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WAVE FUNCTIONS OF BOUND STATES OF A FERMION AND A DIRAC DYON AND MATRIX ELEMENTS IN AN EXTERNAL ELECTROMAGNETIC FIELD

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ABSTRACT

Wave functions of bound states for a fermion and a Dirac dyon with charge $Z_d < Z_d^c$ and for $j \geqslant |q|+\frac{1}{2}$ are obtained. Matrix elements of this system in the external electromagnetic field are calculated and the corresponding selection rules are shown.

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1. - INTRODUCTION

In recent years, the problem of bound states of a fermion in a fixed Dirac monopole [1] or in a 't Hooft-Polyakov monopole [2] has been extensively discussed [3-20]. In this paper the bound system of a fermion and a fixed Dirac dyon will be discussed further [9,17]. As is well known for the system of a fermion and a Dirac monopole, there is the Lipkin-Weisberger-Peshkin (LWP) difficulty [21] in the angular momentum states $j = |q|-\frac{1}{2}$ which shows up in the fermion's radial wave functions at the origin. The radial wave functions of the fermion in the angular momentum state $j = |q|-\frac{1}{2}$ do not vanish at the origin. This means that the fermion in these states goes through the monopole; thus, the Hamiltonian of the system is ill defined at the origin. To avoid this difficulty, an infinitesimal extra magnetic moment is endowed to the fermion by Kazama and Yang [5,6]. In Ref. [9] we showed that for the system of a fermion and a Dirac dyon there is also the LWP difficulty in the angular momentum states $j \geq |q|+\frac{1}{2}$ when the dyon charge Z_d exceeds some critical value Z_d^c. In order to avoid the LWP difficulty, besides the Kazama-Yang term $-(Kq/2Mr^3)\beta\vec{\Sigma}.\vec{r}$, the term $i(KZZ_de^2/2Mr^3)\vec{\gamma}.\vec{r}$ should be considered also. But in the case $Z_d < Z_d^c$, the Hamiltonian of the system is well defined at the origin, so we can solve the bound-state energy for $j \geq |q|+\frac{1}{2}$ without the Kazama-Yang term [5] and the term $i(KZZ_de^2/2Mr^3)\vec{\gamma}.\vec{r}$ [9]. The results show that the bound-state energy is hydrogen-like [17].

In this paper, wave functions of bound states for a fermion and a Dirac dyon with charge $Z_d < Z_d^c$ and for $j \geq |q|+\frac{1}{2}$ are obtained. Using these wave functions, matrix elements of this system in the external electromagnetic field are calculated and the corresponding selection rules are shown.

2. - THE BASIC EQUATION AND ITS SOLUTION

In this section we first review the basic equation to fix our conventions. The Hamiltonian of this system is [17]:

$$H = \vec{\alpha}\cdot(-i\nabla - Ze\vec{A}) + \beta M - \lambda/r, \tag{2.1}$$

where $\vec{A}$ is the vector potential of the dyon. In order to remove the string of singularities, $\vec{A}$ is defined in terms of two or more functions in a corresponding number of overlapping regions [3]. $\lambda = ZZ_de^2$, Z is the electric charge of the fermion which is an integer, and Z_d is the electric charge of the dyon which need not be an integer. For the states $j \geqslant |q|+\frac{1}{2}$ there are two types of simultaneous eigensections of $\vec{J}^2 . J_Z$, and H [4]:

Type A
$$\psi^{(1)}_{jm} = \frac{1}{r}\begin{pmatrix} h_1(r)\,\xi^{(1)}_{jm} \\ -i h_2(r)\,\xi^{(2)}_{jm} \end{pmatrix}, \qquad (j \geqslant |q| + 1/2) \tag{2.2}$$

Type B
$$\psi^{(2)}_{jm} = \frac{1}{r}\begin{pmatrix} h_3(r)\,\xi^{(2)}_{jm} \\ -i h_4(r)\,\xi^{(1)}_{jm} \end{pmatrix}, \qquad (j \geqslant |q| + 1/2) \tag{2.3}$$

where

$$\xi^{(1)}_{jm} = c\,\phi^{(1)}_{jm} - s\,\phi^{(2)}_{jm}, \tag{2.4}$$

$$\xi^{(2)}_{jm} = s\,\phi^{(1)}_{jm} + c\,\phi^{(2)}_{jm}, \tag{2.5}$$

$$c = q\left[(2j+1+2q)^{1/2} + (2j+1-2q)^{1/2}\right] \Big/ 2|q|(2j+1)^{1/2}, \tag{2.6}$$

$$s = q\left[(2j+1+2q)^{1/2} - (2j+1-2q)^{1/2}\right] \Big/ 2|q|(2j+1)^{1/2}, \tag{2.7}$$

