Helly Property, Clique Graphs, Complementary Graph Classes, and Sandwich Problems*

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Abstract

A sandwich problem for property Π *asks whether there exists a sandwich graph of a given pair of graphs which has the desired property* Π. *Graph sandwich problems were first defined in the context of Computational Biology as natural generalizations of recognition problems. We contribute to the study of the complexity of graph sandwich problems by considering the Helly property and complementary graph classes. We obtain a graph class defined by a finite family of minimal forbidden subgraphs for which the sandwich problem is* NP-*complete. A graph is clique-Helly when its family of cliques satisfies the Helly property. A graph is hereditary clique-Helly when all of its induced subgraphs are clique-Helly. The clique graph of a graph is the intersection graph of the family of its cliques. The recognition problem for the class of clique graphs was a long-standing open problem that was recently solved. We show that the sandwich problems for the graph classes: clique, clique-Helly, hereditary clique-Helly, and clique-Helly nonhereditary are all* NP-*complete. We propose the study of the complexity of sandwich problems for complementary graph classes as a mean to further understand the sandwich problem as a generalization of the recognition problem.*

Keywords: Helly property, Clique graphs, Sandwich problems, Computational difficulty of problems.

1. INTRODUCTION

We consider simple, finite, undirected graphs. Given a graph $G = (V, E)$, a *complete set* of G is a subset of V inducing a complete subgraph. A *clique* is a maximal complete set. Let $\mathcal{F}$ be a family of subsets of a set S. We say that $\mathcal{F}$ satisfies the *Helly property* when every subfamily consisting of pairwise intersecting subsets has a non-empty intersection. The Helly property has an important role in Combinatorics and its computational aspects [5]. A graph is *clique-Helly* when its family of cliques satisfies the Helly property. A graph is *hereditary clique-Helly* when all of its induced subgraphs are clique-Helly. This graph class admits a finite family of minimal forbidden subgraphs. A clique-Helly graph that is not hereditary clique-Helly is called *clique-Helly nonhereditary*. The *clique graph of a graph* is the intersection graph of the family of its cliques. G is a *clique graph* if there exists a graph H such that G is the clique graph of H. The recognition problems for the classes of clique-Helly graphs and of hereditary clique-Helly graphs are known to be polynomial. The recognition problem for the class of clique graphs was a long-standing open problem [3, 14, 16, 18] that was recently solved [1]. Clique graphs have been much studied as intersection graphs, and in the context of graph operators, and are included in several books [3, 12, 15].

We consider the following generalization of recognition problems. We say that a graph $G_1 = (V, E_1)$ is a *spanning* subgraph of $G_2 = (V, E_2)$ if $E_1 \subseteq E_2$; and that

a graph $G = (V, E)$ is a *sandwich* graph for the pair G_1, G_2 if $E_1 \subseteq E \subseteq E_2$. We call E_1 the *forced edge set*, $E_2 \setminus E_1$ the *optional edge set*. The GRAPH SANDWICH PROBLEM FOR PROPERTY Π is defined as follows [8]:

GRAPH SANDWICH PROBLEM FOR PROP-
  ERTY Π
Instance: Vertex set V, forced edge set E_1, op-
  tional edge set $E_2 \setminus E_1$.
Question: Is there a graph $G = (V, E)$ such that
  $E_1 \subseteq E \subseteq E_2$ that satisfies property Π?

Graph sandwich problems were defined in the context of Computational Biology and have attracted much attention lately arising from many applications and as a natural generalization of recognition problems [4, 7, 8, 9, 11]. The recognition problem for a class of graphs $\mathcal{C}$ is equivalent to the graph sandwich problem in which the forced edge set $E_1 = E$, the optional edge set $E_2 \setminus E_1 = \emptyset$, $G = (V, E)$ is the graph we want to recognize, and property Π is "to belong to class $\mathcal{C}$".

Golumbic et al. [8] have considered sandwich problems with respect to several subclasses of perfect graphs, and proved that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains in P. On the other hand, they proved that the GRAPH SANDWICH PROBLEM FOR PERMUTATION GRAPHS turns out to be NP-complete.

Clique graphs admit a characterization by a special edge cover. A complete set C *covers* the edge uv if the endvertices u and v belong to C. A *complete edge cover* of G is a family of complete sets covering all edges of G.

