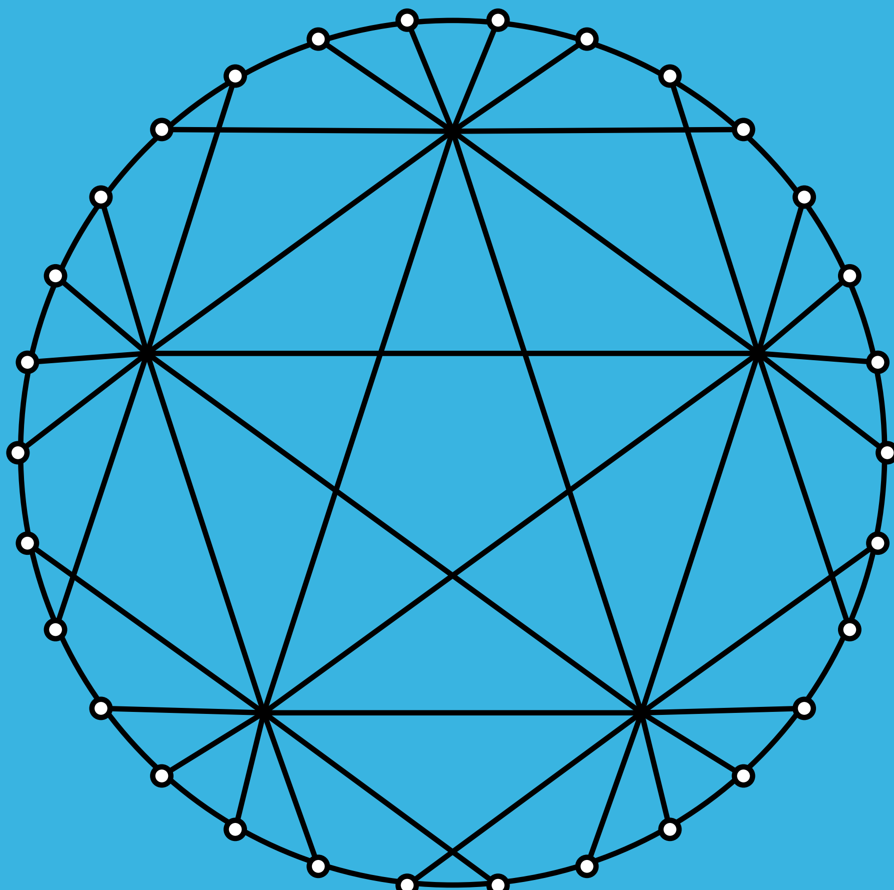
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 83
June 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **83** (2018), Pages 61–76

Schwenk graphs of cages

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In Honor of Professor Allen Schwenk on the Occasion of his Retirement from Western Michigan University

Abstract: The girth of a graph G (with cycles) is the length of a smallest cycle of G and is denoted by $g(G)$. For a connected graph G having girth $2k+1 \geq 5$ for some integer $k \geq 2$, the Schwenk graph G^* of G has the set of all paths of order $k+1$ of G as its vertex set $V(G^*)$, where two vertices P and Q of G^* are adjacent in G^* if $P = (u_1, u_2, \ldots, u_{k+1})$ and $Q = (v_1, v_2, \ldots, v_{k+1})$ such that $u_{k+1} = v_1$, $V(P) \cap V(Q) = \{u_{k+1}\}$ and $u_1 v_{k+1} \in E(G)$. It is shown that the Schwenk graph is triangle-free and for each odd integer $g \geq 5$, there exists a connected graph of girth g whose Schwenk graph contains 4-cycles. Connected graphs of girth 5 whose Schwenk graph contains 4-cycles are characterized. Structural properties of the Schwenk graphs of the unique 5-cage (the Petersen graph) and the unique 7-cage (the McGee graph) are studied. Other results and open questions are presented for the Schwenk graphs of cages.

Key Words: girth, cage, Schwenk graph.

AMS Subject Classification: 05C38, 05C45, 05C60.

Received: 7 November 2017
Accepted: 13 May 2018

1 Introduction

The *girth* of a graph G (with cycles) is the length of a smallest cycle of G and is denoted by $g(G)$. For each pair r, g of integers with $r \geq 2$ and $g \geq 3$, there exists a graph of minimum order that is both r-regular and has girth g (see [2]). Such a graph is called an (r, g)-*cage* or simply a *cage*. The $(3, g)$-cages have been the studied the most and are often referred to as g-*cages*. For $r = 2$ and $g = 5$, the 5-cycle C_5 is the unique $(2, 5)$-cage; while for $r = 3$ and $g = 5$, the Petersen graph P is the unique 5-cage. Furthermore, for $r = 3$ and $g = 7$, the McGee graph is the unique 7-cage (see [4]).

Although (r, g)-cages exist for each pair r, g of integers with $r \geq 2$ and $g \geq 3$, they are not always unique. While for $r = 3$, there is a unique g-cage for $4 \leq g \leq 8$, there are 18 different 9-cages, each of order 58, and there are three different 10-cages, each of order 70. The *Cage Problem* is one of the well-known classical problems in Graph Theory. The goal is to find the minimum order of those graphs having a prescribed girth and degree of regularity and to find all such graphs satisfying these conditions. The study of this problem was initiated by Tutte [6] in 1947. A related problem of determining the minimum order of an r-regular Hamiltonian graph of girth g for given integers r and g was described by Kárteszi [3] in 1960. We refer to the book [1] for graph theory notation and terminology not described in this paper.

