VAN EST'S EXPOSITION OF CARTAN'S PROOF OF LIE'S THIRD THEOREM, EXPLAINED BY

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1. Introduction

In this note, we explain a geometric proof of Lie's third Theorem:

Theorem 1. *A finite-dimensional Lie algebra* $\mathfrak{g}$ *over* $\mathbb{R}$ *is* integrable*, that is, there exists a Lie group* G *with* $\mathrm{Lie}(G) \cong \mathfrak{g}$.

The version stated above could not have been proved by Lie himself, because the concept of a global Lie group was missing in the 19th century. What Lie proved was a local version of the theorem. The standard proof of this result is via Ado's Theorem - stating that $\mathfrak{g}$ has a finite-dimensional faithful representation. Thus $\mathfrak{g}$ is linear, i.e. isomorphic to a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ for n large enough. But a linear Lie algebra is integrable, the group in question is generated by all exponentials of elements of $\mathfrak{g}$ inside $\mathrm{GL}_n(\mathbb{R})$. A standard, but nontrivial fact is that such a group is a Lie group (the topology is possibly different from the subpace topology induced by $\mathrm{GL}_n(\mathbb{R})$. There is an alternative proof in the book [2] that does not involve any structure theory of Lie algebras. In this note, we present a geometric proof that we found in Willem Van Est's paper [4], who in turn ascribes the proof to Elie Cartan. The ingredients of the proof are:

- The integrability of linear Lie algebras.
- A characterization of Lie groups as a manifold equipped with a certain Lie algebra of vector fields.
- A dualization of that characterization in terms of the Chevalley-Eilenberg complex of a Lie algebra.
- The vanishing of $H^2_{dR}(G)$ for a simply-connected Lie group (this is also used in [2]).

This might seem quite involved, but Ado's theorem is a hard result and I found the present proof marvellous enough to write down the details.

1.1. The idea of the proof.

In the first section, we analyze the definition of a Lie group in depth. By definition, the Lie algebra $\mathfrak{g}$ of G is the space of all left-invariant vector fields on G, with Lie bracket given by the commutator of vector fields. For each $g \in G$, the evaluation $\mathfrak{g} \to T_g G$ is an isomorphism of vector spaces; if $g = 1$, this is the well-known identification $\mathfrak{g} \cong T_1 G$.

The group G acts on itself both, from the left and the right and both actions commute. It turns out that G_L is the transformation group that *leaves L invariant*, while G_R is the transformation group that *is generated by L*. This remarkable symmetry has its roots in the double nature of vector fields, as derivations and infinitesimal symmetries.

Thus we are led to consider a manifold M, equipped with a Lie algebra L of vector fields, such that the evaluation map $M \times L \to TM$ is an isomorphism. There are two groups associated with this structure: G_L, which is the group of diffeomorphisms fixing the Lie algebra and the group Γ_L generated by the Lie algebra. In order to make sense out of Γ_L, one needs L to consist of complete vector fields. This condition is guaranteed by the *transitivity* of G_L. Under this condition, it follows that both Γ_L and G_L act transitively and freely on M. An important point is that until now, we do not need to care about topologies on these groups; the setup separates the manifold structure and the group structure. It requires an argument to equip both G_L and Γ_L with Lie group structures; they are mutually isomorphic and their Lie algebra is L. We call this structure an "integral manifold" for $\mathfrak{g}$ if $\mathfrak{g} \cong L$.

Vector fields are good for geometric constructions, but it is much easier to compute with their duals - differential forms. Therefore we then proceed to a dual formulation of the above results, involving the Chevalley-Eilenberg-complex.