Theorem 1 (Roberts and Spencer [16]) *A graph G is a clique graph if and only if there exists a complete edge cover of G satisfying the Helly property.*

Such a family of complete sets is called an RS-family for $G = (V, E)$. If G admits an RS-family, then G admits a *small RS-family* with at most $|E|$ sets, because every edge needs to be covered by only one complete set and any subfamily of a family satisfying the Helly property does satisfy the Helly property too. Therefore, the characterization given in Theorem 1 gives a short certificate for the class of clique graphs because the Helly property for a family of subsets can be tested in polynomial time on the size of the family [2, 16, 18]. On the other hand, recognizing clique graphs was recently shown to be NP-complete [1].

The family of all cliques of G is a complete edge cover of G, so it follows that clique-Helly graphs are clique graphs. A polynomial-time recognition algorithm for the class of clique-Helly graphs was presented in [6] and independently in [17]. This algorithm is a consequence of a characterization of clique-Helly graphs based on the concept of extended triangle. A *triangle* T of a graph G is

a complete set containing exactly three vertices. The *extended triangle of G relative to* the triangle T is defined in [17] as the subgraph induced in G by the vertices adjacent to at least two vertices of T and it is denoted T'. A *universal vertex* in a graph is adjacent to every other vertex of the graph.

Theorem 2 (Dragan [6], Szwarcfiter [17]) *A graph G is a clique-Helly graph if and only if every extended triangle of G contains a universal vertex.*

A polynomial-time recognition algorithm for the class of hereditary clique-Helly graphs was presented in [13]. This algorithm uses a finite family of induced forbidden subgraphs, the so-called *ocular graphs* depicted in Figure 1.

Given a graph G and a triangle $T = \{t_1, t_2, t_3\}$, we say that T *yields an ocular graph in G* if there are vertices s_1, s_2 and s_3 such that vertex s_1 is not adjacent to t_1 and is adjacent to t_2 and t_3, vertex s_2 is not adjacent to t_2 and is adjacent to t_1 and t_3, and vertex s_3 is not adjacent to t_3 and is adjacent to t_1 and t_2. The graph on the left of Figure 1 is known as the *Hajós graph* and the triangle that yields this particular ocular graph is known as the *central triangle* of this Hajós graph. Ocular graphs were defined in [19] as a tool to study the Helly property and were shown in [13] to be the minimal forbidden configurations for hereditary clique-Helly graphs.
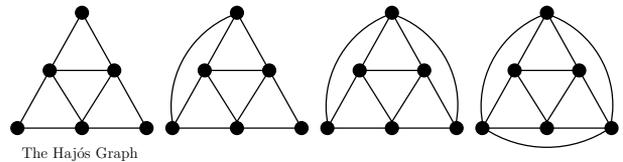

The Hajós Graph

Figure 1. The ocular graphs are the minimal forbidden induced subgraphs for a hereditary clique-Helly graph

Theorem 3 (Prisner [13]) *A graph G is a hereditary clique-Helly graph if and only if G contains none of the four graphs shown in Figure 1 as an induced subgraph.*

Clearly, from Theorems 2 and 3 one can also recognize in polynomial time whether a graph is clique-Helly nonhereditary.

We contribute to the study of the complexity of graph sandwich problems by considering the Helly property and complementary graph classes. Surprisingly, we obtain a graph class defined by a finite family of minimal forbidden subgraphs for which the corresponding sandwich problem is NP-complete. Additionally, we propose the study of the complexity of sandwich problems for complementary graph classes as a mean to further understand

the sandwich problem as a generalization of the recognition problem.

The studies on sandwich problems focus on properties that are neither hereditary nor ancestral on subgraphs, in which case the complexity of the recognition problem and of the corresponding sandwich problem coincide. The graph classes considered in the present paper: clique, clique-Helly, hereditary clique-Helly are neither hereditary nor ancestral on subgraphs. Most properties considered so far are hereditary on induced subgraphs, which means that the corresponding graph class admits a family of minimal forbidden subgraphs. In particular, every studied sandwich problem corresponding to a graph class defined by a finite family of minimal forbidden subgraphs has been classified as Polynomial. Note the classes clique and clique-Helly are not hereditary on induced subgraphs. The class hereditary clique-Helly is the first example of a graph class defined by a finite family of minimal forbidden subgraphs for which the corresponding sandwich problem is NP-complete.