$$\phi^{(1)}_{jm} = \begin{pmatrix} \left(\frac{j+m}{2j}\right)^{1/2} Y_{q,\,j-1/2,\,m-1/2} \\ \left(\frac{j-m}{2j}\right)^{1/2} Y_{q,\,j-1/2,\,m+1/2} \end{pmatrix}. \tag{2.8}$$

$$\phi^{(2)}_{jm} = \begin{pmatrix} -\left(\frac{j-m+1}{2j+2}\right)^{1/2} Y_{q,\,j+1/2,\,m-1/2} \\ \left(\frac{j+m+1}{2j+2}\right)^{1/2} Y_{q,\,j+1/2,\,m+1/2} \end{pmatrix}. \tag{2.9}$$

where $Y_{q,L,M}$ is the monopole harmonic whose basic properties are tabulated in Appendix A [3,22-24].

In (2.2) and (2.3), $h_i(r)$ (i=1,2,3,4) are defined in a rather different way than in Ref. [4]; thus, the system of equations satisfied by $h_i(r)$ is obtained in the compact form which is easily treated. According to Lemma I of Ref. [4], from (2.1)-(2.3), we obtain, for type A:

$$\begin{cases} (M-E-\lambda/r)h_1(r)+(\partial_r+\mu/r)h_2(r)=0, \\ (\partial_r-\mu/r)h_1(r)+(M+E+\lambda/r)h_2(r)=0, \end{cases} \tag{2.10}$$

and for type B:

$$\begin{cases} (M-E-\lambda/r)h_3(r)+(\partial_r-\mu/r)h_4(r)=0, \\ (\partial_r+\mu/r)h_3(r)+(M+E+\lambda/r)h_4(r)=0, \end{cases} \tag{2.11}$$

where

$$\mu=\left[(j+1/2)^2-q^2\right]^{1/2}>0, \tag{2.12}$$

$q = Zeg$, g is the strength of the magnetic monopole; Dirac quantization is that $eg = n/2$ $(n=0,\pm1,\pm2,\ldots)$ [1].

We can solve (2.10) and (2.11) according to the standard treatment in quantum-mechanics textbooks [25]. When $r \to 0$, (2.10) is reduced to

$$\begin{cases} (\lambda/r)h_1(r)-(\partial_r+\mu/r)h_2(r)=0, \\ (\partial_r-\mu/r)h_1(r)+(\lambda/r)h_2(r)=0. \end{cases} \tag{2.13}$$

Setting $h_1(r) = ar^{\nu}$, $h_2(r) = br^{\nu}$, where a and b are constants, from (2.13) we obtain:

$$\begin{cases} \lambda a-(\nu+\mu)b=0 \\ (\nu-\mu)a+\lambda b=0 \end{cases}$$

From conditions of non-zero a and b, the finiteness of $h_1(r)$ and $h_2(r)$ when $r \to 0$, we have

$$\nu=(\mu^2-\lambda^2)^{1/2}=\left[(j+1/2)^2-q^2-(ZZ_de^2)^2\right]^{1/2}>0. \tag{2.14}$$

Setting

$$\rho=2(M^2-E^2)^{1/2}r=2pr, \qquad (p=(M^2-E^2)^{1/2}) \tag{2.15}$$

$$\begin{cases} h_1(\rho)=2p(M+E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_1(\rho)+Q_2(\rho)), \\ h_2(\rho)=2p(M-E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_1(\rho)-Q_2(\rho)). \end{cases} \tag{2.16}$$

we have

$$\begin{cases} \rho Q_1'(\rho)+(\nu-\lambda E/p)Q_1(\rho)-(\mu+\lambda M/p)Q_2(\rho)=0, \\ \rho Q_2'(\rho)+(\nu-\rho+\lambda E/p)Q_2(\rho)-(\mu-\lambda M/p)Q_1(\rho)=0, \end{cases} \tag{2.17}$$

and

$$\rho Q_1''(\rho)+(2\nu+1-\rho)Q_1'(\rho)-(\nu-\lambda E/p)Q_1(\rho)=0, \tag{2.18}$$

$$\rho Q_2''(\rho)+(2\nu+1-\rho)Q_2'(\rho)-(\nu+1-\lambda E/p)Q_2(\rho)=0. \tag{2.19}$$

If we set

$$\begin{cases} h_3(\rho)=2p(M+E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_3(\rho)-Q_4(\rho)), \\ h_4(\rho)=2p(M-E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_3(\rho)+Q_4(\rho)), \end{cases} \tag{2.20}$$

then $Q_3(\rho)$ satisfies (2.18), and $Q_4(\rho)$ satisfies (2.19).

