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PART II

Bend Curvature in MAD-X for the Module PTC

Étienne Forest [1)], Simon C. Leemann [1)] and Frank Schmidt [2)]

[1)] *High Energy Accelerator Research Organization,*
1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
[2)] *SL-AP Group, CERN, Geneva, CH*

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Bend Curvature in MAD-X for the Module PTC

Étienne Forest and Simon C. Leemann

National High Energy Research Organization (KEK)

1-1 Oho, Tsukuba, Ibaraki, 305-0801, Japan

and

Frank Schmidt

SL-AP Group, CERN, Geneva, CH

Abstract

In addition to standard second order fringe effects, MAD has a curvature effect which is partially documented in SLAC-75. In this note we want to document this effect as well as our PTC implementation. It is not easy to make this effect totally exact in the Talman sense.

Contents

A Discussion of the SLAC-75 Expressions

A.1 The SLAC-75 Expressions

The geometry we will consider is depicted in Figure 1. In addition to the vertical focusing effect described

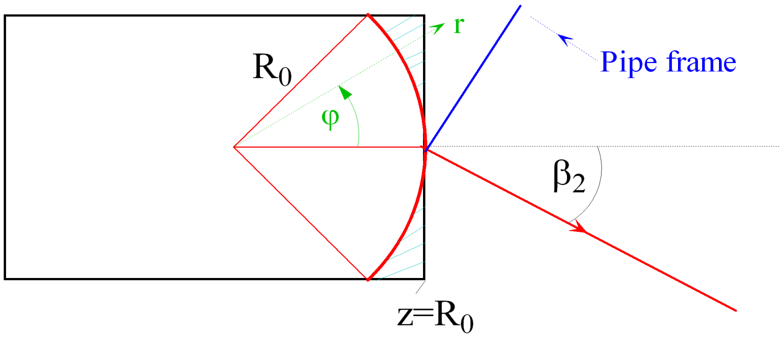

Figure 1: Exit Face of a Bend with a Curved Boundary

in Part I of this study, SLAC-75 also introduces a curved boundary effect. This effect is sextupole-like in nature if expressed in the design frame (the pipe frame of Figure 1). According to SLAC-75, it is given to leading order, by the expression:

$$
\begin{aligned}
p_x^f &= \frac{1}{\cos^3\beta_2}\frac{b_0}{2R_0}\left\{x^2 - y^2\right\} \\
p_y^f &= -\frac{1}{\cos^3\beta_2}\frac{b_0}{R_0}xy
\end{aligned} \tag{1}
$$

A.2 Calling it "Quits"

As in the case of the second order fringe effect, if we assume that the SLAC-75 results are correct (see Sect. C.2) we should be able to concoct a more correct kick-like expression. Here it is more subtle due to the presence of the transverse horizontal variable x. We must distinguish what is "beam pipe" from what is momentum in Equation (1). This is done by taking into account the linear part of an $x - z$ rotation of angle β_2, the PROT of Dragt or the ROT_XZ of the code PTC.

$$
\begin{aligned}
x^{\text{face}} &= \frac{x^{\text{pipe}}}{\cos\beta_2} \\
p_x^{\text{face}} &= \cos\beta_2\; p_x^{\text{pipe}}
\end{aligned} \tag{2}
$$

People unfamiliar to z or s-parameterized dynamics may find the expression for a rotation in Equation (2) puzzling. Well, not only it is true but this is just the linear part of it! This will suffice for our present purpose. Clearly if we want to re-express Equation (1) at the flat surface at $z = R_0$, we must fold in the formulas of Equation (2). The result is simply:

$$
\begin{aligned}
p_x^f &= p_x + \underbrace{\frac{b_0}{2R_0}x^2}_{\text{Geometrical effect}} - \underbrace{\frac{1}{\cos^2\beta_2}\frac{b_0}{2R_0}y^2}_{\text{Maxwellian effect}} \\
p_y^f &= p_y - \underbrace{\frac{1}{\cos^2\beta_2}\frac{b_0}{R_0}xy}_{\text{Maxwellian effect}}
\end{aligned} \tag{3}
$$

The expression in Equation (3) is very similar to that obtained when trying to connect a parallel face bend to a sector bend for example. The geometrical term represent the map due to the additional bending region hashed on Figure 1. For a constant b_0 it is not easy to solve. In the standard wedge case, it can be solved exactly. It is the so-called WEDGE routine of PTC. It is used in MAD as well as in PTC if a certain

LIKEMAD logical is turned on. The second term in Equation (3) is a quasi-solenoidal term in the radial direction. It exists because Maxwell's equation cannot be ignored in leading order at the edge of the magnet.