This paper is organized as follows: in Section 2 we prove that HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM, CLIQUE-HELLY SANDWICH PROBLEM, and CLIQUE GRAPH SANDWICH PROBLEM are NP-complete; in Section 3 we prove that CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM is NP-complete; in Section 4 we have our concluding remarks about complementary graph classes and sandwich problems.

2. CLIQUE, CLIQUE-HELLY, AND HEREDITARY CLIQUE-HELLY SANDWICH PROBLEMS

We prove that the HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM is NP-complete by a reduction from the NP-complete problem 3-SAT. These two decision problems are defined as follows.

3-SAT

Instance: Set $X = \{x_1, \ldots, x_n\}$ of variables, collection $C = \{c_1, \ldots, c_m\}$ of clauses over X such that each clause $c \in C$ has $|c| = 3$ literals.

Question: Is there a truth assignment for X such that each clause in C has at least one true literal?

HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM

Instance: Vertex set V, forced edge set E_1, optional edge set $E_2 \setminus E_1$.

Question: Is there a graph $G = (V, E)$, such that $E_1 \subseteq E \subseteq E_2$ and G is hereditary

clique-Helly?

Theorem 4 *The* HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM *is NP-complete.*

Proof. The HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM is in NP since a short certificate is a sandwich graph G that is hereditary clique-Helly together with the polynomial-time algorithm for hereditary clique-Helly recognition.

In order to reduce 3-SAT to HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM we need to construct in polynomial time a particular instance (V, E_1, E_2) of HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM from a generic instance (X, C) of 3-SAT, such that C is satisfiable if and only if (V, E_1, E_2) admits a sandwich graph $G = (V, E)$ which is hereditary clique-Helly. Recall the notation $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$.

Let (X, C) be a generic instance of 3-SAT. We assume without loss of generality that each variable occurs both as a positive and as a negative literal. For each variable x_i, add to V four *variable vertices*: $X_i, \overline{X}_i, a_i$ and b_i. For each clause $c_j = \{\ell_r, \ell_s, \ell_t\}$, where $\ell_u, u \in \{r, s, t\}$, is a literal of variable x_u, add to V *clause vertices* y_{jr}, y_{js}, y_{jt}. The set V contains $4n$ variable vertices, plus $3m$ clause vertices.
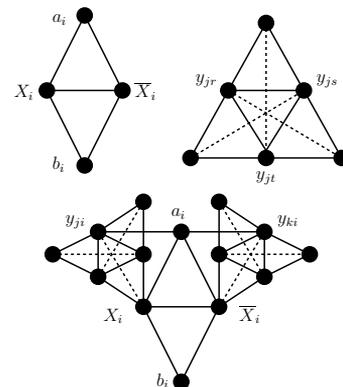

Figure 2. Forced variable gadget; forced Hajós graph corresponding to clause c_j; and Hajós graph in G_2, induced by vertices $X_i, \overline{X}_i, a_i, b_i$, y_{ji} and y_{ki}, corresponding to variable x_i

Please refer to Figure 2, where we have depicted the forced edges as continuous edges, and the optional edges as dotted edges. Each triple of clause vertices corresponding to a clause c_j induce a triangle T_j in G_1, called *clause triangle*. Vertices $a_i, b_i, X_i, \overline{X}_i$ induce a forced variable gadget D_i in G_1, composed by two triangles that share one edge $X_i \overline{X}_i$.

Let y_{jr}, y_{js}, y_{jt} be the clause vertices of a clause triangle T_j. If the clause vertex $y_{ju}, u \in \{r, s, t\}$, corresponds to the positive literal of variable x_u, then add optional

edge $X_u y_{ju}$, add two forced edges $X_u y_{jw}$, for $w \neq u$, and add forced edge $a_u y_{ju}$. If the clause vertex y_{ju}, $u \in \{r, s, t\}$ corresponds to the negative literal of variable x_u, then add optional edge $\overline{X}_u y_{ju}$, add two forced edges $\overline{X}_u y_{jw}$, for $w \neq u$, and add forced edge $a_u y_{ju}$. The construction of the particular instance (V, E_1, E_2) is concluded.