Equations (2.18) and (2.19) are standard confluent hypergeometric equations. Their finite solutions at the origin are the confluent hypergeometric function $F(a,b,\rho)$:

$$\begin{cases} Q_{1,3}(\rho)=A_{1,3}F(\nu-\lambda E/p, 2\nu+1, \rho), \\ Q_{2,4}(\rho)=A_{2,4}F(\nu+1-\lambda E/p, 2\nu+1, \rho). \end{cases} \tag{2.21}$$

From (2.2), (2.16) and (2.21), the radial wave functions are

$$R_{\substack{1\\2}}(\rho)=2p\,h_{\substack{1\\2}}(\rho)/\rho$$
$$=4p^2(M\pm E)^{1/2}e^{-\rho/2}\rho^{\nu-1}\left[A_1F(\nu-\lambda E/p,2\nu+1,\rho)\pm A_2F(\nu+1-\lambda E/p,2\nu+1,\rho)\right] \tag{2.22}$$

When $\rho \to 0$, $F(a,b,\rho) \to 0$, from (2.17), we have

$$A_2=\frac{\nu-\lambda E/p}{\mu+\lambda M/p}A_1. \tag{2.23}$$

Similarly, for A_3 and A_4, we have

$$A_4=\frac{\nu-\lambda E/p}{\mu-\lambda M/p}A_3. \tag{2.24}$$

When $\rho \to \infty$, $F(a,b,\rho) \to e^{\rho}$, so $R_i(\rho)$ is divergent. In order to avoid the divergence, we must set $\nu-\lambda E/p = -n$, $(n=0,1,2,\ldots)$ and $\nu+1-\lambda E/p = -m$, $(m=0,1,2,\ldots)$. When $n = 0$, $F(\nu-\lambda E/p,2\nu+1,\rho)$ is finite, but $F(\nu+1-\lambda E/p,2\nu+1,\rho)$ is still divergent. In order to make $R_i(\rho)$ finite, we must have $n = m+1$, $(m=0,1,2,\ldots)$, so

$$\nu-\lambda E/p=-n, \qquad (n=1,2,3,\cdots) \tag{2.25}$$

Thus we obtain [17]

$$E_{n,q,j}=\pm M\left[1+\frac{1}{(n+\nu)^2/\lambda^2}\right]^{-1/2}$$
$$=\pm M\left(1+\left[\frac{ZZ_de^2}{n+[(j+1/2)^2-q^2-(ZZ_de^2)^2]^{1/2}}\right]^2\right)^{-1/2}, \tag{2.26}$$

where $n = 1,2,3,\ldots$; $j \geq |q|+\frac{1}{2}$; $q = Zeg \neq 0$; $eg = \pm\frac{1}{2},\pm 1,\pm 3/2,\ldots$, $\mu = [(j+\frac{1}{2})^2-q^2]^{\frac{1}{2}} > 0$. $\lambda = ZZ_de^2$. Notice that the total angular momentum j, which is defined in Ref. [4], is different from the total angular momentum in ordinary quantum mechanics. Here, j can take integer as well as half-integer values. The spectrum (2.26) is hydrogen-like, but is different from the atomic or the molecular spectrum. If dyons exist in Nature, (2.26) leads to the possibility to look for dyonic bound states, for example, from astronomical observations.

By the relation

$$L_n^{\nu}(x)=\frac{\Gamma(\nu+1+n)}{n!\,\Gamma(\nu+1)}F(-n,\nu+1,x),$$

and using (2.23) and (2.25), (2.22) is reduced to:

$$R_{\substack{1\\2}}^{n\delta j}(\rho)=4p_{n\delta j}^{2}(M\pm E_{n\delta j})^{1/2}A_1^{n\delta j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\,\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right.$$
$$\left.\mp\frac{n}{\mu+(\mu^2+n^2+2n\nu)^{1/2}}\;\frac{(n-1)!\,\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\}, \tag{2.27}$$

Similarly, we have

$$R_{\substack{3\\4}}^{n\delta j}(\rho)=4p_{n\delta j}^{2}(M\pm E_{n\delta j})^{1/2}A_3^{n\delta j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\,\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right.$$
$$\left.\pm\frac{n}{\mu-(\mu^2+n^2+2n\nu)^{1/2}}\;\frac{(n-1)!\,\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\}. \tag{2.28}$$

Because $\xi_{jm}^{(1)}$ and $\xi_{jm}^{(2)}$ are normalized, so radial wave functions $R_i(\rho)$ satisfy the following normalization condition

$$\int_0^{\infty}\sum_{i=1(3)}^{2(4)}\left|R_i(2p_{n\delta j}r)\right|^2 r^2dr=1.$$

