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FERMIONIC PATH INTEGRALS, THE NICOLAI MAP AND THE WITTEN INDEX

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A B S T R A C T

We present an explicit evaluation of the coherent state fermion path integral and discuss our results in the light of supersymmetric quantum mechanics, the Nicolai map and the Witten index.

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Path integrals have played an important role in the formulation of, and also as a very efficient calculational technique in quantum mechanics, quantum field theory and many other branches of physics[1]. For bosonic degrees of freedom, very detailed and explicit calculations have been carried out using functional integration methods. The same has also been attempted for systems involving fermions by introducing path integrals over Grassmann variables[2]. In most of the calculations, however, the fermionic path integrals have been used rather formally to get the required fermionic determinant without really evaluating the integrals. The determinant is then evaluated by solving an eigenvalue problem imposing appropriate boundary conditions. The purpose of this letter is to present an explicit evaluation of the fermion path integral. This yields, in addition, to an expression for the fermion determinant also the fermion propagator, energy levels and all the transition matrix elements.

To be specific, we restrict ourselves to a Fermi system with a single spin variable and use the coherent state representation. Generalization to many degrees of freedom is straightforward. Our results find a direct application in supersymmetric quantum mechanics[3]. In particular we obtain the fermion determinant precisely needed to cancel the bosonic Jacobian in the Nicolai map[4] and the determinant giving the Witten index[5]. In the space of coherent states the kernel of the time evolution operator is given by ($\hbar$=1) [see Faddeev in Ref. 2)]

$$\mathcal{U}(t_b,\bar{\eta}_b;t_a,\eta_a) = \int\limits_{\substack{\eta(t_a)=\eta_a\\ \bar{\eta}(t_b)=\bar{\eta}_b}} \mathcal{D}\bar{\eta}(t)\,\mathcal{D}\eta(t)\; e^{\bar{\eta}_b\eta(t_b)+i\int_{t_a}^{t_b}dt\left[i\bar{\eta}(t)\dot{\eta}(t)-H(t;\bar{\eta},\eta)\right]}$$

$$= \lim_{N\to\infty}\int d\bar{\eta}_{N-1}d\eta_{N-1}\cdots d\bar{\eta}_1 d\eta_1\; e^{\bar{\eta}_N\eta_N-\sum_{k=1}^{N}\left[\bar{\eta}_k(\eta_k-\eta_{k-1})+i\varepsilon H(t_k;\bar{\eta}_k,\eta_{k-1})\right]} \tag{1}$$

Here $\bar{\eta}_k$ and η_k denote Grassmann variables satisfying $\{\eta_k,\eta_\ell\} = \{\bar{\eta}_k,\bar{\eta}_\ell\} = \{\bar{\eta}_k,\eta_\ell\} = 0$ for all k and ℓ; $\bar{\eta}_k \equiv \bar{\eta}(t_k)$, etc., $t_k = t_a+k\varepsilon$, $\varepsilon = (t_b-t_a)/N$. The boundary conditions are imposed by requiring η to be fixed at $t = t_a$, $\eta_o = \eta_a$, and $\bar{\eta}$ to be fixed at $t = t_b$, $\bar{\eta}_N = \bar{\eta}_b$. $H(t;\bar{\eta},\eta)$ is obtained from the Hamiltonian $H(t;a^+,a)$ in "normal ordered form" in the sense of Faddeev[2] by replacing the fermion creation and annihilation operators a^+,a as $a^+ \to \bar{\eta}$, $a \to \eta$.

It is worthwhile to note here that the continuum version of U in Eq. (1) is sometimes written in a symmetrized form using integration by parts as [see Fadaeev, Ref. 2)]

$$\mathcal{U}(b,a)=\int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\; e^{\frac{1}{2}\left(\bar{\eta}_b\eta(t_b)+\bar{\eta}(t_a)\eta_a\right)+i\int_{t_a}^{t_b}dt\left[\frac{i}{2}\left(\bar{\eta}\dot{\eta}-\dot{\bar{\eta}}\eta\right)-H(t;\bar{\eta},\eta)\right]} \tag{2}$$

In general, such a formal manipulation in path integrals leads to incorrect results due to their stochastic nature. In fact we find that if one takes, say, the forward difference for both time derivatives in Eq. (2) and considers the free case (H=0), for example, the action at each time lattice point no longer remains quadratic but becomes linear. Thus the integral for the Bose case converges no more and hence makes no sense. For the fermion case, even though the integral is finite, one does not recover Eq. (1) which is the correct expression for the kernel on the lattice.