First, we show that, given a sandwich graph G' of G_1 and G_2, if G' contains an ocular graph as induced subgraph, then this ocular graph is in fact a Hajós graph. Recall that a triangle $T = \{t_1, t_2, t_3\}$ yields an ocular graph if there are vertices s_1, s_2 and s_3 such that vertex s_1 is not adjacent to t_1 and is adjacent to t_2 and t_3, vertex s_2 is not adjacent to t_2 and is adjacent to t_1 and t_3, and vertex s_3 is not adjacent to t_3 and is adjacent to t_1 and t_2. Furthermore, we show there are only two cases for a triangle to yield an ocular graph.

Please refer to Figure 2. First, note that if the triangle $T = \{y_{jr}, y_{js}, y_{jt}\}$, formed by the three clause vertices of a clause c_j, yields an ocular graph in G', this subgraph is in fact a Hajós graph, because there are no edges joining two variable vertices of distinct variables. We call such a subgraph of *clause Hajós graph*.

Now, see that if the triangle $T = \{a_i, X_i, \overline{X}_i\}$, formed by three variable vertices of a variable x_i, yields an ocular graph in G', this subgraph is also a Hajós graph, because b_i has degree two in G_2 and there are no edges joining two clause vertices of distinct clauses. We call such a graph of *variable Hajós graph*.

The only other two possibilities of a triangle T are: $T = \{b_i, X_i, \overline{X}_i\}$, for some i; and $T = \{v, l, z\}$, where v is a variable vertex, l is a clause vertex, and z is a variable or a clause vertex. In the first possibility $T = \{b_i, X_i, \overline{X}_i\}$, for some i, since b_i has degree two in G_2, such T cannot yield an ocular graph in G'. In the second possibility $T = \{v, l, z\}$, where v is a variable vertex, l is a clause vertex, and z is a variable or a clause vertex, since a_i and y_{ji} have just one common neighbor in G_2 and since the only common neighbors of y_{jr} and X_s in G_2 are y_{js} and y_{jt} it is clear that such T cannot yield an ocular graph in G'.

We shall use the characterization of Theorem 3. By the construction, it is clear that G_1 and G_2 contain, respectively, a clause Hajós graph and a variable Hajós graph, hence G_1 and G_2 are not hereditary clique-Helly.

Suppose first that C is satisfiable, and consider a truth assignment for X such that each clause c_j in C has at least one true literal. To define the sandwich graph G, if x_i has value true, add to E, for every clause c_j where x_i occurs as positive literal, the optional edge $y_{ji} X_i$; if x_i has value false, add to E, for every clause c_j where x_i occurs as negative literal, the optional edge $y_{ji} \overline{X}_i$. Note that in the constructed sandwich graph G, for each variable x_i, either all optional edges incident to vertex X_i or all

optional edges incident to vertex $\overline{X}_i$ are present. Therefore, each variable Hajós graph in G_2 is avoided in G. In addition, note that in the constructed sandwich graph G, for each clause c_j, at least one optional edge is added to the corresponding forced clause Hajós graph. Therefore, each clause Hajós graph in G_1 is avoided in G.

Conversely, suppose that G is a hereditary clique-Helly sandwich graph for the particular constructed instance (V, E_1, E_2). It follows that G contains none of the forbidden configurations depicted in Figure 1. By the construction of the particular instance (V, E_1, E_2), graph G_1 contains a forced clause Hajós graph corresponding to each clause. Therefore, in the hereditary clique-Helly sandwich graph G, for each forced clause Hajós graph, we must have at least one added optional edge. In addition, G cannot have for a variable x_i, both an optional edge incident to vertex X_i and an optional edge incident to vertex $\overline{X}_i$ present, as this would give a variable Hajós graph. Therefore we can construct a truth assignment by setting variable x_i as true if there exists in G an edge $y_{ji} X_i$ for some clause c_j where x_i occurs as positive literal, and by setting variable x_i as false otherwise. ∎
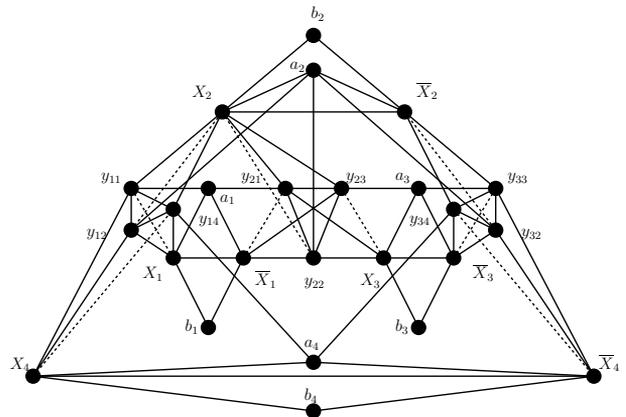