Using the normalization condition of $L_n^{\nu}(x)$

$$\int_0^{\infty}dx\; x^{\nu}e^{-x}L_n^{\nu}(x)\,L_{n'}^{\nu}(x)=\frac{\Gamma(\nu+n+1)}{n!}\delta_{nn'},$$

we have

$$A_{\substack{1\\3}}^{n\delta j}=\frac{1}{2\Gamma(2\nu+1)}\left(Mp_{n\delta j}\frac{n!}{\Gamma(2\nu+n)}\left[\frac{1}{2\nu+n}+\frac{n}{[\mu\pm(\mu^2+n^2+2\nu n)^{1/2}]^2}\right]\right)^{-1/2}. \tag{2.29}$$

3. - TRANSITION MATRIX ELEMENTS IN THE EXTERNAL ELECTROMAGNETIC FIELD

In this section we show the examples of calculation of matrix elements by using wave functions of bound states of a fermion with a Dirac dyon.

If the external electromagnetic field is described by the vector potential

$$\vec{A}(\vec{x},t)=\vec{A}_0 \exp\left[i(\vec{K}\cdot\vec{x}-\omega t)\right], \tag{3.1}$$

then the interaction Hamiltonian of the fermion-Dirac dyon system in the external electromagnetic field is:

$$H_i = -Ze\vec{\alpha}\cdot\vec{A}(\vec{x},t). \tag{3.2}$$

Treat H_i as the perturbation. Suppose that within the scale of the bound system of a fermion and a Dirac dyon, $\vec{K}.\vec{X} << 1$, the $\exp(i\vec{K}.\vec{X}) \approx 1+i\vec{K}.\vec{X}+\ldots$. In order to show the general method of the calculation of matrix elements, we discuss not only the first term, but also the $\vec{K}.\vec{X}$ term in the above expansion. Take $\vec{A}_0$ along the X axis, $\vec{K}$ along the Z axis, and work in the Coulomb gauge. Let N represent the quantum number set (n,q,j,m). We have

$$H'_{N',N} = H'^{(0)}_{N',N} + H'^{(1)}_{N',N}\,, \tag{3.3}$$

where H' represents the part of H_i separated from the time-dependent factor $\exp(-i\omega t)$. In (3.3),

$$H'^{(0)}_{N',N} = -ZeA_0\langle\psi^{(1)}_{N'}\left|\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\right|\psi^{(1)}_{N}\rangle = -ZeA_0\langle\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\rangle_{N',N}, \tag{3.4}$$

$$\begin{aligned}H'^{(1)}_{N',N} &= -iZeA_0K\sqrt{\tfrac{4\pi}{3}}\langle\psi^{(1)}_{N'}\left|rY_{010}\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\right|\psi^{(1)}_{N}\rangle\\ &= -iZeA_0K\sqrt{\tfrac{4\pi}{3}}\langle rY_{010}\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\rangle_{N',N}.\end{aligned} \tag{3.5}$$

For Section A (2.2),

$$H'^{(0)a}_{N',N} = -iZeA_0\left[\Omega^{(2)\,\delta n'j'nj}_{21}(\sigma_1)^{21}_{j'm'jm} - \Omega^{(2)\,\delta n'j'nj}_{12}(\sigma_1)^{12}_{j'm'jm}\right., \tag{3.6}$$

$$H'^{(1)a}_{N',N} = ZeA_0K\sqrt{\frac{4\pi}{3}}\left[\Omega^{(3)gn'j'nj}_{21}(Y_{010}\sigma_1)^{21}_{j'm'jm} - \Omega^{(3)gn'j'nj}_{12}(Y_{010}\sigma_1)^{12}_{j'm'jm}\right], \tag{3.7}$$

where

$$\Omega^{(2)gn'j'nj}_{st} = \int_0^\infty r^2 dr\, R^{n'gj'*}_s(2p_{n'gj'}r)R^{ngj}_t(2p_{ngj}r), \quad (s,t=1,2) \tag{3.8}$$

$$\Omega^{(3)gn'j'nj}_{st} = \int_0^\infty r^3 dr\, R^{n'gj'*}_s(2p_{n'gj'}r)R^{ngj}_t(2p_{ngj}r), \quad (s,t=1,2) \tag{3.9}$$

$$(\sigma_1)^{st}_{j'm'jm} = \int d\Omega\, \zeta^{(s)\dagger}_{j'm'}\sigma_1\zeta^{(t)}_{jm}, \qquad (s,t=1,2) \tag{3.10}$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm} = \int d\Omega\, \zeta^{(s)\dagger}_{j'm'}Y_{010}\sigma_1\zeta^{(t)}_{jm}, \qquad (s,t=1,2) \tag{3.11}$$

From the calculation of matrix elements which will be shown later, the selection rules of $H'^{(0)a}_{N',N}$ are

$$\Delta j = 0,\pm 1, \quad \Delta m = \pm 1. \tag{3.12}$$

In the general co-ordinates, they are $\Delta j = 0,\pm 1$, $\Delta m = 0,\pm 1$.