The actual proof of Lie's third theorem is by induction on the dimension of the center. If the center is trivial, the Lie algebra is linear and hence integrable. An easy algebraic argument shows that it is enough to prove that a central extension $\mathfrak{u} \to \mathfrak{g} \to \mathfrak{h}$ of an integrable Lie algebra $\mathfrak{h}$ is integrable. Starting from an integral manifold for $\mathfrak{h}$, we construct a suitable Lie algebra of vector fields on $M \times \mathbb{R}$. Here we meet the crucial condition that a simply connected finite-dimensional Lie group has trivial second cohomology - this is completely false for infinite-dimensional Lie groups. The proof of the transitivity of the symmetry group is not difficult - this is the reason why the transitivity is emphasized in the basic construction theorem.

2. What is a Lie group?

Let us first fix conventions and notation. If M is a smooth manifold, $\mathcal{V}(M)$ denotes the Lie algebra of vector fields on M; that is, all linear maps $X : C^\infty(M) \to C^\infty(M)$ with $X(ab) = (Xa)b + aXb$. The Lie bracket is the commutator $[X,Y] := XY - YX$. We can identify $\mathcal{V}(M)$ with the space of smooth sections of the tangent bundle $TM \to M$. The value of X at $x \in M$ is denoted X_x. Let G be a Lie group (which we will assume to be connected throughout). Left-multiplication by $g \in G$ is a diffeomorphism $L_g : G \to G$; likewise, right-multiplication is denoted R_g. The subspace $\mathcal{V}(G)^G \subset \mathcal{V}(G)$ of left-invariant vector-fields is a Lie subalgebra. Evaluation at $1 \in G$ gives a linear map $\mathcal{V}(G)^G \to T_1 G$; and this is an isomorphism. More generally, we could take any other element instead of 1. By definition, $\mathrm{Lie}(G) := \mathcal{V}(G)^G$ is the Lie algebra of G; often denoted by $\mathfrak{g}$. Our investigation begins with an axiomatization of the structure that we found. At the beginning of the argument, the Lie algebra structure on L does not play a role.

Definition 1. *Let M be a smooth manifold. A* framing *of M is a subspace $L \subset \mathcal{V}(M)$ such that the evaluation map $M \times L \to TM$; $(x, X) \mapsto X_x$ is a vector bundle isomorphism. The* symmetry group *of the framing is the group of all diffeomorphisms $f : M \to M$ with $f_* X = X$ for all $X \in L$. The framing is* transitive *if G_L acts transitively on M.*

Note that the expression $f_*(X)$ is only defined when f is a diffeomorphism; it is $(f_*(X))(a) := X(a \circ f^{-1})$. We have seen an example of a transitive framing: if G is a Lie group and L the Lie algebra of left-invariant vector fields on G, then L is a framing of G. By definition, the left translation $L_g(x) := gx$ by an element of G is in G_L (the notational coincidence is deliberate). We will soon learn that these are all elements of G_L. By definition, this framing is transitive. It is important to note that G_L is an abstract group, acting by diffeomorphisms on M (no topology is specified on G_L - not yet).

A vector field X on a manifold M has a schizophrenic double nature. First, it serves as a derivation $C^\infty(M) \to C^\infty(M)$. On the other hand, it generates a 1-parameter group $t \to \phi^{tX}$ of diffeomorphisms of M. The relation between both aspects is given by

$$\frac{d}{dt}|_{t=0}(a \circ \phi^{tX}(x)) = (Xa)(x)$$

when $x \in M$ and $a \in C^\infty(M)$. Of course, this is not quite true: the 1-parameter group is defined if and only if the vector field is *complete* in the sense that the flow lines exist for arbitrary times t.

Definition 2. *A framed manifold $(M; L)$ is* complete *if the elements of L are complete vector fields.*

Definition 3. *Let $(M; L)$ be a complete framing. Let Γ_L be the group generated by all diffeomorphisms ϕ^X, $X \in L$. It is called the group* generated by L.

Again, no topology on Γ_L is specified yet. It acts by diffeomorphisms on M, this time not by definition, but by the smooth dependence of solutions of ODEs on initial values.