Figure 3. Example of constructed instance of (V, E_1, E_2) corresponding to clauses $\{x_1, x_2, x_4\}, \{\overline{x}_1, x_2, x_3\}, \{\overline{x}_2, \overline{x}_3, \overline{x}_4\}$

Figure 3 shows an example of an instance constructed in Theorem 4. We have depicted the forced edges as continuous edges, and the optional edges as dotted edges.

In the sequel, we further study the properties of the particular instance constructed in Theorem 4 and prove in Theorem 5 and Theorem 6, respectively, that the CLIQUE-HELLY SANDWICH PROBLEM and CLIQUE SANDWICH PROBLEM are NP-complete.

CLIQUE-HELLY SANDWICH PROBLEM
Instance: Vertex set V, forced edge set E_1, optional edge set $E_2 \setminus E_1$.

Question: Is there a graph $G = (V, E)$, such that $E_1 \subseteq E \subseteq E_2$ and G is clique-Helly?

Theorem 5 *The* CLIQUE-HELLY SANDWICH PROBLEM *is NP-complete.*

Proof. The CLIQUE-HELLY SANDWICH PROBLEM is in NP since a short certificate is a sandwich graph G that is clique-Helly together with the polynomial-time algorithm for clique-Helly recognition.

In order to establish the hardness of the problem, we perform the same reduction from 3-SAT of Theorem 4. It is sufficient to show that every sandwich graph of (V, E_1, E_2) which is clique-Helly is also hereditary clique-Helly. By contradiction, suppose that a sandwich graph G of (V, E_1, E_2) is clique-Helly but is not hereditary clique-Helly. By Theorem 3, G has as induced subgraph one of the graphs of Figure 1. Denote it by G'. By the observations present in the proof of Theorem 4, G' is indeed either a clause Hajós graph or a variable Hajós graph. In both cases there are two variable vertices v_1 and v_2 in G'. Note these two variable vertices can be of a same variable or not. Since G is clique-Helly, by Theorem 2, there must exist a vertex v adjacent to every vertex of G'. Recall every vertex of V is a clause vertex or a variable vertex. Since the degree of a clause vertex is at most 6, and in this case, its 6 neighbors do not induce a Hajós graph, we conclude that v is a variable vertex. We have a contradiction, because there is no variable vertex outside of G' adjacent to v_1 and v_2. ∎

Now, we turn our attention to the CLIQUE GRAPH SANDWICH PROBLEM and we present a simple proof of its NP-completeness.

CLIQUE GRAPH SANDWICH PROBLEM
Instance: Vertex set V, forced edge set E_1, optional edge set $E_2 \setminus E_1$.
Question: Is there a graph $G = (V, E)$, such that $E_1 \subseteq E \subseteq E_2$ and G is clique graph?

First we need an auxiliary lemma. This lemma is a consequence of a more general result established in [10].

Lemma 1 *Let G be a graph containing a Hajós Graph H as an induced subgraph. If the central triangle T of H is not contained in a K_4, then G is not a clique graph.*

Proof. Assume to get a contradiction that we have an RS-family $\mathcal{C}$ for G. We consider two cases with respect to the central triangle T of H to belong to $\mathcal{C}$ or not. Let h_1, h_2, h_3 be the vertices of T, and h_4, h_5, h_6 the other vertices of H, such that $h_4h_2, h_4h_3, h_5h_1, h_5h_2, h_6h_1, h_6h_3 \in E$. In case vertices h_i, h_j, h_k define a triangle, denote this triangle by T_{ijk}. Let H_{pq} be a complete set of $\mathcal{C}$ covering edge h_ph_q.

First, suppose that $T \notin \mathcal{C}$. Since T is not contained in a K_4, the subfamily of distinct complete sets H_{12}, H_{13}, H_{23} of $\mathcal{C}$ does not satisfy the Helly property, a contradiction.

