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AN EXPANSION METHOD FOR CALCULATING WAKE POTENTIALS

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Abstract

We describe how the wake potential of a bunched beam of arbitrary charge distribution can be calculated from the wake potential of a short Gaussian bunch by using the Hermite polynomial expansion.

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1 Introduction

Theoretically, the wake potential of a bunched beam of arbitrary charge distribution can be calculated by using the wake function of a point charge as a Green function.[1] In reality, except for few very simplified geometries, the wake function of a point charge is impossible to obtain analytically. More difficulties emerge when one tries to compute the wake function (delta function wake) numerically because of the singularity in the time domain and the infinite number of resonances in the frequency domain. Approximations in the time domain have been proposed to obtain the wake function from the smooth wake potential of a short bunch by shifting the wake preceding the center of the bunch to the rear of it. Another approach in the frequency domain is to approximate the spectrum semi-analytically at high frequencies[2,3,4]. These approximations may not give satisfactory results for all cases.

The difficulty of evaluating the wake function of a point charge is avoided by computing the wake potential of a non-singular charge distribution of extended dimension. A number of computer programs have been developed for calculating the wake potentials of charged particle bunches in various boundaries of different geometries [5,6,7]. Nonetheless, even with the most advanced computers, a wake potential calculation still requires a significant amount of computer time, hence it is not practical to use these programs to calculate wake potentials repeatedly in a simulation program for beam stability or beam-beam interaction in accelerators. A conventional method for a fast computation is to calculate the effective impedance in the frequency domain from the resonant modes of a structure and then to Fourier transform the results to the time domain[8]. Clearly, the utilization of this method relies on knowing the modes up to very high frequencies. It is important to know the impedance of a structure before it is built, therefore one has to depend on the results from computations. Attempting to calculate the impedance numerically faces the same difficulties mentioned earlier. Trying to "unfold" the effective impedance of a bunch of finite length does not work in practice because of the extremely large weight factors where the effective impedance is small, e.g. for a Gaussian distribution one would have to multiply the effective impedance with the weight factor $\exp(+\omega^2\sigma^2)$. Thus, a better scheme for rapid computation of wake potentials is required.

As will be discussed in the following, some special properties of a Gaussian function and of the Hermite polynomials allow one to calculate directly the wake potential of a bunch with arbitrary distribution which can be expanded into a series of products of these functions. This method is similar to the Green function method; hence for any specific geometry, one needs only to compute the wake potential of a (short) Gaussian bunch once. The wake potentials of each term in the expansion then can be obtained by (numerical) integrations. One may construct tables from the results and use the "table-look-up" technique[8] to increase the computation speed.

2 The Wake Potential of a Bunched Beam

2.1 A Generalization of the Green Function

It is known that the wake function of a point charge $G(x, x')$ can be used as a Green function to calculate the wake potential $W_F(x)$ of a bunch with a charge distribution $F(x)$ by using the relation

$$W_F(x) = \int_{-\infty}^{\infty} G(x, x')F(x')dx' \quad . \tag{1}$$

For infinitely long beam pipes (open boundary conditions), the wake function is a function of the difference of x and x' only; the above equation can be written as

$$\begin{aligned} W_F(x) &= \int_{-\infty}^{\infty} G(x - x')F(x')dx' \\ &= \int_{-\infty}^{\infty} G(x')F(x - x')dx' \quad , \end{aligned} \tag{2}$$

where it is understood that $G(y) = 0$ for $y < 0$. In this note, we shall limit our discussions to the case where Eq.(2) holds, but we shall employ a "generalization" of Eq.(2): if $W_g(x)$ is the wake potential of a known function $g(x)$, and if a charge distribution $F(y)$ can be expressed as a convolution of $g(x)$ and some function $f(x)$, i.e.

$$F(y) = \int_{-\infty}^{\infty} g(x)f(y - x)dx \quad , \tag{3}$$

then one can calculate the wake potential $W_F(t)$ of the distribution $F(y)$ by using the relation

$$W_F(t) = \int_{-\infty}^{\infty} W_g(x)f(t - x)dx \quad . \tag{4}$$

2.2 The Hermite Polynomial Expansion

We now consider the case where $g(x)$ is a Gaussian function and where a solution of the above integral equation exists. For an arbitrary distribution $F(y)$, it is usually impossible to find a closed form for the solution, but one can expand it into orthogonal functions. The fact that the Hermite polynomials' weight function is Gaussian suggests that it might be advantageous to expand $F(y)$ in terms of them. Thus, we write

$$F(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) \sum_{n=0}^{\infty} a_n H_n\Big(\frac{y}{\sqrt{2}\sigma}\Big) \quad . \tag{5}$$