In 2015 Schwenk [5] introduced a new class of derived graphs when he was investigating problems involving cages, and graphs in general having odd girth 5 or more. For a connected graph G having girth $2k + 1 \geq 5$ for some integer $k \geq 2$, the *Schwenk graph* G^* of G has the set of all paths of order $k + 1$ (or $(k + 1)$-paths) of G as its vertex set $V(G^*)$, where two vertices P and Q of G^* are adjacent in G^* if $P = (u_1, u_2, \ldots, u_{k+1})$ and $Q = (v_1, v_2, \ldots, v_{k+1})$ such that $u_{k+1} = v_1$, $V(P) \cap V(Q) = \{u_{k+1}\}$ and $u_1 v_{k+1} \in E(G)$. Since the girth of G is $2k + 1$, it follows that the subgraph of G induced by $V(P) \cup V(Q)$ is $G[V(P) \cup V(Q)] \cong C_{2k+1}$. For the special case where G is a connected graph of girth 5, the Schwenk graph G^* of G is defined as that graph whose vertex set is the set of all 3-paths (paths of order 3) of G, where two vertices P and Q of G^* (two 3-paths P and Q of G) are adjacent in G^* if they have an end-vertex in common but no other vertex in common and the subgraph of G induced by $V(P) \cup V(Q)$ is a 5-cycle. To illustrate this concept, we show that $C_g^* \cong C_g$ for every odd integer $g \geq 5$.

Proposition 1.1. *If G is a cycle of odd order $g \geq 5$, then $G^* \cong C_g$.*

Proof. Let $G = C_g = (v_1, v_2, \ldots, v_g, v_1)$ and let $g = 2k+1$ for some integer $k \geq 2$. A $(k+1)$-path P_{k+1} in G is $(v_i, v_{i+1}, \ldots, v_{k+i})$ for some integer i with $1 \leq i \leq g$, where the subscript of each vertex is expressed as an integer $1, 2, \ldots, g$ modulo g. Thus, the vertex set of G^* is $V(G^*) = \{x_i = (v_i, v_{i+1}, \ldots, v_{k+i}) : 1 \leq i \leq g\}$ and so the order of G^* is g. For each integer $i \in \{1, 2, \ldots, g\}$, the $(k+1)$-paths x_i and x_{i+k} of G have exactly one vertex in common, namely v_{i+k}, and it is an end-vertex of both x_i and x_{i+k} (where x_i is a $v_i - v_{i+k}$ path and x_{i+k} is a $v_{i+k} - v_{i+2k}$ path). Since $G = C_{2k+1}$, it follows that $v_i v_{i+2k} \in E(G)$. Thus, x_i adjacent to x_{i+k} in G^*. Similarly, x_i is adjacent to x_{i-k} in G^*. If $j \neq i \pm k$, then x_i and x_j have at least two vertices in common and so x_i is not adjacent to x_j in G^*. Hence, G^* is a 2-regular graph. Furthermore, $G^* = (x_1, x_{k+1}, x_{2k+1}, x_{3k+1}, \ldots, x_{gk+1} = x_1)$ and so $G^* \cong C_g$. □

By Proposition 1.1, for every connected graph G of odd girth $g \geq 5$, the Schwenk graph G^* of G must contain a g-cycle. However, for no such integer g can G^* contain a triangle.

Proposition 1.2. *If G is a connected graph of odd girth $g \geq 5$, then G^* is triangle-free and so the girth of G^* is at least 4.*

Proof. Let $g = 2k+1$ for some integer $k \geq 2$. Assume, to the contrary, that G^* contains a triangle (a, b, c, a). Let $a = (a_1, a_2, \ldots, a_{k+1})$, $b = (b_1, b_2, \ldots, b_{k+1})$ and $c = (c_1, c_2, \ldots, c_{k+1})$. Since $ab \in E(G^*)$, we may assume that $a_{k+1} = b_1$ and $a_1 b_{k+1} \in E(G)$. Since $bc \in E(G^*)$, it follows that (i) $c_1 = b_{k+1}$ and $b_1 c_{k+1} \in E(G)$ or (ii) $b_1 = c_1$ and $b_{k+1} c_{k+1} \in E(G)$. See Figure 1. First, suppose that (i) occurs. Since $ac \in E(G^*)$ and $a_{k+1} c_{k+1} \in E(G)$, it follows that $a_1 = c_1$. However, because $c_1 = b_{k+1}$ and $a_1 b_{k+1}$ is an edge of G, it is impossible that $a_1 = c_1$, a contradiction. Next, suppose that (ii) occurs. Because $ac \in E(G^*)$ and $a_{k+1} = c_1$, we have $a_1 c_{k+1} \in E(G)$. However then, $(a_1, b_{k+1}, c_{k+1}, a_1)$ is a triangle in G, a contradiction. Therefore, G^* is triangle-free and so $g(G^*) \geq 4$. □

Since the Schwenk graph G^* of a connected graph G of girth $g \geq 5$ must contain a g-cycle and cannot contain a 3-cycle, this brings up the question as to whether G^* contains a 4-cycle. The following result provides an answer to this question.