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For Section B (2.3), we have similar results.

Now we show the calculations of the matrix elements.

1. $\Omega^{(\eta)}_{st}$ $(\eta=2,3)$

We notice that F(a,b,Z) is reduced to a polynomial when a takes 0 or negative integer:

$$F(-n,b,\rho)=\sum_{\ell=0}^{n}\frac{(-1)^{\ell}n(n-1)\cdots(n-\ell+1)}{\ell!\,b(b+1)\cdots(b+\ell-1)}\rho^{\ell},\qquad (n=0,1,2,\cdots)$$

From (3.8) we have

$$\Omega_{st}^{(2)qn'j'nj}=G^{(1)}_{n'qj'(\mp)}G^{(1)}_{nqj(\pm)}2^{\nu+\nu'-2}p_{n'qj'}^{\nu'-1}p_{nqj}^{\nu-1}\cdot\Big[\Big(\sum_{\ell=0}^{n}\sum_{\ell'=0}^{n'}\mp K^{(+)}_{nj}\sum_{\ell=0}^{n-1}\sum_{\ell'=0}^{n'}\pm K^{(+)}_{n'j'}\sum_{\ell=0}^{n}\sum_{\ell'=0}^{n'-1}\mp K^{(+)}_{nj}K^{(+)}_{n'j'}\sum_{\ell=0}^{n-1}\sum_{\ell'=0}^{n'-1}\Big)C^{(n)}_{\nu\ell}C^{(n')}_{\nu'\ell'}I^{\nu\nu'}_{\ell\ell'}\Big],$$

(3.14)

where

$$G^{(1)}_{nqj(\pm)}=4p^{2}_{nqj}(M\pm E_{nqj})^{1/2}A_{1}^{nqj},\qquad K^{(+)}_{nj}=\frac{n}{\mu+(\mu^{2}+n^{2}+2n\nu)^{1/2}},$$

$$C^{(n)}_{\nu\ell}=\frac{(-1)^{\ell}n(n-1)\cdots(n-\ell+1)}{\ell!(2\nu+1)(2\nu+2)\cdots(2\nu+\ell)},$$

$$I^{\nu\nu'}_{\ell\ell'}=\int_{0}^{\infty}dr\,e^{-(p_{nqj}+p_{n'qj'})r}r^{\nu+\nu'+\ell+\ell'}=(p_{nqj}+p_{n'qj'})^{-(\nu+\nu'+\ell+\ell')}\Gamma(\nu+\nu'+\ell+\ell'+1).$$

In (3.14), the upper (lower) sign is taken for st = 12(21).

Similarly, we can calculate $\Omega_{st}^{(3)qn'j'nj}$.

2. $(\sigma_1)^{st}_{j'm'jm}$ and $(Y_{010}\sigma_1)^{st}_{j'm'jm}$

Using the orthogonal and normalization relations of monopole harmonics [(A.2) of Appendix A], from (3.10) and (2.4)-(2.9), we have:

$$(\sigma_1)^{st}_{j'm'jm}=\frac{q}{2j(j+1)}\left[(j+m+1)(j-m)\right]^{1/2},\quad (m=-j,\cdots,j-1),\ \text{when } j'=j,\ m'=m+1;\qquad (3.15a)$$

$$(\sigma_1)^{st}_{j'm'jm}=\frac{q}{2j(j+1)}\left[(j-m+1)(j+m)\right]^{1/2},\quad (m=-(j-1),\cdots,j),\ \text{when } j'=j,\ m'=m-1;\qquad (3.15b)$$

$$(\sigma_1)^{st}_{j'm'jm} = \pm \frac{[(j+m+1)(j+m+2)]^{1/2}}{8(j+1)[(2j+1)(2j+3)]^{1/2}} \left[(2j+3+2\delta)^{1/2} \pm (2j+3-2\delta)^{1/2}\right]\left[(2j+1+2\delta)^{1/2} \pm (2j+1-2\delta)^{1/2}\right],$$

$$(m=-(j+1),\cdots,j), \qquad \text{when } j'=j+1,\ m'=m+1; \qquad (3.15c)$$

$$(\sigma_1)^{st}_{j'm'jm} = \mp \frac{[(j-m+2)(j-m+1)]^{1/2}}{8(j+1)[(2j+1)(2j+3)]^{1/2}} \left[(2j+3+2\delta)^{1/2} \pm (2j+3-2\delta)^{1/2}\right]\left[(2j+1+2\delta)^{1/2} \pm (2j+1-2\delta)^{1/2}\right],$$