As in the case of the parallel/sector discussion, one can talk about a "multipole" dangling at the end of the magnet. Here it is a sextupole-like term rather than a quadrupole. However we can see once more that it is a convenient first order mathematical description and not fundamental physics. If a probe were to be put into the fringe region, as in the quadrupole-like case, only half of the multipole, the Maxwellian term would show up.

Finally, and most importantly for a correct fudge, we must introduce the proper momentum dependence. This is now trivial. We can affirm that the remaining β_2 dependencies are all related to the momentum. Thus a more correct expression for the kick is given by

$$\begin{aligned}
p_x^f &= p_x + \underbrace{\frac{b_0}{2R_0}x^2}_{\text{Geometrical effect}} - \underbrace{\frac{(1+\delta)^2 - p_y^2}{p_z^2}\frac{b_0}{2R_0}y^2}_{\text{Maxwellian effect}} \\
p_y^f &= p_y - \underbrace{\frac{(1+\delta)^2 - p_y^2}{p_z^2}\frac{b_0}{R_0}xy}_{\text{Maxwellian effect}} \qquad (4) \\
\text{since } \frac{1}{\cos^2\beta_2} &= \frac{(1+\delta)^2 - p_y^2}{p_z^2}
\end{aligned}$$

The expression in Equation (4) is hard to compute in a symplectic manner. The first thing we can do, which is mathematically as well as physically convenient, is to split this kick into two maps to be applied in succession. The first one is the geometrical effect:

$$p_x^f = p_x + \frac{b_0}{2R_0}x^2 \qquad (5)$$

In theory the exact expression for this could be computed by solving (numerically) horrible transcendental equations involving forward/backward propagation through a bend all the way to a complex shape interface. Of course it is not worth the trouble and thus only Equation (5) is used in PTC. The second term could be solved by the following generating function:

$$\begin{aligned}
F &= p_x x^f + p_y y^f + \Delta \ell^f - \frac{b_0}{2R_0}\frac{(1-\Delta)^2 - p_y^2}{p_z^2}x^f {y^f}^2 \qquad (6) \\
\text{where } p_z &= \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \text{ and } \Delta = -\delta.
\end{aligned}$$

However this generating function forces us to solve a nasty cubic equation. It is not really worth our trouble and therefore we replace p_z by its value the mid-plane.

$$\begin{aligned}
F &= p_x x^f + p_y y^f + \Delta \ell^f - \frac{b_0}{2R_0}\frac{1+\delta}{p_m^2}x^f {y^f}^2 \qquad (7) \\
\text{where } p_m &= \sqrt{(1+\delta)^2 - p_x^2}\ .
\end{aligned}$$

We have seen that if one believes the result of SLAC-75, it is possible to "reverse engineer" the formulas so as to make them more palatable to a symplectic integrator such as PTC. In the following sections we provide a derivation of these formulas.

B Derivation of the Magnetic Field

As our cursory analysis of the SLAC-75 results indicates, a proper investigation of the curvature effects must include a solution of Maxwell equation at the boundary. The simplest way to introduce the curved geometry is to solve a problem which has cylindrical symmetry, i.e., invariance under the angle φ.

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}f + \frac{\partial^2}{\partial y^2}f = 0 \qquad (8)$$

This potential f must give rise to a constant vertical field b_0 for a radius $r \ll R_0$ and must vanish outside. This is a hockey puck type of geometry. Therefore we start with the following guess for the potential:

$$f = b(r)y + \sum_{n=1}^{\infty} a_n(r)\; y^{2n+1} \tag{9}$$

Substituting into Equation (8) gives the result:

$$a_n = -\frac{1}{2n(2n+1)}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}a_{n-1} \quad \text{with } a_0 = b_0 \tag{10}$$