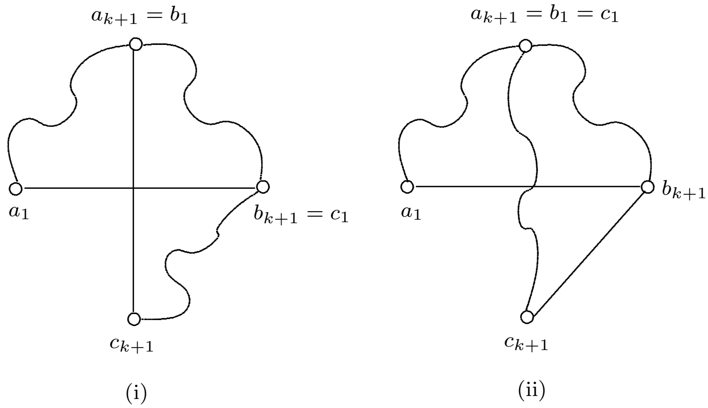

Figure 1: The two situations in the proof of Proposition 1.2

Proposition 1.3. *For each odd integer $g \geq 5$, there exists a connected graph G having girth g such that G^* contains C_4 as a subgraph.*

Proof. Let $g = 2k + 1$ for some integer $k \geq 2$ and let G be the graph of order $4k$ obtained from the cycle $C = (v_1, v_2, \ldots, v_{4k})$ of order $4k$ by adding the two edges v_1v_{2k+1} and $v_{k+1}v_{3k+1}$. For each integer i with $1 \leq i \leq 4k$, let $x_i = (v_i, v_{i+1}, \ldots, v_{i+k})$ be the subpath of order $k + 1$ on C, where the subscripts are expressed as integers $1, 2, \ldots, 4k$ modulo $4k$. Thus, the girth of G is $2k+1$. We now consider the four vertices x_1, x_{k+1}, x_{2k+1} and x_{3k+1} in G^*. Since x_1 and x_{k+1} have the vertex v_{k+1} in common and v_1v_{2k+1} is an edge of G, it follows that x_1 and x_{k+1} are adjacent in G^*. Similarly, x_{k+1} and x_{2k+1} are adjacent in G^*, x_{2k+1} and x_{3k+1} are adjacent in G^* and x_{3k+1} and x_1 are adjacent in G^*. Thus, $C_4 = (x_1, x_{k+1}, x_{2k+1}, x_{3k+1}, x_1)$ is a 4-cycle in the graph G^*. □

For graphs G of girth 5, we know precisely the conditions under which G^* contains a 4-cycle.

Theorem 1.4. *Let G be a connected graph of girth* 5*. Then G^* has a* 4*-cycle if and only if G contains a subgraph isomorphic to the graph H of Figure* 2*.*

Proof. First, suppose that G is a connected graph of girth 5 containing a subgraph isomorphic to the graph H of Figure 2, which is necessarily an induced subgraph of G. Then G^* contains the 4-cycle shown in Figure 2.

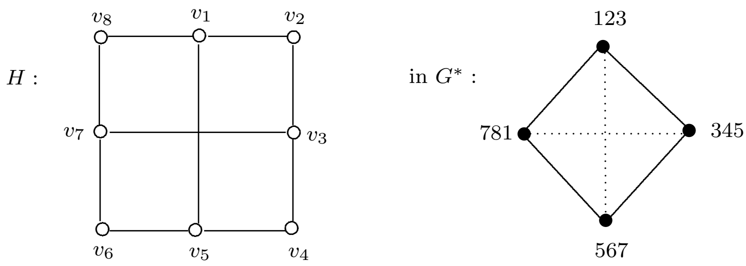

Figure 2: A graph H of girth 5 and the subgraph C_4 in H^*

For the converse, suppose that G is a connected graph of girth 5 such that G^* contains a 4-cycle. We show that G contains a subgraph isomorphic to the graph H of Figure 2. Let $(x_1, x_2, x_3, x_4, x_1)$ be a 4-cycle in G^*. We may assume that $x_1 = (a, b, c)$ and $x_2 = (c, d, e)$ are two 3-paths in G having only the vertex c in common and $ae \in E(G)$. There are two choices for x_3, namely either (i) $x_3 = (c, f, g)$ where $\{d, e\} \cap \{f, g\} = \emptyset$ and $eg \in E(G)$ or (ii) $x_3 = (e, f, g)$ where $\{c, d\} \cap \{f, g\} = \emptyset$ and $cg \in E(G)$. We claim that $f, g \notin \{a, b, c, d, e\}$ in both situations (i) and (ii). Since x_3 is adjacent to $x_2 = (c, d, e)$, it follows that $f, g \notin \{c, d, e\}$. Thus, it remains to show that $f, g \notin \{a, b\}$.

- ⋆ First, suppose that (i) occurs.

  If $f = a$, then (a, b, c, a) is a triangle in G, a contradiction.

  If $f = b$, then (a, b, g, e, a) is a 4-cycle in G, a contradiction.

  If $g = a$, then (a, b, c, f, a) is a 4-cycle in G, a contradiction.

  If $g = b$, then (c, f, b, c) is a triangle in G, a contradiction.

- ⋆ Next, suppose that (ii) occurs.

  If $f = a$, then (a, b, c, g, a) is a 4-cycle in G, a contradiction.

  if $f = b$, then (b, c, g, b) is a triangle in G, a contradiction.

  if $g = a$, then (a, e, f, a) is a triangle in G, a contradiction.

  If $g = b$, then (a, b, f, e, a) is a 4-cycle in G, a contradiction.

Therefore, $f, g \notin \{a, b, c, d, e\}$ in both (i) and (ii). We consider these two cases.

Case 1. $x_3 = (c, f, g)$ *for some vertices* f *and* g *of* G. Then $eg \in E(G)$ and $ag \notin E(G)$. First, suppose that x_4 contains c. Since (1) x_4 is adjacent to

x_1 and x_3 and (2) x_1 and x_3 both contain c, it follows that $x_4 = (c, h, h')$ for some vertices h and h' (distinct from a and g) of G and h' is adjacent to both a and g. However then (a, e, g, h', a) is a 4-cycle in G, which is a contradiction. Hence, x_4 cannot contain c and so $x_4 = (a, h, g)$ for some vertex h of G. However then, (a, e, g, h, a) is a 4-cycle in G, which is impossible.