$$(m=-j,\cdots,j+1), \qquad \text{whe } j'=j+1,\ m'=m-1; \qquad (3.15d)$$

$$(\sigma_1)^{st}_{j'm'jm} = \pm \frac{[(j-m+1)(j-m)]^{1/2}}{8j[(2j-1)(2j+1)]^{1/2}} \left[(2j\pm1+2\delta)^{1/2} \mp (2j\pm1-2\delta)^{1/2}\right]\left[(2j\mp1+2\delta)^{1/2} \mp (2j\mp1-2\delta)^{1/2}\right],$$

$$(m=-j,\cdots,j-1), \qquad \text{when } j'=j-1,\ m'=m+1; \qquad (3.15e)$$

$$(\sigma_1)^{st}_{j'm'jm} = \mp \frac{[(j+m-1)(j+m)]^{1/2}}{8j[(2j-1)(2j+1)]^{1/2}} \left[(2j\pm1+2\delta)^{1/2} \mp (2j\pm1-2\delta)^{1/2}\right]\left[(2j\mp1+2\delta)^{1/2} \mp (2j\mp1-2\delta)^{1/2}\right],$$

$$(m=-(j-1),\cdots,j), \qquad \text{when } j'=j-1,\ m'=m-1; \qquad (3.15f)$$

$$(\sigma_1)^{st}_{j'm'jm} = 0, \qquad \text{when } j'm' \text{ take other values.} \qquad (3.15g)$$

When st = 12(21), the upper (lower) sign is taken.

In order to calculate $(Y_{010}\sigma_1)^{st}_{j'm'jm}$, we need to use some properties of monopole harmonics [3,22-24], especially their addition theorem [22], and Wigner 3-j symbols [26], which are tabulated separately, in Appendices A and B. By the tedious calculation, from (3.11) and (2.4)-(2.9), we obtain

$$(Y_{010}\sigma_1)^{st}_{j'm'jm} = \pm\left[(2j-3+2\delta)^{1/2} \mp (2j-3-2\delta)^{1/2}\right]\left[(2j+1+2\delta)^{1/2} \mp (2j+1-2\delta)^{1/2}\right]$$
$$\cdot\left[32j(j-1)(2j-1)\right]^{-1}\left[(2j-3)(2j+1)\right]^{-1/2}$$
$$\cdot\left[(3/\pi)(j-m-1)(j-m-2)(j-m)(j+m)(2j-1-2\delta)(2j-1+2\delta)\right]^{1/2},$$

$$(m=-(j-1),\cdots,(j-2)). \qquad \text{when } j'=j-2,\ m'=m+1; \qquad (3.16a)$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm}=\left[(2j-3+2g)^{1/2}\mp(2j-3-2g)^{1/2}\right]\left[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}\right]$$
$$\cdot\left[32j(j-1)(2j-1)\right]^{-1}\left[(2j-3)(2j+1)\right]^{-1/2}$$
$$\cdot\left[(3/\pi)(j+m-1)(j+m-2)(j+m)(j-m)(2j-1-2g)(2j-1+2g)\right]^{1/2},$$
$$(m=-(j-2),\cdots,(j-1)),\ \text{when}\ j'=j-2,\ m'=m-1; \tag{3.16b}$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm}=(8j)^{-1}\left[(3/\pi)(j-m)(j-m-1)\right]^{1/2}\left[(2j+1)(2j-1)\right]^{-1/2}$$
$$\cdot\Big\{(j+m)\left[4(j-1)(2j-1)\right]^{-1}\left[(2j-1+2g)(2j-1-2g)\right]^{1/2}\left[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}\right]$$
$$\cdot\left[(2j-1+2g)^{1/2}\pm(2j-1-2g)^{1/2}\right]$$
$$\mp g(2m+1)\left[(2j-1)(2j+1)\right]^{-1}\left[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}\right]$$
$$\cdot\left[(2j-1+2g)^{1/2}\mp(2j-1-2g)^{1/2}\right]$$
$$+(j+m+1)\left[4(j+1)(2j+1)\right]^{-1}\left[(2j+1-2g)(2j+1+2g)\right]^{1/2}\left[(2j\mp1+2g)^{1/2}-(2j\mp1-2g)^{1/2}\right]$$
$$\cdot\left[(2j\pm1+2g)^{1/2}+(2j\pm1-2g)^{1/2}\right]\Big\},$$
$$(m=-j,\cdots,(j-1)),\quad \text{when}\ j'=j-1,\ m'=m+1; \tag{3.16c}$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm}=(8j)^{-1}\left[(3/\pi)(j+m)(j+m-1)\right]^{1/2}\left[(2j+1)(2j-1)\right]^{-1/2}$$
$$\cdot\Big\{(j-m)\left[4(j-1)(2j-1)\right]^{-1}\left[(2j-1+2g)(2j-1-2g)\right]^{1/2}$$
$$\cdot\left[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}\right]\left[(2j-1+2g)^{1/2}\pm(2j-1-2g)^{1/2}\right]$$
$$\pm g(2m-1)\left[(2j-1)(2j+1)\right]^{-1}\left[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}\right]$$
$$\cdot\left[(2j-1+2g)^{1/2}\mp(2j-1-2g)^{1/2}\right]$$
$$+(j-m+1)\left[4(j+1)(2j+1)\right]^{-1}\left[(2j+1-2g)(2j+1+2g)\right]^{1/2}$$
$$\cdot\left[(2j+1+2g)^{1/2}\pm(2j+1-2g)^{1/2}\right]\left[(2j-1+2g)^{1/2}\mp(2j-1-2g)^{1/2}\right]\Big\},$$
$$(m=-(j-1),\cdots,j),\quad \text{when}\ j'=j-1,\ m'=m-1; \tag{3.16d}$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm}=\pm\left[(3/\pi)(j-m)(j+m+1)\right]^{1/2}\left[2(2j+1)^2\right]^{-1}$$
$$\cdot\Big\{\left[(2j+1-2g)(2j+1+2g)\right]^{1/2}\left[8j(j+1)\right]^{-1}\left[(2j+1)^2\pm(2m+1)(2j+1-2g)^{1/2}(2j+1+2g)^{1/2}\right]$$
$$\mp g^2(2m+1)\left[(j(2j-1))^{-1}+((j+1)(2j+3))^{-1}\right]\Big\},$$
$$(m=-j,\cdots,(j-1)),\quad \text{when}\ j'=j,\ m'=m+1; \tag{3.16e}$$