Next, suppose that $T \in \mathcal{C}$. If $h_2 \notin H_{34}$ and $h_3 \notin H_{24}$, then H_{24}, H_{34} and T do not satisfy the Helly property. Therefore, there are complete sets $C_1, C_2, C_3 \in \mathcal{C}$ such that $h_1, h_3, h_6 \in C_1$, $h_1, h_2, h_5 \in C_2$, and $h_2, h_3, h_4 \in C_3$. And C_1, C_2, C_3 contain a common vertex if and only if $T \subset K_4$, a contradiction. ∎

Theorem 6 *The* CLIQUE GRAPH SANDWICH PROBLEM *is NP-complete.*

Proof. The CLIQUE GRAPH SANDWICH PROBLEM is in NP since a short certificate is a sandwich graph G that is a clique graph, and a small RS-family for G together with the polynomial-time algorithm for the Helly property.

Again we perform a reduction from 3-SAT by using the same particular instance of Theorem 4. Let (V, E_1, E_2) be the particular instance constructed. Since, by Theorem 1, every hereditary clique-Helly graph is a clique graph, it is sufficient to show that every sandwich graph of (V, E_1, E_2) which is a clique graph is also hereditary clique-Helly. By contradiction, suppose that the sandwich graph G of (V, E_1, E_2) is a clique graph but is not hereditary clique-Helly. Hence, by Theorem 3 and the observations of Theorem 4, G contains either a clause Hajós graph or a variable Hajós graph as induced subgraph. Denote it by G'. Since G is a clique graph, by Lemma 1, the central triangle of G' must be contained in a K_4. If G' is a clause Hajós graph, then its central triangle is of the form $T = \{y_{kr}, y_{ks}, y_{kt}\}$. Since, given $u \in \{r, s, t\}$, the only vertex adjacent to y_{ku} and not contained in G' is a variable vertex a_u, and a_u is nonadjacent to the other vertices of T, this central triangle is not in a K_4. Finally, if G' is a variable Hajós graph, its central triangle is of the form $\{a_i, X_i, \overline{X}_i\}$. This central triangle is not in a K_4 because there is no vertex adjacent in G_2 to these three vertices. ∎

3. CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM

We cannot conclude from Theorems 4 and 5 that the sandwich problem for clique-Helly nonhereditary graphs is NP-complete. Theorem 7 proves that the CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM is NP-complete by a reduction from the NP-complete problem SAT. These two decision problems are defined as follows.

SAT

Instance: Set $X = \{x_1, \ldots, x_n\}$ of variables, collection $C = \{c_1, \ldots, c_m\}$ of clauses over X.

Question: Is there a truth assignment for X such that each clause in C has at least one true literal?

CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM

Instance: Vertex set V, forced edge set E_1, optional edge set $E_2 \setminus E_1$.

Question: Is there a graph $G = (V, E)$, such that $E_1 \subseteq E \subseteq E_2$ and G is clique-Helly nonhereditary?

Theorem 7 *The* CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM *is NP-complete.*

Proof. The CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM is in NP since a short certificate is a sandwich graph G that is clique-Helly nonhereditary together with the polynomial-time algorithms for clique-Helly and hereditary clique-Helly recognition.

Let (X, C) be a generic instance of SAT. We assume without loss of generality that each variable occurs both as a positive and as a negative literal.

Each clause c_j corresponds to a copy H_j of the Hajós graph in G_1. Each pair of distinct clauses c_j, c_k correspond to an auxiliary vertex d_{jk} adjacent in G_1 to two vertices in each of the central triangles of the corresponding Hajós graphs H_j and H_k (see Figure 4).

As each variable x_i occurs both as a positive and as a negative literal, for each variable x_i, we have three vertices: *literal vertex* x_i represents the occurrences of variable x_i as a positive literal, *literal vertex* $\overline{x}_i$ represents the occurrences of variable x_i as a negative literal, and vertex X_i represents the variable. These three vertices $x_i, \overline{x}_i, X_i$ induce a *variable triangle* T_i in G_1.

We conclude the construction of particular instance (V, E_1, E_2) by defining the optional edge set $E_2 \setminus E_1$. Suppose literal $\ell \in c_j$. It corresponds to literal vertex $\ell \in V$. We add to $E_2 \setminus E_1$ six optional edges ℓz, one for each vertex z of the Hajós graph H_j. In addition, for each auxiliary vertex d_{jk}, add to $E_2 \setminus E_1$ the optional edge ℓd_{jk}.