Case 2. $x_3 = (e, f, g)$ *for some vertices* f *and* g *of* G. Then cg is an edge of G and ag is not an edge of G (for otherwise, G contains a 4-cycle). By the symmetry of the graph C_4, the argument in Case 1 that shows that x_3 cannot contain c can also be used here to show that x_4 cannot contain c. Thus, x_4 must contain a. Thus, there are two possible choices for x_4, according to whether x_4 contains e or x_4 contains g. If $x_4 = (a, h, e)$ for some edge h of G, then (a, e, h, a) is a triangle, a contradiction. If $x_4 = (a, h, g)$, then the subgraph $G[\{a, b, c, d, e, f, g, h\}]$ induced by the set $\{a, b, c, d, e, f, g, h\}$ of eight vertices of G is the graph H of Figure 2. Therefore, G contains H as a subgraph. □

Since the graph H of Figure 2 contains an 8-cycle, the following corollary is an immediate consequence of Theorem 1.4.

Corollary 1.5. *If* G *is a connected graph of girth* 5 *having no* 8*-cycle, then* $g(G^*) = 5$.

The converse of Corollary 1.5 is not true, however. For example, the graph G of the dodecahedron contains 8-cycles. This graph is a 3-regular graph of order 20 and girth 5. Thus, the Schwenk graph G^* has order 60. If C and C' are two distinct 5-cycles in G, then either (i) C and C' have exactly one edge in common or (ii) C and C' are edge-disjoint. Therefore, every 3-path belongs to exactly one 5-cycle in G and so G^* is 2-regular. In fact, G^* consists of twelve 5-cycles and so $g(G^*) = 5$.

2 The Petersen graph: the unique 5-cage

One of the best-known graphs in graph theory is the Petersen graph P, shown in Figure 3. The Schwenk graph P^* of P is a 4-regular graph of order 30 and girth 4. Since the Petersen graph is 3-regular of order 10 and each 3-path corresponds to a pair of adjacent edges in P, the Petersen graph P has $10\binom{3}{2} = 30$ distinct 3-paths and so P^* has order 30. For each 3-path $Q = (u, v, w)$ in the Petersen graph, there are only two paths Q'

with end-vertices u and two 3-paths Q' with end-vertices w such that Q and Q' are edge-disjoint and $P[V(Q) \cup V(Q')] \cong C_5$, so P^* is 4-regular.

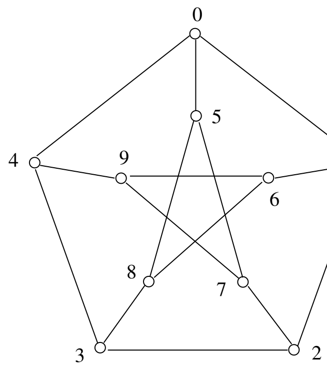

Figure 3: The Petersen graph P

By Proposition 1.2, the graph P^* is triangle-free. However, P^* contains 4-cycles. Figure 4 shows the Schwenk graph P^* embedded in the projective plane where edges that cross the outer circle continue diametrically opposite (due to Schwenk). In this figure, a 3-path (v_i, v_j, v_k) is denoted by ijk where $0 \leq i, j, k \leq 9$ and $|\{i, j, k\}| = 3$.

While the Petersen graph P is not Hamiltonian, the Schwenk graph P^* is Hamiltonian and so P^* contains a Hamiltonian cycle, that is, a cycle C_{30} of order 30. In fact, more can be said. First, we introduce some definitions. A graph G is said to be *decomposable* into the subgraphs H_1, H_2, ..., H_k if $\{E(H_1), E(H_2), \ldots, E(H_k)\}$ is a partition of $E(G)$. Such a partition produces a *decomposition* of G. If $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a decomposition of a graph G such that $H_i \cong H$ for some graph H for each i $(1 \leq i \leq t)$, then $\mathcal{D}$ is an *H-decomposition* of G. If there exists an H-decomposition of a graph G, then G is said to be *H-decomposable*. If each subgraph in $\mathcal{D}$ is a cycle in G, then $\mathcal{D}$ is a *cyclic decomposition* of a graph G. If each subgraph in $\mathcal{D}$ is a Hamiltonian cycle of G, then $\mathcal{D}$ is a *Hamiltonian decomposition* of a graph G. In this case, G is *Hamiltonian-decomposable*. Not only is the Schwenk graph P^* of the Petersen graph P Hamiltonian, this graph is Hamiltonian-decomposable, as we show next.

Proposition 2.1. *The Schwenk graph of the Petersen graph is Hamiltonian-decomposable.*

Proof. The Schwenk graph P^* of the Petersen graph P can be decomposed into two Hamiltonian cycles. For example, let C be the Hamiltonian cycle