$$
\begin{aligned}
(Y_{010}\sigma_1)^{st}_{j'm'jm} = {} & [(3/\pi)(j+m)(j-m+1)]^{1/2}[2(2j+1)^2]^{-1} \\
& \cdot\{[(2j+1-2g)(2j+1+2g)]^{1/2}[8j(j+1)]^{-1}[(2m-1)(2j+1-2g)^{1/2}(2j+1+2g)^{1/2}\mp(2j+1)^2] \\
& -g^2(2m-1)[(j(2j-1))^{-1}+((j+1)(2j+3))^{-1}]\}, \\
& (m=-(j-1),\cdots,j), \qquad \text{when } j'=j,\ m'=m-1;
\end{aligned}
\tag{3.16f}
$$

$$
\begin{aligned}
(Y_{010}\sigma_1)^{st}_{j'm'jm} = {} & [(3/\pi)(j+m+1)(j+m+2)]^{1/2}[8(j+1)]^{-1}[(2j+1)(2j+3)]^{-1/2} \\
& \cdot\{(j-m)[(2j+1-2g)(2j+1+2g)]^{1/2}[4j(2j+1)]^{-1}[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}] \\
& \cdot[(2j+3+2g)^{1/2}\pm(2j+3-2g)^{1/2}] \\
& \mp g(2m+1)[(2j+1)(2j+3)]^{-1}[(2j+1+2g)^{1/2}\pm(2j+1-2g)^{1/2}][(2j+3+2g)^{1/2}\pm(2j+3-2g)^{1/2}] \\
& +(j-m+1)[(2j+3-2g)(2j+3+2g)]^{1/2}[4(j+2)(2j+3)]^{-1}[(2j+1+2g)^{1/2}\pm(2j+1-2g)^{1/2}] \\
& \cdot[(2j+3+2g)^{1/2}\mp(2j+3-2g)^{1/2}]\}, \\
& (|m|\leq j), \qquad \text{when } j'=j+1,\ m'=m+1;
\end{aligned}
\tag{3.16g}
$$