Note that every sandwich graph of (V, E_1, E_2) is not hereditary clique-Helly, because for each clause there is a Hajós graph as induced subgraph of it. Therefore, any sandwich graph of (V, E_1, E_2) is clique-Helly nonhereditary if and only if it is clique-Helly. We shall use the characterization of Theorem 2 for clique-Helly graphs.

Suppose first that C is satisfiable, and consider a truth assignment for X such that each clause c_j in C has at least one true literal. To define the sandwich graph G, add to E,

for each Hajós graph H_j corresponding to clause c_j, the optional edges that make precisely one of the true literals of clause c_j adjacent to every vertex of H_j, and adjacent to every vertex d_{jk}. Let L be the set of the chosen true literals. Note that every added optional edge has one endpoint in L. And for an extended triangle present in G_1, say T', with a vertex z, if ℓz, with $\ell \in L$, was added to E in the construction of G, then $\ell z'$, for every vertex $z' \in V(T')$, was also added to E. This implies that all the extended triangles of G_1 have universal vertices in G. It remains to analyze the extended triangles relative to triangles of G not present in G_1. Every such triangle contains precisely one literal vertex and this vertex is universal with respect to the relative extended triangle.
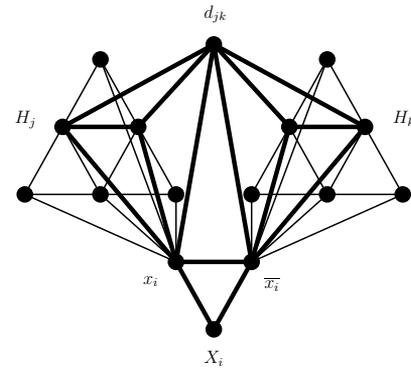

Figure 4. In G_2 extended triangle relative to triangle $\{x_i, \overline{x}_i, d_{jk}\}$ has no universal vertex

Conversely, suppose that G is a clique-Helly sandwich graph for the particular constructed instance (V, E_1, E_2). Thus, every extended triangle of G has a universal vertex. Let c_j be a clause, and consider its corresponding Hajós graph H_j. In G the extended triangle relative to the central triangle of H_j has a universal vertex. This universal vertex is a literal vertex corresponding to a literal of clause c_j. For a variable x_i, suppose that its positive literal occurs in c_j and its negative literal occurs in c_k. If in G both literal vertices x_i and $\overline{x}_i$ are universal with respect to corresponding Hajós graphs H_j and H_k, then the extended triangle relative to $x_i, \overline{x}_i, d_{jk}$ has no universal vertex. See Figure 4. Therefore, for each variable x_i, if its positive literal vertex is universal with respect to a Hajós graph associated to a clause, then set variable x_i as true, otherwise set the variable x_i as false. We have the desired truth assignment. ∎

Figure 5 shows an example of an instance constructed in the proof of Theorem 7. We have depicted the forced edges as continuous edges, and the optional edges incident to d_{jk} as dotted edges. Each Hajós graph H_j is drawn inside a square and we have depicted the six optional edges that make literal $\ell \in c_j$ universal with re-
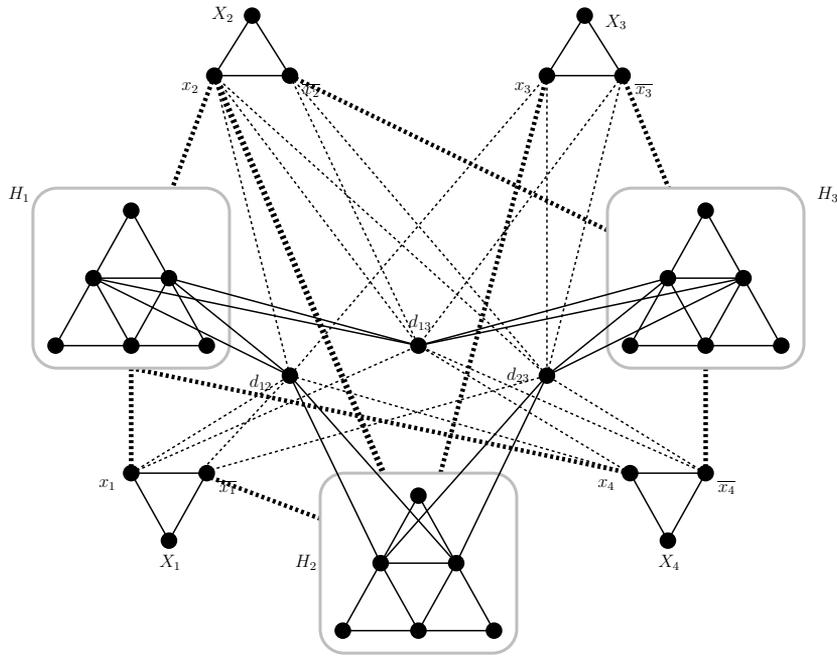
Figure 5. Example of constructed instance of (V, E_1, E_2) corresponding to clauses $\{x_1, x_2, x_4\}, \{\overline{x}_1, x_2, x_3\}, \{\overline{x}_2, \overline{x}_3, \overline{x}_4\}$