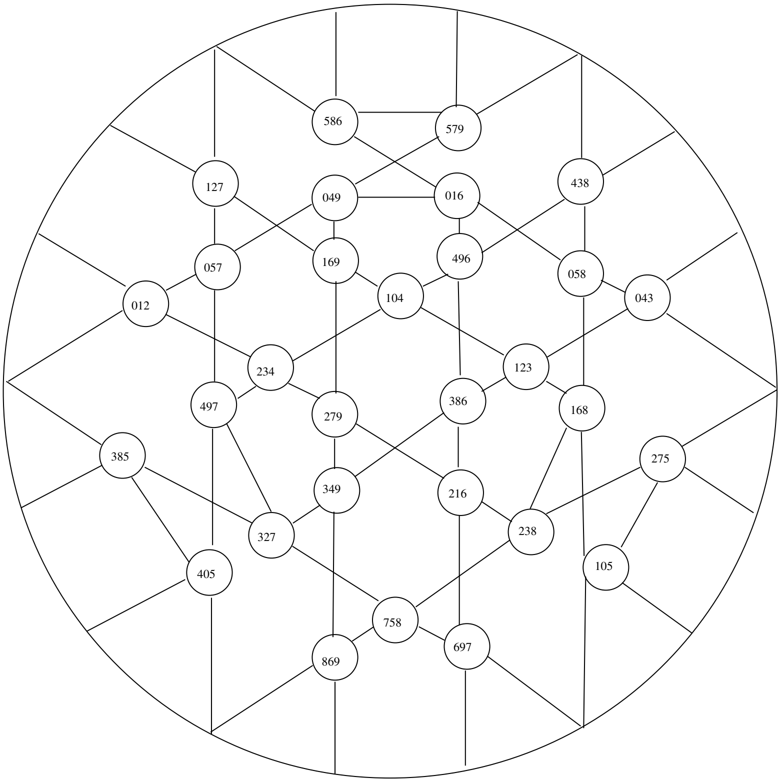

Figure 4: The Schwenk graph of the Petersen graph
embedded in the projective plane

as follows:

$$C = (x_1, x_2, x_3, x_{21}, x_{22}, x_{23}, x_{24}, x_{30}, x_{29}, x_{28}, x_{19}, x_{20}, x_{10}, x_{11}, x_{12}, x_{18}, x_{17}, x_{13}, x_{14}, x_4, x_{25}, x_{26}, x_{16}, x_{15}, x_{27}, x_9, x_8, x_7, x_6, x_5, x_1).$$

Then $C' = G - E(C)$ is another Hamiltonian cycle of P^*. Thus, P^* can be decomposed into C and C'. This is illustrated in Figure 5. Thus, P^* is Hamiltonian-decomposable. $\square$

In fact, the Schwenk graph P^* of the Petersen graph P has a variety of cyclic decompositions. We list some of these:

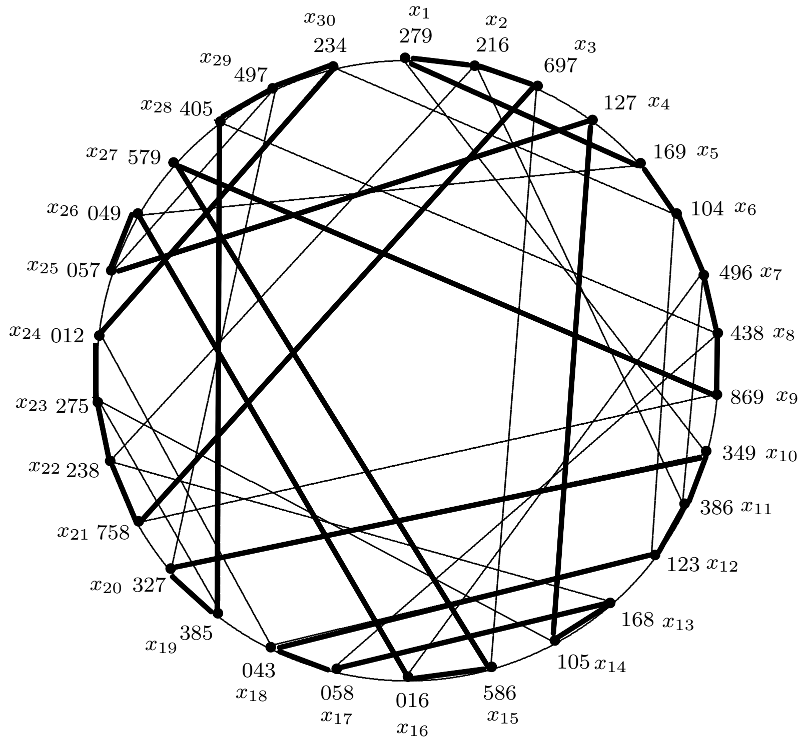

Figure 5: A Hamiltonian decomposition of the Schwenk graph P^*

⋆ The Schwenk graph P^* can be decomposed into three distinct cycles, namely, a 30-cycle, a 25-cycle and a 5-cycle, as follows:

$$\begin{aligned}
C_{30} &= (x_2, x_3, x_4, x_5, x_1, x_{10}, x_9, x_8, x_7, x_6, x_{30}, x_{29}, x_{28},\\
&\quad \ldots, x_{13}, x_{12}, x_{11}, x_2).\\
C_{25} &= (x_2, x_1, x_{30}, x_{24}, x_{18}, x_{12}, x_6, x_5, x_{26}, x_{16}, x_7, x_{11}, x_{10},\\
&\quad x_{20}, x_{29}, x_{25}, x_4, x_{14}, x_{23}, x_{19}, x_{28}, x_8, x_{17}, x_{13}, x_{22}, x_2).\\
C_5 &= (x_3, x_{15}, x_{27}, x_9, x_{21}, x_3)
\end{aligned}$$

Such a decomposition is referred to as an *irregular decomposition*, where no two subgraphs in this decomposition are isomorphic.