Theorem 2. *Let $(M; L)$ be a framed manifold.*

1. *If $(M; L)$ is transitive, then it is is complete.*
2. *If $(M; L)$ is complete and M is connected, then Γ_L acts transitively on M and the actions of G_L and Γ_L commute.*
3. *If $(M; L)$ is transitive, then both groups G_L and Γ_L act transitively and freely on M.*

Proof. Ad 1.) Let $X \in L$ and $x_0 \in M$. Consider an integral curve $c : (-2\epsilon, 2\epsilon) \to M$ to X through x_0. Pick $g \in G_L$ with $gx_0 = c(\epsilon)$. The curve gc is a curve through $c(\epsilon)$. Because the action of G preserves X, gc is an integral curve to X through $c(\epsilon)$. By this process, we have extended c to the interval $(-2\epsilon, 3\epsilon)$. Proceeding in this manner, also to the negative side, we extend it to all of $\mathbb{R}$.

Ad 2.) That the actions of Γ_L and G_L commute should be clear by now (the infinitesimal generators of Γ_L are invariant under G_L!). To prove transitivity of the Γ-action, we prove that for a given $x \in M$, the orbit $\Gamma_L x$ contains an open neighborhood U of x. Thus the orbits are open in M; by formal nonsense they are also closed in M. Since M is connected, the orbit Γx exhaust M, thus the action is transitive. To prove existence of U, let

$$\Phi : \mathbb{R} \times M \times L \to M;\ \Phi(t, x, X) := \phi^{tX}(x) \tag{1}$$

be the map that associates to (t, x, X) the point on the integral curve to X through x at the time t. By the smooth dependance of solutions of ODEs on parameters and initial values, Φ is smooth. We claim that $\Phi|_{1\times x\times L} : L \to M$ is regular at 0. To see this, note that $\Phi(t, x, X) = \Phi(1, x, tX)$ and thus

$$\frac{d}{dt}|_{t=0}\Phi(1,x,tX) = \frac{d}{dt}|_{t=0}\Phi(t,x,X) = X(x).$$

As $L \to T_xM$; $X \mapsto X(x)$ is an isomorphism, regularity of $\Phi|_{1\times x\times L}$ at 0 follows. By the inverse function theorem, the image of $\Phi|_{1\times x\times L}$ contains a small open neighborhood of x; and since $\Phi(\mathbb{R}\times x\times L) \subset \Gamma_L x$ by definition, we are done with the second statement.

Ad 3.) This follows from the first two ones and the (trivial) Lemma below. □

Lemma 1. *Let X be a set and G_0, G_1 two groups that act faithfully on X, such that the actions commute. If one action is transitive, the other one is free.*

Proof. Write the G_0-action from the left, and the G_1-action from the right and let G_1 be transitive. Let $gx = x$ and let $y \in X$. Pick h with $xh = y$. Then $gy = g(xh) = (gx)h = xh = y$. So each $g \in G_0$ with a fixed point acts trivially. □

The previous theorem applies to the framing of a Lie group, because the left action is transitive. Another trivial lemma shows what the group Γ_L is.

Lemma 2. *Let G be a group and Γ be a group acting transitively from the right on the set G. If Γ commutes with $\{L_g | g \in G\}$, then $\Gamma = \{R_g | g \in G\}$.*

Proof. If $f \in \Gamma$ and $g \in G$, we find that $f(g) = f(L_g(1)) = L_g f(1) = R_{f(1)}(g)$, so f is a right translation, and the transitivity implies the claim □

So if we start from a Lie group G, we find that $G_L == \{L_g | g \in G\}$ and $\Gamma_L = \{R_g | g \in G\}$. To finish the geometric part of the proof of Lie's third theorem, we finally show how to recover the Lie group structure from the situation in Theorem 2.