$$
\begin{aligned}
(Y_{010}\sigma_1)^{st}_{j'm'jm} = {} & [(3/\pi)(j-m+1)(j-m+2)]^{1/2}[8(j+1)]^{-1}[(2j+1)(2j+3)]^{-1/2} \\
& \cdot\{(j+m)[(2j+1-2g)(2j+1+2g)]^{1/2}[4j(2j+1)]^{-1}[(2j+1+2g)^{1/2}\mp(2j+1-2g)^{1/2}] \\
& \cdot[(2j+3+2g)^{1/2}\pm(2j+3-2g)^{1/2}] \\
& \pm g(2m-1)[(2j+1)(2j+3)]^{-1}[(2j+1+2g)^{1/2}\pm(2j+1-2g)^{1/2}][(2j+3+2g)^{1/2}\pm(2j+3-2g)^{1/2}] \\
& +(j+m+1)[(2j+3-2g)(2j+3+2g)]^{1/2}[4(j+2)(2j+3)]^{-1} \\
& \cdot[(2j+1+2g)^{1/2}\pm(2j+1-2g)^{1/2}][(2j+3+2g)^{1/2}\mp(2j+3-2g)^{1/2}]\}, \\
& (|m|\leq j), \qquad \text{when } j'=j+1,\ m'=m-1;
\end{aligned}
\tag{3.16h}
$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm} = \pm\left[(3/\pi)(j-m+1)(j+m+1)(j+m+2)(j+m+3)(2j+3-2\gamma)(2j+3+2\gamma)\right]^{1/2}$$
$$\cdot\left[32(j+1)(j+2)(2j+3)\right]^{-1}\left[(2j+5)(2j+1)\right]^{-1/2}$$
$$\cdot\left[(2j+5+2\gamma)^{1/2}\pm(2j+5-2\gamma)^{1/2}\right]\left[(2j+1+2\gamma)^{1/2}\pm(2j+1-2\gamma)^{1/2}\right],$$
$$(m=-(j+1),\cdots,j), \quad \text{when } j'=j+1,\ m'=m+1; \qquad (3.16i)$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm} = \mp\left[(3/\pi)(j+m+2)(j-m+1)(j-m+2)(j-m+3)(2j+3-2\gamma)(2j+3+2\gamma)\right]^{1/2}$$
$$\cdot\left[32(j+1)(j+2)(2j+3)\right]^{-1}\left[(2j+5)(2j+1)\right]^{-1/2}$$
$$\cdot\left[(2j+5+2\gamma)^{1/2}\pm(2j+5-2\gamma)^{1/2}\right]\left[(2j+1+2\gamma)^{1/2}\pm(2j+1-2\gamma)^{1/2}\right],$$
$$(m=-j,\cdots,(j+1)), \quad \text{when } j'=j+2,\ m'=m-1; \qquad (3.16j)$$

$$(Y_{010}\sigma_1)^{st}_{j'm'jm} = 0, \qquad \text{when } j'm' \text{ take other values.} \qquad (3.16k)$$

In the above, when st = 12(21), the upper (lower) sign is taken.

Reference [8] presents analytic approximate results for dyon-fermion binding energies and the corresponding bound-state wave functions for angular momentum $j \geq |q|+\frac{1}{2}$ with the Kazama-Yang term. But their results are only valid in the limit of weak binding, $M-E \ll M$, and for small dyon charges $ZZ_d e^2 \ll 1$.

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APPENDIX A - SOME PROPERTIES OF MONOPOLE HARMONICS [3,22-24]

1. $$\hat{L}^2 Y_{q,L,M} = L(L+1) Y_{q,L,M},$$
$$\hat{L}_z Y_{q,L,M} = M Y_{q,L,M},$$
$$L = |q|, |q|+1, |q|+2, \cdots, \qquad M = -L, -L+1, \cdots, L. \tag{A.1}$$

2. $$\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi \, Y^*_{q,L',M'}(\theta,\varphi) Y_{q,L,M}(\theta,\varphi) = \delta_{L'L}\delta_{M'M}. \tag{A.2}$$

For the fixed q, $Y_{q,L,M}$ is orthogonal and normalized.

3. $$Y_{0,L,M}(\theta,\varphi) = Y_{L,M}(\theta,\varphi) \tag{A.3}$$

which is ordinary harmonics.

4. $$Y_{q,L,M} Y_{q',L',M'} = \sum_{L''} (-1)^{L+L'+L''+q''+M''} \left[\frac{(2L+1)(2L'+1)(2L''+1)}{4\pi}\right]^{1/2}$$
$$\cdot \begin{pmatrix} L & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{pmatrix} L & L' & L'' \\ q & q' & q'' \end{pmatrix} Y_{-q'',L'',-M''}. \qquad \text{(A.4)[22]}$$

where M" = -M-M', q" = -q-q', L" takes all the possible values of coupled $\vec{L}$ and $\vec{L}'$. (A.4) is the addition theorem of monopole harmonics.

APPENDIX B - WIGNER 3-j SYMBOLS AND C-G COEFFICIENTS [26]

1. Definition:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-1/2}\langle j_1m_1j_2m_2|j_1j_2j_3-m_3\rangle \tag{B.1}$$

2. Some symmetry properties

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}, \tag{B.2}$$

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{B.3}$$

3. $$\langle j_a m_a j_b m_b | j_a j_b j m\rangle = (-1)^{j_a+j_b-j}\langle j_b m_b j_a m_a | j_b j_a j m\rangle \tag{B.4}$$

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