⋆ The Schwenk graph P^* can be decomposed into four cycles, namely two 20-cycles and two 5-cycles, as follows:

$$\begin{array}{rcl} C_{25} & = & (x_6, x_7, x_8, x_9, x_{10}, \ldots, x_{30}, x_6) \\ C_{25} & = & (x_1, x_{10}, x_{20}, x_{29}, x_{25}, x_4, x_{14}, x_{23}, x_{19}, x_{28}, x_8, x_{17}, x_{13}, \\ & & x_{22}, x_2, x_{11}, x_7, x_{16}, x_{26}, x_5, x_6, x_{12}, x_{18}, x_{24}, x_{30}, x_1) \\ C_5 & = & (x_9, x_{27}, x_{15}, x_3, x_{21}, x_9) \\ C_5 & = & (x_1, x_2, x_3, x_4, x_5, x_1). \end{array}$$

⋆ There is an irregular cycle decomposition

$$\mathcal{D} = \{C_{25}, C_{13}, C_9, C_8, C_5\}$$

of the Schwenk graph P^* into five cycles of different length as follows:

$$\begin{array}{rcl} C_{25} & = & (x_1, x_{10}, x_{20}, x_{29}, x_{25}, x_4, x_{14}, x_{23}, x_{19}, x_{28}, x_8, x_{17}, x_{13}, \\ & & x_{22}, x_2, x_{11}, x_7, x_{16}, x_{26}, x_5, x_6, x_{12}, x_{18}, x_{24}, x_{30}, x_1) \\ C_{13} & = & (x_{15}, x_{16}, \ldots, x_{26}, x_{27}, x_{15}) \\ C_9 & = & (x_9, x_{10}, \ldots, x_{15}, x_3, x_{21}, x_9) \\ C_8 & = & (x_9, x_{27}, x_{28}, x_{29}, x_{30}, x_6, x_7, x_8, x_9) \\ C_5 & = & (x_1, x_2, \ldots, x_5, x_1) \end{array}$$

⋆ There is an irregular cycle decomposition

$$\mathcal{D} = \{C_{18}, C_{12}, C_{11}, C_8, C_6, C_5\}$$

of the Schwenk graph P^* into six cycles of different length as follows:

$$\begin{array}{rcl} C_{18} & = & (x_3, x_4, x_{25}, x_{24}, x_{18}, x_{12}, x_6, x_{30}, x_{29}, x_{20}, \\ & & x_{10}, x_1, x_5, x_{26}, x_{16}, x_7, x_{11}, x_2, x_3), \\ C_{12} & = & (x_{22}, x_{21}, x_9, x_{27}, x_{15}, x_{14}, x_{23}, x_{19}, x_{28}, x_8, x_{17}, x_{13}, x_{22}), \\ C_{11} & = & (x_4, x_5, \ldots, x_{14}, x_4), \\ C_8 & = & (x_3, x_{15}, x_{16}, \ldots, x_{21}, x_3), \\ C_6 & = & (x_{30}, x_1, x_2, x_{22}, x_{23}, x_{24}, x_{30}), \\ C_5 & = & (x_{25}, x_{26}, \ldots, x_{29}, x_5). \end{array}$$

⋆ There is an isomorphic C_5-decomposition of Schwenk graph P^* into twelve 5-cycles as follows:

$$
\begin{array}{ll}
(x_1, x_2, x_3, x_4, x_5, x_1), & (x_7, x_8, x_9, x_{10}, x_{11}, x_7), \\
(x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{13}), & (x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{19}), \\
(x_{25}, x_{26}, x_{27}, x_{28}, x_{29}, x_{25}), & (x_{30}, x_6, x_{12}, x_{18}, x_{24}, x_{30}), \\
(x_2, x_{11}, x_{12}, x_{13}, x_{22}, x_2), & (x_3, x_{15}, x_{27}, x_9, x_{21}, x_3), \\
(x_4, x_{14}, x_{23}, x_{24}, x_{25}, x_4), & (x_8, x_{17}, x_{18}, x_{19}, x_{28}, x_8), \\
(x_{10}, x_{20}, x_{29}, x_{30}, x_1, x_{10}), & (x_{16}, x_{26}, x_5, x_6, x_7, x_{16}).
\end{array}
$$

Proposition 2.2. *The maximum number of cycles in a cycle decomposition of the Schwenk graph of the Petersen graph is* 15*.*

Proof. Since the girth of P^* is 4, the largest possible number of cycles in a cycle decomposition of P^* is 15. On the other hand, the graph P^* has an isomorphic C_4-decomposition into fifteen 4-cycles as follows:

$$
\begin{array}{lll}
(x_1, x_2, x_{11}, x_{10}, x_1), & (x_4, x_5, x_{26}, x_{25}, x_4), & (x_7, x_{16}, x_{17}, x_8, x_7), \\
(x_{14}, x_{13}, x_{22}, x_{23}, x_{14}), & (x_{20}, x_{19}, x_{28}, x_{29}, x_{20}), & (x_3, x_4, x_{14}, x_{15}, x_3), \\
(x_6, x_7, x_{11}, x_{12}, x_6), & (x_2, x_3, x_{21}, x_{22}, x_2), & (x_8, x_9, x_{27}, x_{28}, x_8), \\
(x_9, x_{10}, x_{20}, x_{21}, x_9), & (x_{16}, x_{15}, x_{27}, x_{26}, x_{16}), & (x_{12}, x_{13}, x_{17}, x_{18}, x_{12}), \\
(x_{18}, x_{19}, x_{23}, x_{24}, x_{18}), & (x_{29}, x_{30}, x_{24}, x_{25}, x_{29}), & (x_{30}, x_1, x_5, x_6, x_{30}).
\end{array}
$$

Therefore, the maximum number of cycles in a cycle decomposition of the Schwenk graph P^* of the Petersen graph P is 15. □

Consequently, the Schwenk graph P^* of the Petersen graph P is C_4-decomposable and C_{30}-decomposable. We thus have the following question.