Thus, to establish equality between the lattice versions of Eqs. (1) and (2), the time derivatives in Eq. (2) have got to be chosen in a specific way by taking a forward difference for one and a backward difference for the other, i.e., $\int dt[\bar{\eta}\dot{\eta}-\dot{\bar{\eta}}\eta] \to \sum_k[\bar{\eta}_k(\eta_k-\eta_{k-1})-(\bar{\eta}_k-\bar{\eta}_{k-1})\eta_{k-1}]$[6]. This observation shows that there is just one unique way of defining Eq. (2) on the lattice. This further implies that the problem emphasized by Schulman [Ref. 1), p. 249] connected with the neglected higher order terms does not really exist.

Now we consider the motion of a quantum spin-½ in a time-dependent magnetic field B(t) along the z-axis coupled to two external time-dependent Grassmann sources J(t), $\bar{J}(t)$ characterized by the Hamiltonian

$$H=-\frac{1}{2}B(t)\sigma_z-\bar{J}(t)\sigma_+-\sigma_-J(t) \tag{3}$$

where $\sigma_\pm=(\sigma_x\pm i\sigma_y)/2$ and $\sigma_{x,y,z}$ are the Pauli matrices. With $\sigma_+=a$, $\sigma_-=a^+$ and $\sigma_z=[a,a^+]$ one obtains the following normal ordered form of the Hamiltonian (3)

$$H(t;a^+,a)=B(t)\left(a^+a-\frac{1}{2}\right)-\bar{J}(t)\,a-a^+J(t) \tag{4}$$

in terms of the fermionic creation and annihilation operators a^+, a satisfying $\{a,a^+\}=1$. Inserting the Hamiltonian (4) into the path integral (1) and carrying out the integrations at every lattice point we obtain

$$\mathcal{U}(t_b,\bar\eta_b;t_a,\eta_a)=\lim_{N\to\infty}\exp\Big\{\bar\eta_N\prod_{k=1}^{N}(1-i\varepsilon B_k)\eta_0\ +i\bar\eta_N\sum_{k=1}^{N}\varepsilon J_k\prod_{\ell=k+1}^{N}(1-i\varepsilon B_\ell)$$

$$+i\sum_{k=1}^{N}\varepsilon\bar J_k\prod_{\ell=1}^{k-1}(1-i\varepsilon B_\ell)\eta_0\ -\sum_{k=2}^{N}\varepsilon\bar J_k\sum_{\ell=1}^{k-1}\varepsilon J_\ell\prod_{m=\ell+1}^{k-1}(1-i\varepsilon B_m)\Big\}\cdot$$

$$\cdot\, e^{\frac{i}{2}\int_{t_a}^{t_b}dt\ B(t)} \tag{5}$$

just by using the standard Gaussian integration rules for the Grassmann variables. The continuum limit of Eq. (5) can be easily written down as

$$\mathcal{U}(t_b,\bar\eta_b;t_a,\eta_a)=\exp\Big\{\bar\eta_b\big[e^{-i\int_{t_a}^{t_b}dt\,B(t)}\big]\eta_a\ +i\bar\eta_b\int_{t_a}^{t_b}dt\ J(t)\,e^{-i\int_{t_a}^{t_b}dt'\,\theta(t'-t)B(t')}$$

$$+i\int_{t_a}^{t_b}dt\ \bar J(t)\ e^{-i\int_{t_a}^{t_b}dt'\ \theta(t-t')B(t')}\ \eta_a$$

$$-\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\ \bar J(t)\Big[\theta(t-t')\,e^{-i\int_{t'}^{t}dt''\,B(t'')}\Big]J(t')\Big\}\ e^{\frac{i}{2}\int_{t_a}^{t_b}dt\ B(t)}\quad. \tag{6}$$