Theorem 3. *Let $(M; L)$ be a transitive framing of a connected manifold. There are unique Lie group structures on G_L and Γ_L such that the actions on M are smooth. A choice of a basepoint determines a diffeomorphism $(G_L, 1) \cong (M; x)$ and an isomorphism $G_L \cong \Gamma_L$. Under these isomorphisms, G_L acts by left-translations and Γ_L by right-translation. The space L is the space of left-invariant vector fields on G_L (in particular, L is a Lie algebra and it is the Lie algebra of G_L).*

Proof. Fix a basepoint $x \in M$. We continue to write the action of G_L from the left and that of Γ_L from the right. Define $A : G_L \to M$ by $A(g) = gx$; this is a bijective map. The relation

$$gx = xF(g)$$

defines a bijection $F : G_L \to \Gamma_L$ (because both actions are free and transitive), which is an isomorphism because

$$xF(gh) = ghx = g(hx) = gxF(g) = xF(h)F(g).$$

The diagram

$$\begin{array}{ccc} G_L \times M \times \Gamma_L & \longrightarrow & M \\ {\scriptstyle 1\times A\times F}\uparrow & & \uparrow{\scriptstyle A} \\ G_L \times G_L \times G_L & \longrightarrow & G_L \end{array}$$

is commutative (the horizontal arrows are the group actions). Now we have to put a smooth structure on G_L; this is done by requiring that A is a diffeomorphism (there is no other choice if the G_L-action on M should be smooth). Moreover, we define the smooth manifold structure on Γ_L by saying that F is a diffeomorphism. It follows that the bijection $B : \Gamma_L \to M$, $h \mapsto xh$ is a diffeomorphism as well, since $B(h) = xh = F^{-1}(h)x = A \circ F^{-1}(h)$.

More or less by definition, the groups G_L and Γ_L act by diffeomorphisms on M. This translates into the statement that left- and right-translations by fixed group elements are diffeomorphisms $G_L \to G_L$. This is not yet enough to prove that G_L is a Lie group, but it reduces the task to proving that there is a neighborhood $1 \in U \subset G_L$ such that the multiplication $\mu : U \times U \to G_L$ is smooth, by the following argument. For $(g, h) \in G_L \times G_L$, the diagram

$$\begin{array}{ccc} gU \times Uh & \xrightarrow{\mu} & G_L \\ \uparrow{\scriptstyle L_g \times R_h} & & \uparrow{\scriptstyle L_g \circ R_h} \\ U \times U & \xrightarrow{\mu} & G_L \end{array}$$

commutes, the vertical arrows are diffeomorphisms and if the the bottom map is smooth, so is the top map. If $G_L \times G_L \to G_L$ is smooth, the smoothness of the inversion follows from the implicit function theorem.

So far we have given the groups G_L and Γ_L smooth manifold structures and a smooth isomorphism $G_L \cong \Gamma_L$. The proof that the multiplication is locally smooth near the origin can therefore be carried out on Γ_L. Recall that Γ_L is generated by the Lie algebra L. In the proof of Theorem 2, we found a map $L \to M$ and a neighborhood V of $0 \in L$ such that V maps diffeomorphically onto an open $U \subset M$. Since $\Gamma_L \to M$ is - by definition - a diffeomorphism, we have a map $e : L \to \Gamma_L$ that is a local diffeomorphism in a neighborhood of 0 (of course, this will be the exponential map of the Lie group G_L) and we redefine U to be $e(V)$ for a suitably small $V \subset L$. The composition

$$V \times V \stackrel{e \times e}{\to} U \times U \to \Gamma_L \cong M$$

is given by

$$(X, Y) \mapsto \phi^Y \phi^X (x)$$

and again the smooth dependence of solutions of ODE on parameters and initial values proves that this map is smooth. Therefore, multiplication in Γ is smooth around the origin, and by the above arguments, the proof that Γ_L and G_L are Lie groups is complete.