We then derive a vector potential using the ansatz

$$\begin{aligned} \vec{a} &= (a_\varphi, 0, a_r) \\ a_r &= -r\varphi b \\ a_\varphi &= \sum_{n=1}^{\infty} \alpha_n\; y^{2n}. \end{aligned} \tag{11}$$

The B field is given by:

$$\begin{aligned} b_\varphi &= \partial_y a_r \\ b_y &= \frac{1}{r}\partial_r r a_\varphi - \partial_\varphi a_r = \frac{1}{r}\partial_r r a_\varphi + b(r) \\ b_r &= -\frac{1}{r}\partial_y r a_\varphi \end{aligned} \tag{12}$$

Using the equation for b_y, we can solve for the coefficients α_n's:

$$a_\varphi = -\sum_{n=1}^{\infty}\frac{1}{2n}\frac{da_{n-1}}{dr}y^{2n} \tag{13}$$

C Derivation of the Cartesian Hamiltonian

C.1 Analytical Treatment

Now that we have a hockey puck field, we first simply project the cylindrical potential along the Cartesian direction:

$$\begin{aligned} a_x &= \cos\varphi\; a_\varphi + \sin\varphi\; a_r \\ a_z &= \cos\varphi\; a_r - \sin\varphi\; a_\varphi \\ \text{where} \quad & \cos\varphi = \frac{z}{\sqrt{x^2+z^2}} \text{ and } \sin\varphi = \frac{x}{\sqrt{x^2+z^2}} \end{aligned} \tag{14}$$

Equation (14) cannot be used because the expression p_x/p_z does not reduce to x' inside the magnet in account of the term $\sin\varphi\; a_r$ in the definition of a_x. Thus we must choose a new gauge or, equivalently, perform a canonical transformation on the Hamiltonian. Thus we add to a_x the gradient of a function namely:

$$\begin{aligned} a_x^{\text{new}} &= \cos\varphi\; a_\varphi \\ a_z^{\text{new}} &= \cos\varphi\; a_r - \sin\varphi\; a_\varphi - \frac{\partial}{\partial z}\int^x \sin\varphi\; a_r\; dx \end{aligned} \tag{15}$$

The final Hamiltonian is given by the equation:

$$H = -\sqrt{(1+\delta)^2 - (p_x - \cos\varphi\; a_\varphi)^2 - p_y^2} - \cos\varphi\; a_r + \sin\varphi\; a_\varphi + \frac{\partial}{\partial z}\int^x \sin\varphi\; a_r\; dx \tag{16}$$

C.2 Numerical Treatment

We use our favorite step function for the numerical check:

$$
\begin{aligned}
b(r) &= b_0 \frac{1 - tanh\left((r - R_0)/\delta_r\right)}{2} \\
b'(r) &= -b_0 \frac{1 - tanh^2\left((r - R_0)/\delta_r\right)}{2\delta_r} \\
b''(r) &= b_0 \frac{tanh\left((r - R_0)/\delta_r\right)\left(1 - tanh^2\left((r - R_0)/\delta_r\right)\right)}{\delta_r^2}
\end{aligned} \tag{17}
$$

We have integrated the Hamiltonian of Equation (16) for a small value of the drop width δ_r, i.e., $\delta_r \ll R_0$,

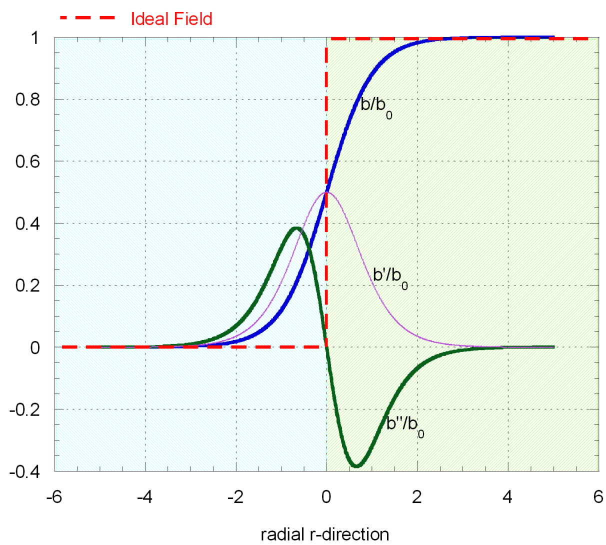

Figure 2: B_y and few relevant derivatives

and using the polymorphic package FPP, we verified SLAC-75's result.