Equation (6) has the following general decomposition

$$\mathcal{U}\ =\ K_{00}\ +\ K_{11}\,\bar\eta_b\eta_a\ +\ \bar\eta_b\,K_{10}+K_{01}\,\eta_a \tag{7}$$

where the coefficients K_{mn} are given by

$$K_{00}=e^{\frac{i}{2}\int_{t_a}^{t_b}dt\,B(t)}\quad e^{-\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\ \bar J(t)\,D_F(t,t')\,J(t)}$$

$$D_F(t,t')=\theta(t-t')\ e^{-i\int_{t'}^{t}dt''\ B(t'')}$$

$$K_{11}\ =\ e^{-i\int_{t_a}^{t_b}dt\ B(t)}\ \cdot K_{00}$$

$$K_{10}=i\int_{t_a}^{t_b}dt\ J(t)\ e^{-i\int_{t}^{t_b}dt'\ B(t')}\ \cdot K_{00}$$

$$K_{01}=i\int_{t_a}^{t_b}dt\ \bar J(t)\,e^{-i\int_{t_a}^{t}dt'\ B(t')}\ \cdot K_{00}\quad. \tag{8}$$

The K_{mn}'s are actually the matrix elements of the time evolution operator in the two-dimensional Fock space spanned by the two vectors $|0\rangle$ and $|1\rangle$ defined by $a|0\rangle = 0$ and $a^+|0\rangle = |1\rangle$;

$$K_{mn} = \langle m | \, T \, e^{-i\int_{t_a}^{t_b} dt \, H(t; a^+, a)} | n \rangle \, . \tag{9}$$

In supersymmetric quantum mechanics[3], one considers the Hamiltonian

$$H^{SUSY} = \frac{p^2}{2} + \frac{1}{2} \left(V(q)\right)^2 + \frac{1}{2} \left[a^+, a\right] V'(q) \tag{10}$$

which leads to the quantum Lagrangian

$$L^{SUSY} = \frac{\dot{q}^2}{2} - \frac{1}{2} \left(V(q)\right)^2 + i \, a^+ \dot{a} - \frac{1}{2} \left[a^+, a\right] V'(q) \, . \tag{11}$$

In calculating the generating functional by path integration, the commutator in the Lagrangian (11) is usually replaced by $\frac{1}{2}[a^+, a] \to \bar{\psi}\psi$ treating ψ, $\bar{\psi}$ as "classical" Grassmann variables, and the fermionic path integral is set equal to the fermion determinant:

$$\int \mathcal{D}\bar{\psi}(t) \, \mathcal{D}\psi(t) \, e^{i\int_{t_a}^{t_b} dt \, \bar{\psi} \left[i\frac{d}{dt} - V'(q(t)) \right] \psi} = \det\left(i\frac{d}{dt} - V'(q(t)) \right) . \tag{12}$$

However, we believe that the correct replacement of the commutator to be used in the fermionic path integral should be $\frac{1}{2}[a^+, a] \to \bar{\psi}\psi - \frac{1}{2}$. The simplest way to see this is to consider the ground state energy of the non-interacting case, $V(q) = q$. The additional contribution of $-\frac{1}{2}$ is precisely needed to cancel the ground state energy $+\frac{1}{2}$ of the Bose oscillator for supersymmetry to be valid.

In the literature, Eq. (12) is essentially used as the defining equation of the fermion determinant which is later evaluated not from the path integral but by solving an eigenvalue problem with appropriate boundary conditions. Since the determinant is finally normalized by hand, one arrives at the correct kernel even though the correct quantum Lagrangian has not been used.

Actually the path integral (12) does not stand well defined without specifying the boundary conditions. If the path integral (12) is used to define the generating functional as is the case with the Nicolai map[4], it must be interpreted as the vacuum-vacuum transition amplitude which is identical to the matrix element K_{oo} of Eq. (7) with $J = \bar{J} = 0$ and $B(t) = V'(q(t))$. Thus we obtain from Eq. (8)

$$\det\left(i\frac{d}{dt}-V'(q(t))\right)_{\text{vacuum}} \equiv K_{00}\Big|_{J=\bar{J}=0}$$

$$=-\int d\bar{\eta}\,d\eta\, e^{-\bar{\eta}\eta}\,\bar{\eta}\left\{\mathcal{U}(t_b,\bar{\eta};t_a,\eta)\Big|_{J=\bar{J}=0}\right\}\eta = e^{\frac{i}{2}\int_{t_a}^{t_b}dt\, V'(q(t))} \tag{13}$$

The result (13) for the fermion determinant is exactly the inverse of the bosonic Jacobian for the Nicolai map as evaluated by Ezawa and Klauder[7] using the Stratonovich prescription. It is, therefore, our contention that the cancellation of the fermion determinant and the bosonic Jacobian under the Nicolai map does not require a modified path integral as proposed by the above authors.