Finally, L is a space of G-invariant vector fields; thus $L \subset \mathrm{Lie}(G_L)$, and equality follows by dimension reasons. □

The nice feature is that we really need both groups to accomplish the goal. Also, the two roles of vector field are related by a certain "duality": L is the Lie algebra of G-invariant vector fields, and it generates Γ. We conclude this section by the answer to the question posed in its title.

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra and M a smooth manifold. A* Maurer-Cartan-$\mathfrak{g}$-structure *on M is a framing L of M, such that L is closed under Lie brackets* *and $L \cong \mathfrak{g}$ as Lie algebras. A Maurer-Cartan-structure is* transitive/complete *if the framing L is transitive/complete.*

What is a Lie group?. *A connected Lie group with Lie algebra $\mathfrak{g}$ is a transitive Maurer-Cartan structure $(M; L)$ with connected M, together with an isomorphism of Lie algebras $L \cong \mathfrak{g}$.*

3. An algebraic reformulation

More notation: If M is a smooth manifold, its de Rham algebra is $\mathcal{A}^*(M)$. For a real vector space V, $V^\vee$ will denote its dual space.

Definition 5. *Let $(M; L)$ be a manifold with a framing. A p-form $\omega \in \mathcal{A}^p(M)$ is* constant *if for all $X_1, \ldots, X_p \in L$, the function $\omega(X_1, \ldots, X_p)$ is a constant function on M.*

By the formula for exterior products, the space of constant forms is closed under wedge products. Moreover, the algebra of constant forms is the same as $\Lambda^*(L^\vee)$.

Lemma 3. *A framing is a Maurer-Cartan structure (i.e., $[L, L] \subset L$) if and only if $d(\Lambda^* L^\vee) \subset \Lambda^* L^\vee$.*

Proof. Recall the following formula for the exterior derivative:

$$dw(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i X_i(w(X_0, \ldots, \hat{X}_i, \ldots, X_p)) + \sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

for each p-form w and vector fields X_i on a manifold. The first sum is zero if $X_i \in L$ and w is constant. Thus if L is a Lie algebra, the above expression is a constant function for all $X_i \in L$ and constant forms w. Vice versa, assume that for each constant form w, dw is again constant. If w is a constant 1-form, we find that

$$dw(X_0, X_1) = w([X_0, X_1])$$

for all $X_i \in L$. Thus $[X_0, X_1]$ is constant. □

The formula in the above proof suggests the following definition:

Definition 6. *Let $\mathfrak{g}$ be a Lie algebra. The* Chevalley-Eilenberg complex *of $\mathfrak{g}$ is $C^*_{CE}(\mathfrak{g}) := \Lambda^* \mathfrak{g}^\vee$, together with $d : \Lambda^p(\mathfrak{g}^\vee) \to \Lambda^{p+1}(\mathfrak{g}^\vee)$ defined by*

$$dw(X_0, \ldots, X_p) = \sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

This definition is due to Chevalley and Eilenberg [1]. It remains to verify that $C^*_{CE}(\mathfrak{g})$ is a d.g.a. (differential graded algebra). First, one verifies that $d(w \wedge v) = (dw) \wedge v + (-1)^{|w|} w \wedge dv$ for all v, w, by a straightforward but slightly tedious calculation. Therefore, it is enough to verify the propery $d^2 w = 0$ when w is a 0-form (this is clear) or a 1-form. But the equation $ddw = 0$ for a 1-form is an immediate consequence of the Jacobi identity.

Definition 7. *Let $\mathfrak{g}$ be a Lie algebra. An* algebraic Maurer-Cartan $\mathfrak{g}$-structure *on a manifold M is a d.g.a. homomorphism $\Theta : C^*_{CE}(\mathfrak{g}) \to \mathcal{A}^*(M)$ such that the evaluation map $M \times \mathfrak{g}^\vee \to T^\vee M$, $(x, w) \mapsto w_x$ is a vector bundle isomorphism. The symmetry group is the group of all diffeomorphisms f with $f^* \circ \Theta = \Theta$.*

We summarize what we have proven so far.