D Operator Derivation of SLAC-75 Curvature Effects

D.1 Preliminary Manipulations

As usual the fringe map is sandwiched between an ideal bend and a drift:

$$
\begin{aligned}
T &= \overbrace{D_{-\varepsilon}}^{\text{drift}} \circ F_{-\varepsilon+R_0\to\varepsilon+R_0} \circ \overbrace{B_{\varepsilon+R_0\to R_0}(b_0)}^{\text{ideal dipole}} \\
\text{or for the Lie maps} \Rightarrow \mathcal{T} &= \mathcal{B}_{\varepsilon+R_0\to R_0}(b_0)\mathcal{F}_{-\varepsilon+R_0\to\varepsilon+R_0}\mathcal{D}_{-\varepsilon}
\end{aligned} \tag{18}
$$

Following Dragt and others, we write an operator equation for the map $\mathcal{F}$:

$$
\frac{d\mathcal{F}}{dz} = \mathcal{F} : -H : \tag{19}
$$

We can use the Heisenberg representation and write $\mathcal{F}$ as follows:

$$
\mathcal{F} = \mathcal{P}\mathcal{D} \tag{20}
$$

Here $\mathcal{D}$ is the drift map and $\mathcal{P}$ is the residual effect of the bend. Obviously $\mathcal{D}$ obeys the equation:

$$\frac{d\mathcal{D}}{dz} = \mathcal{D} : -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} : \tag{21}$$

As for $\mathcal{P}$, it obeys the usual interaction picture equation as in quantum mechanics:

$$\begin{aligned} \frac{d\mathcal{P}}{dz} &= \mathcal{P}\mathcal{D} : -V : \mathcal{D}^{-1} \\ V &= H + \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \\ &= \sum_{n=1}^{\infty} b_0^n V_n \ . \end{aligned} \tag{22}$$

The map is to be integrated from $z = -\varepsilon + R_0$ to a final value[1] of $z = \varepsilon + R_0$, then the map $\mathcal{D}$ in Equation (22) is a drift from $z = -\varepsilon + R_0$ to an arbitrary position z:

$$\begin{aligned} \mathcal{D}x &= x + (z + \varepsilon - R_0)x' \\ \mathcal{D}y &= y + (z + \varepsilon - R_0)y' \end{aligned} \tag{23}$$

In Equation (22) all maps and operators are symplectic, and therefore the following is true:

$$\begin{aligned} \frac{d\mathcal{P}}{dz} &= \mathcal{P} : -V^\dagger : \\ \text{where } V^\dagger(x, y; z) &= V(x + (z + \varepsilon - R_0)x', y + (z + \varepsilon - R_0)y'; z) \end{aligned} \tag{24}$$

To reproduce SLAC-75's curvature results, we need to solve Equation (24) only to first order. This can be done by integrating both sides of the equation from $z = -\varepsilon$ to z:

$$\begin{aligned} \int_{-\varepsilon+R_0}^{z} \frac{d\mathcal{P}}{dz} dz' &= \int_{-\varepsilon+R_0}^{z} \mathcal{P} : -V_{z'}^\dagger : dz' \\ \Rightarrow \mathcal{P} &= 1 + \int_{-\varepsilon+R_0}^{z} \mathcal{P} : -V_{z'}^\dagger : dz' \\ \Longrightarrow \mathcal{P}_{-\varepsilon+R_0 \to \varepsilon+R_0} &= 1 + \int_{-\varepsilon+R_0}^{\varepsilon+R_0} : -V_{z'}^\dagger : dz' \end{aligned} \tag{25}$$

We can rewrite Equation (18) for Lie operators as

$$\begin{aligned} \mathcal{T} &= \mathcal{B}_{\varepsilon+R_0 \to R_0} \mathcal{F}_{-\varepsilon+R_0 \to \varepsilon+R_0} \mathcal{D}_{-\varepsilon} \\ &= \mathcal{B}_{\varepsilon+R_0 \to R_0} \mathcal{P} \mathcal{D}_{2\varepsilon} \mathcal{D}_{-\varepsilon} \\ &= \mathcal{B}_{\varepsilon+R_0 \to R_0} \mathcal{D}_{\varepsilon} \mathcal{S} \end{aligned} \tag{26}$$