Problem 2.3. *For which integers g in addition to $g = 4$, $g = 5$ and $g = 30$, is the Schwenk graph of the Petersen graph C_g-decomposable?*

3 The McGee graph: the Unique 7-Cage

In this final section, we investigate the Schwenk graph of another cage: the 7-cage called the McGee graph M. In fact, the McGee graph M is the unique 7-cage (see [4]). Since M contains 7-cycles, it follows that M^* has

7-cycles. Observe that M contains a subgraph G that is isomorphic to the graph described in the proof of Proposition 1.3 for $g = 7$. This is illustrated in Figure 6 where the vertices of G are indicated as solid vertices. Since G^* contains C_4 as a subgraph, it follows that M^* contains 4-cycles. It then follows by Proposition 1.2 that $g(M^*) = 4$. However, M^* contains neither 5-cycles nor 6-cycles, as we show next.

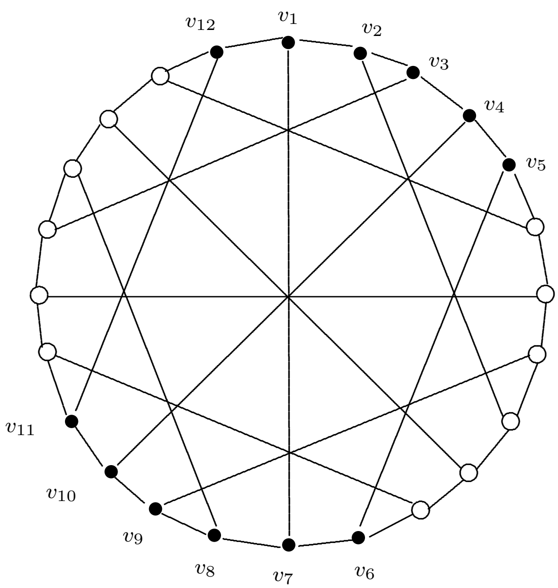

Figure 6: A subgraph G in the 7-cage M

Proposition 3.1. *If M is the McGee graph, then the Schwenk graph M^* contains neither 5-cycles nor 6-cycles.*

Proof. First, we show that M^* is C_5-free. Assume, to the contrary, that M^* contains 5-cycles. Let $C = (a, b, c, d, e, a)$ be a 5-cycle in M^*, where

$$a = (a_1, a_2, a_3, a_4),\ b = (b_1, b_2, b_3, b_4),\ c = (c_1, c_2, c_3, c_4),$$

$$d = (d_1, d_2, d_3, d_4) \text{ and } e = (e_1, e_2, e_3, e_4)$$

are 4-paths in M. Since ab is an edge of M^*, we may assume, without loss of generality, that $a_4 = b_1$ and $a_1 b_4 \in E(M)$. Next, because bc is an edge of M^*, it follows that

(i) $c_1 = b_1$ and $c_4b_4 \in E(M)$ or (ii) $c_1 = b_4$ and $b_1c_4 \in E(M)$.

We consider these two cases.

Case 1. $c_1 = b_1$ *and* $c_4b_4 \in E(M)$. Since $cd \in E(M^*)$, it follows that

(1) $d_1 = c_1$ and $d_4c_4 \in E(M)$ or (2) $d_1 = c_4$ and $c_1d_4 \in E(M)$.

Because $de \in E(M^*)$, we have

(a) $e_1 = d_1$ and $e_4d_4 \in E(M)$ or (b) $e_1 = d_4$ and $d_1e_4 \in E(M)$.

This is illustrated in Figure 7. We now consider the edge ea in M^*.

- ⋆ Suppose that (1) and (a) occur. Since $a_4 = e_1$, it follows that $a_1e_4 \in (M)$. However then, $(a_1, b_4, c_4, d_4, e_4, a_1)$ is a 5-cycle in the 7-cage, which is a contradiction.
- ⋆ Suppose that (1) and (b) occur. Since $a_4e_4 \in E(M)$, it follows that $a_1 = e_1$ and so $(a_1, b_4, c_4, e_1 = a_1)$ is a triangle in the 7-cage, a contradiction.
- ⋆ Suppose that (2) and (a) occur. If $a_1 = e_1$ and $a_4e_4 \in E(M)$, then (a_4, e_4, d_4, a_4) is a triangle in the 7-cage, a contradiction. Hence, $a_1 = e_4$ and $a_4e_1 \in E(M)$. However then, $(c_1, c_2, c_3, c_4, c_1)$ is a 4-cycle in the 7-cage, a contradiction.
- ⋆ Suppose that (2) and (b) occur. Since $e_1a_4 \in E(M)$, it follows that $a_1 = e_4$. However then, $(a_1, b_4, c_4, e_4 = a_1)$ is a triangle in the 7-cage, a contradiction.

Case 2. $c_1 = b_4$ *and* $b_1c_4 \in E(M)$. Since $cd \in E(G^*)$, it follows that

(1) $d_1 = c_1$ and $d_4c_4 \in E(M)$ or (2) $d_1 = c_4$ and $c_1d_4 \in E(M)$.

Since $de \in E(G^*)$, it follows that

(a) $e_1 = d_1$ and $e_4d_4 \in E(M)$ or (b) $e_1 = d_4$ and $d_1e_4 \in E(M)$.