In the Appendix we will show that the above expression can be replaced by one for a smaller value of σ_1. From Eq.(A.5) we get

$$F(y) = \frac{1}{2\pi\sigma_2\sigma_1} \sum_{n=0}^{\infty} a_n \Big(\frac{\sigma}{\sigma_2}\Big)^n \int_{-\infty}^{\infty} H_n\Big(\frac{s}{\sqrt{2}\sigma_2}\Big) \exp\Big(-\frac{s^2}{2\sigma_2^2}\Big) \exp\Big[-\frac{(y-s)^2}{2\sigma_1^2}\Big] ds \quad , \tag{6}$$

where

$$\sigma_2^2 = \sigma^2 - \sigma_1^2 \quad . \tag{7}$$

Substituting the above result into Eq.(2); changing variables from y to $x - x'$ and s to $x - t$, yield the wake potential of the (arbitrary) distribution F

$$W_F(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \sum_{n=0}^{\infty} a_n \Big(\frac{\sigma}{\sigma_2}\Big)^n \int_{-\infty}^{\infty} W_g(t) H_n\Big(\frac{x-t}{\sqrt{2}\sigma_2}\Big) \exp\Big[-\frac{(x-t)^2}{2\sigma_2^2}\Big] dt \quad , \tag{8}$$

where

$$W_g(t) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} G(x') \exp\Big[-\frac{(t-x')^2}{2\sigma_1^2}\Big] dx' \quad , \tag{9}$$

is the wake potential of a bunch with a Gaussian charge distribution with standard deviation $\sigma_1 < \sigma$. One can prove by comparing Eq.(6) with Eq.(3) and changing variables that the solution for $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \sum_{n=0}^{\infty} a_n \Big(\frac{\sigma}{\sigma_2}\Big)^n H_n\Big(\frac{x}{\sqrt{2}\sigma_2}\Big) \exp\Big(-\frac{x^2}{2\sigma_2^2}\Big) \quad . \tag{10}$$

When $F(x)$ is a Gaussian function, then the solution of Eq.(3) is also a Gaussian function. This can be seen by the relation

$$\frac{\sigma}{\sqrt{2\pi}\sigma_1\sigma_2}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\sigma_2^2}\right)\exp\left[-\frac{(y-x)^2}{2\sigma_1^2}\right]dx=\exp\left(-\frac{y^2}{2\sigma^2}\right)\quad . \tag{11}$$

Thus, we obtain the well-known result that the wake potential of the longer Gaussian bunch can be expressed as a superposition of the wake potential of the shorter Gaussian bunch as[9]

$$W_F(x)=\frac{1}{\sqrt{2\pi}\sigma_2}\int_{-\infty}^{\infty}W_g(s)\exp\left[-\frac{(s-x)^2}{2\sigma_2^2}\right]ds\quad . \tag{12}$$

2.3 Applications

For any boundary conditions for which one can obtain the wake potential of a Gaussian bunch, Eqs. (5) and (8) allow us to calculate the wake potential of a longer bunch with arbitrary charge distribution. The advantages of the method presented here compared to other methods described before are better accuracy and higher efficiency. One only needs to use the time consuming wake potential programs once, and to apply the results to any other function. This feature is especially useful in a beam stability or beam-beam interaction simulation where one can increase the computing speed by using this method in conjunction with the table-look-up technique.

3 Conclusions

We have shown that the wake potential of a bunched beam of arbitrary charge distribution can be calculated from the wake potential of a bunch with Gaussian charge distribution by using the Hermite polynomial expansion. Using this method, one can obtain better accuracy and higher computing efficiency particularly in the simulation programs of beam stability or beam-beam interaction in accelerators.

References

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Appendix: Convolutions of Products of Gaussian Functions and Hermite Polynomials

To derive Eq.(6), we first notice the following equality:

$$\frac{\sigma}{\sqrt{2\pi}\sigma_1\sigma_2}\int_{-\infty}^{\infty}\exp\Big[-\Big(u-\frac{s}{\sqrt{2}\sigma_2}\Big)^2\Big]\exp\Big[-\frac{(y-s)^2}{2\sigma_1^2}\Big]ds=\exp\Big[-\Big(v-\frac{y}{\sqrt{2}\sigma}\Big)^2\Big] \quad , \tag{A.1}$$

where

$$\sigma^2=\sigma_1^2+\sigma_2^2 \quad , \tag{A.2}$$

and

$$v=\Big(\frac{\sigma_2}{\sigma}\Big)u \quad . \tag{A.3}$$

Next, from the relation between the Hermite polynomials and their generating function[10], we have

$$e^{-(p-s)^2}=e^{-s^2}\sum_{n=0}^{\infty}\frac{p^n}{n!}H_n(s) \quad . \tag{A.4}$$

One now can evaluate the nth derivative of Eq.(A.1) with respect to u at $u=0$ to obtain

$$\begin{aligned}&\exp\Big(-\frac{y^2}{2\sigma^2}\Big)H_n\Big(\frac{y}{\sqrt{2}\sigma}\Big)\\ &=\frac{1}{\sqrt{2\pi}\sigma_1}\Big(\frac{\sigma}{\sigma_2}\Big)^{n+1}\int_{-\infty}^{\infty}H_n\Big(\frac{s}{\sqrt{2}\sigma_2}\Big)\exp\Big(-\frac{s^2}{2\sigma_2^2}\Big)\exp\Big[-\frac{(y-s)^2}{2\sigma_1^2}\Big]ds \quad .\end{aligned} \tag{A.5}$$

Substituting the above equation into Eq.(5) yields Eq.(6). If $u=0$ in Eq.(A.1) or $n=0$ in Eq.(A.5), then one has

$$\frac{\sigma}{\sqrt{2\pi}\sigma_1\sigma_2}\int_{-\infty}^{\infty}\exp\Big(-\frac{s^2}{2\sigma_2^2}\Big)\exp\Big[-\frac{(y-s)^2}{2\sigma_1^2}\Big]ds=\exp\Big(-\frac{y^2}{2\sigma^2}\Big) \quad . \tag{A.6}$$

Thus the convolution of two Gaussian functions is also a Gaussian function.