$\mathcal{S}$ is given by

$$\begin{aligned} \mathcal{S} = \mathcal{D}_{-\varepsilon} \mathcal{P}_{-\varepsilon+R_0 \to \varepsilon+R_0} \mathcal{D}_{\varepsilon} &= 1 + \int_{-\varepsilon+R_0}^{\varepsilon+R_0} : -\widetilde{V}_{z'} : dz' + \cdots \\ \text{where} \quad \widetilde{V}_z &= V\left(x + (z - R_0)\, x', y + (z - R_0)\, y'; z\right) \end{aligned} \tag{27}$$

D.2 Actual Calculation

We need to expand the Hamiltonian of Equation (16) to leading order:

$$\begin{aligned} V_1 &= -\frac{p_x}{p_z} \cos\varphi \ a_\varphi + \sin\varphi \ a_\varphi + \frac{\partial}{\partial z} \int^x \sin\varphi \ a_r \ dx - \cos\varphi \ a_r \\ &= -\frac{p_x}{p_z} \frac{z}{r} \ a_\varphi + \frac{x}{r} \ a_\varphi + \frac{\partial}{\partial z} \int^x \frac{x}{r} \ a_r \ dx - \frac{z}{r} \ a_r \end{aligned} \tag{28}$$

[1]In the numerical check of Sect. C.2, the variables were ordered as follows: $\delta_r \ll \varepsilon \ll R_0$

To simplify the notation we will use the tilde to denote transformed variables as in Equation (27). We start the calculation with the terms proportional to a_φ. These terms are directly proportional to the derivatives of the B-field, which becomes a Dirac delta function in the limit interesting us.

We begin with the integral of the first term:

$$\begin{aligned}\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \frac{p_x}{p_z}\widetilde{a}_\varphi \frac{z}{\widetilde{r}}dz &= -\frac{p_x}{2p_z}\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \widetilde{b}'\frac{z}{\widetilde{r}}\widetilde{y}^2 dz \\ &= -\frac{p_x}{2p_z}\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \widetilde{b}'\frac{z}{\widetilde{r}}\widetilde{y}^2 \frac{dz}{d\widetilde{r}}d\widetilde{r} = -\frac{p_x}{2p_z}\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} \widetilde{b}'\widetilde{y}^2\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1} d\widetilde{r} \\ &= b_0\frac{p_x}{2p_z}\widetilde{y}^2\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1}\Bigg|_{\widetilde{r}(R_0)} \\ &= b_0\frac{p_x}{2p_z}y^2\left(1-\frac{xx'}{R_0}\right) \ . \end{aligned} \quad (29)$$

We recognize in Equation (29) the usual vertical focusing due to Maxwell equation on a flat boundary and an additive curvature term:

$$\text{First Term} = -\frac{b_0}{2R_0}{x'}^2 xy^2 \ . \quad (30)$$

The second term is obtained with the same approach:

$$\begin{aligned}-\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \widetilde{a}_\varphi \frac{x}{\widetilde{r}}dz &= \frac{1}{2}\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \widetilde{b}'\frac{x}{\widetilde{r}}\widetilde{y}^2 dz \\ &= \frac{1}{2}\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \widetilde{b}'\frac{x}{\widetilde{r}}\widetilde{y}^2\frac{dz}{d\widetilde{r}}d\widetilde{r} = \frac{1}{2}\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} \widetilde{b}'\widetilde{y}^2\frac{x}{z}\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1} d\widetilde{r} \\ &= -b_0\frac{1}{2}\widetilde{y}^2\frac{x}{z}\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1}\Bigg|_{\widetilde{r}(R_0)} \\ &= -\frac{b_0}{R_0}xy^2+\cdots. \end{aligned} \quad (31)$$

Adding both terms, we get the vertical effect:

$$\text{First} + \text{Second Term} = -\frac{b_0}{2R_0}(1+{x'}^2)xy^2 \ . \quad (32)$$