On the other hand, if the path integral (12) is meant to represent the trace of the time-evolution operator as done by Gildener and Patrascioiu and Cooper and Freedman[8], then we easily obtain it from our path integral result (8) putting $J = \bar{J} = 0$ and $B(t) = V'(q(t))$ as

$$\det\left(i\frac{d}{dt}-V'(q(t))\right)_{\text{trace}} \equiv \mathrm{Tr}\left(T e^{-i\int_{t_a}^{t_b}dt\, H_F}\right) = \left[K_{00}+K_{11}\right]_{J=\bar{J}=0}$$

$$=\int d\bar{\eta}\,d\eta\, e^{-\bar{\eta}\eta}\,\mathcal{U}(t_b,-\bar{\eta};t_a,\eta)\Big|_{J=\bar{J}=0} = 2\cos\left[\frac{1}{2}\int_{t_a}^{t_b}dt\, V'(q(t))\right] \tag{14}$$

where H_F is the fermionic part of the Hamiltonian (10). Notice that the argument $-\bar{\eta}$ of U in Eq. (14) is the origin of taking the antiperiodic boundary condition in the evaluation of the fermion determinant in the earlier works. The Euclidean version of our result (14) is identical to the results obtained earlier[8]. Thus we have shown that the fermionic determinant can be directly evaluated by performing path integrations alone.

Evidently, we also easily obtain the generating functional (= vacuum persistence amplitude) for the pure fermionic system,

$$Z_F[J,\bar{J}] = \frac{K_{00}}{K_{00}\big|_{J=\bar{J}=0}} = e^{-\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\,\bar{J}(t)\,D_F(t,t')\,J(t')} \tag{15}$$

One immediately reads off the fermion propagator in the presence of the external field $B(t)$ to be given by the function $D_F(t,t')$ of Eq. (8) which, again, fully agrees with the Euclidean version obtained by Gildener and Patrascioiu[8].

The partition function of the pure fermionic system in the case of a constant magnetic field $B(t) \equiv 2\omega > 0$ can be easily obtained from Eq. (14) leading to identification of two energy levels with energy $E_0 = -\omega$ and $E_1 = +\omega$ as is expected for a spin-½ particle in the presence of a constant magnetic field.

As a measure of supersymmetry breaking, Witten[5] has introduced the index $\Delta = \mathrm{Tr}(-1)^F$ where $(-1)^F = 1-2a^+a$ is the "fermion" number operator. The regularized version is defined as follows,

$$\Delta(\beta) \equiv \mathrm{Tr}\left[(-1)^F e^{-\beta H^{SUSY}} \right]$$

$$= \int_{-\infty}^{\infty} dx \int_{q(0)=x}^{q(\beta)=x} \mathcal{D}q(s)\, e^{-\frac{1}{2}\int_0^\beta ds\left[\dot{q}^2 + V^2(q(s))\right]} \mathrm{Tr}\left[(-1)^F e^{-\beta H_F}\right] \quad (16)$$

where the trace over the bosonic degrees of freedom has been converted to the path integral form. The remaining trace over the fermionic degrees of freedom is immediately obtained from our Eq. (8) as

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H_F}\right] \equiv \det\left(\frac{d}{ds} + V'(q(s))\right)_{S\text{-trace}} = \left[K^E_{00} - K^E_{11}\right]_{J=\bar{J}=0}$$

$$= \int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta}\, \mathcal{U}(-i\beta,\bar{\eta};0,\eta)\Big|_{J=\bar{J}=0} = 2\sinh\left[\frac{1}{2}\int_0^\beta ds\, V'(q(s))\right] \quad (17)$$

where superscript E stands for Euclidean form. It is to be noted that in this case we have kept the arguments of U unchanged which implements periodic boundary condition in calculating the determinant. Inserting Eq. (17) into Eq. (16) the complete expression for Witten index is obtained.

In conclusion, we have shown that the coherent state path integral for fermions yields all the information about the physically important quantities such as fermion determinants, transition matrix elements, energy levels and fermion propagator in a rather straightforward manner. This, we believe, clarifies the results of earlier investigations which exist in the literature as discussed in the text.

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