Theorem 4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is integrable if and only if there exists a connected manifold M, together with an algebraic Maurer-Cartan structure $\Theta : C^*_{CE}(\mathfrak{g}) \to \mathcal{A}^*(M)$ that has a transitive symmetry group. The pair $(M; \Theta)$ is called an* integral manifold *for $\mathfrak{g}$.*

Finally, we reformulate the result in more concrete terms in order to simplify the calculations. To that end, choose a basis $\mathcal{B} = (X_1, \ldots, X_n)$ of $\mathfrak{g}$ and let $(w^1, \ldots, w^n)$ be the dual basis of $\mathfrak{g}^\vee$. The *structure constants* of $\mathfrak{g}$ with respect to $\mathcal{B}$ are given by

$$[X_i, X_j] = c^k_{ij} X_k, \tag{2}$$

using the Einstein summation convention. The axioms for a Lie algebra translate into the equations

$$c^k_{ij} = -c^k_{ji}; \;\; c^l_{jk} c^m_{il} + c^l_{ki} c^m_{jl} + c^l_{ij} c^m_{kl} = 0.$$

One quickly verifies that the differential in $C^*_{CE}(\mathfrak{g})$ has the form

$$dw^l = -c^l_{ij} w^i \wedge w^j. \tag{3}$$

Lemma 4. *To give a d.g.a. morphism $\Theta : C^*_{CE}(\mathfrak{g}) \to \mathcal{A}^*(M)$, it is sufficient and necessary to give 1-forms $\theta^1, \ldots, \theta^n \in \mathcal{A}^1(M)$ such that the structure equation $dw^l = -c^l_{ij} w^i \wedge w^j$ holds.*

This is obvious.

4. Proof of Lie's third theorem, main part

We now prove Lie's theorem. Along the proof, we will meet an obstruction whose vanishing will be shown afterwards. According to our integrability theorem 4, we have to find a certain d.g.a. map $C^*_{CE}(\mathfrak{g}) \to \mathcal{A}^*(M)$. The proof will be by induction on the dimension of the center and it begins with:

Proposition 1. *A Lie algebra with trivial center is integrable.*

Proof. By definition, the center of $\mathfrak{g}$ is the kernel of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, so under our assumption, the adjoint representation is injective and $\mathfrak{g}$ is linear. But any subalgebra of $\mathfrak{gl}(V)$ is integrable. □

Now let $\mathfrak{g}$ be a general Lie algebra, $\mathfrak{z} \subset \mathfrak{g}$ its center. If $\dim(\mathfrak{z}) \geq 1$, take a one-dimensional subspace $\mathfrak{u} \subset \mathfrak{z}$. Being in the center, $\mathfrak{u}$ is an ideal and $\mathfrak{h} := \mathfrak{g}/\mathfrak{u}$ is a Lie algebra whose center is $\mathfrak{z}/\mathfrak{u}$ (look at the adjoint representations) and $\dim(\mathfrak{z}/\mathfrak{u}) < \dim(\mathfrak{z})$. Thus the proof of Lie's theorem will be accomplished by the following result.

Theorem 5. *Let $0 \to \mathfrak{u} \subset \mathfrak{g} \xrightarrow{p} \mathfrak{h} \to 0$ be a short exact sequence of Lie algebras, $\mathfrak{u}$ one-dimensional and central in $\mathfrak{g}$. If $\mathfrak{h}$ is integrable, then so is $\mathfrak{g}$.*

Proof. Let $(M; \Theta_{\mathfrak{h}})$ be an integral manifold for $\mathfrak{h}$. We will construct a $\mathfrak{g}$-Maurer-Cartan-structure on $M \times \mathbb{R}$.