We are left with the two last terms of Equation (28). Before proceeding, we just pause a little and discuss the methodology. We have opted to use a Cartesian frame despite the obvious hockey puck geometry. This is because we need to be in a Cartesian frame in the end of the calculation. This is very convenient in numerical integration since we simply integrate the Cartesian Hamiltonian and spit the results painlessly thanks to the polymorphism package FPP. However the analytical calculations can be somewhat messy, particularly in the horizontal plane. This is because the function $b(r)$ is now a function of the transverse phase space. Therefore the transfer map to second order will depend on derivatives of $b(r)$. This implies that we cannot assume that integrals across the edge containing $b(r)$ smoothly vanish in the limit of a hard edge since derivatives of this function are now needed for the transfer map. In conclusion, by choosing a Cartesian frame, we have transformed physically transparent geometrical transformations (connecting the spherical edge to the flat ideal edge) into subtle integrals involving perhaps the derivatives of the step function. These derivatives will make the Dirac delta function appear and lead to a finite second order matrix. These subtle terms will emerge, in leading order, in the treatment of the last term of Equation (28).

$$\begin{aligned}\text{Third} + \text{Fourth Term} &= \int_{-\varepsilon+R_0}^{\varepsilon+R_0} -\frac{\partial}{\partial z}\int^{\widetilde{x}}\frac{x}{r}\ a_r\ dx\ dz + \int_{-\varepsilon+R_0}^{\varepsilon+R_0}\frac{z}{\widetilde{r}}\ \widetilde{a}_r\ dz \\ &= \underbrace{\int^{\widetilde{x}}\frac{x}{r}\ a_r\ dx\Bigg|_{z=-\varepsilon+R_0}}_{\text{easy term}} + \underbrace{\int_{-\varepsilon+R_0}^{\varepsilon+R_0}\frac{z}{\widetilde{r}}\ \widetilde{a}_r\ dz}_{\text{hard term}} \end{aligned} \quad (33)$$

Let us first look that "easy term" in Equation (33).

$$\int^{\widetilde{x}} \frac{x}{r}\; a_r\; dx\Big|_{z=-\varepsilon+R_0} \;=\; -\int^{\widetilde{x}} x\;\varphi\; b\; dx\Big|_{z=-\varepsilon+R_0}$$
$$= -\int^{\widetilde{x}} \left\{ \frac{x^2}{R_0-\varepsilon}\; b_0 + \; O(x^2)\cdots \right\}\; dx$$
$$\overset{\varepsilon\to 0}{=} -\frac{b_0}{3R_0}x^3 \qquad (34)$$

Let us look that the "hard term":

$$\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \frac{z}{\widetilde{r}}\;\widetilde{a}_r\; dz \;=\; \int_{-\varepsilon+R_0}^{\varepsilon+R_0} -z\;\widetilde{\varphi}\;\widetilde{b}(\widetilde{r})\; dz \qquad (35)$$

In Equation (35) we see the appearance of $\widetilde{b}(\widetilde{r})$. However the integral is performed over z and not r, this means, as we explained before, that we cannot assume that the integral leads to a zero contribution in the limit of a hard edge because the step function depends on the transverse variable. Therefore, in order to evaluate the Taylor series expansion, we will take the x derivative of Equation (35) while keeping all misbehaved functions under the integral sign. Let us look at the derivative of the integrand with respect to x:

$$-\frac{d}{dx}\left\{z\;\widetilde{\varphi}\;\widetilde{b}(\widetilde{r})\right\} \;=\; \left\{-\frac{z^2}{\widetilde{r}^2}\widetilde{b}(\widetilde{r}) - z\;\widetilde{\varphi}\;\widetilde{b}'(\widetilde{r})\;\frac{\widetilde{x}}{\widetilde{r}}\right\} \qquad (36)$$

The second term in Equation (36) is second order in the transverse variables and can be integrated in the limit of a step function:

$$-\int_{-\varepsilon+R_0}^{\varepsilon+R_0} z\;\widetilde{\varphi}\;\widetilde{b}'(\widetilde{r})\;\frac{\widetilde{x}}{\widetilde{r}}dz \;=\; -\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} z\;\widetilde{\varphi}\;\widetilde{b}'(\widetilde{r})\;\frac{\widetilde{x}}{\widetilde{r}}\frac{dz}{d\widetilde{r}}d\widetilde{r}$$
$$= -\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} \widetilde{\varphi}\;\widetilde{b}'(\widetilde{r})\;\frac{\widetilde{x}}{\widetilde{r}}\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1} d\widetilde{r}$$
$$= \frac{b_0}{R_0}x^2 + \cdots = \frac{\partial}{\partial x}\;\frac{b_0}{3R_0}x^3 \qquad (37)$$