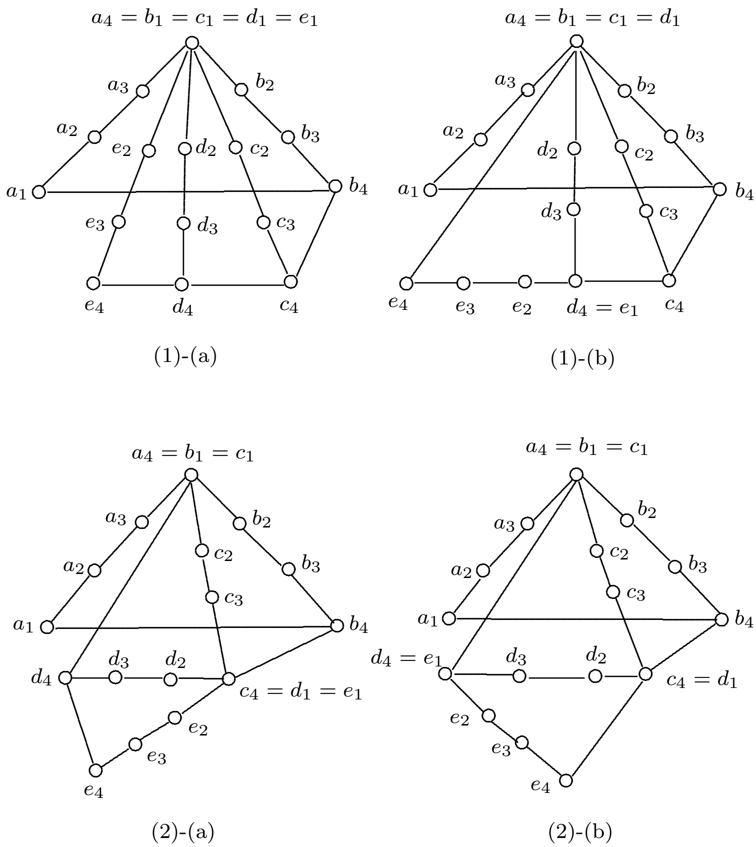

Figure 7: The four situations in Case 1 in the proof of Proposition 3.1

This is illustrated in Figure 8. We now consider the edge ea in M^*.

- ⋆ Suppose that (1) and (a) occur. Since $a_1e_1 \in E(M)$, it follows that $a_4 = e_4$. Hence, $(a_4, b_2, b_3, b_4 = e_1, e_2, e_3, e_4 = a_4)$ is a $6-$cycle in the 7-cage, a contradiction.

- ⋆ Suppose that (1) and (b) occur. If $a_1 = e_1$ and $a_4e_4 \in E(M)$, then (since $d_1 = b_4$ and $d_4 = e_1$), it follows that $(d_1, d_2, d_3, d_4, d_1)$ is a $4-$cycle in the 7-cage, a contradiction. Hence, $a_1 = e_4$ and $a_4e_1 \in E(M)$. However then, $(b_1, c_4, e_1, a_4 = b_1)$ is a triangle in the 7-cage, a contradiction.

- ⋆ Suppose that (2) and (a) occur. Since $a_4e_1 \in E(M)$ and so $a_1 = e_4$, it

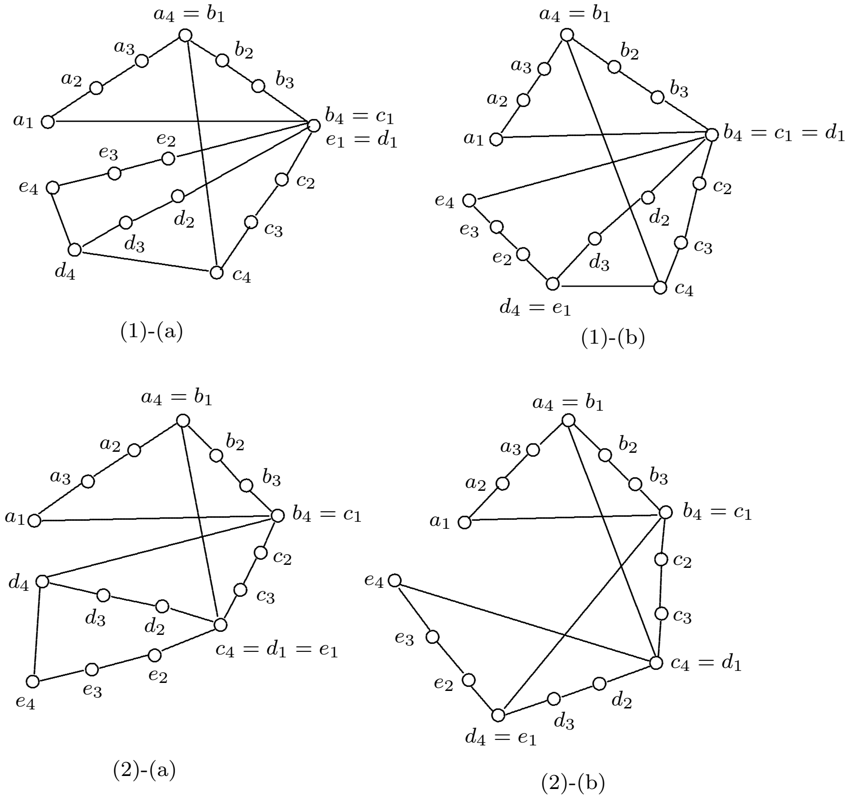

Figure 8: The four situations in Case 2 in the proof of Proposition 3.1

follows that (a_1, b_4, d_4, a_1) is a triangle in the 7-cage, a contradiction.

- ⋆ Suppose that (2) and (b) occur. If $a_1 = e_1$ and $a_4e_4 \in E(M)$, then (e_4, a_4, c_4, e_4) is a triangle in the 7-cage, a contradiction. Hence, $a_1 = e_4$ and $a_4e_1 \in E(M)$. However then, $(a_4, d_1, d_2, d_3, d_4 = e_1, a_4)$ is a $5-$cycle in the 7-cage, a contradiction.

By a similar argument, it can be shown that M^* is C_6-free. □

This brings up the more general question:

If G is a graph of odd girth $g \geq 7$, what smaller cycles can G^* contain?

Acknowledgment. We greatly appreciate the valuable suggestions made by an anonymous referee that resulted in an improved paper.

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