Let $0 \neq X_1 \in \mathfrak{u}$; let $(w^2, \ldots, w^n)$ be a basis of $\mathfrak{h}^\vee$. Let $v_i := p^* w_i$. Pick v_1 with $v_1(X_1) = 1$. Then $(v^1, \ldots, v^n)$ is a basis of $\mathfrak{g}^\vee$, and X_1 is the first element of the dual basis $(X_1, \ldots, X_n)$ of $\mathfrak{g}$. The associated structure constants are denoted c_{ij}^k. The condition that X_1 is in the center of $\mathfrak{g}$ translates into the fact that

$$c_{1i}^k = c_{i1}^k = 0 \quad \text{for all } i, k. \tag{4}$$

Therefore

$$dv^1 = -c_{jk}^1 v^j \wedge v^k$$

is a linear combination of wedge products $v^j \wedge v^k$ for $k, l \geq 2$. Since p^* is injective, there is a unique $\alpha^{'} \in \Lambda^2(\mathfrak{h}^\vee)$ with $p^*\alpha' = dv^1$. Moreover, $d\alpha' = 0$. Let $\alpha := \Theta_{\mathfrak{h}}(\alpha^{'}) \in \mathcal{A}^2(M)$; a closed 2-form. We now have to make a crucial assumption, that will be justified at the end of the proof.

Assumption 1. *The second de Rham cohomology of M is trivial; $H_{dR}^2(M) = 0$.*

By this assumption, we find a 1-form $\tau^1 \in \mathcal{A}^1(M)$ with $d\tau^1 = \alpha$. Moreover, put $\tau^i := \Theta_{\mathfrak{h}}(w^i)$, $i \geq 2$. Let $\pi : M \times \mathbb{R} \to M$ be the projection. Let $\theta^1 := \pi^*\tau^1 + dt$ and $\theta^i := \pi^*\tau^i$ for $i \geq 2$.

Claim 1. *The linear map $\mathfrak{g}^\vee \to \mathcal{A}^*(M \times \mathbb{R})$ given by $v^i \mapsto \theta^i$ extends to an algebraic Maurer-Cartan structure.*

The isomorphism property is clear (because the original $\Theta_{\mathfrak{h}}$ satisfied this condition and because we added the summand dt to the first form. We have to verify the structure equation. We already observed that the centrality of $\mathfrak{u}$ implies that $c_{i1}^k = c_{1i}^k = 0$, which means that the old structure equation in $\mathfrak{h}$ translates to the structure equation in $\mathfrak{g}$ for the forms θ^i, $i \geq 2$ ($d\theta^i$ does not involve θ^1!). Moreover

$$d\theta^1 = \pi^* d\tau^1 = \Theta_{\mathfrak{h}}(\alpha^{'}) = \Theta_{\mathfrak{h}}(-c_{jk}^1 w^j \wedge w^k) = -c_{jk}^1 \theta^j \wedge \theta^k,$$

which verifies the structure equation. According to the integrability theorem, we now have to prove that the automorphism group of this Maurer-Cartan structure acts transitively on $M \times \mathbb{R}$. For a given $y \in \mathbb{R}$, the map

$$(m, x) \mapsto (m, x + y)$$

is clearly an automorphism, since the form $dt \in \mathcal{A}^1(\mathbb{R})$ is translation-invariant. So the automorphism group acts transitively "in the $\mathbb{R}$-direction". Let H be the automorphism group of $(M; \Theta_{\mathfrak{h}})$. We want to lift each $h \in H$ to $\phi_h : M \times \mathbb{R}$ and make the following:

Ansatz 1. *For given $h \in H$, find a function $f : M \to \mathbb{R}$ such that $\phi_{h,f}(m, x) := (hm, x + f(m))$ preserves the differential system.*