Going back to the first term of Equation (36), we take one more derivative:

$$\frac{d}{dx}\left\{-\frac{z^2}{\widetilde{r}^2}\widetilde{b}(\widetilde{r})\right\} \;=\; \left\{-\frac{z^2}{\widetilde{r}^2}\widetilde{b}'(\widetilde{r})\frac{d\widetilde{r}}{dx} + \frac{2xz^2}{\widetilde{r}^4}\widetilde{b}(\widetilde{r})\right\} \qquad (38)$$

In Equation (38), the second term is hopelessly nonlinear. Thus we are left with the first term, which is now ready to go under the integral:

$$-\int_{-\varepsilon+R_0}^{\varepsilon+R_0} \frac{z^2}{\widetilde{r}^2}\widetilde{b}'(\widetilde{r})\frac{d\widetilde{r}}{dx}dz \;=\; -\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} \frac{z^2}{\widetilde{r}^2}\widetilde{b}'(\widetilde{r})\frac{d\widetilde{r}}{dx}\frac{dz}{d\widetilde{r}}d\widetilde{r}$$
$$= -\int_{\widetilde{r}(-\varepsilon+R_0)}^{\widetilde{r}(\varepsilon+R_0)} \frac{z^2x}{\widetilde{r}^3}\widetilde{b}'(\widetilde{r})\left(1+\frac{\widetilde{x}x'}{z}\right)^{-1} d\widetilde{r}$$
$$= \frac{b_0}{R_0}x + \cdots = \frac{\partial^2}{\partial x^2}\;\frac{b_0}{6R_0}x^3 \qquad (39)$$

We are now ready to add all the pieces together, Equations (32), Equation (34), (37), and (39). The final first order Lie function is thus:

$$f \;=\; \frac{b_0}{R_0}\left(\frac{1}{6}x^3 - \frac{1}{2}(1+x'^2)xy^2\right)$$
$$= \frac{b_0}{2R_0}\left(\frac{1}{3}x^3 - \frac{(1+\delta)^2-p_y^2}{\sqrt{(1+\delta)^2-p_x^2-p_y^2}}xy^2\right) \qquad (40)$$

Remarkably the reversed engineered results as well as those derived in Equation (40) are in perfect agreement. In addition, it is clear from our derivation that Maxwell's equation never entered in the horizontal effects as we said before. The reader can actually derive the horizontal results from simple geometry. Here we intended to provide a unified treatment of the analytical and numerical results (with the FPP package) that confirmed the accuracy of SLAC-75.

E PTC Implementation of SLAC-75 Fringe Effects

So, in the light of the previous results, we just implement in PTC the map of Equation (7) in addition to the trivial horizontal kick of Equation (5).

$$\begin{aligned} F &= p_x x^f + p_y y^f + \Delta \ell^f - \Xi x^f {y^f}^2 \\ \text{where } p_m &= \sqrt{(1+\delta)^2 - p_x^2} \text{ and } \Delta = -\delta. \end{aligned} \tag{41}$$

PTC solves this equation at the end of any dipole element. The leading order term is compulsory and the second order term[2], discussed in Part I of this series, is optional. The formulas of this paper are only available for dipole in the exact option. The elements using the expanded Hamiltonian (as in TRACYII or SixTrack) do not use any of the formulas of this paper: they use the infamous quadrupole thin lens which *de facto* incorporates the standard linear term computed in this paper.

The results are

$$\begin{aligned} x^f &= \frac{x}{1 - \frac{\partial \Xi}{\partial p_x} y^2} \\ p_x^f &= p_x - \Xi y^2 \\ p_y^f &= p_y - 2\Xi x^f y \\ \ell^f &= \ell - \frac{\partial \Xi}{\partial \delta} x^f y^2 \\ \text{where } \Xi &= \frac{b_0}{2R_0} \frac{1+\delta}{p_m^2} \end{aligned} \tag{42}$$

The remaining variables y and δ stay constant.

Acknowledgments

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[2] This is the FINT and HGAP term in MAD8 and MAD-X.