spect to the Hajós graph H_j by a dotted line joining ℓ to the square.

4. CONCLUDING REMARKS

We have presented four NP-completeness proofs, establishing that CLIQUE-HELLY SANDWICH PROBLEM, HEREDITARY CLIQUE-HELLY SANDWICH PROBLEM, CLIQUE-HELLY NONHEREDITARY SANDWICH PROBLEM, and CLIQUE GRAPH SANDWICH PROBLEM are NP-complete. We emphasize that the NP-completeness of clique graph recognition is a recently solved problem and directly implies the NP-completeness of CLIQUE GRAPH SANDWICH PROBLEM. The present paper proposed a simple NP-completeness proof for the sandwich version of the problem.

One interesting line of research is to investigate other complementary graph classes with respect to the complexity of the corresponding sandwich problems. It is well-known that one can decide in polynomial time if a graph is chordal and if it is interval. Therefore one can also decide in polynomial time if such a graph is chordal but not interval. The complexity of the corresponding sandwich problems: GRAPH SANDWICH PROBLEM FOR CHORDAL GRAPHS and GRAPH SANDWICH PROBLEM FOR INTERVAL GRAPHS was considered in [8], where both sandwich problems are proved to be NP-complete. In this case we can conclude that the GRAPH SANDWICH PROBLEM FOR CHORDAL NONINTERVAL GRAPHS is also NP-complete. More generally, consider three

graph classes, say $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, such that $\mathcal{C}_2 \subset \mathcal{C}_1$ and $\mathcal{C}_3 = \mathcal{C}_1 \setminus \mathcal{C}_2$. If both $\mathcal{C}_1$ and $\mathcal{C}_2$ are hereditary with respect to *induced* subgraphs and if $\mathcal{C}_1$ corresponds to an NP-complete sandwich problem, then $\mathcal{C}_3$ also corresponds to an NP-complete sandwich problem. Indeed, we can construct a polynomial-time reduction from GRAPH SANDWICH PROBLEM FOR $\mathcal{C}_1$ GRAPHS to GRAPH SANDWICH PROBLEM FOR $\mathcal{C}_3$ GRAPHS by using a graph of class $\mathcal{C}_1$ that is a forbidden subgraph for class $\mathcal{C}_2$. We emphasize that the present paper investigates the case where $\mathcal{C}_1$ is not hereditary with respect to induced subgraphs. Indeed, again consider three graph classes, say $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, such that $\mathcal{C}_2 \subset \mathcal{C}_1$ and $\mathcal{C}_3 = \mathcal{C}_1 \setminus \mathcal{C}_2$. If both $\mathcal{C}_1$ and $\mathcal{C}_2$ correspond to polynomial recognition problems, then $\mathcal{C}_3$ does correspond to a polynomial recognition problem as well. For the corresponding sandwich problems, we propose the study of how the complexity of the sandwich problems for $\mathcal{C}_1$ and $\mathcal{C}_2$ may determine the complexity of the sandwich problem for $\mathcal{C}_3$, even if both $\mathcal{C}_1$ and $\mathcal{C}_2$ correspond to polynomial sandwich problems: given an instance of the sandwich problem for $\mathcal{C}_1$, it may admit a sandwich graph that belongs to $\mathcal{C}_2$ and another sandwich graph that belongs to $\mathcal{C}_3$.

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