Clearly $\pi \circ \phi_{h,f} = h \circ \pi$ and therefore $\phi_{h,f}^* \theta^i = \theta^i$ for $i \geq 2$ (this holds for arbitrary f). Moreover

$$\phi_{h,f}^* dt = \pi^* df + dt.$$

Thus we seek a function f with

$$\phi_{h,f}^* \pi^* \tau^1 = -\pi^* df + \pi^* \tau^1.$$

Because

$$\phi_{h,f}^* \pi^* \tau^1 = \pi^* h^* \tau^1$$

and because π^* is injective, we have to solve the equation $-df + \tau^1 = h^*\tau^1$ or

$$df = \tau^1 - h^*\tau^1. \tag{5}$$

If τ^1 were closed, the existence of such an f follows from the homotopy-invariance of the de Rham cohomology. However, τ^1 is closed only if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$. On the other hand, $d\tau^1$ is in the image of $\Theta_{\mathfrak{h}}$ and therefore invariant under H: $h^* d\tau^1 = d\tau^1$. Thus $d(\tau^1 - h^*\tau^1) = 0$, and we find an f solving 5 provided that $H^1(M) = 0$. This holds if $\pi_1(M) = 1$. Because M is a Lie group and we know that universal coverings of Lie groups are again Lie groups (with canonically isomorphic Lie algebra), this assumption can be made without loss of generality.

This finishes the proof of Lie's third theorem, except that we have to justify the assumption 1. It is a classical result, due to Hopf, that a simply connected Lie group G satisfies $H^2_{dR}(G)) = 0$ (even the stronger fact $\pi_2(G) = 0$ is true). An algebraic topology textbook that covers this result is [3], p. 285, with the little caveat that the proof given there assumes finite-dimensionality of $H^*_{dR}(G)$. This is true, as any Lie group is homotopy equivalent to a maximal compact subgroup, but this latter result is not easy to prove. A proof which takes care of this issue can be found in [2], Theorem 1.14.2. □

5. The failure in infinite dimensions

The following example of a nonintegrable Lie algebra is due to Serre [5]. First observe that Lie's second theorem holds for Banach Lie groups.

Proposition 1. *If $\mathfrak{g}$ is the Lie algebra of the simply-connected Banach Lie group G, $\mathfrak{h} \subset \mathfrak{g}$ a closed ideal and $H \subset G$ the subgroup generated by $\mathfrak{h}$. If $\mathfrak{g}/\mathfrak{h}$ is integrable, then $H \subset G$ is closed.*

Proof. Let K be a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{h}$. By Lie's second theorem, one finds a (continuous) homomorphism $G \to K$ that induces $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ on Lie algebras. The kernel is a closed subgroup and it must be the group H. □

Now let V be a separable Hilbert space of infinite dimension. The group $\mathrm{GL}(V)$ of all continuous linear automorphisms is a Banach Lie group. Now inside $\mathrm{GL}(V) \times \mathrm{GL}(V)$, we find a central subgroup C isomorphic to $\S^1 \times S^1$. Let $H \subset C$ be a dense wind; this is a central connected subgroup in $\mathrm{GL}(V) \times \mathrm{GL}(V)$ that is not closed. Its Lie algebra is an ideal $\mathfrak{h} \subset \mathfrak{gl}(V) \oplus \mathfrak{gl}(V)$. This contradicts the above proposition; therefore $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V))/\mathfrak{h}$ is not integrable.

The role of the condition $H^2(G) = 0$ also becomes transparent in this example. The quotient $G := (\mathrm{GL}(V) \times \mathrm{GL}(V))/C$ is a Lie group. By Kuipers theorem, $\mathrm{GL}(V) \times \mathrm{GL}(V)$ is contractible, and this implies that $\pi_1(G) = 1$ and $H^2(G) \neq 0$. But there is a central extension of Lie algebras

$$\mathbb{R} \to (\mathfrak{gl}(V) \oplus \mathfrak{gl}(V))/\mathfrak{c} \to \mathfrak{g}.$$

Even though $\mathfrak{g}$ is integrable, the middle is not, because $H^2(G)$ fails to vanish.

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